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Mathematically Exact Non-Square-Integrable Solutions in Schrödinger-Equivalent Diffusion Dynamics

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Abstract

We analyze the spherically symmetric complex diffusion and special type of the complex reaction-diffusion equations with the self-similar Ansatz. These equations are form invariant to the free Schrödinger equations and to Schrödinger equations with power-law space dependent potentials. The self-similar Ansatz couples the spatial and temporal variables together instead of the usual separation, therefore new type of solutions can be derived. For both cases analytic solutions are presented which are the Kummer's and the Whittaker functions with complex quadratic arguments. The results are analyzed in depth. In the second case the role of the complex angular momenta is investigated as well.

Keywords: Schrödinger equation; self-similar Ansatz; diffusion

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1. Introduction

One of the simplest transport mechanism is particle diffusion or heat conduction in solids. Such a process is under investigation of physicists, mathematician and engineers in the last two centuries. The corresponding literature is extensive and fulfill libraries hence we just list some recent textbooks [1–4].

It is also clear that the corner stone of non-relativistic quantum mechanics is the Schrödinger equation (SE) which is a complex diffusion equation. SE therefore describes the dispersion of the wave function of quantum particles. Such statements can be found in all basic quantum mechanic textbooks in all decades [5–8]. It is well-known [9] that quantum mechanics was born exactly hundred years ago in 1925 when Max Born first used the German phrase of "Über Quantenmechanik" in one of his paper [10]. The entire quantum theory was worked out in the later years. The main motivation of our present study is to remember this centenary deriving new type of results which lie between the classical and the Schrödinger-like quantum diffusion. We mention that, there are variational principles which lead to equations of diffusion or certain generalizations which are in connection to it [11], and its quantum counterpart may be also derived by [12].

Some of the similarities and differences between real and complex diffusion equations can be found in [13,14]. We follow this path in our next study and present self-similar solutions of the spherically symmetric SE. This analysis is organically connected to our decade-long activity in while we systematically investigated numerous diffusion or hydrodynamic equations and dissipate physical systems presented physically relevant disperse solutions. As an example we just mention one of our previous works [15]. Interestingly, we note that with the self-similar Ansatz we derived hydrodynamic scaling solutions which are consistent with the Hubble parameter measurements and help to describe the rapidly expanding Universe [16]. So the scaling self-similar Ansatz seems to be a powerful tool.

In our former study we analyzed the same problem but it Cartesian symmetry [17]. We found analytic solutions for five power-law potentials $V(x) = ax^i$ where $i = -2, -1, 0, 1, 2$ which can be

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expressed with some special functions like, Whittaker, Kummer or Heun functions. The a/x^2 potential has some peculiarities which are also reflected in our analysis. It came clear that there are three disjunct regimes exist: real diffusion with self-similar symmetry, the usual time-independent Schrödinger equation with additional potentials where temporal and spatial variables are separated and our new complex diffusion equation where temporal and spatial variables remain coupled via the self-similar Ansatz.

The literature of analytic solutions of the Schrödinger equation with additional potentials is widely extended, in our former study we listed numerous cases for the Cartesian space variables. In the following study the solutions of the spherical symmetry will be mentioned only.

2. Theory and Results

2.0.1. The Spherical Real Diffusion Equation

To have a complete overview and analysis let's outline the self-similar solution of the real diffusion equation in curved space:

$$\frac{\partial C(r,t)}{\partial t} = D\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial C(r,t)}{\partial r}\right), \tag{1}$$

where $n = 0, 1$ and 2 means Cartesian, cylindrical and spherical symmetry, respectively. We will use the self-similar Ansatz $C(r,t) = t^{-\alpha}f(r/t^{\beta}) = t^{-\alpha}f(\eta)$ from now on. After some trial algebraic steps get the reduced ordinary differential equation (ODE) of

$$-\alpha f - \frac{\eta f'}{2} = D\left(\frac{nf'}{\eta} + f''\right), \tag{2}$$

with the usual constraints of $\alpha =$ arbitrary real and $\beta = 1/2$. The general solutions read as follows:

$$f(\eta) = e^{-\frac{\eta^2}{4D}}\left[c_1 M\left(\frac{1}{2}+\frac{n}{2}-\alpha, \frac{1}{2}+\frac{n}{2}, \frac{\eta^2}{4D}\right) + c_2 U\left(\frac{1}{2}+\frac{n}{2}-\alpha, \frac{1}{2}+\frac{n}{2}, \frac{\eta^2}{4D}\right)\right], \tag{3}$$

where $M()$ and $U()$ are the Kummer's M and Kummer's U functions [18,19]. These solutions also shows certain non-Gaussian aspects of the classical dynamics [20].

The above relation holds for $n \geq 1$, and Eq. (3) well describes the cylindrical and spherical symmetric solutions. Note that unlike the Cartesian case here all the solutions have even property (symmetrical about the y-axis). For the spherical symmetric case we have the direct form of:

$$f(\eta) = e^{-\frac{\eta^2}{4D}}\left[c_1 M\left(\frac{3}{2}-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right) + c_2 U\left(\frac{3}{2}-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right)\right]. \tag{4}$$

Without additional analysis we just give the global properties depending on the free parameter α For Kummer's M function:

- $\alpha < 0$ the solutions are divergent at large arguments,
- $\alpha = 0$ the solution is constantly 1,
- $0 < \alpha \leq 3/2$ the solutions have a local maxima and a decay to zero at large arguments,
- $3/2 < \alpha$ the solutions have oscillations proportional to the value of α and have quicker and quicker decays to zero at larger α values.

