Article

Not peer-reviewed version

# Degeneracy of the Operator-Valued Poisson Kernel near the Numerical Range Boundary

Shanmu Jin *

Posted Date: 9 February 2026

doi: 10.20944/preprints202602.0563.v2

Keywords: numerical range; operator-valued Poisson kernel; convex domains; functional calculus; boundary integral methods

*Article*

# Degeneracy of the Operator-Valued Poisson Kernel near the Numerical Range Boundary

**Shanmu Jin**

Peking Union Medical College Hospital; jinshanmu1129@pku.edu.cn

**Abstract**

Let $A \in \mathbb{C}^{d\times d}$ and let $W(A)$ denote its numerical range. For a bounded convex domain $\Omega \subset \mathbb{C}$ with $C^1$ boundary containing $\mathrm{spec}(A)$, consider the operator-valued boundary kernel $P_\Omega(\sigma, A) := \mathrm{Re}\Big(n_\Omega(\sigma)\,(\sigma I - A)^{-1}\Big), \sigma \in \partial\Omega$, where $n_\Omega(\sigma)$ is the outward unit normal at $\sigma$. For convex $\Omega$ with $W(A) \subset \Omega$, this kernel is positive definite on $\partial\Omega$ and underlies boundary-integral functional calculi and spectral-set bounds in the sense of Delyon–Delyon and Crouzeix. We analyze the opposite limiting regime $\Omega \downarrow W(A)$. Along any $C^1$ convex exhaustion $\Omega_\varepsilon \downarrow W(A)$, if $\sigma_\varepsilon \in \partial\Omega_\varepsilon$ approaches a non-spectral boundary point $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$ with convergent outward normals $n_{\Omega_\varepsilon}(\sigma_\varepsilon) \to n$, then $\lambda_{\min}(P_{\Omega_\varepsilon}(\sigma_\varepsilon, A)) \to 0$ and the associated min-eigenvector directions converge (up to subsequences and phases) to the canonical subspace $(\sigma_0 I - A)\mathcal{M}(n)$ determined by the maximal eigenspace of $H(n) = \mathrm{Re}(\overline{n}A)$. Quantitatively, we obtain two-sided bounds in terms of the support-gap scalar $\delta(\sigma, n) = \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n))$, yielding a linear degeneracy rate under bounded-resolvent hypotheses and an explicit rate for outer offsets $W(A) + \varepsilon\mathbb{D}$. Under a spectral-isolation hypothesis for $\lambda_{\max}(H(n))$, we characterize the entire collapsing eigenvalue cluster under non-tangential offsets: exactly $m = \dim \mathcal{M}(n)$ eigenvalues decay as $O(\varepsilon)$ with a computable slope spectrum given by the eigenvalues of an explicit Gram matrix $G(n, \sigma_0)^{-1}$, while the remaining eigenvalues stay uniformly bounded away from 0. This yields a rigorous face detector based on counting small eigenvalues, and the rescaled cluster is intrinsic under arbitrary $C^1$ convex exhaustions after normalization by $\delta$. At spectral support points $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$ we obtain a three-scale picture for nonnormal matrices: an exact $1/\varepsilon$ blow-up on $\mathrm{Ker}(\sigma_0 I - A)$, an $O(\varepsilon)$ collapsing cluster on $\mathcal{M}(n) \ominus \mathrm{Ker}(\sigma_0 I - A)$ with an explicit slope spectrum, and an $O(1)$ bulk separated from 0. For normal matrices we compute the spectrum of $P_\Omega(\sigma, A)$ explicitly, recovering a simple dichotomy at spectral support points in terms of whether the supporting face contains multiple eigenvalues. Finally, we include reproducible numerical experiments (Python) validating the predicted slopes and splittings.

**Keywords:** numerical range; operator-valued Poisson kernel; convex domains; functional calculus; boundary integral methods

---

## 1. Introduction

Let $A \in \mathbb{C}^{d\times d}$ and define its numerical range by

$$W(A) := \{x^*Ax : x \in \mathbb{C}^d, \|x\| = 1\}.$$

The Toeplitz–Hausdorff theorem asserts that $W(A)$ is a compact convex subset of $\mathbb{C}$ (see, e.g., [1,2]). Crouzeix conjectured that $W(A)$ is a 2-spectral set for $A$, i.e.,

$$\|p(A)\| \le 2 \max_{z\in W(A)} |p(z)| \qquad \text{for every polynomial } p. \tag{1.1}$$

See [3,4] for the formulation and [5] for the best known universal constant $1 + \sqrt{2}$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



**Background and relation to the convex-domain functional calculus.** Up to normalization conventions, a central tool in the convex-domain approach of Delyon–Delyon and Crouzeix is the operator-valued boundary kernel

$$P_\Omega(\sigma, A) \;:=\; \mathrm{Re}\Big(n_\Omega(\sigma)\,(\sigma I - A)^{-1}\Big), \qquad \sigma \in \partial\Omega, \tag{1.2}$$

defined for a bounded convex domain $\Omega \subset \mathbb{C}$ with $C^1$ boundary and $\mathrm{spec}(A) \subset \Omega$. Here $n_\Omega(\sigma)$ denotes the outward unit normal at $\sigma$. This kernel appears in double-layer potential representations and boundary integral operators used to obtain functional calculus bounds on convex domains [4–8]. For convex $\Omega$ with $W(A) \subset \Omega$, positivity/coercivity of $\sigma \mapsto P_\Omega(\sigma, A)$ on $\partial\Omega$ encodes strict separation of supporting half-planes and serves as a key structural input in such estimates [7–9].

**Motivation: loss of coercivity near $\partial W(A)$.** In applications and numerical implementations of boundary-integral calculi, one often approximates $W(A)$ by $C^1$ convex supersets $\Omega_\varepsilon \downarrow W(A)$. It is therefore natural to ask whether coercivity of the pointwise kernel $P_{\Omega_\varepsilon}(\sigma, A)$ can remain uniform as $\varepsilon \to 0$. The results below show that this is impossible in general: even when the resolvent stays bounded (i.e. at non-spectral boundary points $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$), the smallest eigenvalue of $P_{\Omega_\varepsilon}(\sigma, A)$ must collapse to 0 at boundary points $\sigma \in \partial\Omega_\varepsilon$ approaching $\partial W(A)$ in a fixed supporting direction.

**What is new in this paper.** The existing convex-domain literature primarily exploits positivity of (1.2) for fixed domains $\Omega \supsetneq W(A)$ [4,7–9]. Here we analyze the complementary limiting regime in which $\Omega$ *shrinks* to $W(A)$, and we make explicit the resulting loss of coercivity of the pointwise kernel. The analysis is driven by a congruence identity and by a scalar *support gap* $\delta(\sigma, n) = \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(\mathrm{Re}(\overline{n}A))$, which admits a support-function interpretation in standard convex-geometry terminology.

- We prove a qualitative degeneracy theorem (Theorem 1): along any $C^1$ convex exhaustion $\Omega_\varepsilon \downarrow W(A)$, if $\sigma_\varepsilon \in \partial\Omega_\varepsilon$ approaches a non-spectral boundary point $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$ with convergent outward normals $n_{\Omega_\varepsilon}(\sigma_\varepsilon) \to n$, then $\lambda_{\min}(P_{\Omega_\varepsilon}(\sigma_\varepsilon, A)) \to 0$ and the limiting min-eigenvector directions lie in $(\sigma_0 I - A)\mathcal{M}(n)$, where $\mathcal{M}(n)$ is the maximal eigenspace of $H(n) = \mathrm{Re}(\overline{n}A)$.
- We establish two-sided bounds for $P(\sigma, n)$ in terms of the support gap $\delta(\sigma, n)$, yielding a linear degeneracy rate under bounded-resolvent hypotheses (Lemma 3 and Corollary 3), and compute $\delta$ explicitly for standard outer offsets $W(A) + \varepsilon\mathbb{D}$ (Proposition 2).
- Under a spectral-isolation hypothesis for $\lambda_{\max}(H(n))$, we quantify the *entire collapsing eigenvalue cluster*: exactly $m = \dim \mathcal{M}(n)$ eigenvalues collapse linearly with an explicit *slope spectrum* given by the eigenvalues of a computable $m \times m$ Gram matrix $G(n, \sigma_0)^{-1}$ (Proposition 7), while the remaining $d - m$ eigenvalues stay uniformly bounded away from 0 (Proposition 8). This yields a rigorous "face detector" based on counting eigenvalues below a threshold proportional to $\varepsilon$ (Corollary 5). The same slope spectrum is shown to be *intrinsic* under arbitrary $C^1$ convex exhaustions after normalization by the support gap (Proposition 9).
- We analyze the contrasting *spectral-support* regime $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$. For general matrices we obtain a three-scale splitting under non-tangential offsets: an exact $1/\varepsilon$ blow-up on $\mathrm{Ker}(\sigma_0 I - A)$, an $O(\varepsilon)$ collapsing cluster on $\mathcal{M}(n) \ominus \mathrm{Ker}(\sigma_0 I - A)$ with an explicit slope spectrum, and an $O(1)$ bulk separated from 0 (Proposition 12). For normal matrices we recover a simple degeneracy dichotomy at spectral support points in terms of whether the supporting face contains multiple eigenvalues (Proposition 11 and Corollary 7).
- We include reproducible numerical experiments (Python) validating the predicted slopes, splittings, and direction-dependent sensitivity profiles (Section 4.9).

**Organization.** Section 2 fixes notation and recalls support-function identities. Section 3 introduces $P_\Omega(\sigma, A)$, proves the key congruence identity, and establishes quantitative support-gap bounds together with a geometric interpretation of $\delta$. Section 4 contains the degeneracy theorem, quantitative corollaries, slope spectra and two-/three-scale spectral splittings (including the spectral-support

 doi:10.20944/preprints202602.0563.v2



regime), subspace convergence, explicit examples, and reproducible numerical tests. It concludes with a brief discussion of remaining open questions.

## 2. Preliminaries

We use the standard notation for disks:

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \qquad \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}.$$

Throughout, $A \in \mathbb{C}^{d\times d}$ is fixed. For vectors $x \in \mathbb{C}^d$ we use $\|x\| := (x^*x)^{1/2}$. For matrices $B \in \mathbb{C}^{d\times d}$ we use the induced operator norm $\|B\| := \sup_{\|x\|=1}\|Bx\|$. We write $B^*$ for the conjugate transpose and $\mathrm{Re}(B) := (B+B^*)/2$.

For a Hermitian matrix $B$, we write its eigenvalues in nondecreasing order as

$$\lambda_1^{\uparrow}(B) \le \cdots \le \lambda_d^{\uparrow}(B),$$

and in nonincreasing order as $\lambda_1^{\downarrow}(B) \ge \cdots \ge \lambda_d^{\downarrow}(B)$. In particular, $\lambda_{\min}(B) = \lambda_1^{\uparrow}(B)$ and $\lambda_{\max}(B) = \lambda_1^{\downarrow}(B)$.

**Remark 1** (Spectrum is contained in the numerical range)**.** *One has* $\mathrm{spec}(A) \subset W(A)$. *Indeed, if* $Ax = \lambda x$ *with* $\|x\| = 1$, *then* $x^*Ax = \lambda \in W(A)$. *Consequently,* $W(A) \subset \Omega$ *implies* $\mathrm{spec}(A) \subset \Omega$ *for any open set* $\Omega \subset \mathbb{C}$.

### *2.1. Support Functions and the Hermitian Pencil*

For unimodular $\omega \in \mathbb{C}$ (i.e. $|\omega| = 1$), define the Hermitian matrix

$$H(\omega) \;:=\; \mathrm{Re}(\overline{\omega}\,A) \;=\; \tfrac{1}{2}(\overline{\omega}\,A + \omega A^*). \tag{2.1}$$

We will later write $n \in \mathbb{C}$ (with $|n| = 1$) for outward unit normals on $\partial\Omega$; in the support-function identities below and throughout, such an $n$ simply plays the role of the unimodular direction $\omega$.

Let $\lambda_{\max}(H(\omega))$ denote its largest eigenvalue and let

$$\mathcal{M}(\omega) \;:=\; \mathrm{Ker}\big(\lambda_{\max}(H(\omega))I - H(\omega)\big)$$

denote the corresponding maximal eigenspace.

**Lemma 1** (Support function of the numerical range)**.** *For every unimodular* $\omega \in \mathbb{C}$,

$$\max_{z\in W(A)} \mathrm{Re}(\overline{\omega}\,z) \;=\; \lambda_{\max}(H(\omega)). \tag{2.2}$$

*Moreover, if* $x \in \mathbb{C}^d$ *is a unit eigenvector of* $H(\omega)$ *associated with* $\lambda_{\max}(H(\omega))$, *then* $x^*Ax \in \partial W(A)$ *and*

$$\mathrm{Re}\big(\overline{\omega}\,x^*Ax\big) = \lambda_{\max}(H(\omega)).$$

**Proof.** For $\|x\| = 1$,

$$\mathrm{Re}\big(\overline{\omega}\,x^*Ax\big) = \mathrm{Re}\big(x^*(\overline{\omega}A)x\big) = x^*\ \mathrm{Re}(\overline{\omega}A)\,x = x^*H(\omega)x.$$

Taking the maximum over $\|x\| = 1$ yields (2.2) by Rayleigh–Ritz (see, e.g., [10]). If $x$ is a maximizing unit vector, then $x^*Ax \in W(A)$ attains the support functional in direction $\omega$, hence lies on $\partial W(A)$ and satisfies the stated identity. □

### 2.2. Convex Domains with $C^1$ Boundary and Normals

We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with $C^1$ boundary. Then for each $\sigma \in \partial\Omega$ there is a unique outward unit normal vector. This $C^1$ assumption is used only to guarantee that the outward unit normal $n_\Omega(\sigma)$ exists and is unique at every boundary point, ensuring that $P_\Omega(\sigma, A)$ is well-defined; no higher regularity (e.g. curvature bounds) is used. We represent the normal as a unimodular complex number $n_\Omega(\sigma) \in \mathbb{C}$ with $|n_\Omega(\sigma)| = 1$ so that the supporting half-plane at $\sigma$ is

$$\Pi_\Omega(\sigma) \;=\; \Big\{z \in \mathbb{C} :\; \mathrm{Re}\big(\overline{n_\Omega(\sigma)}(z-\sigma)\big) \le 0\Big\}. \tag{2.3}$$

Equivalently, by convexity one has $\overline{\Omega} \subseteq \Pi_\Omega(\sigma)$ and $\Omega \subset \{z \in \mathbb{C} :\; \mathrm{Re}(\overline{n_\Omega(\sigma)}(z-\sigma)) < 0\}$. Under the identification $\mathbb{C} \simeq \mathbb{R}^2$, the functional $z \mapsto \mathrm{Re}(\overline{n}z)$ is the Euclidean inner product with the unit vector corresponding to $n$.

