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The Explicit Formula for the Chebyshev–Von Mangoldt Function and the Prime Representing Constant

Prime Representing Constant in the Critical Strip

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Abstract

The Prime Representing Constant infinitely generates prime numbers. We discovered how to compute the Prime Representing Constant using the Hadamard product expansion. We also modified the classical Riemann–von Mangoldt explicit formula. While mathematically equivalent to the classical formula, the cosine-phase form is novel in its computational and structural presentation, enabling faster, memory-efficient, and intuitive computation of $\psi(x)$. Combining all the properties will give us a trace-type oscillatory operator that infinitely generates prime numbers.

Keywords: von Mangoldt function; prime representing Hadamard product expansion; constant; Riemann zeta function; Hadamard product expansion; Critical line; Critical Strip

MSC: Primary 11M99; Secondary 11M38; 11M26

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1. Introduction

The Euler product formula can be used to calculate the asymptotic probability that s randomly selected integers are set-wise coprime. The asymptotic probability that s numbers are coprime is given by a product over all primes,

$$\prod_{p\ prime}\left(1-\frac{1}{p^s}\right)=\frac{1}{\zeta(s)}$$

Bernhard Riemann's 1859 article "On the Number of Primes Less Than a Given Magnitude" [1] extended the Euler definition to a complex variable, proved its meromorphic continuation and functional equation, and established a relation between its zeros and the distribution of prime numbers.
The **classical Riemann–von Mangoldt explicit formula** for the Chebyshev function $\psi(x)$ is [2]

$$\psi(x)=x-\sum_{\rho}\frac{x^{\rho}}{\rho}-\log(2\pi)-\frac{1}{2}\log(1-x^{-2}), \tag{1}$$

where the sum runs over all nontrivial zeros ρ of the Riemann zeta function.

2. Method

Assuming the Riemann Hypothesis ($\rho = \frac{1}{2} \pm i\gamma$) and pairing conjugate zeros:

$$\sum_{\rho} \frac{x^\rho}{\rho} = \sum_{\gamma>0} \left(\frac{x^{1/2+i\gamma}}{1/2+i\gamma} + \frac{x^{1/2-i\gamma}}{1/2-i\gamma} \right) \tag{2}$$

$$= 2 \sum_{\gamma>0} \Re\left(\frac{x^{1/2+i\gamma}}{1/2+i\gamma} \right). \tag{3}$$

Using polar form:

$$\frac{x^{1/2+i\gamma}}{1/2+i\gamma} = \frac{x^{1/2}}{\sqrt{1/4+\gamma^2}} \cos\left(\gamma \ln x - \theta_\gamma\right), \qquad \theta_\gamma = \arctan(2\gamma), \tag{4}$$

so that the sum becomes a real cosine series.

$$\psi(x) - x = -x^{1/2} \sum_{\gamma>0} \frac{2\cos\left(\gamma \ln x - \arctan(2\gamma)\right)}{\sqrt{\frac{1}{4}+\gamma^2}} - \log(2\pi) - \frac{1}{2}\log\left(1 - x^{-2}\right)$$

$$\sqrt{\frac{1}{4}+\gamma^2} = \sqrt{\frac{1+4\gamma^2}{4}} = \frac{\sqrt{1+4\gamma^2}}{2} = \frac{1}{2}\sqrt{1+4\gamma^2}$$

$$\psi(x) - x = -x^{1/2} \sum_{\gamma>0} \frac{2\cos\left(\gamma \ln x - \arctan(2\gamma)\right)}{\frac{1}{2}\sqrt{1+4\gamma^2}} - \log(2\pi) - \frac{1}{2}\log\left(1 - x^{-2}\right)$$

$$\boxed{\psi(x) - x = -x^{1/2} \sum_{\gamma>0} \frac{4\cos(\gamma \ln x - \theta_\gamma)}{\sqrt{1+4\gamma^2}} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}), \quad \theta_\gamma = \arctan(2\gamma)} \tag{5}$$

