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Article

Bounds for the Chebyshev Functional Defined on the q-Circles with Application

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Abstract

In this paper, we move beyond the classical setting by redefining the Chebyshev functional in the context of q-circles situated within Minkowski space, rather than the standard Euclidean circles in $\mathbb{R}^2$. This approach introduces a new theoretical framework suitable for non-Euclidean geometries. We derive sharp estimates for the functional when applied to functions on q-circles that adhere to Hölder-type continuity conditions.

Keywords: grüss inequality; chebyshev functional; q-circles; generalized trigonometric functions

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1. Introduction

In 1879, Lundberg [31] introduced a significant generalization of the classical sine and cosine functions, extending trigonometry to broader geometric frameworks. These generalized functions, denoted by $S_{\frac{q-1}{q}} := S_{\frac{q-1}{q}}(\varphi)$ and $C_{\frac{q-1}{q}} := C_{\frac{q-1}{q}}(\varphi)$, arise from the inverses of integrals parameterized by q:

$$\varphi = \int_0^{S_{\frac{q-1}{q}}} \frac{dt}{(1-t^q)^{\frac{q-1}{q}}},$$
$$\varphi = \int_{C_{\frac{q-1}{q}}}^1 \frac{dt}{(1-t^q)^{\frac{q-1}{q}}}.$$

For $q = 2$, these functions reduce to the standard sine and cosine functions, recovering $S_{\frac{1}{2}}(\varphi) = \sin(\varphi)$ and $C_{\frac{1}{2}}(\varphi) = \cos(\varphi)$. These generalization also naturally leads to an extended notion of π, given by [22]:

$$\omega_q = \frac{2\Gamma^2\left(\frac{1}{q}\right)}{q\Gamma\left(\frac{2}{q}\right)}, \quad q \geq 1,$$

where Γ denotes the gamma function, which coincides with π when $q = 2$.

The generalized sine and cosine functions retain key properties of their classical counterparts, including symmetry and periodicity, but exhibit a flexible dependence on q (see [2]). These properties enable their application in various geometric settings, including Minkowski spaces and other non-Euclidean structures [28–30].

Let us consider an arbitrary point (u, v) in the Euclidean plane. The expression

$$|u|^q + |v|^q = \rho^q,$$

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where $\rho > 0$, represents a generalized form of the Pythagorean identity. This formulation extends the classical trigonometric framework and allows for the definition of generalized trigonometric functions. As an illustrative case, examine a right triangle ABC with the right angle at vertex B, and with legs of unit length, i.e., $\overline{AB} = \overline{BC} = 1$. Accordingly to the traditional Pythagorean theorem, the hypotenuse $\overline{AC}$ has length $\sqrt{2}$, and the angles at A and C each measure $\frac{\pi}{4}$.

When this example is interpreted through the eyes of the generalized identity above, the hypotenuse becomes $\overline{AC} = \sqrt[q]{2}$, and the angles adapt accorollarydingly under the generalized angular measure ω_q. Specifically, if we denote the generalized sine and cosine by $S_{\frac{q-1}{q}}$ and $C_{\frac{q-1}{q}}$, respectively, then the equality

$$S^q_{\frac{q-1}{q}}(\varphi) + C^q_{\frac{q-1}{q}}(\varphi) = 1$$

implies that both $\angle ACB$ and $\angle BAC$ equal $\frac{\omega_q}{4}$. This leads to the following identity:

$$S_{\frac{q-1}{q}}\left(\frac{\omega_q}{4}\right) = \frac{1}{\sqrt[q]{2}} = C_{\frac{q-1}{q}}\left(\frac{\omega_q}{4}\right).$$

Expanding on this idea, suppose now that $\overline{AB} = 1$ while $\overline{AC} = 2$. Applying the same generalized Pythagorean principle, the remaining side becomes

$$\overline{BC} = \sqrt[q]{2^q - 1},$$

and the triangle's internal angles satisfy $\angle ACB = \frac{\omega_q}{6}$ and $\angle BAC = \frac{\omega_q}{3}$. This demonstrates how the generalized identity offers a flexible framework to explore trigonometric relationships beyond the classical setting.

In a related development, Lindqvist and Peetre [28] introduced the concept of a q-circle, which generalizes the idea of a circle using the Minkowski q-norm. For two points $\mathbf{y} = (y_1, z_1)$ and $\mathbf{z} = (y_2, z_2)$, the distance between them in this geometry is determined via the q-metric:

$$d_q(\mathbf{y}, \mathbf{z}) = \begin{cases} (|y_2 - y_1|^q + |z_2 - z_1|^q)^{1/q}, & \text{for } 1 \le q < \infty, \\ \max\{|y_2 - y_1|, |z_2 - z_1|\}, & \text{if } q = \infty, \end{cases}$$

where $q \ge 1$. Accordingly, a q-circle centered at (y_0, z_0) with radius ρ is represented by:

$$|y - y_0|^q + |z - z_0|^q = \rho^q.$$

The geometry defined by this equation varies with the value of q: when $q = 2$, it describes the standard Euclidean circle; if $q = 1$, the figure becomes a diamond (rhombus); and for $q = \infty$, the resulting shape is a square. Notably, the total p-length of a q-circle with radius ρ is given by $2\omega_q\rho$, and the corollary responding q-area is $\omega_q\rho^2$, as detailed in [28].

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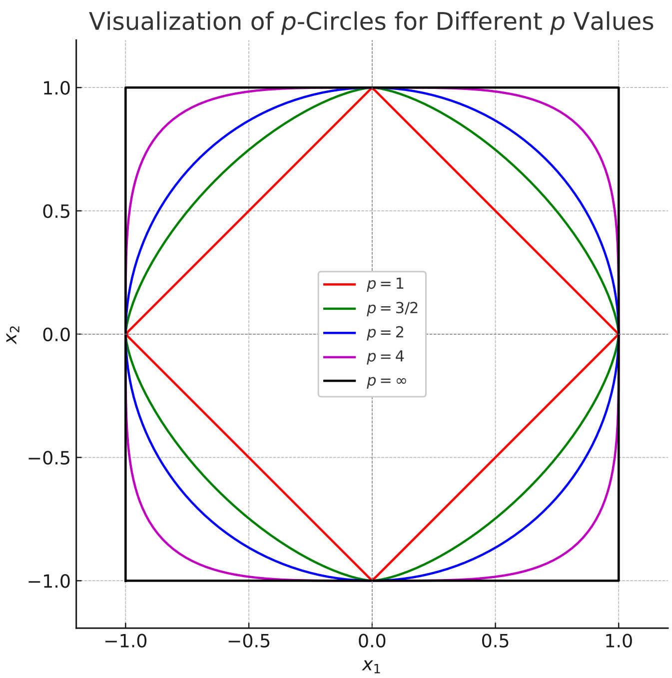

Figure 1. In Minkowski normed spaces, the shape of a q-circle varies based on the value of p. The figure illustrates the different q-circles centered at the origin with radius $\rho = 1$.

The q-circle has several interesting properties that depend on the value of p. Among others, the "*Shape Variability*". i.e,

- $q = 1$ (Red): A **diamond** shape.
- $q = \frac{3}{2}$ (Green): A **rounded diamond**, smoother than $q = 1$.
- $q = 2$ (Blue): A **perfect Euclidean circle**.
- $q = 4$ (Magenta): A **bulging square**, more "boxy" than $q = 2$.
- $q = \infty$ (Black): A **perfect square** with sides parallel to the axes.

To evaluate the deviation between the integral of the product of two functions and the product of their respective integrals, the well-known Chebyshev functional is defined as ([36], p. 296)

$$\mathscr{T}(h_1, h_2) = \frac{1}{d-c}\int_c^d h_1(s)h_2(s)\,ds - \frac{1}{d-c}\int_c^d h_1(s)\,ds \cdot \frac{1}{d-c}\int_c^d h_2(s)\,ds. \tag{1}$$

Over time, significant effort has been devoted to determining sharp bounds for $\mathscr{T}(h_1, h_2)$, using various mathematical techniques. The case where h_1 and h_2 are absolutely continuous functions with derivatives belonging to L_p spaces ($1 \le p \le \infty$) is of particular interest. Notably, four sharp results addressing this case are consolidated in the theorem below.

Theorem 1. *Let $h_1, h_2 : [c, d] \to \mathbb{R}$ be absolutely continuous functions. Then:*

$$|\mathscr{T}(h_1, h_2)| \le \begin{cases} \frac{(d-c)^2}{12}\|h_1'\|_\infty\|h_2'\|_\infty, & \text{if } h_1', h_2' \in L_\infty[c,d] \text{ ([16])}, \\ \frac{d-c}{\pi^2}\|h_2'\|_2\|h_2'\|_2, & \text{if } h_1', h_2' \in H_2[c,d] \text{ ([32])}, \\ \mu(p,q)\|h_1'\|_q\|h_2'\|_q, & \text{if } h_1' \in L_q[a,b], h_2' \in L_q[c,d] \text{ ([36])}, \\ \frac{1}{8}(d-c)(\Psi-\psi)\|h_2'\|_\infty, & \text{if } \psi \le h_1 \le \Psi, h_2' \in L_\infty[c,d] \text{ ([39])}. \end{cases} \tag{2}$$

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The constants $\frac{1}{12}$, $\frac{1}{\pi^2}$, and $\frac{1}{8}$ the values presented above are optimal and cannot be improved, where $\psi := \inf_{t\in[c,d]} h_1(t)$ and $\Psi := \sup_{s\in[c,d]} h_1(s)$.

Beesack et al. provided an improvement to the third inequality in (2), resulting in the sharper bound given below ([36], p. 302).

$$|\mathscr{T}(h_1,h_2)| \leq \frac{d-c}{4}\left(\frac{2^\alpha-1}{\alpha(\alpha+1)}\right)^{1/\alpha}\left(\frac{2^\beta-1}{\beta(\beta+1)}\right)^{1/\beta}\|f'\|_\alpha\|g'\|_\beta, \tag{3}$$

for $\beta > 1$ with $\alpha = \frac{\beta}{\beta-1}$, where the norm is defined as:

$$\|h\|_w := \left(\int_c^d |g(s)|^w\, ds\right)^{1/w}, \quad w > 1.$$

The constant $\mu(\alpha,\beta)$ is explicitly given by:

$$\mu(\alpha,\beta) := \frac{1}{4}\left(\frac{2^\alpha-1}{\alpha(\alpha+1)}\right)^{1/\alpha}\left(\frac{2^\beta-1}{\beta(\beta+1)}\right)^{1/\beta},$$

and satisfies the bounds:

$$\frac{1}{8} \leq \mu(\alpha,\beta) \leq \frac{1}{4}, \quad \text{for all } \alpha > 1,\ \beta = \frac{\alpha}{\alpha-1}. \tag{4}$$

Special Cases of $\mathscr{T}(h_1,h_2)$.