**Definition 1** ($C^1$ convex exhaustion)**.** *A family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is called a $C^1$* convex exhaustion *of a compact convex set $K \subset \mathbb{C}$ if:*

*(i) each $\Omega_\varepsilon \subset \mathbb{C}$ is a bounded open convex set with $C^1$ boundary;*
*(ii) $\Omega_{\varepsilon'} \subset \Omega_\varepsilon$ for $0 < \varepsilon' < \varepsilon$;*
*(iii) $K \subset \Omega_\varepsilon$ for all $\varepsilon > 0$;*
*(iv) $\bigcap_{\varepsilon>0} \overline{\Omega_\varepsilon} = K$.*

**Remark 2** (Subsequence selection for convergent normals)**.** *Let $\varepsilon_k \downarrow 0$ and $\sigma_k \in \partial\Omega_{\varepsilon_k}$ be any sequence. Since each outward normal $n_k := n_{\Omega_{\varepsilon_k}}(\sigma_k)$ is unimodular, the sequence $\{n_k\} \subset \{z \in \mathbb{C} : |z| = 1\}$ lies in a compact set. Hence there is always a subsequence (not relabeled) such that $n_k \to n$ for some unimodular $n$. In particular, the normal convergence hypothesis in Theorem 1 can always be arranged by passing to a subsequence.*

## 3. The Operator-Valued Poisson Kernel

Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with $C^1$ boundary and assume $\mathrm{spec}(A) \subset \Omega$. Then $(\sigma I - A)^{-1}$ exists for all $\sigma \in \partial\Omega$.

**Definition 2** (Operator-valued Poisson kernel)**.** *For $\sigma \in \partial\Omega$, define*

$$P_\Omega(\sigma, A) \;:=\; \mathrm{Re}\Big(n_\Omega(\sigma)\,(\sigma I - A)^{-1}\Big). \tag{3.1}$$

### 3.1. A Congruence Identity

**Lemma 2** (Congruence identity)**.** *Let $\sigma \notin \mathrm{spec}(A)$ and let $n \in \mathbb{C}$ be unimodular. Then*

$$(\sigma I - A)^*\;\mathrm{Re}\big(n(\sigma I - A)^{-1}\big)\,(\sigma I - A) \;=\; \mathrm{Re}\big(\overline{n}(\sigma I - A)\big) \;=\; \mathrm{Re}(\overline{n}\,\sigma)\,I - \mathrm{Re}(\overline{n}A). \tag{3.2}$$

**Proof.** Write $R := (\sigma I - A)^{-1}$. Then $R(\sigma I - A) = I$ and $(\sigma I - A)^* R^* = I$. Using $\mathrm{Re}(X) = \frac{1}{2}(X + X^*)$,

$$\begin{aligned}(\sigma I - A)^*\,\mathrm{Re}(nR)(\sigma I - A) &= \tfrac{1}{2}\Big((\sigma I - A)^*(nR)(\sigma I - A) + (\sigma I - A)^*(\overline{n}\,R^*)(\sigma I - A)\Big)\\ &= \mathrm{Re}\big(\overline{n}(\sigma I - A)\big).\end{aligned}$$

Expanding gives (3.2). □

### 3.2. Support-Gap Bounds

For unimodular $n \in \mathbb{C}$ define the *support gap*

$$\delta(\sigma, n) \;:=\; \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n)), \qquad H(n) = \mathrm{Re}(\overline{n}A).$$

**Lemma 3** (Support-gap characterization and quantitative bounds). *Let $A \in \mathbb{C}^{d\times d}$, let $\sigma \notin \mathrm{spec}(A)$, and let $n \in \mathbb{C}$ be unimodular. Set*

$$P(\sigma,n) \;:=\; \mathrm{Re}\big(n(\sigma I - A)^{-1}\big), \qquad \alpha \;:=\; \mathrm{Re}(\overline{n}\,\sigma), \qquad \delta \;:=\; \alpha - \lambda_{\max}(H(n)).$$

*(This notation emphasizes dependence on the prescribed direction $n$; when $n = n_\Omega(\sigma)$ one has $P(\sigma,n) = P_\Omega(\sigma,A)$.) Then:*

*(a) $P(\sigma,n) \succeq 0$ if and only if $\delta \geq 0$, and $P(\sigma,n) \succ 0$ if and only if $\delta > 0$.*
*(b) If $\delta = 0$, then $P(\sigma,n)$ is singular and*

$$\mathrm{Ker}(P(\sigma,n)) \;=\; (\sigma I - A)\,\mathcal{M}(n), \qquad \mathcal{M}(n) = \mathrm{Ker}(\lambda_{\max}(H(n))I - H(n)).$$

*(c) If $\delta > 0$, then*

$$\frac{\delta}{\|\sigma I - A\|^2} \;\leq\; \lambda_{\min}\big(P(\sigma,n)\big) \;\leq\; \delta\,\|(\sigma I - A)^{-1}\|^2. \tag{3.3}$$

**Proof.** Let $B := \sigma I - A$ and $P := P(\sigma,n)$. By Lemma 2,

$$B^*PB = \mathrm{Re}(\overline{n}B) = \alpha I - \mathrm{Re}(\overline{n}A) = \alpha I - H(n) =: Q.$$

Since $B$ is invertible, congruence by $B$ preserves (semi)definiteness, so $P \succeq 0 \iff Q \succeq 0$ and $P \succ 0 \iff Q \succ 0$. As $Q$ is Hermitian with $\lambda_{\min}(Q) = \alpha - \lambda_{\max}(H(n)) = \delta$, this proves (a).

If $\delta = 0$, then $Q \succeq 0$ is singular with $\mathrm{Ker}(Q) = \mathcal{M}(n)$, and $P \succeq 0$ by (a). For $P \succeq 0$, $x \in \mathrm{Ker}(P) \iff x^*Px = 0$. Writing $x = By$,

$$x^*Px = y^*Qy,$$

so $x \in \mathrm{Ker}(P) \iff y \in \mathrm{Ker}(Q) = \mathcal{M}(n)$, proving (b).

If $\delta > 0$, then $Q \succ 0$ and $P = B^{-*}QB^{-1} \succ 0$. For $\|x\| = 1$ and $y = B^{-1}x$, one has $x = By$ and hence $\|y\| \geq 1/\|B\|$; thus

$$x^*Px = y^*Qy \geq \lambda_{\min}(Q)\|y\|^2 = \delta\|y\|^2 \geq \delta/\|B\|^2,$$

giving the lower bound in (3.3). For the upper bound, take $y$ a unit eigenvector of $Q$ for $\lambda_{\min}(Q) = \delta$ and set $x = By/\|By\|$; then

$$x^*Px = \frac{y^*Qy}{\|By\|^2} = \frac{\delta}{\|By\|^2} \leq \delta\,\|B^{-1}\|^2.$$

□

**Remark 3** (Connection with the convex-domain Poisson kernel literature). *Up to normalization conventions, $P_\Omega(\sigma,A)$ is the operator-valued boundary kernel appearing in the Carl Neumann double-layer potential framework for convex domains; see, e.g., [7–9]. Lemma 3 isolates the dependence of $\lambda_{\min}(P(\sigma,n))$ on the scalar support gap $\delta(\sigma,n)$.*

### 3.3. Strict Positivity when $W(A) \subset \Omega$

**Lemma 4** (Strict separation at a supporting line). *Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with $C^1$ boundary and let $K \subset \Omega$ be compact. Fix $\sigma \in \partial\Omega$ and let $n = n_\Omega(\sigma)$ be the outward unit normal. Then*

$$\max_{z\in K} \mathrm{Re}(\overline{n}\,z) \;<\; \mathrm{Re}(\overline{n}\,\sigma).$$

**Proof.** By (2.3), $\Omega \subset \{z : \mathrm{Re}(\overline{n}(z-\sigma)) < 0\}$, hence $K \subset \{z : \mathrm{Re}(\overline{n}(z-\sigma)) < 0\}$. The continuous function $z \mapsto \mathrm{Re}(\overline{n}(z-\sigma))$ attains its maximum on compact $K$, and this maximum is strictly negative. Rearranging yields the claim. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



**Proposition 1** (Positivity of the Poisson kernel). *Assume $W(A) \subset \Omega$. Then for every $\sigma \in \partial\Omega$,*

$$P_\Omega(\sigma, A) \succ 0.$$

**Proof.** Fix $\sigma \in \partial\Omega$ and set $n := n_\Omega(\sigma)$ and $\alpha := \mathrm{Re}(\overline{n}\,\sigma)$. By Lemma 4 with $K = W(A)$ and Lemma 1,

$$\lambda_{\max}(H(n)) = \max_{z \in W(A)} \mathrm{Re}(\overline{n}z) < \alpha,$$

so $\delta(\sigma, n) = \alpha - \lambda_{\max}(H(n)) > 0$. Now apply Lemma 3 (a). □

*3.4. Geometric Meaning of the Support Gap and Offset Exhaustions*

For a compact convex set $K \subset \mathbb{C}$ and unimodular $n \in \mathbb{C}$, define its *support function*

$$h_K(n) := \max_{z \in K} \mathrm{Re}(\overline{n}\,z).$$

If $\Omega \subset \mathbb{C}$ is a bounded open convex set with $C^1$ boundary and $\sigma \in \partial\Omega$ has outward normal $n = n_\Omega(\sigma)$, then necessarily $\mathrm{Re}(\overline{n}\,\sigma) = h_{\overline{\Omega}}(n)$, i.e. the boundary point lies on the supporting line in direction $n$.

**Lemma 5** (Support gap as a support-function difference). *Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with $C^1$ boundary and $\sigma \in \partial\Omega$. Let $n := n_\Omega(\sigma)$. Then*

$$\delta(\sigma, n) = \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n)) = h_{\overline{\Omega}}(n) - h_{W(A)}(n).$$

*In particular, $\delta(\sigma, n)$ measures the separation between the supporting line of $\overline{\Omega}$ in direction $n$ and the corresponding supporting line of $W(A)$.*

**Proof.** Since $n$ is the outward unit normal at $\sigma \in \partial\Omega$, the supporting half-plane characterization implies $\mathrm{Re}(\overline{n}\,z) \le \mathrm{Re}(\overline{n}\,\sigma)$ for all $z \in \overline{\Omega}$, hence $h_{\overline{\Omega}}(n) = \mathrm{Re}(\overline{n}\,\sigma)$. By Lemma 1, $h_{W(A)}(n) = \lambda_{\max}(H(n))$. Combining gives the claim. □

**Proposition 2** (Outer offsets: $\delta$ is explicit). *Let $K \subset \mathbb{C}$ be compact and convex and fix $\varepsilon > 0$. Define the outer offset (outer parallel set)*

$$K_\varepsilon := K + \varepsilon\mathbb{D} = \{z + w : z \in K, w \in \mathbb{C}, |w| < \varepsilon\}.$$

*Then for every unimodular $n \in \mathbb{C}$,*

$$h_{\overline{K_\varepsilon}}(n) = h_K(n) + \varepsilon.$$

*In particular, taking $K = W(A)$ and $\Omega_\varepsilon := W(A) + \varepsilon\mathbb{D}$, for any boundary point $\sigma \in \partial\Omega_\varepsilon$ with outward normal $n = n_{\Omega_\varepsilon}(\sigma)$ (whenever defined) one has*

$$\delta(\sigma, n) = \varepsilon.$$

*Consequently, since $\sigma \in \partial\Omega_\varepsilon$ implies $\sigma \notin W(A)$ and hence $\sigma \notin \mathrm{spec}(A)$ (Remark 1), Lemma 3 (c) yields*

$$\frac{\varepsilon}{\|\sigma I - A\|^2} \le \lambda_{\min}\big(P_{\Omega_\varepsilon}(\sigma, A)\big) \le \varepsilon \|(\sigma I - A)^{-1}\|^2.$$

**Proof.** Fix unimodular $n$. For any $z \in K$ and $w \in \mathbb{C}$ with $|w| \le \varepsilon$,

$$\mathrm{Re}(\overline{n}(z + w)) = \mathrm{Re}(\overline{n}z) + \mathrm{Re}(\overline{n}w) \le h_K(n) + |w| \le h_K(n) + \varepsilon,$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



so $h_{\overline{K_\varepsilon}}(n) \le h_K(n) + \varepsilon$. On the other hand, choosing $z_\star \in K$ with $\mathrm{Re}(\overline{n} z_\star) = h_K(n)$ and $w_\star = \varepsilon n$ gives $|w_\star| = \varepsilon$ and

$$\mathrm{Re}(\overline{n}(z_\star + w_\star)) = h_K(n) + \varepsilon,$$

so $h_{\overline{K_\varepsilon}}(n) \ge h_K(n) + \varepsilon$. This proves the support-function identity and hence the displayed formula for $\delta$ follows from Lemma 5.