For the Kummer's U functions the properties are a bit different:

- $\alpha < 3/2$, the solution starts at zero has a very sharp and very high positive peak and a very quick decay to zero,
- $\alpha = 3/2$ the solution is unity,
- $3/2 < \alpha < 5/2$ the solutions starts from zero has a very sharp and very high negative peak and a very quick decay to zero,

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- $\alpha = 5/2$ the solutions starts from -1.5 becomes slightly positive and has a slow decay to zero,
- $\alpha > 5/2$ the solutions starts from zero then has a positive or negative very sharp peak (maximum or minimum) then an oscillatory decay, the large the α the larger the number of oscillations.

It is clear to see that all of the resulting functions have odd symmetry. An in-depth analysis of the self-similar solutions of the regular diffusion equation in Cartesian coordinates can be found in [20] and papers with references there.

2.0.2. The Spherical Complex Diffusion Equation

Before we derive and discuss our analytic solutions we have to summarize other available results for the free spherical Schrödinger equation First we give the form of the spherically disperse wave-packet

$$i\hbar\frac{\partial\Psi(r,t)}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{2}{r}\frac{\partial\Psi(r,t)}{\partial r} + \frac{\partial^2\Psi(r,t)}{\partial r^2}\right). \tag{5}$$

With direct derivation and substitution it can be proven that the solution has the form of

$$\Psi(r,t) = \left(\frac{a}{a^2 + (\hbar t/m)^2}\right)^{\frac{3}{2}} e^{-\frac{ar^2}{2(a+i\hbar t/m)}}, \tag{6}$$

and the probability density is given as:

$$P(r,t) = |\Psi|^2 = \Psi^*\Psi = \left(\frac{a}{\sqrt{a^2 + (\hbar t/m)^2}}\right)^3 e^{-\frac{ar^2}{a^2+(\hbar t/m)^2}}, \tag{7}$$

where a is an arbitrary real parameter. The norm is finite and can be calculated analytically for any given parameter set. As an example after fixing $\hbar = m = a = t = 1$ and

$$\int_0^\infty P(r,t)r^2dr = \lim_{r\to\infty} \frac{-re^{-\frac{r^2}{2}}}{2^{3/2}} + \frac{\sqrt{\pi}}{2^{5/2}} erf\left(\frac{r}{\sqrt{2}}\right) \approx 0.443. \tag{8}$$

A more general formula is also available and can be found in ([8]) [1]

$$\psi(r,t) = \left(\frac{2}{\pi w(t)^2}\right)^{3/4} \exp\left(-\frac{r^2}{w(t)^2} + i\frac{mr^2}{2\hbar t} + i\phi(t)\right), \tag{9}$$

with

$$w(t) = w_0\sqrt{1 + (i\hbar t)/(mw_0^2)}. \tag{10}$$

Defining:

$$\psi(r,0) = \left(\frac{2}{\pi w_0^2}\right)^{3/4} \exp\left(-\frac{r^2}{w_0^2} + i\mathbf{k}_0 \cdot \mathbf{r}\right). \tag{11}$$

Understanding the quantum properties of matter via investigating the dynamics of wave packet motion is a popular method with immense literature so we just mention two studies [21,22]. In general the role of time in quantum mechanics attracted remarkable interests and topics of scientific monographs and publications so we also give some references [23–25]. Here we should mention the work of Kleber who summarized the exact solutions for time-dependent phenomena in quantum mechanics [26].

Changing to our method we apply the self-similar Ansatz in the form of $\Psi(r,t) = t^{-\alpha}g\left(\frac{r}{t^\beta}\right) = t^{-\alpha}g(\omega)$ [27] we immediately arrive to a similar ODE of

$$i\left(-\alpha g - \frac{\omega g'}{2}\right) = -D\left(\frac{2g'}{\omega} + g''\right). \tag{12}$$

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At this point we have two possibilities.

If we expect from the function g to be real, then in Eq. (12) the l.h.s, which is complex, may equal the r.h.s., which is real, only in the case if both equal zero:

$$-\alpha g - \frac{\omega g'}{2} = 0, \qquad \frac{2g'}{\omega} + g'' = 0. \tag{13}$$

After some rearrangement the equations have the following differential form

$$\frac{dg}{d\omega} = -\frac{2\alpha}{\omega}, \qquad \omega\frac{df}{d\omega} = -2f, \tag{14}$$

where in the second equation $g' = f$ was used. Solving these two equations, one get

$$g = g_0\left(\frac{\omega_0}{\omega}\right)^{2\alpha}, \qquad f = f_0\left(\frac{\omega_0}{\omega}\right)^2, \tag{15}$$

where g_0, f_0 and ω_0 are constants related to initial conditions $g(\omega_0) = g_0$. Because $g' = f$, integrating the second equation, one finds that

$$g = g_0\left(\frac{\omega_0}{\omega}\right)^{2\alpha}, \qquad g = -\frac{f_0\omega_0^2}{\omega}. \tag{16}$$

These two functions may be the same, if they have the same decay in ω. This condition can be fulfilled, if $\alpha = \frac{1}{2}$, The general solution in this case reads

$$\Psi(r,t) = t^{-\frac{1}{2}} \cdot g(\omega) = t^{-\frac{1}{2}} \cdot \mathrm{Const} \cdot \frac{1}{\omega} = \mathrm{Const} \cdot t^{-\frac{1}{2}} \cdot \frac{t^\beta}{r} = \mathrm{Const} \cdot \frac{t^{-\frac{1}{2}+\beta}}{r}. \tag{17}$$