Notable examples illustrating (3) include the following:

- If $\alpha = \beta = 2$, then

$$|\mathscr{T}(h_1,h_2)| \leq \frac{d-c}{8}\|h_1'\|_2\|h_2'\|_2.$$

- If $\beta \to \infty$, then

$$|\mathscr{T}(h_1,h_2)| \leq \frac{d-c}{4}\|h_1'\|_1\|h_2'\|_\infty.$$

Nevertheless, under these conditions, $\mu(p,q)$ does not provide the sharpest constant. For further developments concerning Grüss-type inequalities in multivariate and higher-dimensional settings, readers may refer to the works in [1–21,23–26,32–48], as well as the citations therein.

This study focuses on broadening the scope of the traditional Chebyshev functional, originally defined for functions over Euclidean disks in $\mathbb{R}^2$, by adapting it to the setting of q-circles in Minkowski space. The proposed generalization establishes a precise formulation suitable for these non-Euclidean geometries, thereby offering an extended framework within Minkowski geometry. Through this approach, the classical Chebyshev functional is not only generalized but also examined under the structural characteristics of Minkowski spaces.

2. Bounds for the Chebyshev Functional on the q-Circle

Consider $B = (b_1,b_2) \in \mathbb{R}^2$ and the q-circle $\sigma_q(B,\rho)$ $(q > 1)$ centered at point B with radius $\rho > 0$. Let

$$\sigma_q(B,\rho) := \left\{(z_1,z_2) \in \mathbb{R}^2 : |z_1-b_1|^q + |z_2-b_2|^q = \rho^q, q \gneqq 1\right\},$$

be the boundary of the q-circle centered at the point B with radius $\rho > 0$.

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Consider the parameterized curve $\tau : [0, 2\omega_q] \to \mathbb{R}^2$ defined as follows:

$$\tau(\vartheta) = \begin{cases} z_1(\vartheta) = b_1 + \rho C_{\frac{q-1}{q}}(\vartheta), \\ z_2(\vartheta) = b_2 + \rho S_{\frac{q-1}{q}}(\vartheta), \end{cases}$$

where $\vartheta \in [0, 2\omega_q]$. This curve describes the boundary of the q-circle, denoted by $\sigma_q(B, \rho)$, such that $\tau([0, 2\omega_q]) = \sigma_q(B, \rho)$.

To compute an integral with respect to the p-arc length (on the boundary of the q-circle), we have:

$$\iint_{\sigma_q(B,\rho)} h(\tau)\, dl(\tau) = \int_0^{2\omega_q} h(z_1(\vartheta), z_2(\vartheta)) \left(|z_1'(\vartheta)|^p + |z_2'(\vartheta)|^p \right)^{\frac{1}{p}} d\vartheta.$$

Substituting the derivatives of z_1 and z_2, we obtain:

$$\iint_{\sigma_q(B,\rho)} h(\tau)\, dl(\tau) = \int_0^{2\omega_q} h(z_1(\vartheta), z_2(\vartheta)) \left(\left| S^{q-1}_{\frac{q-1}{q}}(\vartheta) \right|^p + \left| C^{q-1}_{\frac{q-1}{q}}(\vartheta) \right|^p \right)^{\frac{1}{p}} d\vartheta.$$

Using the property $pq - p = q$, this simplifies further:

$$\iint_{\sigma_q(B,\rho)} h(\tau)\, dl(\tau) = \int_0^{2\omega_q} h(z_1(\vartheta), z_2(\vartheta)) \left(\left| S_{\frac{q-1}{q}}(\vartheta) \right|^{pq-p} + \left| C_{\frac{q-1}{q}}(\vartheta) \right|^{pq-p} \right)^{\frac{1}{q}} d\vartheta.$$

Exploiting the identity $C^q_{\frac{q-1}{q}}(\vartheta) + S^q_{\frac{q-1}{q}}(\vartheta) = 1$, the integral can be rewritten as:

$$\iint_{\sigma_q(B,\rho)} h(\tau)\, dl(\tau) = \rho \int_0^{2\omega_q} h(z_1(\vartheta), z_2(\vartheta)) d\vartheta.$$

Dividing both sides by $2\omega_q \rho$, we obtain:

$$\frac{1}{2\omega_q \rho} \iint_{\sigma_q(B,\rho)} h(\tau)\, dl(\tau) = \frac{1}{2\omega_q} \int_0^{2\omega_q} h(z_1(\vartheta), z_2(\vartheta)) d\vartheta.$$

This expression represents the mean value of the function h_1 over the boundary of the q-circle.

Next, define the Chebyshev functional as:

$$\mathscr{T}(h_1, h_2) := \frac{1}{2\omega_q \rho} \iint_{\sigma_q(B,\rho)} h_1(\tau) h_2(\tau)\, dl(\tau) - \frac{1}{2\omega_q \rho} \iint_{\sigma_q(B,\rho)} h_1(\tau)\, dl(\tau) \cdot \frac{1}{2\omega_q \rho} \iint_{\sigma_q(B,\rho)} h_2(\tau)\, dl(\tau).$$

This functional quantifies the interaction of the functions h_1 and h_2 over the boundary of the q-circle.

Let us begin with following q-polar version of the Korkine identity:

Lemma 1. *Let $q = \frac{p}{p-1}$ $(p > 1)$ and $h_1, h_2 : \sigma_q(B, \rho) \to \mathbb{R}$ be two measurable functions defined on the q-circle $\sigma_q(B, \rho)$. Then*

$$\begin{aligned} &\frac{1}{2\omega_q \rho} \iint_{\sigma_q(B,\rho)} h_1(\tau) h_2(\tau) dl(\tau) - \frac{1}{2\omega_q \rho} \iint_{\sigma_q(B,\rho)} h_1(\tau) dl(\tau) \cdot \frac{1}{2\omega_q \rho} \iint_{\sigma_q(B,\rho)} h_2(\tau) dl(\tau) \\ &= \frac{1}{8\omega_q^2} \int_0^{2\omega_q} \int_0^{2\omega_q} (h_1(z_1(\varphi), z_2(\varphi)) - h_1(z_1(\vartheta), z_2(\vartheta))) \\ &\qquad \times (h_2(z_1(\varphi), z_2(\varphi)) - h_2(z_1(\vartheta), z_2(\vartheta))) d\vartheta d\varphi. \end{aligned}$$

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Proof. It is easy to observe that

$$\begin{aligned}
&\frac{1}{2}\int_0^{2\omega_q}\int_0^{2\omega_q}(h_1(z_1(\varphi),z_2(\varphi))-h_1(z_1(\vartheta),z_2(\vartheta)))\\
&\qquad\qquad\times(h_2(z_1(\varphi),z_2(\varphi))-h_2(z_1(\vartheta),z_2(\vartheta)))d\vartheta d\varphi\\
&=\frac{1}{2}\int_0^{2\omega_q}\int_0^{2\omega_q}h_1(z_1(\varphi),z_2(\varphi))h_2(z_1(\varphi),z_2(\varphi))d\vartheta d\varphi\\
&\quad-\frac{1}{2}\int_0^{2\omega_q}\int_0^{2\omega_q}h_1(z_1(\varphi),z_2(\varphi))h_2(z_1(\vartheta),z_2(\vartheta))d\vartheta d\varphi\\
&\quad-\frac{1}{2}\int_0^{2\omega_q}\int_0^{2\omega_q}h_1(z_1(\vartheta),z_2(\vartheta))h_2(z_1(\varphi),z_2(\varphi))d\vartheta d\varphi\\
&\quad+\frac{1}{2}\int_0^{2\omega_q}\int_0^{2\omega_q}h_1(z_1(\vartheta),z_2(\vartheta))h_2(z_1(\vartheta),z_2(\vartheta))d\vartheta d\varphi\\
&=\frac{1}{2}\cdot 2\omega_q\int_0^{2\omega_q}h_1(z_1(\varphi),z_2(\varphi))h_2(z_1(\varphi),z_2(\varphi))d\varphi\\
&\quad-\frac{1}{2}\left(\int_0^{2\omega_q}h_1(z_1(\varphi),z_2(\varphi))d\varphi\right)\times\left(\int_0^{2\omega_q}h_2(z_1(\vartheta),z_2(\vartheta))d\vartheta\right)\\
&\quad-\frac{1}{2}\left(\int_0^{2\omega_q}h_2(z_1(\varphi),z_2(\varphi))d\varphi\right)\times\left(\int_0^{2\omega_q}h_1(z_1(\vartheta),z_2(\vartheta))d\vartheta\right)\\
&\quad+\frac{1}{2}\cdot 2\omega_q\int_0^{2\omega_q}h_1(z_1(\vartheta),z_2(\vartheta))h_2(z_1(\vartheta),z_2(\vartheta))d\vartheta\\
&=2\omega_q\int_0^{2\omega_q}h_1(z_1(\varphi),z_2(\varphi))h_2(z_1(\varphi),z_2(\varphi))d\varphi\\
&\quad-\left(\int_0^{2\omega_q}h_1(z_1(\varphi),z_2(\varphi))d\varphi\right)\times\left(\int_0^{2\omega_q}h_2(z_1(\vartheta),z_2(\vartheta))d\vartheta\right)\\
&=\frac{2\omega_q}{\rho}\iint_{\sigma_q(B,\rho)}h_1(\tau)h_2(\tau)dl(\tau)-\frac{1}{\rho}\iint_{\sigma_q(B,\rho)}h_1(\tau)dl(\tau)\cdot\frac{1}{\rho}\iint_{\sigma_q(B,\rho)}h_2(\tau)dl(\tau).
\end{aligned}$$

Dividing each sides by $4\omega_q^2$, we get the required identity. □

The subsequent Grüss type inequality is valid.