The final eigenvalue bounds are an immediate substitution of $\delta = \varepsilon$ into (3.3). □

**Remark 4** (Smoothness versus offsets). *If $K$ has flat faces, then $\partial(K + \varepsilon\mathbb{D})$ is typically only $C^{1,1}$ (curvature may jump at transitions between translated faces and rounded arcs). Proposition 2 is therefore best viewed as a geometric model illustrating how the support gap scales with the outer distance parameter $\varepsilon$. For the purposes of Definition 1, one may replace $K + \varepsilon\mathbb{D}$ by any convex domain with $C^1$ boundary whose support function differs from $h_K$ by a quantity comparable to $\varepsilon$; the same interpretation of $\delta$ then applies. For example, one may take Minkowski sums with a fixed smooth strictly convex unit ball (instead of $\mathbb{D}$) or smooth the support function to obtain a genuine $C^1$ (indeed smooth) convex exhaustion with the same first-order support-gap scaling.*

*3.5. Hausdorff Distance and Support-Function Control of the Support Gap*

For a nonempty compact set $K \subset \mathbb{C}$ and $z \in \mathbb{C}$, write

$$\mathrm{dist}(z, K) := \inf_{w \in K} |z - w|.$$

For nonempty compact sets $K, L \subset \mathbb{C}$, define the (Euclidean) Hausdorff distance

$$d_{\mathrm{H}}(K, L) := \max\Big\{\sup_{z \in K} \mathrm{dist}(z, L),\ \sup_{w \in L} \mathrm{dist}(w, K)\Big\}.$$

**Lemma 6** (Hausdorff distance via support functions). *Let $K, L \subset \mathbb{C}$ be nonempty compact convex sets and let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$. Then*

$$d_{\mathrm{H}}(K, L) = \sup_{|n|=1} \big|h_K(n) - h_L(n)\big|.$$

*If moreover $K \subseteq L$, then $h_K(n) \le h_L(n)$ for all $|n| = 1$ and hence*

$$d_{\mathrm{H}}(L, K) = \sup_{|n|=1} \big(h_L(n) - h_K(n)\big).$$

**Proof.** For $t \ge 0$ and a nonempty compact set $K$, the Minkowski sum

$$K + t\overline{\mathbb{D}} = \{z + w : z \in K,\ |w| \le t\}$$

is the closed $t$-neighborhood of $K$, i.e. $K + t\overline{\mathbb{D}} = \{u \in \mathbb{C} : \mathrm{dist}(u, K) \le t\}$. Consequently,

$$d_{\mathrm{H}}(K, L) = \inf\Big\{t \ge 0 : K \subseteq L + t\overline{\mathbb{D}} \text{ and } L \subseteq K + t\overline{\mathbb{D}}\Big\}.$$

For compact convex sets $M, N \subset \mathbb{C}$ one has $M \subseteq N$ if and only if $h_M(n) \le h_N(n)$ for all $|n| = 1$. (Indeed, the forward direction is immediate; conversely, if $x \in M \setminus N$, a separating supporting line for the convex compact set $N$ yields a unimodular $n$ with $\mathrm{Re}(\overline{n}\,x) > \max_{z \in N} \mathrm{Re}(\overline{n}\,z) = h_N(n)$, hence $h_M(n) \ge \mathrm{Re}(\overline{n}\,x) > h_N(n)$.)

Moreover, support functions add under Minkowski sums, and $h_{t\overline{\mathbb{D}}}(n) = t$ for $|n| = 1$; hence

$$h_{K + t\overline{\mathbb{D}}}(n) = h_K(n) + t.$$

 doi:10.20944/preprints202602.0563.v2



Therefore, $K \subseteq L + t\mathbb{D}$ is equivalent to $h_K(n) \le h_L(n) + t$ for all $|n| = 1$, and similarly $L \subseteq K + t\mathbb{D}$ is equivalent to $h_L(n) \le h_K(n) + t$ for all $|n| = 1$. Thus $d_{\mathrm{H}}(K, L)$ is the smallest $t$ such that $|h_K(n) - h_L(n)| \le t$ for all $|n| = 1$, i.e.

$$d_{\mathrm{H}}(K, L) = \sup_{|n|=1} |h_K(n) - h_L(n)|.$$

If $K \subseteq L$, then $h_K \le h_L$, so the absolute value may be dropped, giving the second identity. $\square$

**Corollary 1** (Support gap bounded by the Hausdorff approximation error)**.** *Assume $W(A) \subset \Omega$, and set*

$$\Delta(\Omega) \;:=\; d_{\mathrm{H}}(\overline{\Omega}, W(A)) \;=\; \sup_{|n|=1} \big(h_{\overline{\Omega}}(n) - h_{W(A)}(n)\big).$$

*Then for every $\sigma \in \partial\Omega$ with outward normal $n = n_\Omega(\sigma)$,*

$$\delta(\sigma, n) = \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n)) = h_{\overline{\Omega}}(n) - h_{W(A)}(n) \;\le\; \Delta(\Omega).$$

*Consequently, since $\sigma \notin \mathrm{spec}(A)$ for $\sigma \in \partial\Omega$, Lemma 3 (c) yields*

$$\lambda_{\min}\big(P_\Omega(\sigma, A)\big) \;\le\; \delta(\sigma, n)\,\|(\sigma I - A)^{-1}\|^2 \;\le\; \Delta(\Omega)\,\|(\sigma I - A)^{-1}\|^2.$$

*Moreover, there exists $\sigma_\star \in \partial\Omega$ such that*

$$\delta(\sigma_\star, n_\Omega(\sigma_\star)) = \Delta(\Omega),$$

*and for this point one has the two-sided estimate*

$$\frac{\Delta(\Omega)}{\|\sigma_\star I - A\|^2} \;\le\; \lambda_{\min}\big(P_\Omega(\sigma_\star, A)\big) \;\le\; \Delta(\Omega)\,\|(\sigma_\star I - A)^{-1}\|^2.$$

**Proof.** The identity $\delta(\sigma, n) = h_{\overline{\Omega}}(n) - h_{W(A)}(n)$ is Lemma 5, and the bound $\delta(\sigma, n) \le \Delta(\Omega)$ follows from the definition of $\Delta(\Omega)$. The eigenvalue bounds are then immediate from Lemma 3 (c).

Finally, the function $n \mapsto h_{\overline{\Omega}}(n) - h_{W(A)}(n)$ is continuous on the unit circle, so it attains its maximum at some unimodular $n_\star$. Choose $\sigma_\star \in \overline{\Omega}$ such that $\mathrm{Re}(\overline{n_\star}\,\sigma_\star) = h_{\overline{\Omega}}(n_\star)$; then $\sigma_\star \in \partial\Omega$ and the supporting line $\{z : \mathrm{Re}(\overline{n_\star}z) = \mathrm{Re}(\overline{n_\star}\sigma_\star)\}$ is a supporting line for $\overline{\Omega}$ at $\sigma_\star$. Since $\partial\Omega$ is $C^1$, the outward unit normal at $\sigma_\star$ is uniquely defined and equals $n_\star$, and hence

$$\delta(\sigma_\star, n_\Omega(\sigma_\star)) = h_{\overline{\Omega}}(n_\star) - h_{W(A)}(n_\star) = \Delta(\Omega).$$

$\square$

## 4. Degeneracy Along a $C^1$ Convex Exhaustion

### *4.1. Qualitative Degeneracy and Limiting Kernel Directions*

**Theorem 1** (Degeneracy of the operator-valued Poisson kernel)**.** *Let $A \in \mathbb{C}^{d\times d}$ and let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a $C^1$ convex exhaustion of $W(A)$ (Definition 1). For $\sigma \in \partial\Omega_\varepsilon$, set*

$$P_{\Omega_\varepsilon}(\sigma, A) \;:=\; \mathrm{Re}\Big(n_{\Omega_\varepsilon}(\sigma)\,(\sigma I - A)^{-1}\Big).$$

*Fix any sequence $\varepsilon_k \downarrow 0$ and points $\sigma_k \in \partial\Omega_{\varepsilon_k}$ such that*

$$\sigma_k \to \sigma_0 \in \partial W(A), \qquad n_k := n_{\Omega_{\varepsilon_k}}(\sigma_k) \to n \in \mathbb{C}, \quad |n| = 1.$$

*(After passing to a subsequence, the convergence $n_k \to n$ is automatic; see Remark 2.) Assume $\sigma_0 \notin \mathrm{spec}(A)$. Let $H(n) = \mathrm{Re}(\overline{n}A)$ and $\mathcal{M}(n) = \mathrm{Ker}(\lambda_{\max}(H(n))I - H(n))$.*

*Then:*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



*(1)* (Vanishing) $\lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \to 0$ *as* $k \to \infty$.

*(2)* (Limiting directions) *If* $u_k$ *is any unit eigenvector of* $P_{\Omega_{\varepsilon_k}}(\sigma_k, A)$ *for* $\lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big)$, *then every accumulation point* $u_0$ *of* $\{u_k\}$ *satisfies*

$$u_0 \in (\sigma_0 I - A)\,\mathcal{M}(n).$$

*(3)* (One-dimensional case) *If* $\dim \mathcal{M}(n) = 1$, *then there exist phases* $\theta_k \in \mathbb{R}$ *such that*

$$e^{i\theta_k} u_k \longrightarrow \frac{(\sigma_0 I - A)v}{\|(\sigma_0 I - A)v\|} \qquad (k \to \infty),$$

*where* $v$ *is any unit vector spanning* $\mathcal{M}(n)$.

**Proof.** Set $B_k := \sigma_k I - A$ and $R_k := B_k^{-1}$, and define

$$P_k := \mathrm{Re}(n_k R_k), \qquad \alpha_k := \mathrm{Re}(\overline{n_k}\sigma_k).$$

Define also $B_0 := \sigma_0 I - A$, $R_0 := B_0^{-1}$, $P_0 := \mathrm{Re}(nR_0)$, $\alpha_0 := \mathrm{Re}(\overline{n}\,\sigma_0)$.

**Step 1: Congruence identities.** By Lemma 2,

$$B_k^* P_k B_k = \alpha_k I - H(n_k), \qquad B_0^* P_0 B_0 = \alpha_0 I - H(n), \tag{4.1}$$

where $H(n_k) = \mathrm{Re}(\overline{n_k} A)$.

**Step 2:** $\alpha_0 = \lambda_{\max}(H(n))$**.** Since $n_k$ is the outward normal at $\sigma_k \in \partial\Omega_{\varepsilon_k}$, the supporting half-plane property gives $\mathrm{Re}(\overline{n_k} z) \le \alpha_k$ for all $z \in \Omega_{\varepsilon_k}$ and hence for all $z \in W(A)$. Passing to the limit yields $\mathrm{Re}(\overline{n}\,z) \le \alpha_0$ for all $z \in W(A)$. Because $\sigma_0 \in W(A)$, equality holds at $z = \sigma_0$, so $\alpha_0 = \max_{z \in W(A)} \mathrm{Re}(\overline{n} z)$. Lemma 1 now gives

$$\alpha_0 = \lambda_{\max}(H(n)), \qquad \mathrm{Ker}(\alpha_0 I - H(n)) = \mathcal{M}(n),$$

so $\alpha_0 I - H(n) \succeq 0$ is singular.

**Step 3:** $P_k \to P_0$ **in operator norm.** Since $\sigma_0 \notin \mathrm{spec}(A)$, $B_0$ is invertible. Write

$$B_k = B_0 + (\sigma_k - \sigma_0)I = B_0(I + E_k), \qquad E_k := (\sigma_k - \sigma_0)R_0.$$

Then $\|E_k\| \to 0$, so for large $k$, $I + E_k$ is invertible and

$$R_k = B_k^{-1} = (I + E_k)^{-1} R_0, \qquad \|R_k - R_0\| \to 0.$$

Therefore,

$$\|P_k - P_0\| = \|\,\mathrm{Re}(n_k R_k - nR_0)\| \le |n_k - n|\,\|R_k\| + \|R_k - R_0\| \to 0.$$

**Step 4:** $\lambda_{\min}(P_0) = 0$ **and** $\lambda_{\min}(P_k) \to 0$**.** Since $P_k, P_0$ are Hermitian, Weyl's inequality (see, e.g., [10]) yields

$$\big|\lambda_{\min}(P_k) - \lambda_{\min}(P_0)\big| \le \|P_k - P_0\| \to 0,$$

so $\lambda_{\min}(P_k) \to \lambda_{\min}(P_0)$. By (4.1) and Step 2,

$$B_0^* P_0 B_0 = \alpha_0 I - H(n) \succeq 0 \;\text{ is singular.}$$

Since $B_0$ is invertible, $P_0 \succeq 0$ is singular, hence $\lambda_{\min}(P_0) = 0$, proving (1). Moreover,

$$\mathrm{Ker}(P_0) = B_0\,\mathrm{Ker}(\alpha_0 I - H(n)) = (\sigma_0 I - A)\mathcal{M}(n)$$

 doi:10.20944/preprints202602.0563.v2



by Lemma 3 (b) (with $\delta = 0$).

**Step 5: Limiting eigenvectors.** Let $u_k$ be unit min-eigenvectors: $P_k u_k = \lambda_{\min}(P_k)u_k$. Along a convergent subsequence, $u_k \to u_0$. Then

$$u_0^* P_0 u_0 = \lim_{k\to\infty} u_k^* P_0 u_k = \lim_{k\to\infty} \left(u_k^* P_k u_k + u_k^*(P_0 - P_k)u_k\right) = \lim_{k\to\infty} \left(\lambda_{\min}(P_k) + o(1)\right) = 0.$$

Since $P_0 \succeq 0$, this implies $u_0 \in \operatorname{Ker}(P_0) = (\sigma_0 I - A)\mathcal{M}(n)$, proving (2).