If the function g may be a general complex function - it is not required to be real - than the usual constraints of α = arbitrary real, and $\beta = \frac{1}{2}$ holds, with diffusion constant $D = \frac{\hbar}{2m}$. The solutions of (12) are:

$$g(\omega) = c_1 M\left(\alpha, \frac{3}{2}, \frac{i\omega^2}{4D}\right) + c_2 U\left(\alpha, \frac{3}{2}, \frac{i\omega^2}{4D}\right), \tag{18}$$

where $M()$ and $U()$ are still the Kummer's function [18,19]. Note, the two relevant difference to the Cartesian solutions, are the lack of the extra ω dependence and the shift in the first argument of the Kummer's functions. It is useful to evaluate the first few terms of the Taylor series of the function which read as:

$$\begin{aligned} g(\omega) = c_1\left(1 + \frac{i\alpha\omega^2}{6D} - \frac{\alpha[\alpha+1]\omega^4}{120D^2} - \frac{i\alpha[\alpha+1][\alpha+2]\omega^6}{5040D^3} + ...\right) + \\ c_2\left(\frac{2\sqrt{\pi}}{\sqrt{\frac{i}{D}}\Gamma[\alpha]} \cdot \frac{1}{\omega} - \frac{2\sqrt{\pi}}{\Gamma\left[-\frac{1}{2}+\alpha\right]} + \frac{i(-i+2\alpha)\sqrt{\pi}}{2\sqrt{\frac{i}{D}}D^2\Gamma[\alpha]} \cdot \omega - \frac{i\alpha\sqrt{\pi}}{3\Gamma\left[-\frac{1}{2}+\alpha\right]D} \cdot \omega^2 - \right. \\ \left. \frac{[-1+4\alpha^2]\sqrt{\pi}}{48\sqrt{\frac{i}{D}}D^2\Gamma[\alpha]} \cdot \omega^3 + ...\right). \end{aligned} \tag{19}$$

Note that the Kummer's M function has even symmetry. The Kummer's U function however has no even or odd symmetry for general αs. However, due to the Gamma function for negative αs the odd terms are not undefined, and for negative half integer values the even terms are undefined.

For completeness we give final analytic solution in the form of:

$$\Psi(r,t) = t^{-\alpha}\left(c_1 M\left[\alpha, \frac{3}{2}, \frac{ir^2}{4Dt}\right] + c_2 U\left[\alpha, \frac{3}{2}, \frac{ir^2}{4Dt}\right]\right). \tag{20}$$

To understand the properties of this solution we have to make a regular parameter study, namely how the solution depends on the free parameter α. Figures 1 and 2 show the real, imaginary and absolute value of the shape functions for different αs, for the Kummer's M and for the Kummer's U functions. The first figure shows the solutions which are regular at the origin. The real and the imaginary parts are look similar to each other and hard to say which α value divides the different branches of solutions. The absolute value figures however help us to resolve the issue unambiguously. Positive αs define shape functions with decaying and oscillatory properties. The $\alpha = 0$ means the constant solution, and negative αs define divergent solutions. The second figure present the irregular Kummer's U function which (most of them) are infinite in the origin. Positive αs define solution which are divergent in the origin and has a strong decay to zero.

For completeness Figure 3 presents two possible radial particle density functions $r^2|\Psi(r,t)|^2$ expressions. Note, that the Kummer's M function shows some temporal wavy structure which is familiar from the "ordinary" quantum mechanic solutions. The Kummer's U functions (which are the irregular solutions) has a very quick decay in space and time with a high value in the origin. To investigate the possible far relationship to quantum mechanics we numerically integrated the $r^2|\Psi(r,t)|^2$ quantity for numerous αs and parameters. In the former Cartesian case study [17] we found an α parameter region, where the spatial numerical integral (which is the norm of the wave function) become convergent for the Kummer's U function. There was no numerical convergence found for the Kummer's M functions. Now - for spherical symmetric case - with interval doubling we can easily proof the same property. There is no convergence for the Kummer's M function at any α parameter. For Kummer's U function we found that for $\alpha > 0.8$ the convergence is easy to see. We cannot say at which α value lies the convergence radius.

