Technical Note

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Modeling of Flexible Multi-Body Systems: A Tutorial Overview

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Abstract

This review paper explores two primary methodologies for modeling flexible multi-body systems, namely, the Assumed Mode Method and the Lumped Parameter Method. Flexible multi-body systems models, which involve small elastic deformations, are critical for simulating and optimizing the behavior of structures in various engineering applications, such as robotic arms, space structures, and rotating machinery. The Assumed Mode Method decomposes a system's motion into rigid-body movements and elastic deformations using predefined mode shapes, providing an efficient alternative to the Finite Element Method. The Lumped Parameter Method simplifies flexible systems by modeling them as rigid segments connected by springs and dampers, which capture elasticity and damping effects. This review focuses on the basic use, implementation of these two approaches. Additionally, a carne with flexible pendulum load is modeled using both approaches to demonstrate their effectiveness in capturing system dynamics.

Keywords: flexible multibody system; assumed mode method; lumped parameter method

1. Introduction

Multibody dynamics, with a history rooted in analytical and continuum mechanics, computer science, and applied mathematics (Shabana, 1997; Rahnejat, 2000; Nobuyuki SHIMIZO, 2003; Schiehlen, 2005; Jobs, 2016); serves as the cornerstone for analyzing and virtually prototyping innovative applications across a wide range of modern engineering fields. Equations of motion are typically derived using a combination of D'Alembert, Newton, Euler, Hamilton or Lagrange principles. Researchers have long sought efficient automated procedures for obtaining and solving these governing equations (Margolis and Karnopp, 1978; Kamman and Huston, 1984; Rosenthal and Sherman, 1986; Jain, 1991; Yen, Masada and Chan, 1993; Karnopp, 1997; Lennartsson, 1999; Suleman, 2004; Behzadipour and Khajepour, 2006; Schiehlen, 2006; Müller, 2018; Honein and O'Reilly, 2021; Sarkar and Fitzgerald, 2021; Park, 2021; Yap *et al.*, 2022).

The importance of multibody dynamics continues to grow as industries demand more accurate models and simulations. Whether for mechanical or biomechanical systems, the analysis, design, and control of interconnected bodies require advanced techniques to optimize performance. Despite the advancements, it remains common to simplify simulations by assuming bodies are rigid, which makes multibody analysis a quick and effective tool for testing and optimization. However, in many real-world scenarios, deformations must be considered, particularly in systems involving compliant materials or dynamic loads. In response, flexible multibody dynamics has been extensively studied, with applications in lightweight robotic arms (Tokhi and Azad, 1996; Kim *et al.*, 1996; Matsuno and Kasai, 1998; Feliu, Feliu and Cerrada, 1999; Liu and Yuan, 2003; KIM and UCHIYAMA, 2003; Dwivedy and Eberhard, 2006; Korayem, Nikoobin and Azimirad, 2009; Dupac and Noroozi, 2014; Cambera and Feliu-Batlle, 2017; Giorgio and Del Vescovo, 2018a, 2019; Pucher, Gattringer and Müller, 2019; Banerjee *et al.*, 2021; Tang and Zhao, 2022; Yue *et al.*, 2024), soft robotics(Novelia, 2018; Santina and Rus, 2020; Schiller, Seibel and Schlattmann, 2020; Troise *et al.*, 2021), rotating machinery

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(Bauchau, Bottasso and Nikishkov, 2001; Hu, Mourelatos and Vlahopoulos, 2003; Tadeo and Cavalca, 2003; Milind and Mitra, 2016; Pappalardo and Guida, 2018; Jie Wang, Zhiqiang Huang, Tao Li, Gang Li, 2024), helicopter rotor with flexible blades (Goulos, Pachidis and Pilidis, 2015; Ozturan, 2019), large flexible space structures(Santiago, Lange and Jamshidi, 1985; Zhang *et al.*, 2024).

Modeling flexible multibody systems is often achieved through discretized methods like finite element methods (FEM), assumed mode methods (AMM), or lumped parameter models Sometimes known as a finite segment approach . The model of the whole multibody system is then derived using the Lagrange formalism or any other approach. This paper focuses on two approaches: the Lumped-Parameter method and the Assumed Mode method. The Lumped-Parameter method, where a body is divided into rigid segments connected by pseudo joints of springs and dampers, offers a balance of simplicity, computational efficiency, and accuracy (J. D. Connelly, 1994; Ge, Lee and Zhu, 1997; Seo *et al.*, 2005; Sun *et al.*, 2009; Miller *et al.*, 2017; Giorgio and Del Vescovo, 2018b; Nchekwube, 2019; Wanner and Sawodny, 2019; Subedi, Ilya Tyapin and Hovland, 2020; Yap *et al.*, 2022);). However, determining appropriate spring and damper constants remains a challenge.

The Assumed Mode method (Green and Sasiadek, 2003; Pan and Liu, 2012; Sharifnia and Akbarzadeh, 2017; Fan, Zhang and Shen, 2020; Subedi, Tyapin and Hovland, 2021; Mishra and Singh, 2022), is conceptually similar (AMM typically assumes a linear combination of pre-defined modes and focuses on capturing small deformations. FFRF, on the other hand, explicitly separates large rigid-body motion from small elastic deformation and handles the combination of both in a more structured manner) to the Floating Frame of Reference formulation (Shabana and Schwertassek, 1998). Here, motion is decomposed into the nonlinear motion of a reference frame and linear elastic deformation relative to that frame. Both approaches enable realistic simulations of deformations without the computational burden of a full FEM analysis.