Theorem 2. *Let $q=\frac{p}{p-1}$ $(p>1)$ and $h_1,h_2:\sigma_q(B,\rho)\to\mathbb{R}$ be two measurable functions defined on the q-circle $\sigma_q(B,\rho)$ satisfying the conditions*

$$|h_1(z_1(\vartheta),z_2(\vartheta))-h_1(z_1(\varphi),z_2(\varphi))|\le H_1\rho^{\kappa_1}\left|C_{\frac{q-1}{q}}(\vartheta)\right|^{\kappa_2}+H_2\rho^{\kappa_3}\left|S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa_4} \tag{5}$$

and

$$|h_2(z_1(\vartheta),z_2(\vartheta))-h_2(z_1(\varphi),z_2(\varphi))|\le H_3\rho^{\kappa_5}\left|C_{\frac{q-1}{q}}(\vartheta)\right|^{\kappa_6}+H_4\rho^{\kappa_7}\left|S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa_8} \tag{6}$$

for all $\vartheta,\varphi\in[0,2\omega_q]$, some constants $H_1,H_2,H_3,H_4>0$, and $\kappa_j\in[0,\infty)$ $(i=\overline{1,8})$. Then

$$\begin{aligned}
|\mathscr{T}(h_1,h_2)|\le\frac{M}{p\omega_q^2}\cdot\Bigg[&\omega_q\rho^{\kappa_1+\kappa_5}\mathrm{B}\left(\frac{\kappa_2+\kappa_6+1}{q},\frac{1}{q}\right)\\
&+\frac{2\rho^{\kappa_1+\kappa_7}}{q}\mathrm{B}\left(\frac{\kappa_2+1}{q},\frac{1}{q}\right)\cdot\mathrm{B}\left(\frac{\kappa_8+1}{q},\frac{1}{q}\right)\\
&+\frac{2\rho^{\kappa_3+\kappa_5}}{q}\mathrm{B}\left(\frac{\kappa_4+1}{q},\frac{1}{q}\right)\cdot\mathrm{B}\left(\frac{\kappa_6+1}{q},\frac{1}{q}\right)\\
&+\omega_q\rho^{\kappa_3+\kappa_7}\mathrm{B}\left(\frac{\kappa_4+\kappa_8+1}{q},\frac{1}{q}\right)\Bigg],
\end{aligned} \tag{7}$$

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where $\mathrm{B}(\cdot,\cdot)$ *is the Euler Beta function, and* $M := \max\{H_1H_3, H_1H_4, H_2H_3, H_2H_4\}$

Proof. By employing Lemma 1, along with the assumptions in (5)–(6), the triangle inequality, and an application of integration by parts, one obtains the following:

$$
\begin{aligned}
&\frac{1}{2}\Big|\int_0^{2\omega_q}\int_0^{2\omega_q}(h_1(z_1(\varphi),z_2(\varphi))-h_1(z_1(\vartheta),z_2(\vartheta)))\\
&\qquad\qquad \times\ (h_2(z_1(\varphi),z_2(\varphi))-h_2(z_1(\vartheta),z_2(\vartheta)))d\vartheta d\varphi\Big|\\
&\le \frac{1}{2}\int_0^{2\omega_q}\int_0^{2\omega_q}|h_1(z_1(\varphi),z_2(\varphi))-h_1(z_1(\vartheta),z_2(\vartheta))|\\
&\qquad\qquad \times |h_2(z_1(\varphi),z_2(\varphi))-h_2(z_1(\vartheta),z_2(\vartheta))|d\vartheta d\varphi\\
&= \frac{1}{2}\int_0^{2\omega_q}\int_0^{2\omega_q}|h_1(z_1(\vartheta),z_2(\vartheta))-h_1(z_1(\varphi),z_2(\varphi))|\\
&\quad \times |h_2(z_1(\vartheta),z_2(\vartheta))-h_2(z_1(\varphi),z_2(\varphi))|d\vartheta d\varphi\\
&\le \frac{1}{2}H_1H_3\rho^{\kappa_1+\kappa_5}\int_0^{2\omega_q}\int_0^{2\omega_q}\left|C_{\frac{q-1}{q}}(\vartheta)\right|^{\kappa_2+\kappa_6}d\vartheta d\varphi\\
&\quad +\frac{1}{2}H_1H_4\rho^{\kappa_1+\kappa_7}\int_0^{2\omega_q}\left|C_{\frac{q-1}{q}}(\vartheta)\right|^{\kappa_2}d\vartheta\int_0^{2\omega_q}\left|S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa_8}d\varphi\\
&\quad +\frac{1}{2}H_2H_3\rho^{\kappa_3+\kappa_5}\int_0^{2\omega_q}\left|S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa_4}d\varphi\int_0^{2\omega_q}\left|C_{\frac{q-1}{q}}(\vartheta)\right|^{\kappa_6}d\vartheta\\
&\quad +\frac{1}{2}H_2H_4\rho^{\kappa_3+\kappa_7}\int_0^{2\omega_q}\int_0^{2\omega_q}\left|S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa_4+\kappa_8}d\vartheta d\varphi
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}H_1H_3\rho^{\kappa_1+\kappa_5}\cdot 2\omega_q\cdot 4\int_0^{\frac{\omega_q}{2}}\left(C_{\frac{q-1}{q}}(\vartheta)\right)^{\kappa_2+\kappa_6}d\vartheta\\
&\quad +\frac{1}{2}H_1H_4\rho^{\kappa_1+\kappa_7}\cdot 4\int_0^{\frac{\omega_q}{2}}\left(C_{\frac{q-1}{q}}(\vartheta)\right)^{\kappa_2}d\vartheta\cdot 4\int_0^{\frac{\omega_q}{2}}\left(S_{\frac{q-1}{q}}(\varphi)\right)^{\kappa_8}d\varphi\\
&\quad +\frac{1}{2}H_2H_3\rho^{\kappa_3+\kappa_5}\cdot 4\int_0^{\frac{\omega_q}{2}}\left(S_{\frac{q-1}{q}}(\varphi)\right)^{\kappa_4}d\varphi\cdot 4\int_0^{\frac{\omega_q}{2}}\left(C_{\frac{q-1}{q}}(\vartheta)\right)^{\kappa_6}d\vartheta\\
&\quad +\frac{1}{2}H_2H_4\rho^{\kappa_3+\kappa_7}\cdot 2\omega_q\cdot 4\int_0^{\frac{\omega_q}{2}}\left(S_{\frac{q-1}{q}}(\varphi)\right)^{\kappa_4+\kappa_8}d\varphi\\
&= H_1H_3\cdot\frac{4\omega_q\rho^{\kappa_1+\kappa_5}}{q}\mathrm{B}\left(\frac{\kappa_2+\kappa_6+1}{q},\frac{1}{q}\right)\\
&\quad + H_1H_4\cdot\frac{8\rho^{\kappa_1+\kappa_7}}{q^2}\mathrm{B}\left(\frac{\kappa_2+1}{q},\frac{1}{q}\right)\cdot\mathrm{B}\left(\frac{\kappa_8+1}{q},\frac{1}{q}\right)\\
&\quad + H_2H_3\cdot\frac{8\rho^{\kappa_3+\kappa_5}}{q^2}\mathrm{B}\left(\frac{\kappa_4+1}{q},\frac{1}{q}\right)\cdot\mathrm{B}\left(\frac{\kappa_6+1}{q},\frac{1}{q}\right)\\
&\quad + H_2H_4\cdot\frac{4\omega_q\rho^{\kappa_3+\kappa_7}}{q}\mathrm{B}\left(\frac{\kappa_4+\kappa_8+1}{q},\frac{1}{q}\right).
\end{aligned}
$$

Dividing both sides by $4\omega_q^2$, we get the required result in (7), and this completes the proof. □

Corollary 1. *Let* $q=\frac{p}{p-1}$ $(p>1)$ *and* $h_1,h_2:\sigma_q(B,\rho)\to\mathbb{R}$ *be two measurable functions defined on the q-circle* $\sigma_q(B,\rho)$ *satisfying the conditions*

$$|h_1(z_1(\vartheta),z_2(\vartheta))-h_1(z_1(\varphi),z_2(\varphi))|\le H_1\rho^{\alpha}\left|C_{\frac{q-1}{q}}(\vartheta)\right|^{\alpha}+H_2\rho^{\kappa}\left|S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa}$$

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and

$$|h_2(z_1(\vartheta),z_2(\vartheta))-h_2(z_1(\varphi),z_2(\varphi))|\le H_3\rho^{\gamma}\left|C_{\frac{q-1}{q}}(\vartheta)\right|^{\gamma}+H_4\rho^{\delta}\left|S_{\frac{q-1}{q}}(\varphi)\right|^{\delta}$$

for all $\vartheta,\varphi\in[0,2\omega_q]$, *some constants* $H_1,H_2,H_3,H_4>0$, *and* $\alpha,\kappa,\gamma,\delta\in[0,\infty)$. *Then, we have*

$$\begin{aligned}&|\mathscr{T}(h_1,h_2)| \\ &\le \frac{M}{p\omega_q^2}\cdot\left[\omega_q\rho^{\alpha+\gamma}\mathrm{B}\left(\frac{\alpha+\gamma+1}{q},\frac{1}{q}\right)+\frac{2\rho^{\alpha+\delta}}{q}\mathrm{B}\left(\frac{\alpha+1}{q},\frac{1}{q}\right)\cdot\mathrm{B}\left(\frac{\delta+1}{q},\frac{1}{q}\right)\right.\\ &\quad\left.+\frac{2\rho^{\kappa+\gamma}}{q}\mathrm{B}\left(\frac{\kappa+1}{q},\frac{1}{q}\right)\cdot\mathrm{B}\left(\frac{\gamma+1}{q},\frac{1}{q}\right)+\omega_q\rho^{\kappa+\delta}\mathrm{B}\left(\frac{\kappa+\delta+1}{q},\frac{1}{q}\right)\right],\end{aligned} \tag{8}$$

where $\mathrm{B}(\cdot,\cdot)$ *is the Euler Beta function, and* $M:=\max\{H_1H_3,H_1H_4,H_2H_3,H_2H_4\}$.

Proof. Setting $\kappa_1=\kappa_2=\alpha$, $\kappa_3=\kappa_4=\kappa$, $\kappa_5=\kappa_6=\gamma$, and $\kappa_7=\kappa_8=\delta$ in (7) we get the required result. □

Corollary 2. *Let* $h_1,h_2:\sigma(B,\rho)\to\mathbb{R}$ *be two measurable functions defined on the (the classical) circle* $\sigma(B,\rho)$ *satisfying the conditions*

$$\begin{aligned}&\left|h_1\left(b_1+\rho C_{\frac12}(\vartheta),b_2+\rho S_{\frac12}(\vartheta)\right)-h_1\left(b_1+\rho C_{\frac12}(\varphi),b_2+\rho S_{\frac12}(\varphi)\right)\right|\\ &\qquad\le H_1\rho^{\alpha}\left|C_{\frac12}(\vartheta)\right|^{\alpha}+H_2\rho^{\kappa}\left|S_{\frac12}(\varphi)\right|^{\kappa}\end{aligned}$$

and

$$\begin{aligned}&\left|h_2\left(b_1+\rho C_{\frac12}(\vartheta),b_2+\rho S_{\frac12}(\vartheta)\right)-h_2\left(b_1+\rho C_{\frac12}(\varphi),b_2+\rho S_{\frac12}(\varphi)\right)\right|\\ &\qquad\le H_3\rho^{\gamma}\left|C_{\frac12}(\vartheta)\right|^{\gamma}+H_4\rho^{\delta}\left|S_{\frac12}(\varphi)\right|^{\delta}\end{aligned}$$

for all $\vartheta,\varphi\in[0,2\pi]$, *some constants* $H_1,H_2,H_3,H_4>0$, *and* $\alpha,\kappa,\gamma,\delta\in[0,\infty)$. *Then*

$$\begin{aligned}&|\mathscr{T}(h_1,h_2)| \\ &\le \frac{M}{2\pi^2}\cdot\left[\pi\rho^{\alpha+\gamma}\mathrm{B}\left(\frac{\alpha+\gamma+1}{2},\frac{1}{2}\right)+\rho^{\alpha+\delta}\mathrm{B}\left(\frac{\alpha+1}{2},\frac{1}{2}\right)\cdot\mathrm{B}\left(\frac{\delta+1}{2},\frac{1}{2}\right)\right.\\ &\quad\left.+\rho^{\kappa+\gamma}\mathrm{B}\left(\frac{\kappa+1}{2},\frac{1}{2}\right)\cdot\mathrm{B}\left(\frac{\gamma+1}{2},\frac{1}{2}\right)+\pi\rho^{\kappa+\delta}\mathrm{B}\left(\frac{\kappa+\delta+1}{2},\frac{1}{2}\right)\right],\end{aligned} \tag{9}$$

where $M:=\max\{H_1H_3,H_1H_4,H_2H_3,H_2H_4\}$.