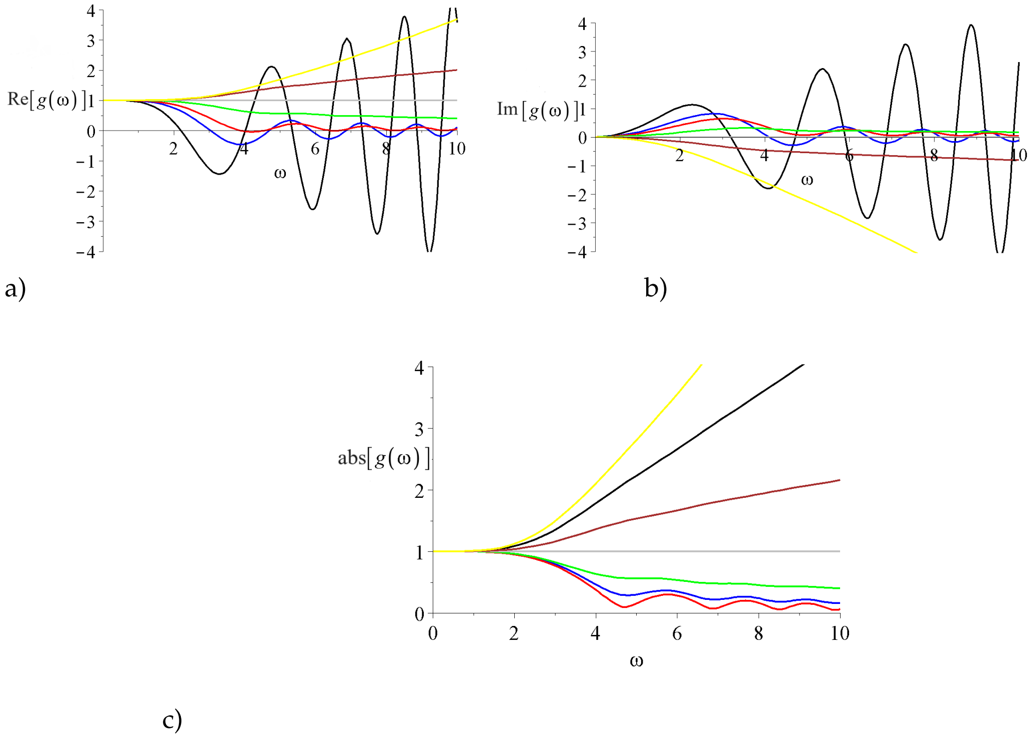

Figure 1. The $a), b)$ and $c)$ are the real, the complex and the absolute values of the Kummer's M function in Eq. (18). The black,blue,red,green, gray, brown and yellow lines are for $\alpha = 1, 1/2, 1/4, 0, -1/4, -1/2$ and -1, respectively.

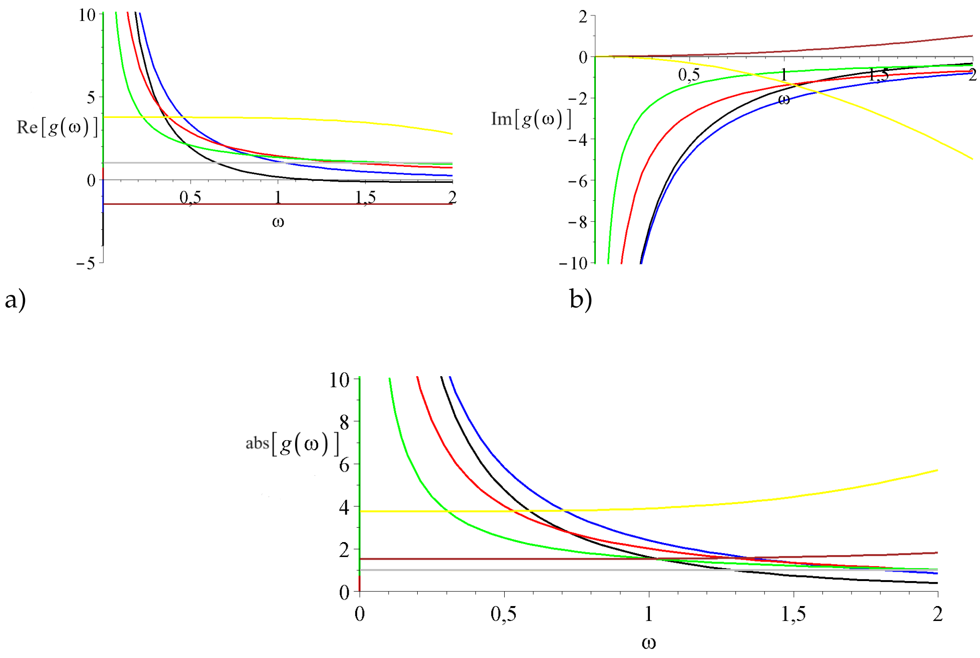

Figure 2. The $a), b)$ and $c)$ are the real, the complex and the absolute values of the Kummer's U function in Eq. (18). The black,blue,red,green, gray, brown and yellow lines are for $\alpha = 1, 1/2, 1/4, 0, -1/4, -1/2$ and -1, respectively.

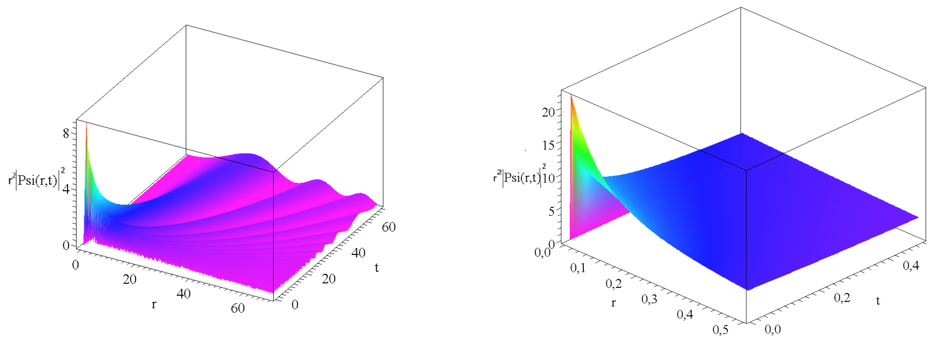

Figure 3. The $r^2|\Psi(r,t)|^2$ with Eq. (20) for $\alpha = 2/3$ and $\hat{D} = 1/2$ the subfigure $a)$ shows the Kummer's M and $b)$ the Kummer's U function, respectively.