**Step 6: One-dimensional case.** If $\dim \mathcal{M}(n) = 1$, then $\dim \operatorname{Ker}(P_0) = 1$, so the smallest eigenvalue of $P_0$ is simple. By the Davis–Kahan $\sin\Theta$ theorem for invariant subspaces (see [11]), the corresponding one-dimensional eigenspaces of $P_k$ converge to $\operatorname{Ker}(P_0)$ in gap metric, hence there exist phases $\theta_k$ such that $e^{i\theta_k} u_k \to u_\star$, where $u_\star$ spans $\operatorname{Ker}(P_0) = (\sigma_0 I - A)\mathcal{M}(n)$. This gives (3). □

**Remark 5** (Why $\sigma_0 \notin \operatorname{spec}(A)$ is essential)**.** *The hypothesis $\sigma_0 \notin \operatorname{spec}(A)$ ensures that $(\sigma I - A)^{-1}$ remains bounded near $\sigma_0$, so $P_0$ is a finite Hermitian matrix. When $\sigma_0 \in \operatorname{spec}(A)$, the resolvent diverges and the behavior of $\lambda_{\min}(P_{\Omega_\varepsilon}(\sigma_\varepsilon, A))$ depends on the spectral geometry; see Proposition 11 below and Section 4.11.*

**Corollary 2** (Global coercivity collapse along a $C^1$ convex exhaustion)**.** *Assume that $A$ is not a scalar multiple of the identity (equivalently, $W(A)$ is not a singleton). Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a $C^1$ convex exhaustion of $W(A)$ and define the global coercivity constant*

$$c(\varepsilon) \;:=\; \inf_{\sigma\in\partial\Omega_\varepsilon} \lambda_{\min}(P_{\Omega_\varepsilon}(\sigma, A)), \qquad P_{\Omega_\varepsilon}(\sigma, A) = \operatorname{Re}\Big(n_{\Omega_\varepsilon}(\sigma)\,(\sigma I - A)^{-1}\Big).$$

*Then*

$$\liminf_{\varepsilon\downarrow 0} c(\varepsilon) \;=\; 0.$$

*In particular, there do not exist $\varepsilon_0 > 0$ and $c_0 > 0$ such that $P_{\Omega_\varepsilon}(\sigma, A) \succeq c_0 I$ for all $0 < \varepsilon < \varepsilon_0$ and all $\sigma \in \partial\Omega_\varepsilon$.*

**Proof.** Since $A$ is not scalar, the compact convex set $W(A)$ contains more than one point, hence $\partial W(A)$ is infinite, whereas $\operatorname{spec}(A)$ is finite. Choose $\sigma_0 \in \partial W(A) \setminus \operatorname{spec}(A)$.

Fix any sequence $\varepsilon_k \downarrow 0$. We claim that $\operatorname{dist}(\sigma_0, \partial\Omega_{\varepsilon_k}) \to 0$. Indeed, if not, then there exist $\delta > 0$ and a subsequence (not relabeled) such that $\operatorname{dist}(\sigma_0, \partial\Omega_{\varepsilon_k}) \geq \delta$ for all $k$, hence the open ball $B(\sigma_0, \delta)$ is contained in $\Omega_{\varepsilon_k}$ for all $k$. Taking closures and intersecting over $k$ yields $B(\sigma_0, \delta) \subset \bigcap_k \overline{\Omega_{\varepsilon_k}} = W(A)$, contradicting $\sigma_0 \in \partial W(A)$.

Therefore we may choose $\sigma_k \in \partial\Omega_{\varepsilon_k}$ with $\sigma_k \to \sigma_0$. By compactness of the unit circle, after passing to a subsequence we have $n_{\Omega_{\varepsilon_k}}(\sigma_k) \to n$ for some unimodular $n$. Theorem 1 then gives

$$\lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \longrightarrow 0.$$

Since $c(\varepsilon_k) \leq \lambda_{\min}(P_{\Omega_{\varepsilon_k}}(\sigma_k, A))$, it follows that $\liminf_{\varepsilon\downarrow 0} c(\varepsilon) = 0$. □

### *4.2. Quantitative Degeneracy Rate*

**Corollary 3** (Linear rate in terms of the support gap)**.** *In the setting of Theorem 1, define*

$$\delta_k := \operatorname{Re}(\overline{n_k}\sigma_k) - \lambda_{\max}(H(n_k)), \qquad H(n_k) = \operatorname{Re}(\overline{n_k} A).$$

*Then $\delta_k > 0$ for each $k$ and $\delta_k \to 0$. Moreover, for all sufficiently large $k$,*

$$\frac{\delta_k}{4\|\sigma_0 I - A\|^2} \;\leq\; \lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \;\leq\; 4\,\delta_k\,\|(\sigma_0 I - A)^{-1}\|^2. \tag{4.2}$$

*In particular, $\lambda_{\min}(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)) = \Theta(\delta_k)$.*

 doi:10.20944/preprints202602.0563.v2



**Proof.** Since $W(A) \subset \Omega_{\varepsilon_k}$ and $\sigma_k \in \partial\Omega_{\varepsilon_k}$ with normal $n_k$, Lemma 4 and Lemma 1 imply $\lambda_{\max}(H(n_k)) < \operatorname{Re}(\overline{n_k}\sigma_k)$, so $\delta_k > 0$.

As $n_k \to n$ and $\sigma_k \to \sigma_0$, $\operatorname{Re}(\overline{n_k}\sigma_k) \to \operatorname{Re}(\overline{n}\,\sigma_0)$. Also $H(n_k) \to H(n)$ in operator norm, hence $\lambda_{\max}(H(n_k)) \to \lambda_{\max}(H(n))$. By Step 2 in the proof of Theorem 1, $\operatorname{Re}(\overline{n}\,\sigma_0) = \lambda_{\max}(H(n))$, so $\delta_k \to 0$.

Set $B_k = \sigma_k I - A$ and $B_0 = \sigma_0 I - A$. Since $B_k \to B_0$ and $B_0$ is invertible, for large $k$ one has $\|B_k\| \le 2\|B_0\|$ and $\|B_k^{-1}\| \le 2\|B_0^{-1}\|$. Applying Lemma 3 (c) to $(\sigma, n) = (\sigma_k, n_k)$ gives

$$\frac{\delta_k}{\|B_k\|^2} \le \lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \le \delta_k \|B_k^{-1}\|^2,$$

and the stated constants follow. □

### 4.3. Sharpness and Refined Local/Global Bounds

Lemma 3 (c) bounds $\lambda_{\min}(P(\sigma,n))$ in terms of the scalar support gap $\delta(\sigma,n)$ and the global operator norms $\|\sigma I - A\|$, $\|(\sigma I - A)^{-1}\|$. We first record that these norm-based bounds are optimal in the strongest possible sense, and then derive refinements that capture the correct constant in the degeneracy regime $\delta \downarrow 0$.

**Proposition 3** (Optimality of the norm-based support-gap bounds)**.** *The constants in the two-sided inequality (3.3) are sharp and cannot be improved uniformly. More precisely, for $d = 1$ both inequalities in (3.3) hold with equality.*

**Proof.** Let $d = 1$ and write $A = [a]$ with $a \in \mathbb{C}$. Then $B = \sigma - a$ is a nonzero scalar and

$$P(\sigma, n) = \operatorname{Re}\Big(\frac{n}{\sigma - a}\Big) = \frac{\operatorname{Re}(\overline{n}(\sigma - a))}{|\sigma - a|^2}.$$

Moreover $\delta(\sigma,n) = \operatorname{Re}(\overline{n}\sigma) - \operatorname{Re}(\overline{n}a) = \operatorname{Re}(\overline{n}(\sigma - a))$, and $\|B\| = |\sigma - a|$, $\|B^{-1}\| = 1/|\sigma - a|$. Therefore

$$\lambda_{\min}(P(\sigma,n)) = \frac{\delta(\sigma,n)}{\|B\|^2} = \delta(\sigma,n)\,\|B^{-1}\|^2,$$

so (3.3) is an equality in both directions. □

**Proposition 4** (Generalized eigenvalue characterization)**.** *Let $\sigma \notin \operatorname{spec}(A)$ and $|n| = 1$. Set $B := \sigma I - A$, $\alpha := \operatorname{Re}(\overline{n}\sigma)$, $H(n) = \operatorname{Re}(\overline{n}A)$ and $Q := \alpha I - H(n)$. Then*

$$\lambda_{\min}(P(\sigma,n)) = \min_{y \ne 0} \frac{y^* Q y}{\|By\|^2} = \lambda_{\min}(Q,\, B^*B), \tag{4.3}$$

*where $\lambda_{\min}(Q, B^*B)$ denotes the smallest generalized eigenvalue of the Hermitian definite pencil $(Q, B^*B)$.*

**Proof.** By Lemma 2, $P(\sigma,n) = B^{-*}QB^{-1}$. For any $x \ne 0$ write $x = By$ (bijective since $B$ is invertible). Then

$$\frac{x^* P(\sigma,n) x}{\|x\|^2} = \frac{y^* Q y}{\|By\|^2}.$$

Taking the minimum over $x \ne 0$ is equivalent to taking the minimum over $y \ne 0$, which gives the first equality in (4.3); the second is the standard variational characterization of generalized eigenvalues. □

**Proposition 5** (Refined bounds via restriction to the maximal eigenspace)**.** *Let $\sigma \notin \operatorname{spec}(A)$ and $|n| = 1$, and define $B, Q$ as in Proposition 4. Let $\lambda_{\max}(H(n))$ have maximal eigenspace $\mathcal{M}(n)$ and set*

$$\delta := \lambda_{\min}(Q) = \alpha - \lambda_{\max}(H(n)), \qquad \beta(\sigma,n) := \big\|B|_{\mathcal{M}(n)}\big\| = \max_{\substack{y \in \mathcal{M}(n) \\ \|y\| = 1}} \|By\|.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



*Assume $\delta > 0$ (equivalently $P(\sigma, n) \succ 0$). Then:*

*(a)* (Refined upper bound)

$$\lambda_{\min}(P(\sigma, n)) \;\le\; \frac{\delta}{\beta(\sigma, n)^2}. \tag{4.4}$$

*(b)* (Asymptotically sharp two-sided bound) *Assume in addition that $\lambda_{\max}(H(n))$ is spectrally isolated with gap*

$$\gamma_H(n) := \lambda_{\max}(H(n)) - \lambda^{\downarrow}_{m+1}(H(n)) \;>\; 0, \qquad m := \dim \mathcal{M}(n). \tag{4.5}$$

*Then*

$$\frac{\delta}{\beta(\sigma, n)^2 + \frac{\|B\|^2}{\gamma_H(n)}\,\delta} \;\le\; \lambda_{\min}(P(\sigma, n)) \;\le\; \frac{\delta}{\beta(\sigma, n)^2}. \tag{4.6}$$

*In particular, as $\delta \downarrow 0$ with $B$ boundedly invertible, $\lambda_{\min}(P(\sigma, n)) = \frac{\delta}{\beta(\sigma,n)^2} + O(\delta^2)$.*

**Proof.** By Proposition 4,

$$\lambda_{\min}(P(\sigma, n)) = \min_{y \neq 0} \frac{y^* Q y}{\|By\|^2}.$$

*(a)* On $\mathcal{M}(n)$ one has $H(n)y = \lambda_{\max}(H(n))y$, hence $Qy = (\alpha - \lambda_{\max}(H(n)))y = \delta y$. Choose $y \in \mathcal{M}(n)$ with $\|y\| = 1$ attaining $\|By\| = \beta(\sigma, n)$. Then

$$\lambda_{\min}(P(\sigma, n)) \le \frac{y^* Q y}{\|By\|^2} = \frac{\delta}{\beta(\sigma, n)^2},$$

which is (4.4).

*(b)* Decompose $y = y_M + y_\perp$ with $y_M \in \mathcal{M}(n)$ and $y_\perp \perp \mathcal{M}(n)$. Writing $Q = Q_0 + \delta I$ with $Q_0 := \lambda_{\max}(H(n))I - H(n) \succeq 0$, one has $Q_0 y_M = 0$ and, by the gap hypothesis (4.5), $Q_0 \succeq \gamma_H(n) I$ on $\mathcal{M}(n)^\perp$. Hence

$$y^* Q y \;\ge\; \delta \|y_M\|^2 + \gamma_H(n)\|y_\perp\|^2.$$

Moreover, using $\|By_M\| \le \beta(\sigma, n)\|y_M\|$ and $\|By_\perp\| \le \|B\|\,\|y_\perp\|$ gives

$$\|By\| \le \beta(\sigma, n)\|y_M\| + \|B\|\,\|y_\perp\|.$$

By Cauchy–Schwarz, for $a = \|y_M\|$ and $b = \|y_\perp\|$,

$$\big(\beta(\sigma, n)a + \|B\|b\big)^2 \le \left(\frac{\beta(\sigma, n)^2}{\delta} + \frac{\|B\|^2}{\gamma_H(n)}\right)(\delta a^2 + \gamma_H(n) b^2),$$

so combining with the two displays above gives

$$\frac{y^* Q y}{\|By\|^2} \;\ge\; \frac{\delta}{\beta(\sigma, n)^2 + \frac{\|B\|^2}{\gamma_H(n)}\,\delta}.$$

Taking the minimum over $y \neq 0$ yields the lower bound in (4.6); the upper bound is part (a). Dividing (4.6) by $\delta$ and letting $\delta \downarrow 0$ gives the stated first-order expansion. □

**Corollary 4** (Sharp first-order constant for outer offsets). *Fix $|n| = 1$ and let $z_0 \in \partial W(A)$ be a support point in direction $n$, i.e.* $\mathrm{Re}(\overline{n} z_0) = \lambda_{\max}(H(n))$. *Assume $z_0 \notin \mathrm{spec}(A)$ and $\gamma_H(n) > 0$ as in (4.5). For $\varepsilon > 0$ set $\sigma_\varepsilon := z_0 + \varepsilon n$ and*

$$P_\varepsilon(n) := \mathrm{Re}\Big(n(\sigma_\varepsilon I - A)^{-1}\Big).$$

*Then*

$$\lim_{\varepsilon \downarrow 0} \frac{\lambda_{\min}(P_\varepsilon(n))}{\varepsilon} = \Big\|(z_0 I - A)\big|_{\mathcal{M}(n)}\Big\|^{-2}. \tag{4.7}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



*In particular,* $\lambda_{\min}(P_\varepsilon(n)) = \varepsilon\,\|(z_0 I - A)|_{\mathcal{M}(n)}\|^{-2} + O(\varepsilon^2)$.