Proof. Setting $q=2$ in (8), we get required result. □

Corollary 3. *Let* $h_1,h_2:\sigma(B,\rho)\to\mathbb{R}$ *be two measurable functions defined on the (the classical) circle* $\sigma(B,\rho)$ *satisfying the conditions*

$$\begin{aligned}&\left|h_1\left(b_1+\rho C_{\frac12}(\vartheta),b_2+\rho S_{\frac12}(\vartheta)\right)-h_1\left(b_1+\rho C_{\frac12}(\varphi),b_2+\rho S_{\frac12}(\varphi)\right)\right|\\ &\qquad\le H_1\rho\left|C_{\frac12}(\vartheta)\right|+H_2\rho\left|S_{\frac12}(\varphi)\right|\end{aligned}$$

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and

$$\left|h_2\left(b_1+\rho C_{\frac{1}{2}}(\vartheta), b_2+\rho S_{\frac{1}{2}}(\vartheta)\right)-h_2\left(b_1+\rho C_{\frac{1}{2}}(\varphi), b_2+\rho S_{\frac{1}{2}}(\varphi)\right)\right| \leq H_3\rho\left|C_{\frac{1}{2}}(\vartheta)\right|+H_4\rho\left|S_{\frac{1}{2}}(\varphi)\right|$$

for all $\vartheta,\varphi\in[0,2\pi]$. *Then*

$$|\mathscr{T}(h_1,h_2)|\leq M\rho^2\cdot\left(\frac{1}{2}+\frac{4}{\pi^2}\right), \tag{10}$$

where $M:=\max\{H_1H_3, H_1H_4, H_2H_3, H_2H_4\}$.

Proof. Setting $\alpha=\kappa=\gamma=\delta=1$, $q=2$ and $M=\max\{H_1H_3, H_1H_4, H_2H_3, H_2H_4\}$ in (9), we get required result. □

Theorem 3. *Let* $q=\frac{p}{p-1}$ $(p>1)$ *and* $h_1,h_2:\sigma_q(B,\rho)\to\mathbb{R}$ *be two measurable functions defined in the* q*-circle* $\sigma_q(B,\rho)$ *satisfying the conditions*

$$|h_1(z_1(\vartheta),z_2(\vartheta))-h_1(z_1(\varphi),z_2(\varphi))|\leq H_1\rho^{\kappa_1}\left|C_{\frac{q-1}{q}}(\vartheta)\right|^{\kappa_2}\left|S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa_3} \tag{11}$$

and

$$|h_2(z_1(\vartheta),z_2(\vartheta))-h_2(z_1(\varphi),z_2(\varphi))|\leq H_2\rho^{\kappa_4}\left|C_{\frac{q-1}{q}}(\vartheta)\right|^{\kappa_5}\left|S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa_6} \tag{12}$$

for all $\vartheta,\varphi\in[0,2\omega_q]$, *some constants* $H_1,H_2>0$, *and* $\kappa_j\in(0,\infty)$ $(i=\overline{1,6})$. *Then*

$$|\mathscr{T}(h_1,h_2)| \leq H_1H_2\left(\frac{2\rho^{\kappa_1+\kappa_4}}{q^2\omega_q^2}\right)\mathrm{B}^{\frac{1}{\eta}}\left(\frac{\eta\kappa_2+1}{q},\frac{1}{q}\right)\mathrm{B}^{\frac{1}{\eta}}\left(\frac{\eta\kappa_3+1}{q},\frac{1}{q}\right)\times\mathrm{B}^{\frac{1}{\nu}}\left(\frac{\nu\kappa_5+1}{q},\frac{1}{q}\right)\mathrm{B}^{\frac{1}{\nu}}\left(\frac{\nu\kappa_6+1}{q},\frac{1}{q}\right), \tag{13}$$

for all $\eta,\nu>1$ *such that* $\frac{1}{\eta}+\frac{1}{\nu}=1$.

Proof. Following a similar approach to that used in the proof of Theorem 4, and utilizing the Hölder inequality together with the conditions in (11)–(12), the result follows:

$$\begin{aligned}
&\frac{1}{2}\bigg|\int_0^{2\omega_q}\int_0^{2\omega_q}(h_1(z_1(\varphi),z_2(\varphi))-h_1(z_1(\vartheta),z_2(\vartheta)))\\
&\qquad\times(h_2(z_1(\varphi),z_2(\varphi))-h_2(z_1(\vartheta),z_2(\vartheta)))d\vartheta d\varphi\bigg|\\
&=\frac{1}{2}\int_0^{2\omega_q}\int_0^{2\omega_q}|h_1(z_1(\varphi),z_2(\varphi))-h_1(z_1(\vartheta),z_2(\vartheta))|\\
&\qquad\times|h_2(z_1(\varphi),z_2(\varphi))-h_2(z_1(\vartheta),z_2(\vartheta))|d\vartheta d\varphi\\
&\leq\frac{1}{2}\left(\int_0^{2\omega_q}\int_0^{2\omega_q}|h_1(z_1(\varphi),z_2(\varphi))-h_1(z_1(\vartheta),z_2(\vartheta))|^{\eta}d\vartheta d\varphi\right)^{1/\eta}\\
&\qquad\times\left(\int_0^{2\omega_q}\int_0^{2\omega_q}|h_2(z_1(\varphi),z_2(\varphi))-h_2(z_1(\vartheta),z_2(\vartheta))|^{\nu}d\vartheta d\varphi\right)^{1/\nu}\\
&=\frac{1}{2}\left(\int_0^{2\omega_q}\int_0^{2\omega_q}|h_1(z_1(\vartheta),z_2(\vartheta))-h_1(z_1(\varphi),z_2(\varphi))|^{\eta}d\vartheta d\varphi\right)^{1/\eta}\\
&\quad\times\left(\int_0^{2\omega_q}\int_0^{2\omega_q}|h_2(z_1(\vartheta),z_2(\vartheta))-h_2(z_1(\varphi),z_2(\varphi))|^{\nu}d\vartheta d\varphi\right)^{1/\nu}
\end{aligned}$$

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$$= \frac{1}{2} H_1 \left(\int_0^{2\omega_q} \int_0^{2\omega_q} \rho^{\eta\kappa_1} \left| C_{\frac{q-1}{q}}(\vartheta) \right|^{\eta\kappa_2} \left| S_{\frac{q-1}{q}}(\varphi) \right|^{\eta\kappa_3} d\vartheta d\varphi \right)^{1/\eta} \times H_2 \left(\int_0^{2\omega_q} \int_0^{2\omega_q} \rho^{\nu\kappa_4} \left| C_{\frac{q-1}{q}}(\vartheta) \right|^{\nu\kappa_5} \left| S_{\frac{q-1}{q}}(\varphi) \right|^{\nu\kappa_6} d\vartheta d\varphi \right)^{1/\nu}. \tag{14}$$

Now, since

$$\begin{aligned} &\int_0^{2\omega_q} \int_0^{2\omega_q} \rho^{\eta\kappa_1} \left| C_{\frac{q-1}{q}}(\vartheta) \right|^{\eta\kappa_2} \left| S_{\frac{q-1}{q}}(\varphi) \right|^{\eta\kappa_3} d\vartheta d\varphi \\ &= \rho^{\eta\kappa_1} \int_0^{2\omega_q} \left| C_{\frac{q-1}{q}}(\vartheta) \right|^{\eta\kappa_2} d\vartheta \int_0^{2\omega_q} \left| S_{\frac{q-1}{q}}(\varphi) \right|^{\eta\kappa_3} d\varphi \\ &= \rho^{\eta\kappa_1} \times 4 \cdot \int_0^{\frac{\omega_q}{2}} \left(C_{\frac{q-1}{q}}(\vartheta) \right)^{\eta\kappa_2} d\vartheta \times 4 \cdot \int_0^{\frac{\omega_q}{2}} \left(S_{\frac{q-1}{q}}(\varphi) \right)^{\eta\kappa_3} d\varphi \\ &= \rho^{\eta\kappa_1} \frac{4}{q} \mathrm{B}\left(\frac{\eta\kappa_2 + 1}{q}, \frac{1}{q} \right) \times \frac{4}{q} \mathrm{B}\left(\frac{\eta\kappa_3 + 1}{q}, \frac{1}{q} \right). \end{aligned} \tag{15}$$

Similarly, we can obtain that

$$\int_0^{2\omega_q} \int_0^{2\omega_q} \rho^{\nu\kappa_4} \left| C_{\frac{q-1}{q}}(\vartheta) \right|^{\nu\kappa_5} \left| S_{\frac{q-1}{q}}(\varphi) \right|^{\nu\kappa_6} d\vartheta d\varphi = \rho^{\nu\kappa_4} \frac{4}{q} \mathrm{B}\left(\frac{\nu\kappa_5 + 1}{q}, \frac{1}{q} \right) \times \frac{4}{q} \mathrm{B}\left(\frac{\nu\kappa_6 + 1}{q}, \frac{1}{q} \right). \tag{16}$$

Substituting (15) and (16) into (14), and then dividing both sides by $4\omega_q^2$, we get the required result in (13), and this completes the proof. □