2. Assumed Mode Method

The Assumed Mode Method (AMM) is commonly used to model flexible bodies that undergo large overall rigid-body motions while experiencing small elastic deformations. In this method, the flexibility of the links is represented by a combination of spatial mode shapes and time-varying generalized coordinates. Typically, the modal series is truncated to a finite number of modes, as the first few low-frequency modes primarily influence the dynamics and overall motion of the links. Choosing appropriate boundary conditions is crucial when applying the AMM to ensure accurate modelling.

2.1. Modeling a Flexible Pendulum on a Cart Using Assumed Mode

Figure 1 shows a flexible pendulum on a cart. There are two coordinate frames, the global fixed frame (X_0, Y_0) and the local moving frame ($\hat{x}$, $\hat{y}$) attached to the pendulum. The x-axis of the moving frame is aligned along the undeformed neutral axis of the pendulum. The lateral deflection of any point P on the flexible pendulum as measured on the global frame is given by

$$\vec{P} = \begin{Bmatrix} x_c + x\sin(\theta) + w\cos(\theta) \\ -x\cos(\theta) + w\sin(\theta) \end{Bmatrix} \quad (1)$$

where, x is the projection of arc OP measured along the moving $\hat{x}$-axis, $w(x,t) = \sum N_i(x) q_{fi}(t)$ N_i is the i^{th} mode shape and $q_{fi}(t)$ is the time dependent coordinate associated with the i^{th} mode.

The velocity of the point P with respect to the global frame is

$$\dot{\vec{P}} = \begin{Bmatrix} \dot{x}_c + x\dot{\theta}\cos(\theta) + \dot{w}\cos(\theta) - w\dot{\theta}\sin(\theta) \\ x\,\dot{\theta}\sin(\theta) + \dot{w}\sin(\theta) + w\dot{\theta}\cos(\theta) \end{Bmatrix} \quad (2)$$

The kinetic energy of the flexible pendulum on the crane is

$$K.E. = \frac{1}{2} M_c \left\|\dot{\vec{x}}_c\right\|^2 + \frac{1}{2}\rho A \int_0^L \dot{\vec{P}}^T . \dot{\vec{P}} dx$$

where M_c is the mass of the crane, ρ is the density of the pendulum, A is the cross-sectional area of the pendulum and L is the length of the pendulum.

$$\left\|\vec{P}\right\|^2 = \dot{x}_C^2 + x^2\dot{\theta}^2 + \dot{w}^2 + w^2\dot{\theta}^2 + 2\dot{x}_c x cos(\theta)\dot{\theta} + 2\dot{x}_c\dot{w}cos(\theta) - 2\dot{x}_c w sin(\theta)\dot{\theta} + 2x\dot{w}\dot{\theta} \quad (3)$$

The potential energy of the system is obtained by adding the gravitational potential energy to the strain energy as follows

$$P.E. = \frac{1}{2}EI\int_0^L\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \rho A\int_0^L[0 \quad g].\vec{P}\,dx \quad (4)$$

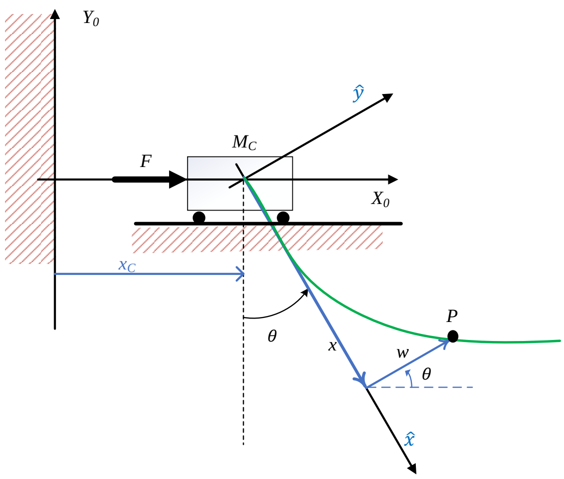

Figure 1. Modeling of pendulum on a cart using Assumed Mode Method.

In this case the Lagrangian is as follows

$$L = \frac{1}{2}M_C\dot{x}_c{}^2 + \frac{1}{2}\rho A\int_0^L\{\dot{x}_C^2 + x^2\dot{\theta}^2 + \dot{w}^2 + w^2\dot{\theta}^2 + 2\dot{x}_c x cos(\theta)\dot{\theta} + 2\dot{x}_c\dot{w}cos(\theta) - 2\dot{x}_c w sin(\theta)\dot{\theta} + 2x\dot{w}\dot{\theta}\}dx - \frac{1}{2}EI\int_0^L\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx - \rho A\int_0^L[0 \quad g].\vec{P}\,dx \quad (5)$$