2.1. The Complex Spherical Reaction-Diffusion Equation

Carry on you analysis with the three dimensional spherically symmetric complex reaction diffusion equation in the usual Schödinger form

$$i\hbar\frac{\partial\Psi(r,\theta,\varphi)}{\partial t} = -\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2 sin(\theta)}\right)\left[sin(\theta)\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(sin(\theta)\frac{\partial\Psi}{\partial\Theta}\right) + \frac{1}{sin(\theta)}\frac{\partial^2\Psi}{\partial\varphi^2}\right] + V(r)\Psi(r,\Theta,\varphi), \quad (21)$$

where μ is the reduced mass of the two-body problem and $V(r)$ is the applied potential. To come closer to the time dependent diffusion equation consider the $E \sim i\hbar\frac{\partial}{\partial t}$ operator substitution from now on. With the separation of the temporal variables it is possible to reduce to the pure radial coordinate, which has the final form of

$$i\hbar\frac{\partial\Phi(r,t)}{\partial t} = -\frac{\hbar^2}{2\mu}\left(\frac{2}{r}\frac{\partial\Phi(r,t)}{\partial r} + \frac{\partial^2\Phi(r,t)}{\partial r^2}\right) + V(r)\Phi(r,t) + \frac{l(l+1)\hbar^2}{2\mu r^2}\Phi(r,t), \tag{22}$$

the technical details can be found in basic textbooks [5–8]. (For completeness we just mention that one of the authors investigated the single- and double-ionization of helium atoms with the time-dependent coupled-channel methods in a mixed Slater and regular Coulomb wave packet basis in heavy ion collisions [28] and in short intense laser fields [29] for many years. So we practically know the standard methods, how a real quantum mechanical problems should be handled to derive observable quantities.)

At this point without completeness we try to summarize the literature of analytic solutions for spherical potentials.

Beckers *et al.* [30] investigated in non-relativistic quantum mechanical equations with the sub-groups of the Euclidean group, where the Schrödinger or the the Pauli equations were examined with different scalar and vector potentials.

The solutions for the most common interactions like finite-depth square well, Woods-Saxon, spherical oscillator, Coulomb, Hulthén, Kratzer's molecular, Morse, Yukawa and exponential potentials can be found in the book of Flügge [31]. Study related to certain features of solutions of biharmonic nonlinear Schrödinger equation one may find in ref. [32].

For the inverse-square-root potential $V(r) = -a/r^{-\frac{1}{2}}$ was solved by Li and Dai [33] in 2016 and derived the biconfluent Heun functions as solutions. [34] A class of exactly solvable rationally extended non-central potentials in two and three dimensions were investigated by Kumari *et al.* [34]

In addition to the Coulomb and the harmonic oscillator problem the $V(x) = -a/x^2$ potential has exotic properties shows some anomalies in quantum mechanics [35]. The problem has a remarkable literature therefore we mention just the relevant studies [36–40]. The potential has direct applications in classical celestial mechanics [41] or even in cosmology matter near horizon of a black hole [42,43]. This potential appears in such physical problems as the Efimov effect [44], an electron near a bipolar molecule [45–47], and a neutral atom in the electric field of a thin charged wire [48,49]. It is crucial to emphasize that these studies do not mention our time dependent self-similar solutions.

To have a later comparison to our self-similar solution we now show the regular quantum mechanical solution of the a/r^2 potential to non-zero angular momenta. The potential does not depend on time so the energy of the system is conserved, therefore the next time-independent Schrödinger equation (which is an ordinary differential equation) should be solved:

$$-D\left(\frac{2}{r}\frac{\partial\Phi(r)}{\partial r} + \frac{\partial^2\Phi(r)}{\partial r^2}\right) + \frac{a\Phi(r)}{r^2} + \frac{l(l+1)}{r^2}\Phi(r) = E\Phi(r,t), \tag{23}$$

where E is the energy of the system, a is the strength of the interaction potential (could be attractive or repulsive as well)and l is the angular momenta now with positive integer values. The solutions are well known as the Bessel functions in the first (J) and second (Y) kind with the form of:

$$\Phi(r) = c_1\sqrt{r}J\left(\frac{\sqrt{D+4a+4l^2+4l}}{2\sqrt{D}}, \sqrt{\frac{E}{D}}r\right) + c_2\sqrt{r}Y\left(\frac{\sqrt{D+4a+4l^2+4l}}{2\sqrt{D}}, \sqrt{\frac{E}{D}}r\right), \tag{24}$$

where c_1, c_2 are the usual real integration constants.

Let's consider now Eq. (22) as a complex diffusion equation and solve it with the self-similar Ansatz. In our case the constraints for α and β exponents dictates that only the $V(r) = \frac{a}{r^2}$ static potential available for to derive a self-similar ODE in the form of

$$i\left(-\alpha h - \frac{\omega h'}{2}\right) = -D\left(\frac{2h'}{\omega} + h''\right) + \frac{a + l(l+1)b}{\omega^2}h, \tag{25}$$

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where $b = \frac{\hbar^2}{2\mu}$, we let another free parameter a which is the potential strength.

(We have to make an important statement at this point. Our reduction mechanism makes it possible that any kind of time-dependent power-law type potential in the form of $V_n(r,t) = \hat{a}r^n t^{-n-2}$ where $\hat{a}, n \in \mathbb{R}$ can be reduced to an ODE similar to Eq. (25). We found analytic solutions for the $n = -2, -1, 0, 1$ and 2 exponents only. The other solutions can be expressed with the Whittaker, Kummer's and Heun functions.

We concentrate now on the $n = -2$ case exclusively. The question of the Coulomb problem - which also has an analytic solution contains the Heun functions multiplied by an exponential and a power law functions - could be the task of a future study.)