**Proof.** For $\sigma_\varepsilon = z_0 + \varepsilon n$ one has $\delta(\sigma_\varepsilon, n) = \varepsilon$. Applying Proposition 5 (b) to $(\sigma, n) = (\sigma_\varepsilon, n)$ gives a two-sided bound of the form

$$\frac{\varepsilon}{\beta(\sigma_\varepsilon, n)^2 + O(\varepsilon)} \le \lambda_{\min}(P_\varepsilon(n)) \le \frac{\varepsilon}{\beta(\sigma_\varepsilon, n)^2}.$$

Since $z_0 \notin \operatorname{spec}(A)$, $B_\varepsilon := \sigma_\varepsilon I - A \to B_0 := z_0 I - A$ in operator norm and $B_0$ is invertible. Hence $\beta(\sigma_\varepsilon, n) = \|B_\varepsilon|_{\mathcal{M}(n)}\| \to \|B_0|_{\mathcal{M}(n)}\|$, and dividing by $\varepsilon$ and letting $\varepsilon \downarrow 0$ yields (4.7). □

**Proposition 6** (Global first-order slope for direction-sampled offsets). *Assume that one can choose support points* continuously*, i.e. there exists a continuous map* $z_0 : \{n \in \mathbb{C} : |n| = 1\} \to \partial W(A)$ *with* $\operatorname{Re}(\overline{n} z_0(n)) = \lambda_{\max}(H(n))$ *and such that:*

*(i)* $z_0(n) \notin \operatorname{spec}(A)$ *(bounded resolvent on* $\partial W(A)$*), and*
*(ii) the top eigenspace* $\mathcal{M}(n)$ *is spectrally isolated with a uniform gap*

$$\inf_{|n|=1} \gamma_H(n) > 0. \tag{4.8}$$

*For each* $\varepsilon > 0$ *define the direction-sampled coercivity constant*

$$\tilde{c}(\varepsilon) := \inf_{|n|=1} \lambda_{\min}(P_\varepsilon(n)), \qquad P_\varepsilon(n) = \operatorname{Re}\Big(n\big((z_0(n) + \varepsilon n)I - A\big)^{-1}\Big).$$

*Set*

$$\beta_0(n) := \big\|(z_0(n) I - A)\big|_{\mathcal{M}(n)}\big\|, \qquad \beta_{\max} := \sup_{|n|=1} \beta_0(n).$$

*Then*

$$\lim_{\varepsilon \downarrow 0} \frac{\tilde{c}(\varepsilon)}{\varepsilon} = \frac{1}{\beta_{\max}^2}. \tag{4.9}$$

**Proof.** Under the uniform gap assumption (4.8), the spectral projector onto $\mathcal{M}(n)$ depends continuously on $n$. Since $n \mapsto z_0(n)$ is continuous by assumption, so is $B_0(n) := z_0(n) I - A$, and hence

$$\beta_0(n) = \big\|B_0(n)\big|_{\mathcal{M}(n)}\big\| = \|B_0(n)\Pi(n)\|$$

is continuous (where $\Pi(n)$ is the orthogonal projector onto $\mathcal{M}(n)$). In particular, $\beta_0$ attains its maximum $\beta_{\max}$.

For $\varepsilon > 0$ set $\sigma_\varepsilon(n) := z_0(n) + \varepsilon n$ and $B_\varepsilon(n) := \sigma_\varepsilon(n) I - A$. Since $z_0(\cdot)$ is continuous on the compact unit circle and avoids $\operatorname{spec}(A)$, there exists $\varepsilon_0 > 0$ such that $\sigma_\varepsilon(n) \notin \operatorname{spec}(A)$ for all $|n| = 1$ and all $0 \le \varepsilon \le \varepsilon_0$. Proposition 5 (b) applied at $(\sigma, n) = (\sigma_\varepsilon(n), n)$ (where $\delta(\sigma_\varepsilon(n), n) = \varepsilon$) gives, for all $0 < \varepsilon \le \varepsilon_0$ and all $|n| = 1$,

$$\frac{1}{\beta_\varepsilon(n)^2 + C\varepsilon} \le \frac{\lambda_{\min}(P_\varepsilon(n))}{\varepsilon} \le \frac{1}{\beta_\varepsilon(n)^2}, \qquad \beta_\varepsilon(n) := \big\|B_\varepsilon(n)\big|_{\mathcal{M}(n)}\big\|,$$

with a finite constant

$$C := \sup_{\substack{|n|=1 \\ 0 \le \varepsilon \le \varepsilon_0}} \frac{\|B_\varepsilon(n)\|^2}{\gamma_H(n)} < \infty$$

(using (4.8) and boundedness of $z_0$ on the unit circle). Since $(\varepsilon, n) \mapsto \beta_\varepsilon(n)$ is continuous on $[0, \varepsilon_0] \times \{n : |n| = 1\}$, it is uniformly continuous, and hence $\beta_\varepsilon(n) \to \beta_0(n)$ uniformly in $n$ as $\varepsilon \downarrow 0$. Moreover, $\beta_0(n) > 0$ for all $n$, so by continuity on the compact unit circle there exists $\beta_{\min} > 0$ with $\beta_0(n) \ge \beta_{\min}$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



for all $|n| = 1$, and therefore the same uniform convergence holds for the reciprocals. Therefore $\lambda_{\min}(P_\varepsilon(n))/\varepsilon \to 1/\beta_0(n)^2$ uniformly in $n$, and taking infima over $|n| = 1$ yields

$$\lim_{\varepsilon\downarrow 0} \frac{\tilde{c}(\varepsilon)}{\varepsilon} = \inf_{|n|=1} \frac{1}{\beta_0(n)^2} = \frac{1}{\sup_{|n|=1} \beta_0(n)^2} = \frac{1}{\beta_{\max}^2},$$

which is (4.9). □

*4.4. Slope Spectra, Two-Scale Splitting, and Face Detection at Non-Spectral Support Points*

The bounds and slope constants in Section 4.3 focus on $\lambda_{\min}$. Under a spectral-gap hypothesis for the supporting pencil $H(n)$, one can quantify the *entire* cluster of eigenvalues that collapses to 0 as the support gap closes. This yields a two-scale spectral splitting (an $O(\varepsilon)$ cluster plus an $O(1)$ bulk) and leads to a simple computational "face detector" based on counting small eigenvalues.

4.4.1. A Rigid Non-Tangential Offset Model

Fix a unimodular direction $n \in \mathbb{C}$ and assume that $\lambda_{\max}(H(n))$ is isolated with multiplicity

$$m := \dim \mathcal{M}(n), \qquad \gamma_H := \lambda_{\max}(H(n)) - \lambda^{\downarrow}_{m+1}(H(n)) > 0 \qquad (m < d), \tag{4.10}$$

where $\mathcal{M}(n) = \mathrm{Ker}(\lambda_{\max}(H(n))I - H(n))$ is the maximal eigenspace of $H(n) = \mathrm{Re}(\overline{n}A)$. Choose any unit vector $v \in \mathcal{M}(n)$ and define the associated numerical-range support point

$$\sigma_0 := v^* A v \in \partial W(A), \qquad \mathrm{Re}(\overline{n}\,\sigma_0) = \lambda_{\max}(H(n)) \tag{4.11}$$

(Lemma 1). We impose the bounded-resolvent hypothesis

$$\sigma_0 \notin \mathrm{spec}(A). \tag{4.12}$$

For $\varepsilon > 0$, define the outer offset point and the offset kernel

$$\sigma_\varepsilon := \sigma_0 + \varepsilon n, \qquad P_\varepsilon := P(\sigma_\varepsilon, n) = \mathrm{Re}\Big(n\big(\sigma_\varepsilon I - A\big)^{-1}\Big). \tag{4.13}$$

Let $V \in \mathbb{C}^{d\times m}$ have orthonormal columns spanning $\mathcal{M}(n)$ and set

$$B_0 := \sigma_0 I - A, \qquad G := V^* B_0^* B_0 V \in \mathbb{C}^{m\times m}. \tag{4.14}$$

**Proposition 7** (Offset slope spectrum for the collapsing eigenvalue cluster)**.** *Under* (4.10)–(4.12), $G \succ 0$ *and*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\, \lambda^{\uparrow}_j(P_\varepsilon) \;=\; \lambda^{\uparrow}_j(G^{-1}), \qquad j = 1, \ldots, m. \tag{4.15}$$

*Equivalently, if* $0 < g_1 \le \cdots \le g_m$ *are the eigenvalues of* $G$*, then*

$$\lambda^{\uparrow}_j(P_\varepsilon) \;=\; \frac{\varepsilon}{g_{m+1-j}} + o(\varepsilon), \qquad j = 1, \ldots, m.$$

*In particular, if* $m = 1$ *and* $\mathcal{M}(n) = \mathrm{span}\{v\}$*, then*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\lambda_{\min}(P_\varepsilon) \;=\; \frac{1}{\|(\sigma_0 I - A)v\|^2},$$

*which recovers Corollary 4 for this rigid offset family.*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



**Proof.** **Step 1:** $G \succ 0$. Since $\sigma_0 \notin \mathrm{spec}(A)$, the matrix $B_0 = \sigma_0 I - A$ is invertible. The $d \times m$ matrix $V$ has full column rank (orthonormal columns), hence $B_0 V$ has full column rank. Therefore the Gram matrix

$$G = (B_0 V)^*(B_0 V) = V^* B_0^* B_0 V$$

is Hermitian positive definite.

**Step 2: Congruence for the offset family.** Set $B_\varepsilon := \sigma_\varepsilon I - A = B_0 + \varepsilon n I$ and

$$Q_0 := \lambda_{\max}(H(n)) I - H(n) \succeq 0.$$

By Lemma 2 (applied at $(\sigma, n) = (\sigma_\varepsilon, n)$) and $\mathrm{Re}(\overline{n}\sigma_\varepsilon) = \mathrm{Re}(\overline{n}\sigma_0) + \varepsilon = \lambda_{\max}(H(n)) + \varepsilon$, we obtain

$$B_\varepsilon^* P_\varepsilon B_\varepsilon = \mathrm{Re}(\overline{n}\sigma_\varepsilon) I - H(n) = Q_0 + \varepsilon I. \tag{4.16}$$

For sufficiently small $\varepsilon$, $B_\varepsilon$ remains invertible, hence $P_\varepsilon \succ 0$.

**Step 3: Work with the inverse.** From (4.16),

$$P_\varepsilon^{-1} = B_\varepsilon \left(Q_0 + \varepsilon I\right)^{-1} B_\varepsilon^*. \tag{4.17}$$

Let $\Pi := VV^*$ be the orthogonal projector onto $\mathcal{M}(n) = \mathrm{Ker}(Q_0)$. On $\mathcal{M}(n)$ one has $(Q_0 + \varepsilon I)^{-1} = (1/\varepsilon) I$. On $\mathcal{M}(n)^\perp$ the gap assumption gives $Q_0 \succeq \gamma_H (I - \Pi)$ and hence

$$\|(Q_0 + \varepsilon I)^{-1}(I - \Pi)\| \le \frac{1}{\gamma_H}.$$

Thus

$$\varepsilon (Q_0 + \varepsilon I)^{-1} = \Pi + \varepsilon R_\varepsilon, \qquad R_\varepsilon := (Q_0 + \varepsilon I)^{-1}(I - \Pi), \quad \|R_\varepsilon\| \le \gamma_H^{-1}.$$

Multiplying (4.17) by $\varepsilon$ yields

$$\varepsilon\, P_\varepsilon^{-1} = B_\varepsilon \Pi B_\varepsilon^* + \varepsilon\, B_\varepsilon R_\varepsilon B_\varepsilon^*. \tag{4.18}$$

Since $B_\varepsilon \to B_0$ and $\{R_\varepsilon\}$ is uniformly bounded, we conclude that

$$\varepsilon\, P_\varepsilon^{-1} \longrightarrow S := B_0 \Pi B_0^* \qquad (\varepsilon \downarrow 0) \tag{4.19}$$

in operator norm.

**Step 4: Identify the nonzero spectrum of the limit.** Since $S = B_0 V V^* B_0^* = (B_0 V)(B_0 V)^*$, $S \succeq 0$ has rank $m$. Its nonzero eigenvalues coincide with the eigenvalues of

$$(B_0 V)^*(B_0 V) = V^* B_0^* B_0 V = G.$$

Writing eigenvalues in nondecreasing order,

$$\lambda^\uparrow_{d-m+i}(S) = \lambda^\uparrow_i(G), \qquad i = 1, \ldots, m, \qquad \lambda^\uparrow_j(S) = 0 \;\; (j \le d - m).$$

**Step 5: Pass to eigenvalues and invert.** By Weyl's inequality and (4.19),

$$\lambda^\uparrow_{d-m+i}(\varepsilon P_\varepsilon^{-1}) \to \lambda^\uparrow_i(G), \qquad i = 1, \ldots, m.$$

Since $P_\varepsilon \succ 0$, the eigenvalues of $P_\varepsilon$ and $P_\varepsilon^{-1}$ are reciprocal and reversed:

$$\lambda^\uparrow_j(P_\varepsilon) = \frac{1}{\lambda^\uparrow_{d+1-j}(P_\varepsilon^{-1})}.$$

Hence for $j = 1, \dots, m$,

$$\frac{1}{\varepsilon}\lambda_j^{\uparrow}(P_\varepsilon) = \frac{1}{\lambda_{d+1-j}^{\uparrow}(\varepsilon P_\varepsilon^{-1})} \longrightarrow \frac{1}{\lambda_{d+1-j}^{\uparrow}(S)}.$$

Now $d+1-j = d-m+(m+1-j)$, so $\lambda_{d+1-j}^{\uparrow}(S) = \lambda_{m+1-j}^{\uparrow}(G)$ and therefore

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\lambda_j^{\uparrow}(P_\varepsilon) = \frac{1}{\lambda_{m+1-j}^{\uparrow}(G)} = \lambda_j^{\uparrow}(G^{-1}),$$

which is (4.15). The case $m = 1$ follows from $G = \|B_0 v\|^2$. □

**Remark 6** (Basis invariance)**.** *Although G is defined using a particular orthonormal basis V of $\mathcal{M}(n)$, its eigenvalues (and hence the slope spectrum $\lambda_j^{\uparrow}(G^{-1})$) depend only on the subspace $\mathcal{M}(n)$: if $\widetilde{V} = VU$ for unitary $U \in \mathbb{C}^{m\times m}$, then $\widetilde{G} = U^*GU$ has the same spectrum.*

**Proposition 8** (Two-scale spectral splitting and a uniform $O(1)$ lower bound)**.** *Assume the hypotheses of Proposition 7. Let $\mu_1 \le \cdots \le \mu_m$ be the eigenvalues of $G^{-1}$ (the slope spectrum). Then:*

*(i)* $\lambda_j^{\uparrow}(P_\varepsilon) = \mu_j\,\varepsilon + o(\varepsilon)$ *for* $j = 1, \dots, m$.