Corollary 4. *Under the assumptions of Theorem 5, we have*

$$|\mathscr{T}(h_1, h_2)| \le H_1 H_2 \left(\frac{2\rho^{\kappa_1+\kappa_4}}{q^2 \omega_q^2} \right) \mathrm{B}^{\frac{1}{2}}\left(\frac{2\kappa_2 + 1}{q}, \frac{1}{q} \right) \mathrm{B}^{\frac{1}{2}}\left(\frac{2\kappa_3 + 1}{q}, \frac{1}{q} \right) \times \mathrm{B}^{\frac{1}{2}}\left(\frac{2\kappa_5 + 1}{q}, \frac{1}{q} \right) \mathrm{B}^{\frac{1}{2}}\left(\frac{2\kappa_6 + 1}{q}, \frac{1}{q} \right) \tag{17}$$

Proof. Setting $\eta = \nu = 2$ in (13). □

Corollary 5. *Under the assumptions of Theorem 5, we have*

$$|\mathscr{T}(h_1, h_2)| \le H_1 H_2 \left(\frac{2\rho^{\alpha+\kappa}}{q^2 \omega_q^2} \right) \mathrm{B}^{\frac{1}{2}}\left(\frac{2\alpha + 1}{q}, \frac{1}{q} \right) \mathrm{B}^{\frac{1}{2}}\left(\frac{2\alpha + 1}{q}, \frac{1}{q} \right) \times \mathrm{B}^{\frac{1}{2}}\left(\frac{2\kappa + 1}{q}, \frac{1}{q} \right) \mathrm{B}^{\frac{1}{2}}\left(\frac{2\kappa + 1}{q}, \frac{1}{q} \right) \tag{18}$$

Proof. Setting $\kappa_1 = \kappa_2 = \kappa_3 = \alpha$ and $\kappa_4 = \kappa_5 = \kappa_6 = \kappa$ in (17), we get the required result. □

Corollary 6. *Under the assumptions of Theorem 5, we have*

$$|\mathscr{T}(h_1, h_2)| \le H_1 H_2 \left(\frac{2\rho^2}{q^2 \omega_q^2} \right) \mathrm{B}^2\left(\frac{3}{q}, \frac{1}{q} \right). \tag{19}$$

Proof. Setting $\alpha = \kappa = 1$ in (18) we get the required result. □

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Corollary 7. *Let $h_1, h_2 : D_2(B,\rho) \to \mathbb{R}$ be two measurable functions defined on the circle $D_2(B,\rho)$. Then*

$$|\mathscr{T}(h_1,h_2)| \leq \frac{1}{8}H_1H_2\rho^2. \tag{20}$$

Proof. Setting $q=2$ in (19), we get the required result. □

Theorem 4. *Let $q=\frac{p}{p-1}$ $(p>1)$ and $h_1, h_2 : \sigma_q(B,\rho) \to \mathbb{R}$ be functions defined in the q-circle $\sigma_q(B,\rho)$ satisfying the conditions*

$$\begin{aligned}&|h_1(z_1(\vartheta),z_2(\vartheta)) - h_1(z_1(\varphi),z_2(\varphi))| \\ &\quad\leq H_1\rho^{\kappa_1}\left|C_{\frac{q-1}{q}}(\vartheta) - S_{\frac{q-1}{q}}(\vartheta)\right|^{\kappa_2} + H_2\rho^{\kappa_3}\left|C_{\frac{q-1}{q}}(\varphi) - S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa_4}\end{aligned} \tag{21}$$

and

$$\begin{aligned}&|h_2(z_1(\vartheta),z_2(\vartheta)) - h_2(z_1(\varphi),z_2(\varphi))| \\ &\quad\leq H_3\rho^{\kappa_5}\left|C_{\frac{q-1}{q}}(\vartheta) - S_{\frac{q-1}{q}}(\vartheta)\right|^{\kappa_6} + H_4\rho^{\kappa_7}\left|C_{\frac{q-1}{q}}(\varphi) - S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa_8}\end{aligned} \tag{22}$$

for all $\vartheta, \varphi \in [0, 2\omega_q]$, some constants $H_1, H_2, H_3, H_4 > 0$, $\kappa_1, \kappa_3, \kappa_5, \kappa_7 \in [0,\infty)$, and positive even integers $\kappa_2, \kappa_4, \kappa_6, \kappa_8$. Then

$$\begin{aligned}&|\mathscr{T}(h_1,h_2)| \\ &\leq \frac{H_1H_3\rho^{\kappa_1+\kappa_5}}{4\omega_q}\sum_{\ell=0}^{\kappa_2+\kappa_6}(-1)^\ell\binom{\kappa_2+\kappa_6}{\ell}\frac{3+(-1)^\ell}{q}\mathrm{B}\left(\frac{\kappa_2+\kappa_6-\ell+1}{q},\frac{\ell+1}{q}\right) \\ &\quad+\frac{H_1H_4\rho^{\kappa_1+\kappa_7}}{8\omega_q^2}\left(\sum_{\ell=0}^{\kappa_2}(-1)^\ell\binom{\kappa_2}{\ell}\frac{3+(-1)^\ell}{q}\mathrm{B}\left(\frac{\kappa_2-\ell+1}{q},\frac{\ell+1}{q}\right)\right) \\ &\qquad\times\left(\sum_{\ell=0}^{\kappa_8}(-1)^\ell\binom{\kappa_8}{\ell}\frac{3+(-1)^\ell}{q}\mathrm{B}\left(\frac{\kappa_8-\ell+1}{q},\frac{\ell+1}{q}\right)\right) \\ &\quad+\frac{H_2H_3\rho^{\kappa_3+\kappa_5}}{8\omega_q^2}\left(\sum_{\ell=0}^{\kappa_4}(-1)^\ell\binom{\kappa_4}{\ell}\frac{3+(-1)^\ell}{q}\mathrm{B}\left(\frac{\kappa_4-\ell+1}{q},\frac{\ell+1}{q}\right)\right) \\ &\qquad\times\left(\sum_{\ell=0}^{\kappa_6}(-1)^\ell\binom{\kappa_6}{\ell}\frac{3+(-1)^\ell}{q}\mathrm{B}\left(\frac{\kappa_6-\ell+1}{q},\frac{\ell+1}{q}\right)\right) \\ &\quad+\frac{H_2H_4\rho^{\kappa_3+\kappa_7}}{4\omega_q}\sum_{\ell=0}^{\kappa_4+\kappa_8}(-1)^\ell\binom{\kappa_4+\kappa_8}{\ell}\frac{3+(-1)^\ell}{q}\mathrm{B}\left(\frac{\kappa_4+\kappa_8-\ell+1}{q},\frac{\ell+1}{q}\right).\end{aligned} \tag{23}$$

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Proof. Utilizing the integral version of the triangle inequality along with the results from equations (21) through (22), we arrive at the following conclusion:

$$\begin{aligned}
&\frac{1}{2}\Big|\int_0^{2\omega_q}\int_0^{2\omega_q}(h_1(z_1(\varphi),z_2(\varphi))-h_1(z_1(\vartheta),z_2(\vartheta))) \\
&\qquad\qquad \times (h_2(z_1(\varphi),z_2(\varphi))-h_2(z_1(\vartheta),z_2(\vartheta)))d\vartheta d\varphi\Big| \\
&\leq \frac{1}{2}\int_0^{2\omega_q}\int_0^{2\omega_q}|h_1(z_1(\varphi),z_2(\varphi))-h_1(z_1(\vartheta),z_2(\vartheta))| \\
&\qquad\qquad \times |h_2(z_1(\varphi),z_2(\varphi))-h_2(z_1(\vartheta),z_2(\vartheta))|d\vartheta d\varphi \\
&= \frac{1}{2}\int_0^{2\omega_q}\int_0^{2\omega_q}|h_1(z_1(\vartheta),z_2(\vartheta))-h_1(z_1(\varphi),z_2(\varphi))| \\
&\quad \times |h_2(z_1(\vartheta),z_2(\vartheta))-h_2(z_1(\varphi),z_2(\varphi))|d\vartheta d\varphi \\
&\leq \frac{1}{2}H_1H_3\int_0^{2\omega_q}d\varphi\int_0^{2\omega_q}\Big|C_{\frac{q-1}{q}}(\vartheta)-S_{\frac{q-1}{q}}(\vartheta)\Big|^{\kappa_2+\kappa_6}d\vartheta \\
&\quad +\frac{1}{2}H_1H_4\int_0^{2\omega_q}\Big|C_{\frac{q-1}{q}}(\vartheta)-S_{\frac{q-1}{q}}(\vartheta)\Big|^{\kappa_2}d\vartheta\int_0^{2\omega_q}\Big|C_{\frac{q-1}{q}}(\varphi)-S_{\frac{q-1}{q}}(\varphi)\Big|^{\kappa_8}d\varphi \\
&\quad +\frac{1}{2}H_2H_3\int_0^{2\omega_q}\Big|C_{\frac{q-1}{q}}(\varphi)-S_{\frac{q-1}{q}}(\varphi)\Big|^{\kappa_4}d\varphi\int_0^{2\omega_q}\Big|C_{\frac{q-1}{q}}(\vartheta)-S_{\frac{q-1}{q}}(\vartheta)\Big|^{\kappa_6}d\vartheta \\
&\quad +\frac{1}{2}H_2H_4\int_0^{2\omega_q}d\vartheta\int_0^{2\omega_q}\Big|C_{\frac{q-1}{q}}(\varphi)-S_{\frac{q-1}{q}}(\varphi)\Big|^{\kappa_4+\kappa_8}d\varphi.
\end{aligned} \tag{24}$$

Let $j = 2,4,6,8$. From the binomial theorem, we deduce that

$$\begin{aligned}
\Big|C_{\frac{q-1}{q}}(\vartheta)-S_{\frac{q-1}{q}}(\vartheta)\Big|^{\kappa_j} &= \Big(C_{\frac{q-1}{q}}(\vartheta)-S_{\frac{q-1}{q}}(\vartheta)\Big)^{\kappa_j} \qquad \text{(since } \kappa_j \text{ is an even integer)} \\
&= \sum_{\ell=0}^{\kappa_j}(-1)^{\ell}\binom{\kappa_j}{\ell}\Big(C_{\frac{q-1}{q}}(\vartheta)\Big)^{\kappa_j-k}\Big(S_{\frac{q-1}{q}}(\vartheta)\Big)^{\ell}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^{2\omega_q}\Big(C_{\frac{q-1}{q}}(\vartheta)-S_{\frac{q-1}{q}}(\vartheta)\Big)^{\kappa_j}d\vartheta \\
&\qquad = \sum_{\ell=0}^{\kappa_j}(-1)^{\ell}\binom{\kappa_j}{\ell}\underbrace{\int_0^{2\omega_q}\Big(C_{\frac{q-1}{q}}(\vartheta)\Big)^{\kappa_j-\ell}\Big(S_{\frac{q-1}{q}}(\vartheta)\Big)^{\ell}d\vartheta}_{J}.
\end{aligned} \tag{25}$$