2.1. Main Benefits of the Cosine-Phase Formula

1. **Fully real arithmetic:**
   - Uses only real numbers and cosines.
   - Avoids complex exponentials and divisions.
   - Reduces CPU operations per zero, giving $\sim 4\times$ speedup for small-to-medium zero sets.
2. **Explicit oscillatory structure:**
   - Each zero contributes a real cosine wave with amplitude $4/\sqrt{1+4\gamma^2}$ and phase $\theta_\gamma = \arctan(2\gamma)$.
   - Makes interference patterns explicit.
   - Ideal for visualization, peak detection, and oscillation analysis.
3. **Improved numerical computation:**
   - Pairing zeros ensures real output.
   - Term magnitudes $O(x^{1/2}/\gamma)$ give natural truncation and error estimation.
   - Compatible with Kahan or compensated summation for cancellation.
4. **Lower memory usage:**
   - Stores only 1 real per zero instead of 2 for complex numbers.
   - Improves cache efficiency, enabling larger zero sets in RAM.
5. **Vectorization-friendly:**
   - Cosines and multiplications can be fully vectorized (SIMD, GPU, multi-core).
   - Simpler to implement in high-performance numerical libraries.
6. **Visualization and peak analysis:**

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- Peaks correspond to constructive interference of zeros.
- Directly interpretable, unlike the classical complex sum.

Feature	Classical Formula	Cosine-Phase Formula
Arithmetic type	Complex	Real only
Oscillatory structure	Hidden in complex plane	Explicit cosine waves
Numerical speed	Moderate	Faster ($\sim 4\times$)
Memory usage	Higher	Lower
Vectorization	Possible	Easier
Peak detection	Hard	Easy and intuitive
Error control	Requires complex analysis	Natural truncation bounds via amplitudes

2.2. *Amplitude and Phase*

Each zero γ contributes with amplitude

$$A_\gamma = \frac{4}{\sqrt{1+4\gamma^2}}$$

and phase

$$\theta_\gamma = \arctan(2\gamma),$$

making the effect of individual zeros transparent. Each term in the classical explicit formula has the form:

$$\frac{x^\rho}{\rho} = \frac{x^{1/2+i\gamma}}{1/2+i\gamma}.$$

Writing $1/2 + i\gamma$ in polar form:

$$1/2 + i\gamma = \underbrace{\sqrt{(1/2)^2+\gamma^2}}_{\text{magnitude}}\, e^{\overbrace{i\arctan(2\gamma)}^{\text{phase}}}.$$

The magnitude $\sqrt{1/4+\gamma^2}$ contributes to the amplitude in the cosine sum. The phase $\arg(1/2 + i\gamma) = \arctan(2\gamma)$ shifts the argument of the cosine.

$$\text{Rotation angle: } \gamma \ln x - \arctan(2\gamma).$$

The phase $\arctan(2\gamma)$ aligns the vector correctly so that the sum of all vectors reproduces $\psi(x) - x$ accurately. Without this phase shift, the oscillations would be misaligned and the formula would be incorrect.

The main uses of $\arctan(2\gamma)$ are:

- **Phase correction:** aligns each zero's contribution in the cosine sum.
- **Correct interference:** ensures the sum over zeros reproduces the oscillatory behavior of $\psi(x) - x$.
- **Magnitude-phase decomposition:** represents the argument of $1/2 + i\gamma$ in polar form.
  Without phase shift $\arctan(2\gamma)$ the vectors misaligned, strong cancellations and slower convergence. The prime contributions are misrepresented or even completely destroyed.

2.3. Prime Representing Constant

The sequence 2, 3, 2, 3, 2, 5, 2,.., is a sequence of smallest primes that do not divide n. The probability of P_k is the smallest prime that does not divide n, for some natural number can be written as[3]:

$$P(P_k \text{ does not divide n and} P_1, P_2, ..., P_{k-1} \text{divide n }) = \left(1 - \frac{1}{P_k}\right) \prod_{i=1}^{k-1} (\frac{1}{P_i})$$

We can prove $\arctan(2\gamma)$ infinitely by the Prime Representing Constant. The constant of the phase shift is 2. The probability of 2 is the smallest prime that does not divide equals to 1/2.

$$\sum_{k=1}^{\infty} \left(1 - \frac{1}{P_k}\right) \prod_{i=1}^{k-1} \frac{1}{P_i} = \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k} \frac{1}{P_i} - \prod_{i=1}^{k-1} \frac{1}{P_i}\right)$$