*(ii) The remaining eigenvalues stay bounded away from 0: there exist $\varepsilon_0 > 0$ and $c_\star > 0$ such that*

$$\lambda_{m+1}^{\uparrow}(P_\varepsilon) \;\ge\; c_\star \qquad \textit{for all } 0 < \varepsilon \le \varepsilon_0.$$

*A concrete bound is*

$$c_\star := \frac{\gamma_H}{2\|B_0\|^2}, \qquad \varepsilon_0 := \min\left\{\frac{1}{2\|B_0^{-1}\|}, \frac{\gamma_H}{4\|B_0\|^2\|B_0^{-1}\|^2}\right\}. \tag{4.20}$$

**Proof.** Part (i) is exactly Proposition 7.

For (ii), define the $\varepsilon = 0$ kernel

$$P_0 := P(\sigma_0, n) = \mathrm{Re}\big(n(\sigma_0 I - A)^{-1}\big),$$

which is well-defined since $\sigma_0 \notin \mathrm{spec}(A)$. By Lemma 2 at $(\sigma, n) = (\sigma_0, n)$ and the support identity $\mathrm{Re}(\overline{n}\sigma_0) = \lambda_{\max}(H(n))$,

$$B_0^* P_0 B_0 = \lambda_{\max}(H(n))I - H(n) = Q_0 \succeq 0.$$

The eigenvalues of $Q_0$ are 0 (multiplicity $m$) and at least $\gamma_H$ on $\mathcal{M}(n)^\perp$ by the gap assumption. By the Courant–Fischer min–max principle (see, e.g., [10]) with the change of variables $x = B_0 y$,

$$\lambda_{m+1}^{\uparrow}(P_0) = \min_{\dim S = m+1}\ \max_{\substack{x\in S\\ \|x\|=1}} x^* P_0 x = \min_{\dim S = m+1}\ \max_{\substack{y \in B_0^{-1}S\\ y\neq 0}} \frac{y^* Q_0 y}{\|B_0 y\|^2} \ge \frac{1}{\|B_0\|^2}\lambda_{m+1}^{\uparrow}(Q_0) = \frac{\gamma_H}{\|B_0\|^2}.$$

Now compare $P_\varepsilon$ to $P_0$ in operator norm. For $\varepsilon \le 1/(2\|B_0^{-1}\|)$, the Neumann series gives $B_\varepsilon^{-1} = (B_0 + \varepsilon n I)^{-1}$ well-defined and

$$\|B_\varepsilon^{-1}\| \le 2\|B_0^{-1}\|, \qquad \|B_\varepsilon^{-1} - B_0^{-1}\| = \|B_\varepsilon^{-1}(B_0 - B_\varepsilon)B_0^{-1}\| \le 2\varepsilon\,\|B_0^{-1}\|^2.$$

Therefore

$$\|P_\varepsilon - P_0\| = \|\,\mathrm{Re}(n(B_\varepsilon^{-1} - B_0^{-1}))\| \le \|B_\varepsilon^{-1} - B_0^{-1}\| \le 2\varepsilon\,\|B_0^{-1}\|^2.$$

Weyl's inequality gives

$$\lambda_{m+1}^{\uparrow}(P_\varepsilon) \ge \lambda_{m+1}^{\uparrow}(P_0) - \|P_\varepsilon - P_0\| \ge \frac{\gamma_H}{\|B_0\|^2} - 2\varepsilon\|B_0^{-1}\|^2.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



If $\varepsilon \le \gamma_H/(4\|B_0\|^2\|B_0^{-1}\|^2)$, then the last term is $\le \gamma_H/(2\|B_0\|^2)$, and we obtain

$$\lambda^{\uparrow}_{m+1}(P_\varepsilon) \ge \frac{\gamma_H}{2\|B_0\|^2} = c_\star,$$

with the explicit choices (4.20). □

**Corollary 5** (A rigorous "face detector" threshold). *Assume the hypotheses of Proposition 7. Let* $\mu_{\max} := \mu_m = \lambda_{\max}(G^{-1})$. *Fix any* $\tau > \mu_{\max}$. *Then there exists* $\varepsilon_\tau > 0$ *such that for all* $0 < \varepsilon \le \varepsilon_\tau$,

$$\#\Big\{ j : \ \lambda^{\uparrow}_j(P_\varepsilon) \ \le \ \tau\,\varepsilon \Big\} = m.$$

**Proof.** By Proposition 7, for each $j \le m$, $\lambda^{\uparrow}_j(P_\varepsilon)/\varepsilon \to \mu_j \le \mu_{\max} < \tau$, so for sufficiently small $\varepsilon$ one has $\lambda^{\uparrow}_j(P_\varepsilon) \le \tau\varepsilon$ for all $j \le m$. By Proposition 8 (ii), $\lambda^{\uparrow}_{m+1}(P_\varepsilon) \ge c_\star > 0$ for small $\varepsilon$, so $\lambda^{\uparrow}_{m+1}(P_\varepsilon) > \tau\varepsilon$ for all sufficiently small $\varepsilon$. This implies the stated count. □

#### 4.4.2. Intrinsic Rescaled Clusters Under General $C^1$ Convex Exhaustions

The offset analysis above uses the rigid non-tangential approach $\sigma_\varepsilon = \sigma_0 + \varepsilon n$. Along a general $C^1$ convex exhaustion $\Omega_\varepsilon \downarrow W(A)$, the natural small parameter is the support gap $\delta = \mathrm{Re}(\overline{n_\Omega(\sigma)}\,\sigma) - \lambda_{\max}(H(n_\Omega(\sigma)))$. After rescaling by $\delta$, the slope spectrum is intrinsic (independent of the exhaustion).

**Proposition 9** (Intrinsic rescaled collapsing cluster under arbitrary $C^1$ exhaustions). *Let* $\{\Omega_\varepsilon\}$ *be a* $C^1$ *convex exhaustion of* $W(A)$. *Let* $\varepsilon_k \downarrow 0$ *and let* $\sigma_k \in \partial\Omega_{\varepsilon_k}$ *satisfy*

$$\sigma_k \to \sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A), \qquad n_k := n_{\Omega_{\varepsilon_k}}(\sigma_k) \to n, \qquad |n_k| = |n| = 1.$$

*Assume that* $\lambda_{\max}(H(n))$ *is isolated with multiplicity*

$$m := \dim \mathcal{M}(n), \qquad \mathcal{M}(n) = \mathrm{Ker}(\lambda_{\max}(H(n))I - H(n)),$$

*and gap*

$$\gamma_H := \lambda_{\max}(H(n)) - \lambda^{\downarrow}_{m+1}(H(n)) > 0.$$

*Define the support gap*

$$\delta_k := \mathrm{Re}(\overline{n_k}\sigma_k) - \lambda_{\max}(H(n_k)) \ \ge \ 0,$$

*and assume* $\delta_k > 0$ *for all sufficiently large* $k$.

*Let* $V \in \mathbb{C}^{d\times m}$ *have orthonormal columns spanning* $\mathcal{M}(n)$ *and set*

$$B_0 := \sigma_0 I - A, \qquad G := V^* B_0^* B_0 V.$$

*Then* $G \succ 0$ *and for each* $j = 1, \dots, m$,

$$\lim_{k\to\infty} \frac{1}{\delta_k}\lambda^{\uparrow}_j(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)) = \lambda^{\uparrow}_j(G^{-1}). \tag{4.21}$$

*Moreover, the remaining eigenvalues stay uniformly bounded away from 0: there exist* $c_\star > 0$ *and* $k_0$ *such that*

$$\lambda^{\uparrow}_{m+1}(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)) \ge c_\star \qquad \text{for all } k \ge k_0.$$

**Proof.** Set $P_k := P_{\Omega_{\varepsilon_k}}(\sigma_k, A)$. Since $\sigma_0 \notin \mathrm{spec}(A)$ and $\mathrm{spec}(A)$ is finite, $\mathrm{dist}(\sigma_0, \mathrm{spec}(A)) > 0$ and thus $\sigma_k \notin \mathrm{spec}(A)$ for all sufficiently large $k$. In particular, $B_k := \sigma_k I - A$ is invertible for large $k$.

**Step 1: Positivity of** $G$. As in Proposition 7, $B_0$ is invertible and $V$ has full column rank, hence $B_0 V$ has full column rank and $G = (B_0V)^*(B_0V) \succ 0$.

**Step 2: Congruence at $(\sigma_k, n_k)$ and a useful bound $\delta_k \to 0$.** Because $\Omega_{\varepsilon_k}$ is $C^1$ convex and $n_k$ is the outward unit normal at $\sigma_k \in \partial\Omega_{\varepsilon_k}$,

$$P_k = \mathrm{Re}\big(n_k(\sigma_k I - A)^{-1}\big) =: P(\sigma_k, n_k).$$

Let $H_k := H(n_k) = \mathrm{Re}(\overline{n_k}A)$ and define

$$Q_k := \lambda_{\max}(H_k)I - H_k \succeq 0.$$

Applying Lemma 2 with $(\sigma, n) = (\sigma_k, n_k)$ yields

$$B_k^* P_k B_k = \mathrm{Re}(\overline{n_k}\sigma_k)I - H_k = Q_k + \delta_k I. \tag{4.22}$$

Since $\sigma_k \to \sigma_0 \in W(A)$, we have the estimate

$$0 \le \delta_k = \mathrm{Re}(\overline{n_k}\sigma_k) - \max_{z\in W(A)} \mathrm{Re}(\overline{n_k}z) \le \mathrm{Re}(\overline{n_k}\sigma_k) - \mathrm{Re}(\overline{n_k}\sigma_0) \le |\sigma_k - \sigma_0|, \tag{4.23}$$

hence $\delta_k \to 0$.

**Step 3: Uniform gap persistence and convergence of spectral projectors.** Since $n_k \to n$ and $H(\cdot)$ is continuous, $H_k \to H := H(n)$ in operator norm. By Weyl's inequality for Hermitian matrices,

$$|\lambda_j(H_k) - \lambda_j(H)| \le \|H_k - H\| \qquad (j = 1, \dots, d),$$

so for all sufficiently large $k$ the top eigenvalue cluster of $H_k$ has the same multiplicity $m$ and a uniform gap

$$\gamma_k := \lambda_{\max}(H_k) - \lambda^{\downarrow}_{m+1}(H_k) \;\ge\; \gamma_H/2. \tag{4.24}$$

Let $\Pi_k$ be the orthogonal projector onto $\mathcal{M}(n_k) = \mathrm{Ker}(Q_k)$ and $\Pi := VV^*$ the orthogonal projector onto $\mathcal{M}(n)$. Standard Riesz projector arguments (cf. [10]) yield $\|\Pi_k - \Pi\| \to 0$.

**Step 4: Work with the inverse and rescale by the support gap.** From (4.22) and invertibility of $B_k$,

$$P_k^{-1} = B_k\,(Q_k + \delta_k I)^{-1}\,B_k^*. \tag{4.25}$$

On $\mathrm{Ker}(Q_k) = \mathrm{ran}(\Pi_k)$ one has $(Q_k + \delta_k I)^{-1} = (1/\delta_k)I$, so

$$\delta_k(Q_k + \delta_k I)^{-1}\Pi_k = \Pi_k.$$

On $\mathrm{ran}(I - \Pi_k)$, the gap bound (4.24) implies $Q_k \succeq \gamma_k(I - \Pi_k)$ and hence

$$\|(Q_k + \delta_k I)^{-1}(I - \Pi_k)\| \le \frac{1}{\gamma_k} \le \frac{2}{\gamma_H}.$$

Therefore

$$\delta_k(Q_k + \delta_k I)^{-1} = \Pi_k + \delta_k R_k, \qquad R_k := (Q_k + \delta_k I)^{-1}(I - \Pi_k), \qquad \|R_k\| \le \frac{2}{\gamma_H}.$$

Multiplying (4.25) by $\delta_k$ gives

$$\delta_k\, P_k^{-1} = B_k\Pi_k B_k^* + \delta_k\, B_k R_k B_k^*. \tag{4.26}$$

**Step 5: Take the limit $k \to \infty$ and identify the spectrum.** Since $B_k \to B_0$ and $\Pi_k \to \Pi$ in operator norm, we have

$$\delta_k\, P_k^{-1} \longrightarrow S := B_0 \Pi B_0^* \qquad \text{in operator norm.}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



As in Proposition 7, $S = (B_0V)(B_0V)^*$ has rank $m$ and its nonzero eigenvalues coincide with those of $G = (B_0V)^*(B_0V)$.