To evaluate the integral J, we need to represent the Euler Beta function in terms of the new sine $S_{\frac{q-1}{q}}$ and cosine $C_{\frac{q-1}{q}}$, which is slightly different from the well known formula

$$\mathrm{B}(u,v) = 2\int_0^{\frac{\pi}{2}}\sin^{2u-1}(\vartheta)\cos^{2v-1}(\vartheta)d\vartheta.$$

To do this, recall that since

$$\mathrm{B}(u,v) = \int_0^1 w^{u-1}(1-w)^{v-1}dw,$$

by substituting $w = s^q$, we get

$$\mathrm{B}(u,v) = p\int_0^1 s^{qu-1}(1-s^q)^{v-1}ds.$$

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Substituting $s = S_{\frac{q-1}{q}}(\vartheta)$, so $ds = C^{q-1}_{\frac{q-1}{q}}(\vartheta)d\vartheta$, and simplifying we get

$$\mathrm{B}(u,v) = q\int_0^{\frac{\omega_q}{2}} \left(S_{\frac{q-1}{q}}(\vartheta)\right)^{qu-1} \left(C_{\frac{q-1}{q}}(\vartheta)\right)^{qv-1} d\vartheta, \tag{26}$$

This representation generalizes the traditional expression in the specific case where $q = 2$, offering significant benefits for the subsequent analytical process. Returning to the evaluation of the integral J, we now employ an effective approach designed to streamline the computations that follow.

$$\begin{aligned}
&\int_0^{2\omega_q} \left(S_{\frac{q-1}{q}}(\vartheta)\right)^{qu-1} \left(C_{\frac{q-1}{q}}(\vartheta)\right)^{qv-1} d\vartheta \\
&= \int_0^{\frac{\omega_q}{2}} \left(S_{\frac{q-1}{q}}(\vartheta)\right)^{qu-1} \left(C_{\frac{q-1}{q}}(\vartheta)\right)^{qv-1} d\vartheta \\
&\quad + \int_{\frac{\omega_q}{2}}^{\omega_q} \left(S_{\frac{q-1}{q}}(\vartheta)\right)^{qu-1} \left(C_{\frac{q-1}{q}}(\vartheta)\right)^{qv-1} d\vartheta \\
&\quad + \int_{\omega_q}^{\frac{3\omega_q}{2}} \left(S_{\frac{q-1}{q}}(\vartheta)\right)^{qu-1} \left(C_{\frac{q-1}{q}}(\vartheta)\right)^{qv-1} d\vartheta \\
&\quad + \int_{\frac{3\omega_q}{2}}^{2\omega_q} \left(S_{\frac{q-1}{q}}(\vartheta)\right)^{qu-1} \left(C_{\frac{q-1}{q}}(\vartheta)\right)^{qv-1} d\vartheta.
\end{aligned}$$

To simplify the second, third, and fourth integrals, we perform the variable changes $\varphi = \omega_q - \vartheta$, $\varphi = \frac{3\omega_q}{2} - \vartheta$, and $\varphi = 2\omega_q - \vartheta$, respectively. After implementing these substitutions, the angle sum and difference identities associated with the generalized trigonometric functions $S_{\frac{q-1}{q}}$ and $C_{\frac{q-1}{q}}$ are utilized to reduce the expressions. These transformations naturally yield the result stated in equation (26).

$$\begin{aligned}
&\int_0^{2\omega_q} \left(S_{\frac{q-1}{q}}(\vartheta)\right)^{qu-1} \left(C_{\frac{q-1}{q}}(\vartheta)\right)^{qv-1} d\vartheta \\
&= \frac{1}{q}\mathrm{B}(u,v) - \int_{\frac{\omega_q}{2}}^{0} \left(S_{\frac{q-1}{q}}(\omega_q+\varphi)\right)^{qu-1} \left(C_{\frac{q-1}{q}}(\omega_q+\varphi)\right)^{qv-1} d\varphi \\
&\quad - \int_{\frac{\omega_q}{2}}^{0} \left(S_{\frac{q-1}{q}}\left(\frac{3\omega_q}{2}+\varphi\right)\right)^{qu-1} \left(C_{\frac{q-1}{q}}\left(\frac{3\omega_q}{2}+\varphi\right)\right)^{qv-1} d\varphi \\
&\quad - \int_{\frac{\omega_q}{2}}^{0} \left(S_{\frac{q-1}{q}}(2\omega_q+\varphi)\right)^{qu-1} \left(C_{\frac{q-1}{q}}(2\omega_q+\varphi)\right)^{qv-1} d\varphi \\
&= \frac{1}{q}\mathrm{B}(u,v) + \int_0^{\frac{\omega_q}{2}} \left(-S_{\frac{q-1}{q}}(\varphi)\right)^{qu-1} \left(-C_{\frac{q-1}{q}}(\varphi)\right)^{qv-1} d\varphi \\
&\quad + \int_0^{\frac{\omega_q}{2}} \left(-C_{\frac{q-1}{q}}(\varphi)\right)^{qu-1} \left(S_{\frac{q-1}{q}}(\varphi)\right)^{qv-1} d\varphi \\
&\quad + \int_0^{\frac{\omega_q}{2}} \left(S_{\frac{q-1}{q}}(\varphi)\right)^{qu-1} \left(C_{\frac{q-1}{q}}(\varphi)\right)^{qv-1} d\varphi \\
&= \frac{2}{q}\mathrm{B}(u,v) + \frac{(-1)^{qu-1+qv-1}}{q}\mathrm{B}(u,v) + \frac{(-1)^{qu-1}}{q}\mathrm{B}(u,v).
\end{aligned}$$

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But for the integral J we have, $qu-1=\kappa_j-\ell\in\mathbb{N}$ and $qv-1=\ell\in\mathbb{N}$, or we write $u=\frac{\kappa_j-\ell+1}{q}$ and $v=\frac{\ell+1}{q}$. Thus,

$$\begin{aligned} J &= \frac{2}{q}\mathrm{B}\left(\frac{\kappa_j-\ell+1}{q},\frac{\ell+1}{q}\right)+\frac{(-1)^{\kappa_j}}{q}\mathrm{B}\left(\frac{\kappa_j-\ell+1}{q},\frac{\ell+1}{q}\right) \\ &\quad+\frac{(-1)^{\kappa_j-\ell}}{q}\mathrm{B}\left(\frac{\kappa_j-\ell+1}{q},\frac{\ell+1}{q}\right) \\ &= \frac{3+(-1)^{\kappa_j-\ell}}{q}\mathrm{B}\left(\frac{\kappa_j-\ell+1}{q},\frac{\ell+1}{q}\right), \qquad \text{for } j=2,4,6,8 \end{aligned} \tag{27}$$

since κ_2 is even.

Moreover, replacing κ_j by $\kappa_2+\kappa_6$ and then by $\kappa_4+\kappa_8$ in (27), we find that

$$\int_0^{2\omega_q}\left|C_{\frac{q-1}{q}}(\vartheta)-S_{\frac{q-1}{q}}(\vartheta)\right|^{\kappa_2+\kappa_6}d\vartheta=\frac{3+(-1)^{\kappa_2+\kappa_6-k}}{q}\mathrm{B}\left(\frac{\kappa_2+\kappa_6-\ell+1}{q},\frac{\ell+1}{q}\right), \tag{28}$$

and

$$\int_0^{2\omega_q}\left|C_{\frac{q-1}{q}}(\varphi)-S_{\frac{q-1}{q}}(\varphi)\right|^{\kappa_4+\kappa_8}d\varphi=\frac{3+(-1)^{\kappa_4+\kappa_8-k}}{q}\mathrm{B}\left(\frac{\kappa_4+\kappa_8-\ell+1}{q},\frac{\ell+1}{q}\right). \tag{29}$$

To conclude the proof, we first insert equations (27)–(29) into equation (25), and then substitute the resulting expression into equation (24). After carrying out the necessary simplifications, we combine the terms as outlined in equation (23) and divide the entire expression by $4\omega_q^2$. This yields the final result as claimed. □

Corollary 8. *Under the assumptions of Theorem 4, we have*

$$\begin{aligned} |\mathcal{T}(h_1,h_2)| \le{}& \frac{4M\rho^4}{q\omega_q}\left(\mathrm{B}\left(\frac{5}{q},\frac{1}{q}\right)-2\mathrm{B}\left(\frac{4}{q},\frac{2}{q}\right)+3\mathrm{B}\left(\frac{3}{q},\frac{3}{q}\right)\right) \\ &+\frac{4M\rho^4}{q^2\omega_q^2}\left(\mathrm{B}\left(\frac{3}{q},\frac{1}{q}\right)-\mathrm{B}\left(\frac{2}{q},\frac{2}{q}\right)+\mathrm{B}\left(\frac{3}{q},\frac{1}{q}\right)\right)^2 \end{aligned} \tag{30}$$

where $\mathrm{B}(\cdot,\cdot)$ *is the Euler Beta function.*

Proof. Setting $\kappa_j=2$ $(j=1,\cdots,8)$ in Theorem 4. □

Corollary 9. *Let* $f:D_2(B,\rho)\to\mathbb{R}$ *be a function defined in the circle* $D_2(B,\rho)$*. Then*

$$|\mathcal{T}(h_1,h_2)|\le M\rho^4\left[\left(1-\frac{1}{\pi}\right)\left(3-\frac{1}{\pi}\right)-\frac{1}{2}\right].$$

Proof. Setting $q=2$ and $M=\max\{H_1H_3,H_1H_4,H_2H_3,H_2H_4\}$ in (30). □

Additional bounds, which take into account the smoothness of h_1 and the boundedness of h_2, are explored in the subsequent series of results, which further relax the conditions established in Theorem 4.