This is a telescoping sum:

$$= \left(1 - \frac{1}{P_1}\right) + \left(\frac{1}{P_1} - \frac{1}{P_1 P2}\right) + ...$$

$$= 1 + \prod_{i=1}^{\infty} \frac{1}{P_i}$$

$\prod_{i=1}^{\infty} \frac{1}{P_i}$, tends to 0

$$= 1 + 0$$

$$= 1$$

if k = 1, then $P_1 = 2$

$$\left(1 - \frac{1}{P_1}\right) \prod_{i=1}^{1-1} \frac{1}{P_i}$$

$$\left(1 - \frac{1}{2}\right) \prod_{i=1}^{0} \frac{1}{P_i} = 1/2$$

$$\left(1 - \frac{1}{2}\right) = 1/2$$

Then ,

$$\sum_{k=2}^{\infty} \left(1 - \frac{1}{P_k}\right) \prod_{i=1}^{k-1} \frac{1}{P_i} = 1/2$$

$$\sum_{k=1}^{\infty} \left(1 - \frac{1}{P_k}\right) \prod_{i=1}^{k-1} \frac{1}{P_i} = 1/2 + 1/2 = 1$$

The smallest prime that does not divide P_2 up to P_∞ is 2. The probability of 2 is the smallest prime that does not divide n is 1/2. The sum of the probabilities of P_2 up to P_∞ is the smallest prime that does not divide n is also 1/2.

In the paper Prime Representing Constant in the Critical Strip shows the relation of the Prime Representing Constant to the Riemann zeta function. It also introduce a new identity of the Prime Representing Constant [4]. If we divide Prime Representing into 2.

$$\sum_{k=1}^{\infty} \frac{P_k - 1}{\prod_{i=1}^{k-1} P_i} = 2.920050977316...$$

$$\sum_{k=1}^{\infty} \frac{P_k - 1}{\prod_{i=1}^{k-1} P_i} = 1.460025488658... + 1.460025488658...$$

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3. Result

Summing over the n non-trivial zeros yields

$$\sum_{i=1}^{n} \frac{\mathrm{Re}(x^{\rho_i})}{x^{1/2}} = \sum_{i=1}^{n} \cos(\gamma_i \ln x).$$

In the explicit formula for prime-counting functions (like $\psi(x)$ or $\pi(x)$), these contributions appear with a minus sign:

$$\boxed{\psi(x) - x = -\sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) \sim -\sum_{i=1}^{n} \cos(\gamma_i \ln x)}.$$

The minus sign is essential to accurately reflect the direction of the correction in the distribution of primes.

$$-\sum_{i=1}^{n} \cos(\gamma_i \ln x),$$

which represents the oscillatory correction to the smooth main term caused by the first n non-trivial zeros of $\zeta(s)$. This expression represents waves whose frequencies are given by the imaginary parts of the nontrivial zeros. The oscillations occur in the logarithmic variable ln x and form the oscillatory component appearing in explicit formulas relating nontrivial zeros to the distribution of prime numbers.

If we change the sign, then

$$-\sum_{k=1}^{\infty} \frac{P_k - 1}{\prod_{i=1}^{k-1} P_i} = -1.460025488658... - 1.460025488658...$$

$$-1.460025488658... = \zeta(0.4999161...)$$

Hadamard gave the infinite product expansion [5]

$$\zeta(\mathbf{s}) = \frac{e^{\left(\log(2\pi) - 1 - \frac{\gamma}{2}\right)s}}{2(s-1)\Gamma(1+\frac{s}{2})} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$$

$$\zeta(\mathbf{0.4999161...}) = \frac{e^{\left(\log(2\pi) - 1 - \frac{\gamma}{2}\right)0.4999161...}}{2(0.4999161... - 1)\Gamma(1+\frac{0.4999161...}{2})} \prod_{\rho} \left(1 - \frac{0.4999161...}{\rho}\right) e^{\frac{0.4999161...}{\rho}}$$

$$\prod_{\rho} \left(1 - \frac{0.4999161...}{\rho}\right) e^{\frac{0.4999161...}{\rho}} = 1.005...$$