$$\begin{aligned} L = \frac{1}{2}\left(M_C + \int_0^L m dx\right)\dot{x}_c{}^2 \\ + \frac{1}{2}\int_0^L m\{x^2 dx\}\dot{\theta}^2 \\ + \frac{1}{2}\sum\int_0^L m\{N(x)_i^2 dx\}\dot{q}_{fi}^2 \\ + \frac{1}{2}\dot{\theta}^2\sum\int_0^L m\{N(x)_i^2 dx\}q_{fi}^2 + \int_0^L\{mxdx\}\dot{x}_c\dot{\theta}cos(\theta) \\ + \left\{\sum\int_0^L mN(x)_i\dot{q}_{fi}dx\right\}\dot{x}_c c(\theta) - \left\{\sum\int_0^L mN(x)_i q_{fi}dx\right\}\dot{x}_c s(\theta)\dot{\theta} \\ + \left\{\sum\int_0^L mxN(x)_i\dot{q}_{fi}dx\right\}\dot{\theta} - \frac{1}{2}EI\int_0^L\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \\ - \int_0^L\left\{mg\left(-x\cos(\theta) + \sum N(x)_i q_{fi}\,sin(\theta)\right)\right\}dx \end{aligned} \quad (6)$$

In which m is the mass per unit length, and $w(x,t) = \sum N_i(x)q_{fi}(t)$ as defined before.

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$$L = \frac{1}{2}(I_0)\dot{x}_c{}^2 + \frac{1}{2}I_2\dot{\theta}^2 + \frac{1}{2}\sum I_4\dot{q}_{fi}{}^2 + \frac{1}{2}\dot{\theta}^2\sum I_4\, q_{fi}^2 + +I_1\dot{x}_c\dot{\theta}cos(\theta)$$
$$+\ \dot{x}_c cos(\theta)\sum I_{3i}\dot{q}_{fi} - \left\{\sum I_{3i}q_{fi}\right\}\dot{x}_c sin(\theta)\dot{\theta} + \dot{\theta}\sum I_{5i}\dot{q}_{fi} - \frac{1}{2}I_6$$
$$- I_1 g cos(\theta) + \left\{\sum I_{3i}q_{fi}\right\}\sin(\theta) \quad (7)$$

where, now, $I_0 = M_C + \int_0^L m dx$, $I_1 = \int_0^L mx dx$, $I_2 = \int_0^L mx^2 dx$,
$I_3 = \int_0^L mN(x)dx$, $I_4 = \int_0^L mN(x)^2 dx$, $I_5 = \int_0^L mxN(x)dx$, $I_6 = EI\int_0^L \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$

The mode shapes of a cantilever beam can be found in any standard vibration textbook and can expressed as follows

$$N_i(x) = L(\cosh(\lambda_i x) - \cos(\lambda_i x) - k_i[\sinh(\lambda_i x) - \sin(\lambda_i x)]) \quad (8)$$

Where, $k_i = \frac{\cosh(\lambda_i L)+\cos(\lambda_i L)}{\sinh(\lambda_i L)+\sin(\lambda_i L)}$, and λ_i are the roots of the equation $1 + \cosh(\lambda_i L)\cos(\lambda_i L) = 0$, the mode shapes satisfy the following orthogonality conditions

$$\int_0^L mN_i(x).N_j(x)dx = mL^3\delta_{ij}$$

where $\delta_{ij} = 0\ for\ i \neq j$

$$\int_0^L EIN_i''(x).N_j''(x)dx = mL^3\omega_i^2\delta_{ij}$$

$$\int_0^L mxN_i(x)dx = \frac{2mL^3}{\lambda_i^2}$$

with frequencies ω_i related to λ_i the by $\omega_i = \sqrt{\frac{EI}{m}}\lambda_i^2$

Therefore,

$I_0 = M_C + \int_0^L m dx = M_c + m_p$, where $m_p = mL$

$$I_1 = \int_0^L mx dx = m\frac{x^2}{2}\bigg|_0^L = m\frac{L^2}{2}$$

$$I_2 = \int_0^L mx^2 dx = m\frac{x^3}{3}\bigg|_0^L = m\frac{L^3}{3}$$

$$I_3 = \int_0^L mN(x)dx, \text{ can be computed numerically}$$

$$I_4 = \int_0^L mN(x).N(x)dx = mL^3\delta_{ij}$$

$$I_5 = \int_0^L mxN(x)dx\ \text{, can be computed numerically}$$

$$I_6 = EI\int_0^L \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = mL^3\omega_i^2\delta_{ij}$$

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We can apply the Lagrange equation to derive the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \quad (9)$$