**Step 6: Invert the corresponding eigenvalues.** By Weyl's inequality and the reciprocity of eigenvalues of $P_k$ and $P_k^{-1}$, one obtains (4.21) exactly as in the offset proof. The uniform lower bound for $\lambda^{\uparrow}_{m+1}(P_k)$ follows by the Courant–Fischer principle and the uniform gap bound $\gamma_k \geq \gamma_H/2$, together with boundedness of $\|B_k\|$ for large $k$. □

**Corollary 6** (Exhaustion-invariant face-detector threshold)**.** *Assume the hypotheses of Proposition 9 and let $\mu_{\max} := \lambda_{\max}(G^{-1})$. Fix any $\tau > \mu_{\max}$. Then there exists $k_\tau$ such that for all $k \geq k_\tau$,*

$$\#\Big\{j : \lambda^{\uparrow}_j(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)) \ \leq \ \tau\,\delta_k\Big\} = m.$$

**Proof.** By (4.21), for each $j \leq m$, $\lambda^{\uparrow}_j(P_k)/\delta_k \to \lambda^{\uparrow}_j(G^{-1}) \leq \mu_{\max} < \tau$, hence $\lambda^{\uparrow}_j(P_k) \leq \tau\delta_k$ for all sufficiently large $k$. The uniform bound in Proposition 9 gives $\lambda^{\uparrow}_{m+1}(P_k) \geq c_\star > 0$ for large $k$. Since $\delta_k \to 0$, eventually $\tau\delta_k < c_\star$, hence $\lambda^{\uparrow}_{m+1}(P_k) > \tau\delta_k$ for large $k$. This yields the count. □

**Remark 7** (Geometric meaning of the support gap)**.** *For a convex exhaustion $\Omega_\varepsilon \downarrow W(A)$, the scalar*

$$\delta_k = \mathrm{Re}(\overline{n_k}\sigma_k) - \lambda_{\max}(H(n_k))$$

*is the support-function mismatch between $\Omega_{\varepsilon_k}$ and $W(A)$ in direction $n_k$. Support functions and their stability properties are classical in convex geometry; see, e.g., [12].*

### 4.5. Convergence of the near-Kernel Subspace

**Proposition 10** (Convergence of the near-kernel spectral projector)**.** *Assume the setting of Theorem 1 and set $m := \dim \mathcal{M}(n)$. Assume that $\lambda_{\max}(H(n))$ is isolated with multiplicity $m$, i.e.*

$$\gamma_H := \lambda_{\max}(H(n)) - \lambda^{\downarrow}_{m+1}(H(n)) \ > \ 0. \tag{4.27}$$

*Let*

$$P_0 := \mathrm{Re}\big(n(\sigma_0 I - A)^{-1}\big), \qquad \mathcal{K}_0 := \mathrm{Ker}(P_0) = (\sigma_0 I - A)\mathcal{M}(n), \qquad \Pi_0 : \mathbb{C}^d \to \mathcal{K}_0$$

*be the orthogonal projector onto $\mathcal{K}_0$.*

*For each $k$, let $P_k := P_{\Omega_{\varepsilon_k}}(\sigma_k, A)$ and let $\Pi_k$ be the orthogonal projector onto the direct sum of the eigenspaces of $P_k$ corresponding to its $m$ smallest eigenvalues. Then $\|\Pi_k - \Pi_0\| \to 0$ as $k \to \infty$.*

*Moreover, writing $B_0 = \sigma_0 I - A$, one has the explicit spectral-gap bound*

$$\lambda^{\uparrow}_{m+1}(P_0) \ \geq \ \frac{\gamma_H}{\|B_0\|^2}, \tag{4.28}$$

*and consequently, for all sufficiently large $k$,*

$$\|\Pi_k - \Pi_0\| \ \leq \ \frac{2\|P_k - P_0\|}{\lambda^{\uparrow}_{m+1}(P_0)} \ \leq \ \frac{2\|B_0\|^2}{\gamma_H}\,\|P_k - P_0\|. \tag{4.29}$$

**Proof.** By Lemma 2 and Step 2 of Theorem 1,

$$B_0^* P_0 B_0 = Q_0 := \lambda_{\max}(H(n))I - H(n) \succeq 0.$$

The eigenvalues of $Q_0$ are 0 with multiplicity $m$ and at least $\gamma_H$ on $\mathcal{M}(n)^\perp$, so $\lambda^{\uparrow}_{m+1}(Q_0) = \gamma_H$.

Using the Courant–Fischer characterization with the change of variables $x = B_0 y$, one obtains for every $j$

$$\lambda_j^{\uparrow}(P_0) = \min_{\dim S = j} \max_{\substack{x \in S \\ \|x\|=1}} x^* P_0 x = \min_{\dim S = j} \max_{\substack{y \in B_0^{-1} S \\ y \neq 0}} \frac{y^* Q_0 y}{\|B_0 y\|^2} \geq \frac{1}{\|B_0\|^2} \lambda_j^{\uparrow}(Q_0).$$

Taking $j = m+1$ gives (4.28).

Next, Theorem 1 gives $\|P_k - P_0\| \to 0$. Since $P_0$ has an isolated cluster of $m$ eigenvalues at 0 separated by the gap $\lambda_{m+1}^{\uparrow}(P_0) > 0$, the Davis–Kahan $\sin\Theta$ theorem for invariant subspaces [11] yields (4.29), and hence $\|\Pi_k - \Pi_0\| \to 0$. □

*4.6. The Spectral-Support Regime for Normal Matrices*

**Proposition 11** (Normal matrices: explicit eigenvalues near a spectral support point)**.** *Let $A$ be normal with eigenvalues $\lambda_1, \dots, \lambda_d$ (listed with algebraic multiplicity). Fix $\sigma \notin \operatorname{spec}(A)$ and unimodular $n \in \mathbb{C}$. Then*

$$P(\sigma, n) := \operatorname{Re}\big(n(\sigma I - A)^{-1}\big)$$

*is unitarily diagonalizable and its eigenvalues are the scalars*

$$p_j(\sigma, n) := \operatorname{Re}\left(\frac{n}{\sigma - \lambda_j}\right) = \frac{\operatorname{Re}(\overline{n}(\sigma - \lambda_j))}{|\sigma - \lambda_j|^2}, \qquad j = 1, \dots, d. \tag{4.30}$$

*Now fix $\sigma_0 \in \operatorname{spec}(A)$ and let*

$$J_0 := \{j \in \{1, \dots, d\} : \lambda_j = \sigma_0\}.$$

*Let $\sigma_k \notin \operatorname{spec}(A)$ and unimodular $n_k$ satisfy*

$$\sigma_k \to \sigma_0, \qquad n_k \to n.$$

*Write $p_{k,j} := p_j(\sigma_k, n_k)$. Then:*

*(i) For every $j \notin J_0$,*

$$p_{k,j} \longrightarrow p_j(\sigma_0, n) = \operatorname{Re}\left(\frac{n}{\sigma_0 - \lambda_j}\right).$$

*(ii) For every $j \in J_0$ one has the* exact *identity*

$$p_{k,j} = \operatorname{Re}\left(\frac{n_k}{\sigma_k - \sigma_0}\right) = \frac{\operatorname{Re}(\overline{n_k}(\sigma_k - \sigma_0))}{|\sigma_k - \sigma_0|^2}.$$

*In particular, if there exists $c > 0$ such that*

$$\operatorname{Re}(\overline{n_k}(\sigma_k - \sigma_0)) \geq c\,|\sigma_k - \sigma_0| \qquad \text{for all sufficiently large } k, \tag{4.31}$$

*then $p_{k,j} \to +\infty$ for every $j \in J_0$.*

*(iii) Assume in addition that $n$ is a supporting direction for $W(A) = \operatorname{conv}\{\lambda_1, \dots, \lambda_d\}$ at $\sigma_0$, i.e.*

$$\operatorname{Re}(\overline{n}\,\lambda_j) \leq \operatorname{Re}(\overline{n}\,\sigma_0), \qquad j = 1, \dots, d. \tag{4.32}$$

*Then for every $j \notin J_0$,*

$$p_j(\sigma_0, n) = \frac{\operatorname{Re}(\overline{n}(\sigma_0 - \lambda_j))}{|\sigma_0 - \lambda_j|^2} = \frac{\operatorname{Re}(\overline{n}\,\sigma_0) - \operatorname{Re}(\overline{n}\,\lambda_j)}{|\sigma_0 - \lambda_j|^2} \geq 0,$$

*and* $p_j(\sigma_0, n) = 0$ *if and only if* $\lambda_j$ *lies on the same supporting line* $\{z : \mathrm{Re}(\overline{n}z) = \mathrm{Re}(\overline{n}\sigma_0)\}$. *If moreover* (4.31) *holds (so that all* $p_{k,j} \to +\infty$ *for* $j \in J_0$*), then*

$$\lambda_{\min}\big(P(\sigma_k, n_k)\big) \longrightarrow \min_{j \notin J_0} p_j(\sigma_0, n),$$

*which is strictly positive if and only if no eigenvalue* $\lambda_j \neq \sigma_0$ *lies on the supporting line* $\{z : \mathrm{Re}(\overline{n}z) = \mathrm{Re}(\overline{n}\sigma_0)\}$.

**Proof.** Since $A$ is normal, $A = U\,\mathrm{diag}(\lambda_1, \dots, \lambda_d)\,U^*$ for some unitary $U$, hence

$$(\sigma I - A)^{-1} = U\,\mathrm{diag}\bigg(\frac{1}{\sigma - \lambda_1}, \dots, \frac{1}{\sigma - \lambda_d}\bigg)U^*.$$

Therefore,

$$\begin{aligned} P(\sigma, n) &= \mathrm{Re}\big(n(\sigma I - A)^{-1}\big) \\ &= U\ \mathrm{Re}\bigg(\mathrm{diag}\bigg(\frac{n}{\sigma - \lambda_1}, \dots, \frac{n}{\sigma - \lambda_d}\bigg)\bigg)U^* \\ &= U\,\mathrm{diag}\bigg(\mathrm{Re}\bigg(\frac{n}{\sigma - \lambda_1}\bigg), \dots, \mathrm{Re}\bigg(\frac{n}{\sigma - \lambda_d}\bigg)\bigg)U^*, \end{aligned}$$

which proves (4.30). The limit in (i) follows by continuity of the map $(\sigma, n) \mapsto \mathrm{Re}\big(n/(\sigma - \lambda_j)\big)$ when $\sigma_0 \neq \lambda_j$. For (ii), if $\lambda_j = \sigma_0$ then

$$\mathrm{Re}\bigg(\frac{n_k}{\sigma_k - \sigma_0}\bigg) = \mathrm{Re}\bigg(\frac{n_k\overline{\sigma_k - \sigma_0}}{|\sigma_k - \sigma_0|^2}\bigg) = \frac{\mathrm{Re}\big(\overline{n_k}(\sigma_k - \sigma_0)\big)}{|\sigma_k - \sigma_0|^2},$$

and (4.31) implies $p_{k,j} \geq c/|\sigma_k - \sigma_0| \to +\infty$.

Finally, (4.32) implies $\mathrm{Re}(\overline{n}(\sigma_0 - \lambda_j)) \geq 0$ for all $j$, giving the nonnegativity (and the characterization of equality) in (iii). If additionally (4.31) holds, then $p_{k,j} \to +\infty$ for all $j \in J_0$ while $p_{k,j} \to p_j(\sigma_0, n) \in [0, \infty)$ for $j \notin J_0$, so for large $k$ the minimum eigenvalue is attained among indices $j \notin J_0$, yielding the stated limit and positivity criterion. □

**Definition 3** (Poisson degeneracy exponent for normal matrices)**.** *Let* $A$ *be normal and let* $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$ *be a spectral support point with supporting direction* $n$, *i.e.* $|n| = 1$ *and* $\mathrm{Re}(\overline{n}\sigma_0) = \lambda_{\max}(H(n))$. *For the non-tangential offset family* $\sigma_\varepsilon = \sigma_0 + \varepsilon n$ *(*$\varepsilon > 0$*), define the* Poisson degeneracy exponent

$$\alpha(\sigma_0, n) := \lim_{\varepsilon \downarrow 0} \frac{\log \lambda_{\min}\big(P(\sigma_\varepsilon, n)\big)}{\log \varepsilon},$$

*whenever the limit exists (note that* $\log \varepsilon \to -\infty$*).*

**Corollary 7** (Normal matrices: a dichotomy for the exponent)**.** *Let* $A$ *be normal with eigenvalues* $\{\lambda_j\}_{j=1}^d$. *Fix a spectral support point* $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$ *and a supporting direction* $n$. *Let* $J_0 := \{j : \mathrm{Re}(\overline{n}\lambda_j) = \lambda_{\max}(H(n))\}$ *be the set of eigenvalues on the supporting line. Then:*

*(i)* *If* $J_0 = \{j_0\}$ *and* $\lambda_{j_0} = \sigma_0$ *(i.e. no other eigenvalue lies on the supporting line), then* $\lambda_{\min}(P(\sigma_0 + \varepsilon n, n)) \to c_0 > 0$ *as* $\varepsilon \downarrow 0$ *and* $\alpha(\sigma_0, n) = 0$.

*(ii)* *If there exists* $j \in J_0$ *with* $\lambda_j \neq \sigma_0$ *(equivalently, the supporting face contains at least two eigenvalues), then*

$$\lambda_{\min}\big(P(\sigma_0 + \varepsilon n, n)\big) = C\,\varepsilon + o(\varepsilon), \qquad C = \min_{\substack{j \in J_0 \\ \lambda_j \neq \sigma_0}} \frac{1}{|\sigma_0 - \lambda_j|^2},$$

*and* $\alpha(\sigma_0, n) = 1$.

**Proof.** By Proposition 11, for $\sigma = \sigma_0 + \varepsilon n$ the eigenvalues of $P(\sigma, n)$ are the scalars

$$p_j(\sigma, n) = \frac{\mathrm{Re}(\overline{n}(\sigma - \lambda_j))}{|\sigma - \lambda_j|^2}.$$

If $J_0 = \{j_0\}$ with $\lambda_{j_0} = \sigma_0$, then for all $j \neq j_0$ one has $\mathrm{Re}(\overline{n}(\sigma_0 - \lambda_j)) > 0$, hence $p_j(\sigma_0, n) > 0$. Continuity gives $\min_{j \neq j_0} p_j(\sigma_0 + \varepsilon n, n) \to \min_{j \neq j_0} p_j(\sigma_0, n) =: c_0 > 0$, while $p_{j_0}(\sigma_0 + \varepsilon n, n) = 1/\varepsilon \to \infty$. Thus $\lambda_{\min}(P(\sigma_0 + \varepsilon n, n)) \to c_0$ and $\alpha(\sigma_0, n) = 0$.