The solution of Eq. (25) reads:

$$h(\omega) = \frac{1}{\eta^{3/2}} \left(c_1 e^{\frac{i\omega^2}{8D}} \mathbf{M}_{\frac{3}{4}-\alpha, \frac{\sqrt{(4l^2+4l)b+D+4a}}{4\sqrt{D}}} \left[\frac{i\omega^2}{4D} \right] + c_2 e^{\frac{i\omega^2}{8D}} \mathbf{W}_{\frac{3}{4}-\alpha, \frac{\sqrt{(4l^2+4l)+D+4a}}{4\sqrt{D}}} \left[\frac{i\omega^2}{4D} \right] \right), \tag{26}$$

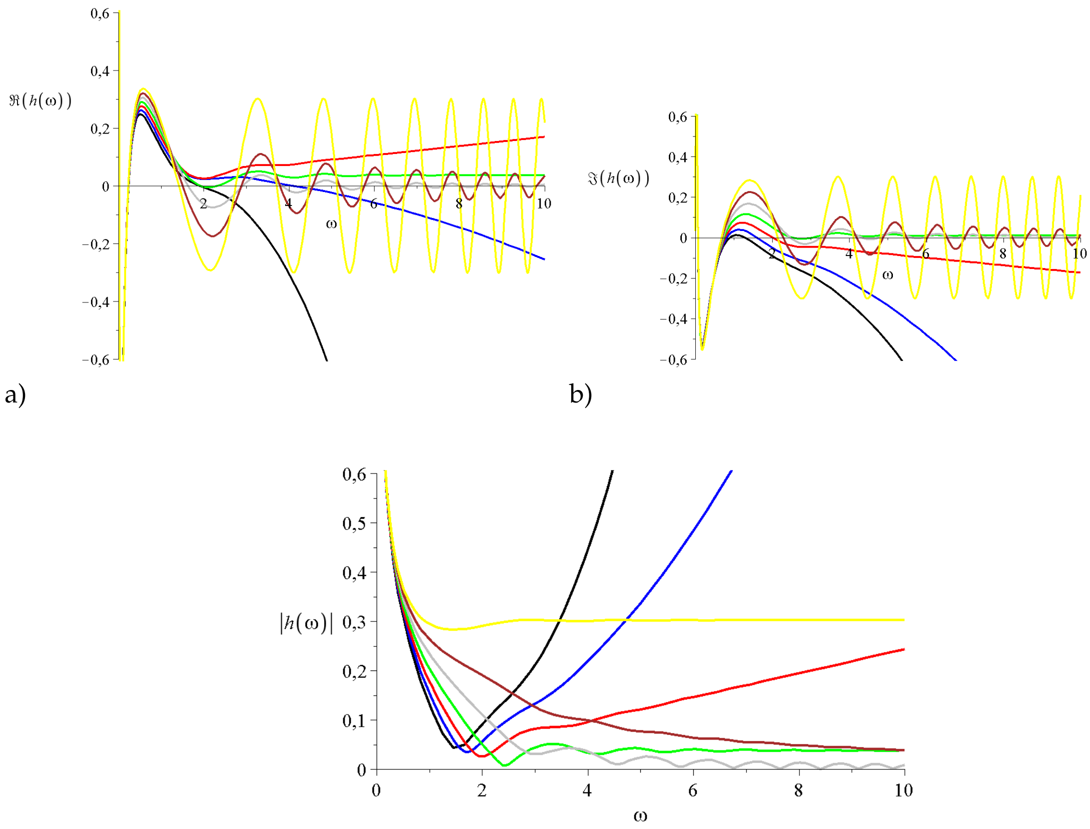
here $\mathbf{M}_{,}()$ and $\mathbf{W}_{,}()$ are the Whittaker functions. For more information please consult [18,19]. Let's analyze (26) first in details.

Note, that both the Whittaker function and the Gaussian prefactor have complex quadratic arguments. Let's analyze our results in details. First consider the the $l = 0$ case and study the α dependence.

Figures 4 and 5 present the real, imaginary and absolute value of the shape function $h(\omega)$ for the Whittaker M and Whittaker W functions for various αs for zero angular momenta. It is again clear to see that αs about $1/2$ have the most reasonable decaying properties. Figure 6 presents the absolute value squared of both Whittaker functions. these are in a sense similar to Figure 3. The Wittaker M function shows more wavy-like structure than the Whittaker W function.

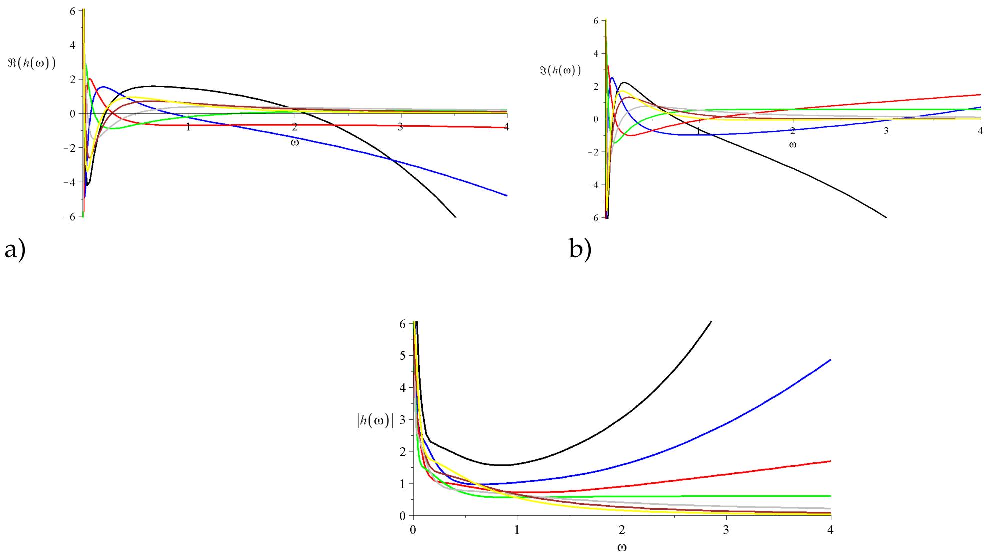
We could not find convincing convergence for the numerical integral of the absolute squared Whittaker W and Whittaker M functions at any α.