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Lemma 2. *Let* $q = \frac{p}{p-1}$ $(p > 1)$ *and* $h_1, h_2 : \sigma_q(B,\rho) \to \mathbb{R}$ *be two measurable functions defined on the q-circle* $\sigma_q(B,\rho)$*. Then, we have*

$$\mathscr{T}(h_1,h_2) = \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} \left[h_1(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_1(\tau)dl(\tau) \right] \times \left[h_2(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \right] dl(\mu)$$

Proof. One can observe that

$$\begin{aligned}
&\mathscr{T}(h_1,h_2) \\
&= \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_1(\mu)h_2(\mu)dl(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_1(\mu)dl(\mu) \cdot \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \\
&\quad - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_1(\tau)dl(\tau) \cdot \iint\limits_{\sigma_q(B,\rho)} h_2(\mu)dl(\mu) \\
&\quad\quad + \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_1(\tau)dl(\tau) \cdot \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \\
&= \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_1(\mu)h_2(\mu)dl(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_1(\tau)dl(\tau) \cdot \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \\
&= \mathscr{T}(h_1,h_2),
\end{aligned}$$

which gives the required identity. □

Theorem 5. *Let* $p = \frac{q}{q-1}$*,* $q > 1$*, and* $B = (b_1, b_2) \in \mathbb{R}^2$ *be any point. Let* $h_1, h_2 : \sigma_q(B,\rho) \to \mathbb{R}$ *be two measurable functions defined on the q-circle* $\sigma_q(B,\rho)$ *such that* $\mathrm{N}_1 \le h_2(z_1,z_2) \le \mathrm{M}_1$ *and* h_1 *satisfying the Hölder condition*

$$\begin{aligned}
&|h_1(z_1(\vartheta), z_2(\vartheta)) - h_1(z_1(\varphi), z_2(\varphi))| \\
&\le H_1\rho^{\kappa_1} \left| C_{\frac{q-1}{q}}(\varphi) \right|^{\kappa_2} \left| C_{\frac{q-1}{q}}(\vartheta) \right|^{\kappa_3} + H_2\rho^{\kappa_4} \left| S_{\frac{q-1}{q}}(\varphi) \right|^{\kappa_5} \left| S_{\frac{q-1}{q}}(\vartheta) \right|^{\kappa_6} \qquad (31)
\end{aligned}$$

for all $(z_1,z_2) \in \sigma_q(B,\rho) \backslash \{B\}$*, some constants* $H_1, H_2 > 0$*, and* $\kappa_j \in [0,\infty)$ $(i = \overline{1,6})$*. Then we have*

$$\begin{aligned}
&|\mathscr{T}(h_1,h_2)| \\
&\le \frac{1}{p\omega_q} \cdot \left[H_1\rho^{\kappa_1} \mathrm{B}\left(\frac{\kappa_3+1}{q}, \frac{1}{q} \right) + H_2\rho^{\kappa_4} \mathrm{B}\left(\frac{\kappa_6+1}{q}, \frac{1}{q} \right) \right] \cdot (\mathrm{M}_1 - \mathrm{N}_1), \qquad (32)
\end{aligned}$$

for all $\vartheta, \varphi \in [0, 2\omega_q]$*.*

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Proof. Applying the triangle inequality to the Lemma 1, we get

$$
\begin{aligned}
&|\mathscr{T}(h_1,h_2)| \\
&= \left| \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} \left[h_1(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_1(\tau)dl(\tau) \right] \right. \\
&\qquad \left. \times \left[h_2(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \right] dl(\mu) \right| \\
&\leq \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} \left| h_1(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_1(\tau)dl(\tau) \right| \\
&\qquad \times \left| h_2(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \right| dl(\mu) \\
&\leq \sup_{\sigma_q(B,\rho)} \left| h_1(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_1(\tau)dl(\tau) \right| \\
&\qquad \times \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} \left| h_2(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \right| dl(\mu).
\end{aligned} \tag{33}
$$

Let us define

$$
J = \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} \left| h_2(\mu) - \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \right|^2 dl(\mu),
$$

and this implies that

$$
\begin{aligned}
J &= \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} g^2(\mu)dl(\mu) - 2 \cdot \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\mu)dl(\mu) \cdot \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \\
&\quad + \left(\frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \right)^2 \\
&= \frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} g^2(\mu)dl(\mu) - \left(\frac{1}{2\omega_q\rho} \iint\limits_{\sigma_q(B,\rho)} h_2(\tau)dl(\tau) \right)^2.
\end{aligned}
$$

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On the other hand, we have

$$\begin{aligned}J &= \frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}\left(h_2(\mu)-\frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}h_2(\tau)dl(\tau)\right)^2 dl(\mu)\\ &= \left(\mathrm{M}_1-\frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}h_2(\tau)dl(\tau)\right)\left(\frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}h_2(\tau)dl(\tau)-\mathrm{N}_1\right)\\ &\quad-\frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}(\mathrm{M}_1-h_2(\tau))(h_2(\tau)-\mathrm{N}_1)dl(\tau).\end{aligned}$$

But since $\mathrm{N}_1 \leq h_2(t_1,t_2) \leq \mathrm{M}_1$, for all $(t_1,t_2)\in\sigma_q(B,\rho)$, we have

$$J \leqslant \left(\mathrm{M}_1-\frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}h_2(\tau)dl(\tau)\right)\left(\frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}h_2(\tau)dl(\tau)-\mathrm{N}_1\right). \tag{34}$$

Let us recall that

$$(\mu-v)(v-\varsigma)\leqslant\frac{1}{4}(\mu-\varsigma)^2, \qquad \forall\mu,v,\varsigma\in\mathbb{R}. \tag{35}$$

Applying (35) to (34), we get

$$J\leq\frac{1}{4}(\mathrm{M}_1-\mathrm{N}_1)^2, \tag{36}$$

from the Cauchy-Buniakowski-Schwarz integral inequality, it follows that

$$J\geq\left[\frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}\left|h_2(\mu)-\frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}h_2(\tau)dl(\tau)\right|dl(\mu)\right]^2,$$

which gives by (36) that

$$\frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}\left|h_2(\mu)-\frac{1}{2\omega_q\rho}\iint\limits_{\sigma_q(B,\rho)}h_2(\tau)dl(\tau)\right|dl(\mu)\leq\frac{1}{2}(\mathrm{M}_1-\mathrm{N}_1). \tag{37}$$

Now, let $\tau,\mu:[0,2\omega_q]\to\mathbb{R}^2$ be the parameterized curve

$$\tau(\vartheta)=\begin{cases}z_1(\vartheta)=b_1+\rho C_{\frac{q-1}{q}}(\vartheta),\\ z_2(\vartheta)=b_2+\rho S_{\frac{q-1}{q}}(\vartheta),\end{cases}$$

and

$$\mu(\varphi)=\begin{cases}y_1(\varphi)=b_1+\rho C_{\frac{q-1}{q}}(\varphi),\\ y_2(\varphi)=b_2+\rho S_{\frac{q-1}{q}}(\varphi),\end{cases}$$

such that $\vartheta,\varphi\in[0,2\omega_q]$.

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By applying the condition (31) we get

$$
\begin{aligned}
&\sup_{\sigma_q(B,\rho)}\left|h_1(\mu)-\frac{1}{2\omega_q\rho}\iint_{\sigma_q(B,\rho)} h_1(\tau)dl(\tau)\right| \\
&=\frac{1}{2\omega_q\rho}\cdot\sup_{\sigma_q(B,\rho)}\left|\iint_{\sigma_q(B,\rho)}(h_1(\mu)-h_1(\tau))dl(\tau)\right| \\
&\leq\frac{1}{2\omega_q\rho}\cdot\sup_{\sigma_q(B,\rho)}\iint_{\sigma_q(B,\rho)}|h_1(\mu)-h_1(\tau)|dl(\tau) \\
&\leq\frac{1}{2\omega_q\rho}\cdot\iint_{\sigma_q(B,\rho)}\sup_{\sigma_q(B,\rho)}|h_1(\mu)-h_1(\tau)|dl(\tau) \\
&=\frac{1}{2\omega_q\rho}\cdot\rho\int_0^{2\omega_q}\sup_{\varphi\in[0,2\omega_q]}\Big|h_1\Big(b_1+\rho C_{\frac{q-1}{q}}(\varphi),b_1+\rho S_{\frac{q-1}{q}}(\varphi)\Big) \\
&\qquad\qquad -h_1\Big(b_1+\rho C_{\frac{q-1}{q}}(\vartheta),b_1+\rho S_{\frac{q-1}{q}}(\vartheta)\Big)\Big|d\vartheta \\
&\leq\frac{1}{2\omega_q}\cdot\int_0^{2\omega_q}\sup_{\varphi\in[0,2\omega_q]}\Big[H_1\rho^{\kappa_1}\Big|C_{\frac{q-1}{q}}(\varphi)\Big|^{\kappa_2}\Big|C_{\frac{q-1}{q}}(\vartheta)\Big|^{\kappa_3} \\
&\qquad\qquad +H_2\rho^{\kappa_4}\Big|S_{\frac{q-1}{q}}(\varphi)\Big|^{\kappa_5}\Big|S_{\frac{q-1}{q}}(\vartheta)\Big|^{\kappa_6}\Big]d\vartheta \\
&\leq\frac{1}{2\omega_q}\cdot\int_0^{2\omega_q}\Big[H_1\rho^{\kappa_1}\Big|C_{\frac{q-1}{q}}(\vartheta)\Big|^{\kappa_3}+H_2\rho^{\kappa_4}\Big|S_{\frac{q-1}{q}}(\vartheta)\Big|^{\kappa_6}\Big]d\vartheta. \\
&=\frac{1}{2\omega_q}\cdot\Big[H_1\rho^{\kappa_1}\int_0^{2\omega_q}\Big|C_{\frac{q-1}{q}}(\vartheta)\Big|^{\kappa_3}d\vartheta+H_2\rho^{\kappa_4}\int_0^{2\omega_q}\Big|S_{\frac{q-1}{q}}(\vartheta)\Big|^{\kappa_6}d\vartheta\Big] \\
&=\frac{1}{2\omega_q}\cdot\Big[H_1\rho^{\kappa_1}\cdot 4\cdot\int_0^{\frac{\omega_q}{2}}\Big(C_{\frac{q-1}{q}}(\vartheta)\Big)^{\kappa_3}d\vartheta+H_2\rho^{\kappa_4}\cdot 4\cdot\int_0^{\frac{\omega_q}{2}}\Big(S_{\frac{q-1}{q}}(\vartheta)\Big)^{\kappa_6}d\vartheta\Big] \\
&=\frac{2}{p\omega_q}\cdot\Big[H_1\rho^{\kappa_1}\mathrm{B}\Big(\frac{\kappa_3+1}{q},\frac{1}{q}\Big)+H_2\rho^{\kappa_4}\mathrm{B}\Big(\frac{\kappa_6+1}{q},\frac{1}{q}\Big)\Big].
\end{aligned} \tag{38}
$$

Combining the inequalities established in (37), (38) and (33), we obtain the required result. □

Corollary 10. *Under the assumptions of Theorem 5, we have*

$$
|\mathscr{T}(h_1,h_2)|\leq K\cdot\frac{2\rho}{p\omega_q}(\mathrm{M}_1-\mathrm{N}_1)\mathrm{B}\Big(\frac{2}{q},\frac{1}{q}\Big), \tag{39}
$$

where $K=\max\{H_1,H_2\}$.