$$\zeta(\mathbf{0.4999161...}) = \frac{1.315989...}{-0.9065632...} 1.005...$$

$$\zeta(\mathbf{0.4999161...}) = -1.460025488658...$$

$$2\zeta(\mathbf{s}) = \frac{e^{\left(\log(2\pi) - 1 - \frac{\gamma}{2}\right)s}}{(s-1)\Gamma(1+\frac{s}{2})} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$$

$$2\zeta(\mathbf{0.4999161...}) = \frac{e^{\left(\log(2\pi) - 1 - \frac{\gamma}{2}\right)0.4999161...}}{(0.4999161... - 1)\Gamma(1+\frac{0.4999161...}{2})} \prod_{\rho} \left(1 - \frac{0.4999161...}{\rho}\right) e^{\frac{0.4999161...}{\rho}}$$

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$$2\zeta(\mathbf{0.4999161...}) = -2.920050977316...$$

This solution proves that the Prime Representing Constant can be computed using the product of nontrivial zeros of the Riemann zeta function.

4. Discussion

The cosine-phase formula offers several advantages over the classical explicit formula. It is a fully real, explicit oscillatory structure, improved numerical computation, lower memory usage, vectorized, systematic visualization, and peak analysis. We can prove it using the Prime Representing Constant. The probability of P_k is the smallest prime that does not divide n, for some natural number can be written as:

$$P(P_k \text{ does not divide n and} P_1, P_2, ..., P_{k-1} \text{divide n }) = \left(1 - \frac{1}{P_k}\right) \prod_{i=1}^{k-1} (\frac{1}{P_i})$$

The average of this equation give us the Prime Representing Constant.

$$\sum_{k=1}^{\infty} \frac{P_k - 1}{\prod_{i=1}^{k-1} P_i} = 2.920050977316...$$

$$-\sum_{k=1}^{\infty} \frac{P_k - 1}{\prod_{i=1}^{k-1} P_i} = -1.460025488658... - 1.460025488658...$$

$$-\sum_{k=1}^{\infty} \frac{P_k - 1}{\prod_{i=1}^{k-1} P_i} = \zeta(0.4999161...) + \zeta(0.4999161...)$$

$$0.4999161... + 0.4999161... \approx 1$$

$$\sum_{k=1}^{\infty} \left(1 - \frac{1}{P_k}\right) \prod_{i=1}^{k-1} \frac{1}{P_i} = \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k} \frac{1}{P_i} - \prod_{i=1}^{k-1} \frac{1}{P_i}\right) = 1$$

This is a telescoping sum.

$$\sum_{k=1}^{\infty} \left(1 - \frac{1}{P_k}\right) \prod_{i=1}^{k-1} \frac{1}{P_i} = 1/2 + 1/2 = 1$$

The smallest prime that does not divide P_2 up to P_∞ is 2. The probability of 2 is the smallest prime that does not divide n is 1/2. The sum of the probabilities ofP_2 up to P_∞ is the smallest prime that does not divide n, which is also 1/2. The sum of 1/2 and 1/2 equals 1.

We infinitely prove the probability that the sum of the P_k is the smallest prime that does not divide n is equal to 1. We also infinitely prove the value of the Prime Representing Constant because the value of the Prime Representing Constant is based on the value of the probability of P_k , which is the smallest prime that does not divide n. The Prime Representing Constant divided by 2 and changing the sign to negative will give us

$$-1.460025488658... = \zeta(0.4999161...)$$

We can verify it on any formula for the Riemann zeta function. Especially the Hadamard product expansion formula, which uses the product of the nontrivial zeros. Using the formula presented below and substituting s = 0.4999161... the result is equal to the value of the Prime Representing Constant.

$$2\zeta(\mathbf{s}) = \frac{e^{\left(\log(2\pi) - 1 - \frac{\gamma}{2}\right)s}}{(s-1)\Gamma(1+\frac{s}{2})} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$$

There exists a linear oscillatory operator $\boldsymbol{T}$ whose distributional trace reproduces the cosine-phase

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explicit formula for $\psi(x)$. The operator encodes prime powers through its trace, and its normalization is fixed by the Prime Representing Constant. The Prime Representing Constant infinitely generates prime numbers and reconstructs the prime counting function.

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