The generalized coordinates are

$$\vec{q} = [\theta \quad \vec{q}_f]$$

Where q_f stands for the flexible coordinates

The equations of motion will be as follows

$$I_0\ddot{x}_c - \dot{\theta}\left\{\sin(\theta)\,\overrightarrow{I_3}\overrightarrow{q_f} + \sin(\theta)\,I_1\dot{\theta} + \dot{\theta}\cos(\theta)\,\overrightarrow{I_3}\overrightarrow{q_f}\right\} + \ddot{\theta}\left\{\cos(\theta)\,I_1 - \sin(\theta)\,\overrightarrow{I_3}\overrightarrow{q_f}\right\} + \cos(\theta)\,\overrightarrow{I_3}\overrightarrow{\ddot{q}_f} - \dot{\theta}\sin(\theta)\,\overrightarrow{I_3}\overrightarrow{\dot{q}_f} = F$$

$$I_1\ddot{x}_c\cos(\theta) + I_1\,\mathrm{g}\sin(\theta) + \overrightarrow{I_5}\overrightarrow{\ddot{q}_f} + I_2\ddot{\theta} + \ddot{\theta}\overrightarrow{I_4}\overrightarrow{q_f} + \cos(\theta)g\,\overrightarrow{I_3}\overrightarrow{q_f} - \sin(\theta)\ddot{x}_c\,\overrightarrow{I_3}\overrightarrow{q_f} + 2I_4\,\dot{\theta}\,\overrightarrow{q_f}\overrightarrow{\dot{q}_f} = 0$$

$$-I_4\dot{\theta}^2 q_{f1} + I_4\ddot{q}_{f1} + I_{51}\ddot{\theta} + I_6 q_{f1} + I_{31}\cos(\theta)\,\ddot{x}_c + I_{31}g\sin(\theta) = 0$$

$$-I_4\dot{\theta}^2 q_{f2} + I_4\ddot{q}_{f2} + I_{52}\ddot{\theta} + I_6 q_{f2} + I_{32}\cos(\theta)\,\ddot{x}_c + I_{32}g\sin(\theta) = 0$$

$$-I_4\dot{\theta}^2 q_{f3} + I_4\ddot{q}_{f3} + I_{53}\ddot{\theta} + I_6 q_{f3} + I_{33}\cos(\theta)\,\ddot{x}_c + I_{33}g\sin(\theta) = 0 \quad (10)$$

3. Lumped Parameter Model

The lumped parameter or lumped segment approach is a simple approach for modeling the behavior of flexible bodies. This technique simplifies the complex problem of flexibility by dividing the body into discrete, rigid segments that are interconnected by joints. These joints are equipped with springs and dampers, which allow for deformation by providing the necessary degrees of freedom. The springs and dampers model the body's elasticity and damping characteristics, while the rigid segments simplify the calculation of inertial effects.

This approach is particularly advantageous in simulation environments that utilize multibody dynamics, as each segment can be treated as an independent rigid body, making the method straightforward to implement. However, this segmentation also increases the system's degrees of freedom. The stiffness and damping properties of the joints are derived from the physical properties of the material, and these parameters are incorporated into the equations of motion to accurately capture the flexible body's dynamic behavior.

A beam of length L_t, mass M_t and moment of inertia J_t can be represented as a series of lumped segments of mass m and moment of inertia J. The beam's elasticity and internal friction are represented by torsional/linear springs and dampers to approximate the beam's bending/shear deformation, respectively. For simplicity let us assume that we have a uniform beam of length L_t. Figure 2 shows a flexible beam with its equivalent Lumped-Parameter model.

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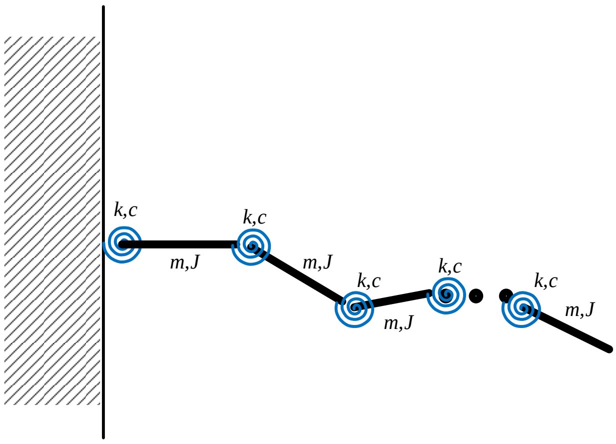

Figure 2. equivalent Lumped-Parameter model of a flexible beam.

The length of each segment is $L = {}^{L_t}/_{n_s}$, where L is the length of each segment assuming a uniform beam that is divide into segments of equal lengths The approach can easily handle nonuniform bodies with complex geometry and hence having segments with different parameters $(L,\ m,\ J,\ldots)$. The mass of each segment is $m = {}^{M_t}/_{n_s}$, the moment of inertia J, can be computed depending on the segment's geometry. Considering E and I as Young's modulus and second moment of area of each segment the equivalent stiffness k of each segment can be computed as follows, $k = \alpha \frac{EI}{L}$, where α can be computed using standard beam theory that relates the deflection to the assumed loading such that deflection is stiffness times loading. The damping coefficient c can be computed experimentally by matching the simulation results with the experimental results of a given system. In this paper we will assume that it is proportional to the stiffness and is related to the discretization level of the flexible body i.e. $c = \beta k$ (Miller *et al.*, 2017).

3.1. Flexible Pendulum on Cart

Now consider the cart and flexible pendulum system shown in Figure 3. The flexible pendulum of length L_t and has a mass of M_t is attached to a cart of mass M_c that moves horizontally by applying an external force to it. We will use the Lumped-Parameter method to derive the dynamics of the system.