For completeness we investigate the role of the angular momenta as well for the more relevant self-similar exponent of $\alpha = 1/2$. To enhance transparency and reduce the number of presented figures and curves we just present the angular dependence of the absolute value of the Whittaker M and Whittaker W functions. We know from the physics of the Coulomb problem ($V(r) = -a/r$) that the larger the angular momentum the larger and the more expansive the wave function in space. Figure 7 presents these two functions for $l = 0, 1, 2, 3$ and 4. The trend is clear to see.



c)

Figure 4. The $a), b)$ and $c)$ are the real, the complex and the absolute values of the Whittaker M shape function in Eq. (26) with $l = 0$. The black,blue,red,green, gray, brown and yellow lines are for $\alpha = -3/2, -1, -1/2, 0, 1/2, 1$ and $3/2$, respectively.



c)

Figure 5. The $a), b)$ and $c)$ are the real, the complex and the absolute values of the Whittaker W shape function in Eq. (26) with $l = 0$. The black,blue,red,green, gray, brown and yellow lines are for $\alpha = -3/2, -1, -1/2, 0, 1/2, 1$ and $3/2$, respectively.

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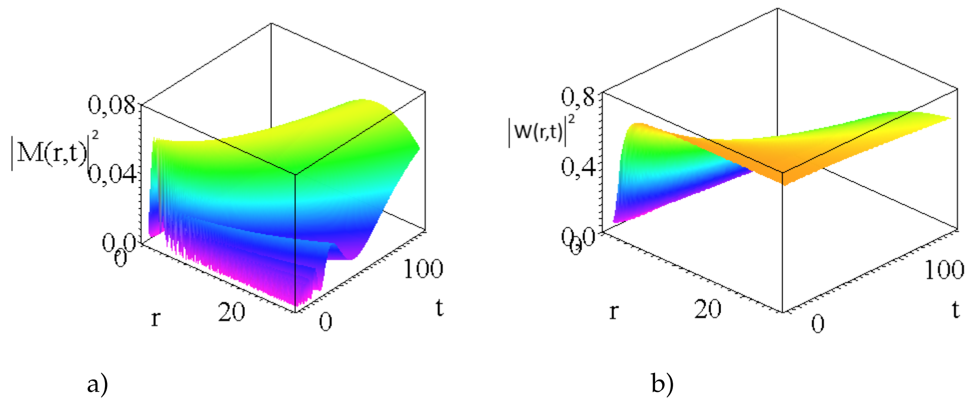

Figure 6. The absolute value square of the Whittaker M a) and Whittaker W functions of Eq. (26) respectively. Other parameters like $D = 1/2, \alpha = 1/2, l = 0, a = -1, b = 1$.

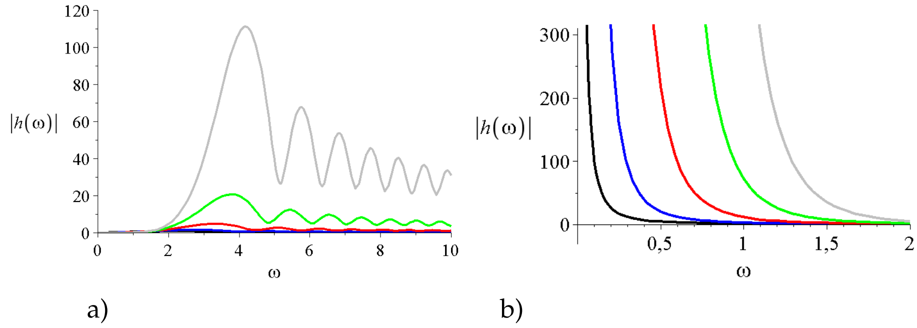

Figure 7. The absolute values of the Whittaker M a) and Whittaker W functions shape functions, b) the black,blue, red, green and gray lines represent $l = 0, 1, 2, 3, 4$, respectively. Other parameters like $D = 1/2$, $\alpha = 1/2$, and $a = b = 1$.