Proof. Setting $\kappa_j=1$ $(i=1,\cdots,6)$ in (32). □

Corollary 11. *Under the assumptions of Theorem 5, we have*

$$
|\mathscr{T}(h_1,h_2)|\leq\frac{1}{2\pi}\cdot\Big[H_1\rho^{\kappa_1}\mathrm{B}\Big(\frac{\kappa_3+1}{q},\frac{1}{q}\Big)+H_2\rho^{\kappa_4}\mathrm{B}\Big(\frac{\kappa_6+1}{2},\frac{1}{2}\Big)\Big](\mathrm{M}_1-\mathrm{N}_1). \tag{40}
$$

Proof. Setting $q=2$ in (32). □

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Corollary 12. *Under the assumptions of Theorem 5, we have*

$$|\mathscr{T}(h_1,h_2)| \leq K \cdot \frac{2\rho}{\pi}(\mathrm{M}_1 - \mathrm{N}_1), \tag{41}$$

where $K = \max\{H_1, H_2\}$

Proof. Setting $\kappa_j = 1$ $(i = 1, \cdots, 6)$ in (40). □

3. Application: Signal Approximation and Error Bounds on Circular Domains

In many applied fields such as physics, engineering, and data science, signals are often defined over circular or periodic domains—such as antennas, rotating machinery, or wavefronts. In this section, we demonstrate how the Chebyshev functional and its associated bound (20) (for example) can be used to analyze signal interactions and derive meaningful error estimates in such settings.

3.1. Physical and Geometric Insight

The Chebyshev functional

$$\mathscr{T}(h_1,h_2) = \frac{1}{2\pi\rho}\int_{\sigma(B,\rho)} h_1(\tau)h_2(\tau)\,dl(\tau) - \left(\frac{1}{2\pi\rho}\int_{\sigma(B,\rho)} h_1(\tau)\,dl(\tau)\right)\left(\frac{1}{2\pi\rho}\int_{\sigma(B,\rho)} h_2(\tau)\,dl(\tau)\right) \tag{42}$$

can be interpreted as a measure of non-uniform interaction between two signals h_1 and h_2 distributed along the boundary of the circle $\sigma(B,\rho)$. Geometrically, it quantifies the deviation between the average value of the product and the product of averages—serving as an indicator of correlation or variation in signal alignment across the boundary.

If h_1 and h_2 represent physical quantities such as temperature and pressure, or signal amplitude and phase, then $\mathscr{T}(h_1,h_2)$ reflects how these quantities interact or fluctuate together along the circular domain.

3.2. Signal Smoothness and Domain Radius

Let h_1 and h_2 be real-valued signals defined on the circular domain, satisfying the following smoothness condition:

$$|h_i(z(\theta)) - h_i(z(\varphi))| \leq H_i\rho|\cos(\theta)||\sin(\varphi)|, \quad \text{for } i = 1,2.$$

This inequality indicates that the signal variation is controlled by the geometry of the domain and the smoothness constants H_1, H_2. Signals with lower H_i vary less and are therefore smoother on the circular boundary.

Given this assumption, we established the sharp Chebyshev-type inequality:

$$|\mathscr{T}(h_1,h_2)| \leq \frac{H_1H_2\rho^2}{8}.$$

This bound provides a direct link between signal smoothness and domain radius: larger ρ allows greater potential variation, whereas smaller H_1, H_2 reflect smoother signals, minimizing the interaction error.

Results:

- Chebyshev functional value: $\mathscr{T}(h_1,h_2) \approx 0.0028$
- Theoretical upper bound: $\frac{H_1H_2\rho^2}{8} = 0.003125$

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3.3. Signal Reference on Circular Domain

To illustrate the signal behavior, we consider two example functions:

$$h_1(x,y) = \left(\frac{x-b_1}{\rho}\right)^2, \quad h_2(x,y) = \left(\frac{y-b_2}{\rho}\right)^2,$$

defined on the circle $\sigma(B,\rho)$, representing radial-symmetric energy-like signals. Their values are evaluated over the parameterized circle:

$$z(\theta) = (b_1 + \rho\cos(\theta), b_2 + \rho\sin(\theta)), \quad \theta \in [0, 2\pi].$$

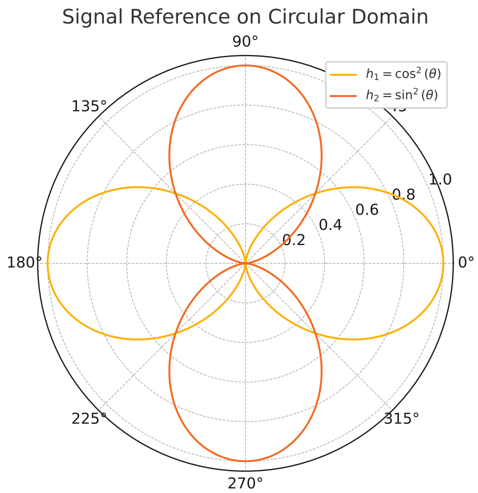

Figure 2. Signal Reference: $h_1 = \cos^2(\theta)$, $h_2 = \sin^2(\theta)$ on a circular domain.

3.4. Product Interaction — Chebyshev Functional

We now visualize the product $h_1(z(\theta)) \cdot h_2(z(\theta))$ and contrast it with the average product and the product of averages. This difference illustrates the interaction captured by the Chebyshev functional.

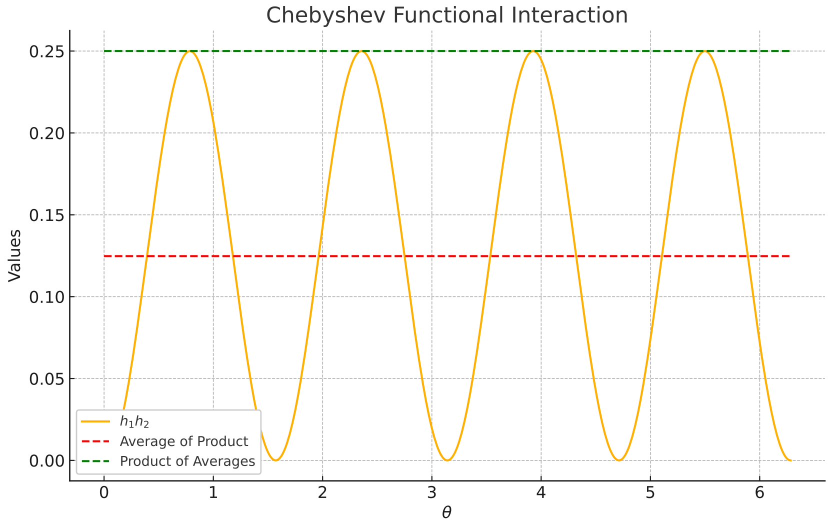

Figure 3. Product interaction: $h_1 h_2$, average of the product, and product of the averages.

3.5. Numerical Case Study: Noisy Signal Approximation

We now consider a numerical experiment where a noisy signal $s(\theta)$ is defined on the boundary:

$$s(\theta) = \cos^2(\theta) + \epsilon(\theta), \quad \epsilon(\theta) \sim \mathcal{N}(0, 0.05^2),$$

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and we approximate it using a smoother base function:

$$\tilde{s}(\theta) = \cos^2(\theta).$$

Let $h_1 = s$, $h_2 = \tilde{s}$, and compute the Chebyshev functional numerically via discretization of the integral.

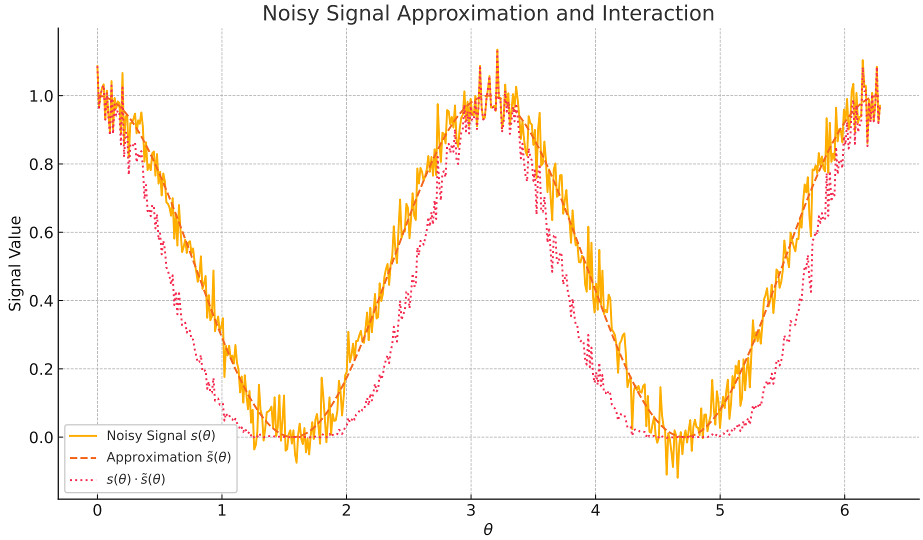

Figure 4. Noisy signal $s(\theta)$ vs. approximation $\tilde{s}(\theta)$ and their interaction.

3.6. Physical Interpretation and Observed Behavior of the Chebyshev Functional

The Chebyshev-type functional provides a powerful analytical framework for assessing the interaction between signals defined on circular domains, which commonly arise in physical and engineering systems such as rotating sensors, circular membranes, and wave propagation in cylindrical coordinates. This functional measures the deviation between the average of the product of two functions and the product of their averages over the boundary of a circle, effectively acting like a covariance measure for periodic or angular signals. A near-zero value implies weak or no correlation, while larger values indicate strong mutual dependence. The inequality $|\mathscr{T}(h_1, h_2)| \leq \frac{1}{8} H_1 H_2 \rho^2$ offers a theoretical bound on this interaction under regularity conditions on the functions, and serves as a type of energy constraint: it quantifies how much deviation or noise in one signal can influence its interaction with another, ensuring predictability and robustness in analytical models or filtering applications.

In our numerical experiments, we evaluated the Chebyshev functional for a clean signal $\tilde{s}(\theta) = \cos^2(\theta)$ and its noisy version $s(\theta) = \cos^2(\theta) + \epsilon(\theta)$, where $\epsilon(\theta)$ is additive Gaussian noise. The plot revealed that while the noisy signal introduces localized perturbations—particularly near the peaks of the base signal—the overall structure remains intact. This was quantitatively confirmed by the small value of the Chebyshev functional, approximately 0.0028, tightly bounded by the theoretical estimate of $\frac{1}{8} H_1 H_2 \rho^2 = 0.003125$ (for $H_1 = H_2 = \rho = 1$). The closeness between the computed and theoretical values highlights both the sharpness and relevance of the bound. Furthermore, the product curve $s(\theta)\tilde{s}(\theta)$ showed how noise distorts the interaction most visibly near the signal's extrema, reinforcing the importance of signal fidelity in modulation or reconstruction tasks. This behavior underscores the practical utility of the Chebyshev functional in noisy environments and confirms its role as a robust diagnostic and bounding tool for signal integrity on circular domains.

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