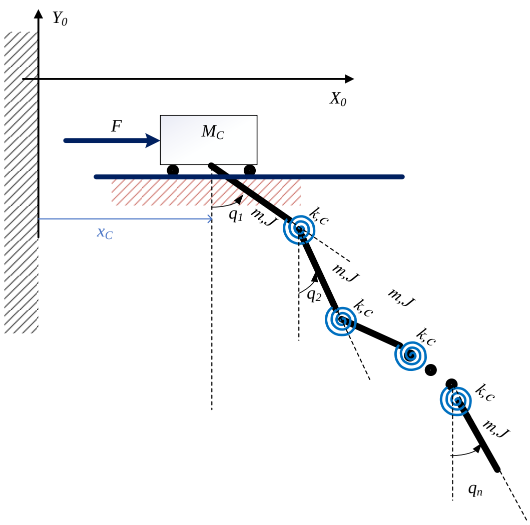

Figure 3. Modeling of cart and flexible pendulum system using Lumped-Parameter method.

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The aluminum pendulum can be thought as composed of three different connected rigid links of equal lengths so each segment i will have a length of

$$L = \frac{L_t}{n_s}$$

with a unit mass of

$$m = \frac{M_t}{n_s}$$

Then we calculate the moment of each segment of the pendulum as we did previously

$$J = \frac{mL^2}{12}$$

For this example, we will derive the equations of motion using Langrage formulation. Assuming that the origin of the system is at $P_0 = (0,0)$, the position of center of mass of the cart can be described by:

$$\vec{P}_{cart} = \vec{P}_0 + x_c\hat{\imath} \tag{11}$$

and the position of center of mass of first segment will be:

$$\vec{P}_{G1} = \vec{P}_{cart} + \frac{L}{2}(sin\, q_1\, \hat{\imath} - cos\, q_1\, \hat{\jmath}) \tag{12}$$

then the position of center of mass of the remaining segments are ($i = 2 \ldots n_s$):

$$\vec{P}_{Gi} = \vec{P}_{i-1} + \frac{L}{2}(sin\, q_i\, \hat{\imath} - cos\, q_i\, \hat{\jmath}) \tag{13}$$

Where,

$$\vec{P}_i = \vec{P}_{i-1} + L(sin\, q_i\, \hat{\imath} - cos\, q_i\, \hat{\jmath})$$

Hence, the linear velocity of each link can be derived, and kinetic energy and potential energy of the system can be expressed as follows:

$$KE = \frac{1}{2}M_{cart}\dot{x}_c^2 + \frac{1}{2}\sum_{i=1}^{n} J\dot{q}_i^2 + \frac{1}{2}\sum_{i=1}^{n} m\left\|\dot{\vec{P}}_{Gi}\right\|^2 \tag{14}$$

$$PE = \sum_{i=1}^{n} mgP_{yi} + \sum_{i=2}^{n}\frac{1}{2}k(q_i - q_{i-1})^2 \tag{15}$$

The Lagrange equation will then be:

$$\mathcal{L} = \frac{1}{2}M_{cart}\dot{x}_c^2 + \frac{1}{2}\sum_{i=1}^{n} Jq_i^2 + \frac{1}{2}\sum_{i=1}^{n} m\left\|\dot{\vec{P}}_{Gi}\right\|^2 - \sum_{i=1}^{n} mgP_{yi} - \sum_{i=2}^{n}\frac{1}{2}k(q_i - q_{i-1})^2 \tag{16}$$

The Rayleigh dissipation function for this system will be defined as follows and its derivative with respect to angular velocity of each segment is included as external force acting the system:

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$$R(t) = \sum_{i=2}^{n} \frac{1}{2} c\left(\dot{q}_i(t) - \dot{q}_{i-1}(t)\right)^2$$

(17)

To obtain the equations of motion, we will use Lagrange equation and apply it in the following formula:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = u - \frac{\partial R}{\partial \dot{q}}$$

(18)

Using the MATLAB code, as shown in the appendix, to perform the necessary calculations, the equations of motion that represent the dynamics of the system can be expressed as the follows:

$$M(\vec{\dot{q}})\vec{\ddot{q}} + D(\vec{q}, \vec{\dot{q}}) = \vec{u}$$

(19)

Mass matrix M(q) is:

$$M(q) = \begin{bmatrix} M_{cart} + 3m & \frac{5Lm\cos(q_1)}{2} & \frac{3Lm\cos(q_2)}{2} & \frac{Lm\cos(q_3)}{2} \\ \frac{5Lm\cos(q_1)}{2} & \frac{9mL^2}{4} + J & \frac{3L^2m\cos(q_1 - q_2)}{2} & \frac{L^2m\cos(q_1 - q_3)}{2} \\ \frac{3Lm\cos(q_2)}{2} & \frac{3L^2m\cos(q_1 - q_2)}{2} & \frac{5mL^2}{4} + J & \frac{L^2m\cos(q_2 - q_3)}{2} \\ \frac{Lm\cos(q_3)}{2} & \frac{L^2m\cos(q_1 - q_3)}{2} & \frac{L^2m\cos(q_2 - q_3)}{2} & \frac{mL^2}{4} + J \end{bmatrix}$$