Finally, we might right down and analyze the most sophisticated case, the ODE for arbitrary complex angular momenta where $\hat{l} = i \cdot p + q$ where p is real number responsible for the imaginary part, and q is a second real number responsible for the real part:

$$i\left(-\alpha k - \frac{\omega k'}{2}\right) = -D\left(\frac{2k'}{\omega} + k''\right) + \frac{a + (i \cdot p + q)(i \cdot p + q + 1)b}{\omega^2}k, \tag{27}$$

The solution looks very similar, but due to the two new free parameters p, q it becomes more elaborate:

$$k(\omega) = \frac{1}{\eta^{3/2}}\left(c_1 e^{\frac{i\omega^2}{8D}} \mathbf{M}_{\frac{3}{4}-\alpha,\theta}\left[\frac{i\omega^2}{4D}\right] + c_2 e^{\frac{i\omega^2}{8D}} \mathbf{W}_{\frac{3}{4}-\alpha,\theta}\left[\frac{i\omega^2}{4D}\right]\right), \tag{28}$$

for a better transparency we use the θ abbreviation for the second parameter of the Whittaker function.

$$\theta = \frac{\sqrt{(-4p^2 + [8Iq + 4I]p + 4q^2 + 4q)b + D + 4a}}{4\sqrt{D}}. \tag{29}$$

The theory of complex angular momentum in quantum mechanics was introduced by Regge in 1959 [50]. To learn more about the subject might start with the older books like from Collins [51], Gribov[52], Frautschi [53] or from Omnés and Froissart [54]. The general literature of the Regge theory of the spherical inverse square potential was worked out by Mastalir in a series of publications [55–57]. He investigated a special $1/r^2$ potential which is regularized in the origin to a numerical value of $-V_0$ and at infinity to $-V_2$. The distribution of the Regge poles are given at low energies with analytic formulas.
We cannot define now the S matrix, or the Jost function which are essential for the investigation of the

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Regge poles, we can just check how our solution depends on the complex angular momenta at a fixed point.

If we fix the numerical values of the parameters α, D, a, b and the temporal and and spatial variables than we can study the role of the real and complex part of the angular momenta. (Usually, we choose $\alpha = 1/2, D = 1/2$ the inverse square potential is attractive $a = -1$ the angular momentum term is positive $b = +1$. Figure 8 shows how the solutions Eq. (28) depend on the real and imaginary part of the angular momenta (notated with p and q). Unfortunately, we found no drastic dependence and no poles on the complex plane. As the two independent variables grow, so grows the Whittaker W function. The Whittaker M function behaves a bit different it grows with enhancing q variable at $p = 0$. We could not fine and extra angular momentum dependence feature.

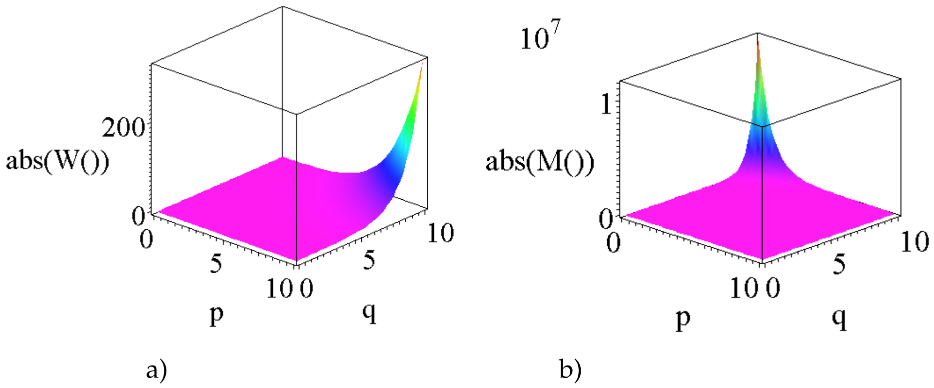

Figure 8. Graphs of the angular momentum dependence of Eq. (28) with the parameter set of: $\alpha = 1/2, D = 1/2, a = -1, b = 1, r = 5, t = 1$. The first subfigure a) presents the absolute value of the Whittaker W and the second subfigure b) for the Whittaker M function. Where p is the real and q is the imaginary value of the angular momenta, respectively.

3. Summary and Outlook

We presented self-similar solutions for the one dimensional complex diffusion equation in spherical coordinate system. The solutions can be expressed with the Kummer's M and Kummer's U function with complex and quadratic arguments. For some α values - which is the new parameter of the solutions - even L^2 integrability can be achieved, which mimic a kind of quantum mechanical interpretability. In the second part of the study some power-law type of potentials were added to the right hand side of the equations the solutions remain analytic expressible with the Kummer's, Whittaker or Heun functions. We concentrate on the inverse square potential which is the only time-independent potential for the self-similar Ansatz. No numerical integrability was found for the absolute square of the solutions for any kind of α parameter. As final generalization even the complex angular momenta was taken into account a together with the $U(r) = \frac{a}{r^2}$ potential and the derived results remained analytic expressible with the Whittaker functions. The second parameter of the the Whittaker function became complicated and can have even complex value. We investigated if integer imaginary angular momenta can cause any kind of singularities in the solutions as in the S-matrix in regular quantum mechanics [50]. No such feature was found the solutions were smooth and increase strictly monotonically for larger values of angular momenta.
Unfortunately, we cannot give physical interpretation or any other physical application to our self-similar solutions neither in quantum mechanics nor in complex diffusion.
We believe that our new type of solutions might prove useful in the far distant future for currently speculative theories, such as quantum consciousness, originally introduced by Penrose and Hameroff [58], as well as for other refined or extended theories discussed later in [59]. The possible connection between mind and quantum mechanics is large interdisciplinary and speculative field with exhaustive literature we just mention two summary study books of [60,61].
As future straightforward additional improvements we mention variable reduction Ansätze (which

describes non-classical symmetries) are available too [62] and will be tested to these equations. We might derive new type of solutions with compact support. As a second way of investigation the complexification of the self-similar Ansatz which is planned to be studied in the future. A final third idea is to study the traveling wave solutions of the complex diffusion equation with harmonic driving terms.

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