(20)

$$D(q,\dot{q}) = \begin{bmatrix} \frac{5Lmsin(q_1)\dot{q}_1^2}{2} + \frac{3Lmsin(q_2)\dot{q}_2^2}{2} + \frac{Lmsin(q_3)\dot{q}_3^2}{2} \\ -c\dot{q}_1 + c\dot{q}_2 - kq_1 + kq_2 - \frac{3L^2msin(q_1 - q_2)\dot{q}_2^2}{2} - \frac{L^2msin(q_1 - q_3)\dot{q}_3^2}{2} - \frac{5Lmgsin(q_1)}{2} \\ c\dot{q}_1 - 2c\dot{q}_2 + c\dot{q}_3 + kq_1 - 2kq_2 + kq_3 + \frac{3L^2msin(q_1 - q_2)\dot{q}_1^2}{2} - \frac{L^2msin(q_2 - q_3)\dot{q}_3^2}{2} - \frac{3Lmgsin(q_2)}{2} \\ c\dot{q}_2 - c\dot{q}_3 + kq_2 - kq_3 + \frac{L^2msin(q_1 - q_3)\dot{q}_1^2}{2} - \frac{L^2msin(q_2 - q_3)\dot{q}_2^2}{2} - \frac{Lmgsin(q_3)}{2} \end{bmatrix}$$

(21)

And the external force acting on the cart and each joint is defined on the following matrix:

$$\vec{u} = \begin{bmatrix} F \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4. Control of Flexible Multi-Dynamics Systems

In applications involving flexible structures—such as antennas, space solar panels, space structures, soft robotic manipulators and flexible-link mechanisms—controllers are primarily tasked with input tracking, vibration suppression, and precise positioning. However, these structures exhibit significant nonlinearity, with time-varying structural parameters and often poorly modeled dynamics. As the demand for automatically controlled mechanical systems with increasing speed and precision grows, accounting for the elastic behavior of these systems becomes crucial.

This review paper centers on the two primary methods for deriving the equations of motion in flexible multibody systems. Although various control strategies are discussed in the literature to address the challenges posed by these systems, our primary focus is on modeling rather than control. The discussion of control strategies serves to illustrate some key approaches but is not intended to be exhaustive, nor does it suggest that the cited literature fully addresses the complexities of managing elastic behavior in these systems.

For the general discussion of control of flexible structures one can refer to, (Korkmaz, 2011), (Alkhatib and Golnaraghi, 2003), (Wani *et al.*, 2022), for more focus on light and flexible manipulators and mechanisms one can refer to (Jonker, Aarts and Van Dijk, 2009), (Li, Chen and George, 2017),(Subedi, Tyapin and Hovland, 2020), (Alandoli and Lee, 2020), (Fraser and Ron W. Daniel, 1991), (Yigit and Ulsoy, 1989). The following article (De Sapio, Khatib and Delp, 2006) focuses on task-level approaches for the control of constrained multibody systems including ones with flexibility.

Some approaches (Komatsu and Modi, 2003; Etxebarria, Sanz and Lizarraga, 2005; Hu and Ng, 2005; Ishijima, Tzeranis and Dubowsky, 2005; Muller and Liu, 2014; Nchekwube, 2019; Rauscher and Sawodny, 2021) divide the control problem into two parts a nominal controller to address (in most cases) the rigid body motion (or a simpler model of the system) and another "loop" or part to take care of the uncertainty (or in most cases the flexible body motion). A good example is the perturbation approach, in which a nominal system model is assumed combined with an efficient way to predict and accommodate changes in the system (parameter changes, loading changes, flexibility,…) a good reference here is (Fraser and Ron W. Daniel, 1991) other references to refer to are (Yen and Kwak, 1999), (Jonker and Aarts, 2001), (Sun-Wook Kim, 1999),(Kim and Croft, 2019). Additionally, robust control techniques, such as LQG/LTR and H-infinity methods, are considered to address uncertainties in the dynamics (Grewal and Modi, 2000), (Caracciolo, Richiedei and Trevisani, 2008), (Grewal and Modi, 1996). Given the nonlinear nature of multibody systems and the additional degrees of freedom introduced by elastic behavior, the literature shows significant interest in nonlinear control methods (Boscariol, Scalera and Gasparetto, 2021), including feedback linearization (Moberg, 2010), (Zhang and Ge, 2014), and sliding mode control (SMC) (Allen, Bernelli-Zazzera and Scattolini, 2000), (Parker *et al.*, 1996), (Wilson *et al.*, 2002),(Jingxin Shi, 2010; Guan *et al.*, 2024). Moreover, dynamic dissipative compensators and passivity-based controllers are also explored (Atul G. Kelkar, Joshi and Alberts, 1995), (Kelkar and Joshi, 1997), (Kelkar, Alberts and Joshi, 1995).

5. Simulation Results for a Flexible Pendulum on a Crane System

Since the focus of this paper is on modeling and not control an LQR controller based on the rigid dynamics of the system will be used. The equations of the rigid dynamics of the system can be linearized and an LQR control can be computed as $F = -k_1(x_c - x_d) - k_2\theta - k_3\dot{x}_c - k_4\dot{\theta}$. In this case the linearized model (around $\theta = \dot{x}_c = \dot{\theta} = 0$), is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m_p g L^2}{M_c m_p L^2 + 4J_p(m_p + M_c)} & 0 & 0 \\ 0 & -\dfrac{2Lgm_p(m_p + M_c)}{M_c m_p L^2 + 4J_p(m_p + M_c)} & 0 & 0 \end{bmatrix}$$

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$$B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{m_p L^2 + 4J_p}{M_c m_p L^2 + 4J_p(m_p + M_c)} \\ \dfrac{-2m_p L}{M_c m_p L^2 + 4J_p(m_p + M_c)} \end{bmatrix}$$

0

If we chose the gains of the controller such that we minimize $J = \int (x^T Q x + u^T R u) dt$, with Q= $\begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and $R = 0.01$, and $R=0.0001$(gains of the controller are higher)

With parameters of the models as follows: $M_c=1kg$; $L=1m$; $m_p=10g$; $J_p = \frac{m_p L^2}{12}$; $E = 7 \times 10^9$ (Aluminum); $I_A = \frac{\pi}{4}\left(\frac{D}{2}\right)^2$ which the 2nd moment of area.

The simulation results are as shown below in Figure 4, Figure 5, Figure 6 and Figure 7 (for cart position *x* in *m* and pendulum angle in *rad*) for the ideal rigid pendulum versus the flexible pendulum modeled here using assumed mode method. With higher values of the gain there is more "excitation" of the flexible mode coordinates, as seen in Figure 8, which in this case (having a stable system) just alter the response as compared to the ideal rigid pendulum. If, however one is trying say to swing the pendulum up and stabilize it this could be an issue if not handled wisely.

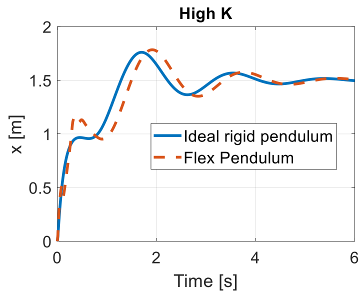

Figure 4. Cart position, x [m].

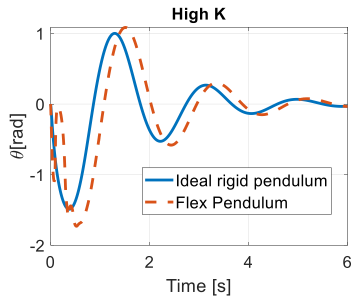

Figure 5. Pendulum angle[rad].

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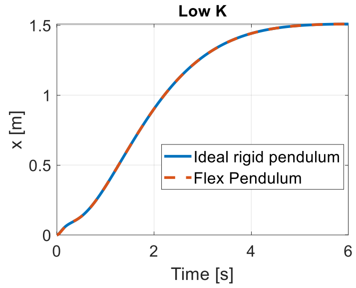

Figure 6. Cart Position, x [m].

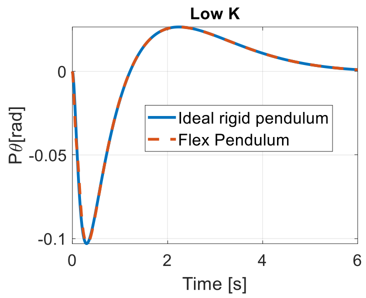

Figure 7. Angle Position [rad].

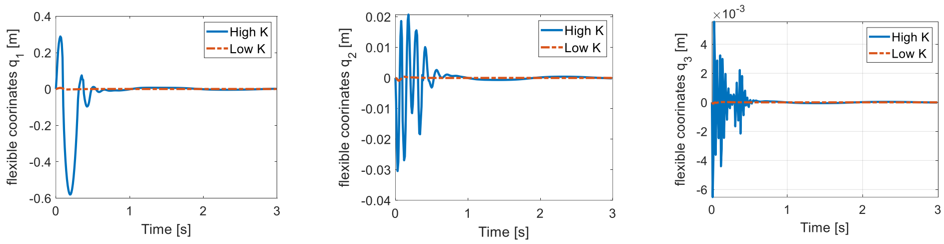

Figure 8. Flexible Modes Response.

The LPM and the AMM simulation models did behave similarly when the higher modes were not excited, but the results did differ otherwise slightly, as seen in Figure 9 and Figure 10, this is due to the way the flexibility is incorporated as well as the way the angle of the pendulum is taken in the LPM (the pendulum is divided into segments and the angle of the 1st segment was taken as the pendulum angle). Ideally the parameters of the model (stiffness and damping) will be tuned to match the experimental results. In the simulation results above the damping for the AMM was $c = \beta\omega_i$, where β is a constant and ω_i, is the natural frequency of the ith mode. while for the LPM $k = \frac{8EI_A}{L_s}$, $c = \alpha k$, again α is a tuning factor.

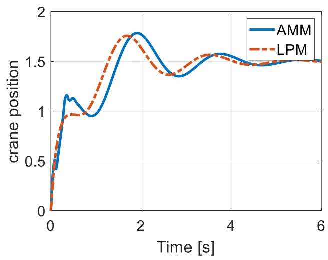

Figure 9. Crane position (AMM,LPM).

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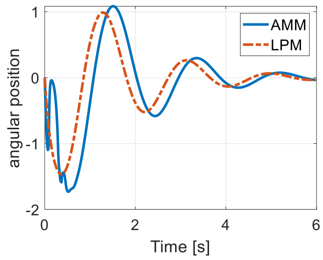

Figure 10. Pendulum Angular Position(AMM,LPM).

The interaction between the control effort and the ignored higher modes which have not been taken when designing the controller leads to the spillover effect The spillover effect refers to unintended or undesirable dynamics that arise when a control system or approximation model fails to capture all the significant modes or degrees of freedom of a flexible system. The control effort may excite these uncontrolled and ignored modes which may severely degrade the system performance. Let us consider the swing up and stabilizing of an inverted pendulum on cart (same model). The control is based on a simple strategy for swinging up the pendulum based on energy control, since the focus here is on the modeling and not control the paper will not "formally" introduce the approach, for more detailed understanding of the method one can refer to (Yoshida, 1999; Åström and Furuta, 2000; Romeo Ortega *et al.*, 2001; Romeo; Ortega *et al.*, 2001; Chatterjee, Patra and Joglekar, 2002; SAKURAMA, HARA and NAKANO, 2007; Banerjee and Pal, 2018; Chandra, Tamba and Sadiyoko, 2019; Kahar, 2019; Rodríguez-Cortés, 2019).

One way to swing the pendulum up is to force it to have the energy corresponding to the upright position, using a strategy as the one seen in

$$u = sat_{umax}(k(E - E_0)sign(\dot{\theta}\, cos(\theta))) \qquad (22)$$

where E is the energy of the pendulum, E_0 is the energy at the upright position, k is a constant, and sat_{umax} is a linear function which is saturated at *umax*. Once the pendulum is close to the upright position a linear controller will be switched on to stabilize it. Figure 11 and Figure 12 shows the response.

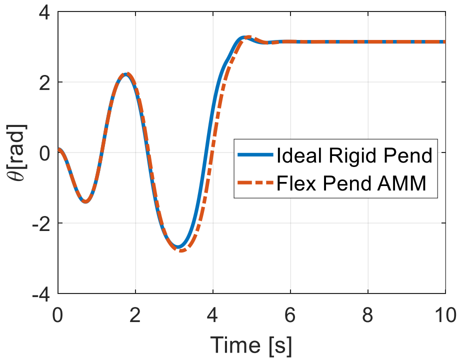

Figure 11. Angular Position of Pendulum.

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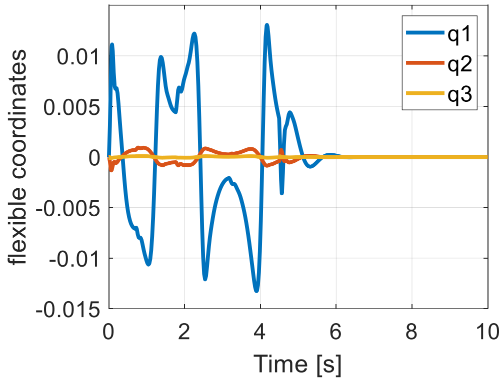

Figure 12. Flexible coordinates response.

if k, $umax$ or parameters of the controller are not tuned properly the system may not behave well because of the excitation of the flexible modes as seen in Figure 13, where for the same parameters the cart with the ideal rigid pendulum was stabilized, but the controller failed to stabilize the pendulum in the model in which a flexible beam model was used.

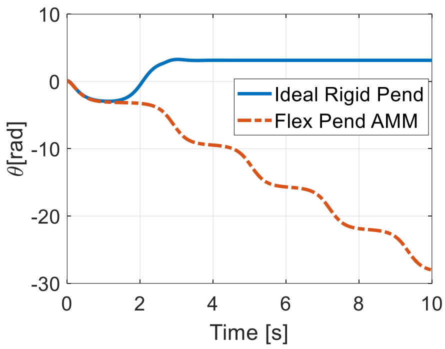

Figure 13. The response if the tuning of the parameters is not done properly.

The focus here was on modeling. For a formal control design, one can consider some of the approaches mentioned in section 4.

6. Conclusion

Modeling flexible structure dynamics is crucial for understanding the behavior of flexible bodies under various conditions. In this tutorial review, we explored two widely used methods for modeling flexible bodies, namely the assumed mode method (AMM) and the lumped parameters method (LPM). Each method relies on different assumptions and computational techniques, leading to varying capabilities in capturing the dynamics of flexible systems. The AMM uses a limited number of mode shapes to represent structural deformations, with these shapes selected based on expected deformation patterns and boundary conditions. Therefore, this method requires solving a number of differential equations with time-varying coefficients. On the other hand, the lumped parameters method (LPM) models the flexible structure by dividing it into a limited number of rigid segments and representing their flexibility with some flexible joints that consist of a spring to represent flexibility and damping elements to dissipate energy. These flexible joints connect the rigid segments and thus allow relative movement between them. This method involves solving a set of equations of motion for each segment and joint. Our comparison of the AMM and LPM in modeling the flexibility of a pendulum on a cart demonstrated that both methods effectively capture the system's dynamics, with only minor differences in results.

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