If there exists $j \in J_0$ with $\lambda_j \neq \sigma_0$, then $\mathrm{Re}(\overline{n}(\sigma_0 - \lambda_j)) = 0$ and

$$p_j(\sigma_0 + \varepsilon n, n) = \frac{\varepsilon}{|\sigma_0 - \lambda_j + \varepsilon n|^2} = \frac{\varepsilon}{|\sigma_0 - \lambda_j|^2} + o(\varepsilon).$$

All $j \notin J_0$ give $p_j(\sigma_0, n) > 0$, hence contribute $O(1)$ values as $\varepsilon \downarrow 0$. Therefore the minimum is attained among $j \in J_0$ with $\lambda_j \neq \sigma_0$, yielding the stated $C$ and $\alpha(\sigma_0, n) = 1$. □

**Example 1** (Nondegeneracy at a spectral support point). *Let $A = \mathrm{diag}(0, 1)$, so $W(A) = [0, 1]$. Take $\sigma = 1 + \varepsilon$ with $\varepsilon > 0$ and $n = 1$. Then*

$$P(\sigma, n) = \mathrm{Re}\big((\sigma I - A)^{-1}\big) = \mathrm{diag}\Big(\frac{1}{1+\varepsilon}, \frac{1}{\varepsilon}\Big),$$

*so $\lambda_{\min}(P(\sigma, n)) = \frac{1}{1+\varepsilon} \to 1$ as $\varepsilon \downarrow 0$. Thus the smallest eigenvalue does* not *degenerate when the limiting support point is spectral and unique on the support face.*

**Example 2** (Degeneracy at a spectral point with a flat support face). *Let $A = \mathrm{diag}(1, 1 + i)$ and take $\sigma = 1 + \varepsilon$, $n = 1$. Then*

$$P(\sigma, n) = \mathrm{diag}\Big(\frac{1}{\varepsilon}, \mathrm{Re}\Big(\frac{1}{\varepsilon - i}\Big)\Big) = \mathrm{diag}\Big(\frac{1}{\varepsilon}, \frac{\varepsilon}{\varepsilon^2 + 1}\Big),$$

*so $\lambda_{\min}(P(\sigma, n)) = \frac{\varepsilon}{\varepsilon^2+1} \sim \varepsilon \downarrow 0$. Here the supporting functional $\mathrm{Re}(z)$ is maximized by more than one eigenvalue, and degeneracy persists at $\sigma_0 = 1 \in \mathrm{spec}(A)$.*

### *4.7. Spectral Support Points for Nonnormal Matrices: A Three-Scale Splitting*

We now turn to the *spectral-support* regime, where the boundary point is itself an eigenvalue:

$$\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A).$$

In contrast to the non-spectral case, $P(\sigma, n)$ may develop both a collapsing cluster and an *exact blow-up* as $\sigma \to \sigma_0$ along non-tangential offsets.

Fix a supporting direction $n$ with $|n| = 1$ and

$$\mathrm{Re}(\overline{n}\sigma_0) = \lambda_{\max}(H(n)), \qquad H(n) = \mathrm{Re}(\overline{n}A), \tag{4.33}$$

and assume the same spectral-isolation hypothesis (4.10) for $\lambda_{\max}(H(n))$ with multiplicity $m = \dim \mathcal{M}(n)$ and gap $\gamma_H > 0$. For $\varepsilon > 0$ define $\sigma_\varepsilon = \sigma_0 + \varepsilon n$ and $P_\varepsilon = P(\sigma_\varepsilon, n)$ as in (4.13).

Let

$$E := \mathrm{Ker}(\sigma_0 I - A), \qquad r := \dim E \geq 1.$$

**Lemma 7** (Spectral eigenspace inclusion). *Under (4.33), one has $E \subseteq \mathcal{M}(n)$.*

**Proof.** Let $0 \neq v \in E$, so $Av = \sigma_0 v$. Then

$$v^* H(n) v = \mathrm{Re}(\overline{n}\, v^* A v) = \mathrm{Re}(\overline{n}\sigma_0)\|v\|^2 = \lambda_{\max}(H(n))\|v\|^2.$$

Since $\lambda_{\max}(H(n))$ is the maximal Rayleigh quotient of $H(n)$, this forces $v \in \mathcal{M}(n)$. $\square$

Set $F := \mathcal{M}(n) \ominus E$, so $\dim F = m - r$ (possibly 0), and choose matrices with orthonormal columns $V_E \in \mathbb{C}^{d\times r}$ spanning $E$ and $V_F \in \mathbb{C}^{d\times(m-r)}$ spanning $F$. Define

$$B_0 := \sigma_0 I - A, \qquad G_F := V_F^* B_0^* B_0 V_F \in \mathbb{C}^{(m-r)\times(m-r)}. \tag{4.34}$$

**Proposition 12** (Three-scale splitting at a spectral support point). *Assume (4.33) and the gap hypothesis (4.10) for $H(n)$. For $\varepsilon > 0$ sufficiently small, $P_\varepsilon \succ 0$ and:*

*(i)* ***Exact blow-up on the geometric eigenspace.*** *For every $v \in E$,*

$$P_\varepsilon v = \frac{1}{\varepsilon} v. \tag{4.35}$$

*In particular, $P_\varepsilon$ has an eigenvalue $1/\varepsilon$ with multiplicity at least $r$.*

*(ii)* $O(\varepsilon)$ ***collapsing cluster on*** $\mathcal{M}(n) \ominus E$. *If $m > r$, then $G_F \succ 0$ and*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \lambda_j^{\uparrow}(P_\varepsilon) = \lambda_j^{\uparrow}(G_F^{-1}), \qquad j = 1, \dots, m-r. \tag{4.36}$$

*Equivalently, the $m - r$ smallest eigenvalues collapse linearly with slopes given by the eigenvalues of $G_F^{-1}$.*

*(iii)* $O(1)$ ***bulk separated from*** 0. *There exist $\varepsilon_0 > 0$ and $c_\star > 0$ such that*

$$\lambda_{m-r+1}^{\uparrow}(P_\varepsilon) \;\ge\; c_\star \qquad \text{for all } 0 < \varepsilon \le \varepsilon_0.$$

**Proof.** **(i)** If $v \in E$, then $(\sigma_\varepsilon I - A)v = (\sigma_0 + \varepsilon n - \sigma_0)v = \varepsilon n\, v$ and hence $(\sigma_\varepsilon I - A)^{-1} v = (1/(\varepsilon n))v$. Therefore

$$P_\varepsilon v = \mathrm{Re}\Big(n \cdot \frac{1}{\varepsilon n} v\Big) = \frac{1}{\varepsilon} v,$$

which is (4.35).

**(ii)** Assume $m > r$. Since $F \cap \mathrm{Ker}(B_0) = \{0\}$, the restriction of $B_0$ to $F$ is injective and $B_0 V_F$ has full column rank, hence $G_F \succ 0$.

Let $\Pi := V_E V_E^* + V_F V_F^*$ be the orthogonal projector onto $\mathcal{M}(n) = E \oplus F$. The congruence identity (Lemma 2) at $(\sigma, n) = (\sigma_\varepsilon, n)$ yields

$$B_\varepsilon^* P_\varepsilon B_\varepsilon = \mathrm{Re}(\overline{n}\sigma_\varepsilon) I - H(n) = (\lambda_{\max}(H(n))I - H(n)) + \varepsilon I = Q_0 + \varepsilon I,$$

where $B_\varepsilon = \sigma_\varepsilon I - A$ and $Q_0 \succeq 0$ has kernel $\mathcal{M}(n)$. Exactly as in (4.18)–(4.19), one obtains the operator-norm limit

$$\varepsilon P_\varepsilon^{-1} \longrightarrow S := B_0 \Pi B_0^* \qquad (\varepsilon \downarrow 0).$$

Since $B_0$ annihilates $E$ and is injective on $F$, $\mathrm{rank}(S) = m - r$ and the nonzero eigenvalues of $S$ coincide with the eigenvalues of

$$(B_0 V_F)^*(B_0 V_F) = V_F^* B_0^* B_0 V_F = G_F.$$

The eigenvalue convergence (4.36) then follows by the same inversion/reversal argument as in Proposition 7.

**(iii)** The lower bound is obtained by the Courant–Fischer principle exactly as in Proposition 8 (ii), using that the multiplicity of the kernel of $Q_0$ is $m$ and the gap is $\gamma_H > 0$ on $\mathcal{M}(n)^\perp$. $\square$

**Remark 8** (Defective eigenvalues and higher-order blow-up). *Proposition 12 isolates the contribution of the* geometric *eigenspace $E = \mathrm{Ker}(\sigma_0 I - A)$, producing an exact $1/\varepsilon$ blow-up along non-tangential offsets. If $\sigma_0$ is* defective *(nontrivial Jordan chains), generalized eigenvectors can lead to stronger growth (typically $1/\varepsilon^p$*

*along a Jordan block of length p), and the interaction between Jordan structure and the Hermitian support pencil is a pseudospectral phenomenon; see, e.g., [13].*

### 4.8. A Fully Explicit $2 \times 2$ Example: A Nilpotent Jordan Block

**Example 3** (Exact Poisson kernel and exact degeneracy rate for a disk exhaustion)**.** *Let*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

*Then* $W(A) = \{z \in \mathbb{C} : |z| \le \frac{1}{2}\}$. *For* $r > \frac{1}{2}$, *let* $\Omega_r := \{z \in \mathbb{C} : |z| < r\}$ *and choose* $\sigma = re^{it} \in \partial\Omega_r$. *The outward normal at* $\sigma$ *is* $n_{\Omega_r}(\sigma) = e^{it}$ *and*

$$(\sigma I - A)^{-1} = \begin{pmatrix} \frac{1}{\sigma} & \frac{1}{\sigma^2} \\ 0 & \frac{1}{\sigma} \end{pmatrix}.$$

*Hence*

$$P_{\Omega_r}(\sigma, A) = \mathrm{Re}\big(e^{it}(\sigma I - A)^{-1}\big) = \begin{pmatrix} \frac{1}{r} & \frac{e^{-it}}{2r^2} \\ \frac{e^{it}}{2r^2} & \frac{1}{r} \end{pmatrix},$$

*whose eigenvalues are* $\lambda_\pm(r) = \frac{2r \pm 1}{2r^2}$. *In particular,*

$$\lambda_{\min}\big(P_{\Omega_r}(\sigma, A)\big) = \frac{2r-1}{2r^2} = \frac{r - \frac{1}{2}}{r^2},$$

*so the degeneracy is* linear *as* $r \downarrow \frac{1}{2}$.

*Moreover, a min-eigenvector is* $u(r,t) \propto (-e^{-it}, 1)^\top$ *(independent of r). For the support direction* $n = e^{it}$,

$$H(n) = \mathrm{Re}(\overline{n}A) = \frac{1}{2}\begin{pmatrix} 0 & e^{-it} \\ e^{it} & 0 \end{pmatrix}, \quad \mathcal{M}(n) = \mathrm{span}\{(e^{-it}, 1)^\top\}.$$

*At* $\sigma_0 = \frac{1}{2}e^{it} \in \partial W(A)$,

$$(\sigma_0 I - A)(e^{-it}, 1)^\top = \tfrac{1}{2}(-1, e^{it})^\top \propto (-e^{-it}, 1)^\top,$$

*in agreement with Theorem 1.*

### 4.9. Numerical Experiments

This section provides numerical illustrations of: (i) the linear degeneracy predicted by Corollary 3 (and, in offset form, Proposition 2), (ii) local sharpness and improved bounds (Proposition 5 and Corollary 4), (iii) global coercivity collapse and its first-order slope under direction sampling (Corollary 2 and Proposition 6), and (iv) the contrasting behavior at spectral support points for normal matrices (Proposition 11 and Examples 1–2).

**Sampling model for an "outer offset" exhaustion.** Fix a unimodular direction $n \in \mathbb{C}$. Let $H(n) = \mathrm{Re}(\overline{n}A)$ and let $v \in \mathcal{M}(n)$ be a unit vector in the maximal eigenspace of $H(n)$ (Lemma 1). The corresponding numerical-range support point is

$$z_0(n) \;:=\; v^* A v \;\in\; \partial W(A), \qquad \mathrm{Re}(\overline{n}\, z_0(n)) = \lambda_{\max}(H(n)).$$

For $\varepsilon > 0$ we define the offset boundary point

$$\sigma_\varepsilon(n) \;:=\; z_0(n) + \varepsilon n.$$

doi:10.20944/preprints202602.0563.v2



Then $\mathrm{Re}(\overline{n}\,\sigma_\varepsilon(n)) = \lambda_{\max}(H(n)) + \varepsilon$, so the support gap equals $\delta(\sigma_\varepsilon(n), n) = \varepsilon$ (cf. Proposition 2). Moreover, $\sigma_\varepsilon(n) \notin W(A)$, hence $\sigma_\varepsilon(n) \notin \mathrm{spec}(A)$ (because $\mathrm{spec}(A) \subset W(A)$; Remark 1), so the resolvent is well-defined.

We evaluate the pointwise kernel

$$P_\varepsilon(n) \;:=\; \mathrm{Re}\Big(n\big(\sigma_\varepsilon(n)I - A\big)^{-1}\Big),$$

and track $\lambda_{\min}(P_\varepsilon(n))$ as $\varepsilon \downarrow 0$. In the generic (bounded-resolvent) regime $z_0(n) \notin \mathrm{spec}(A)$, Corollary 3 predicts the linear scaling $\lambda_{\min}(P_\varepsilon(n)) = \Theta(\varepsilon)$ and convergence of min-eigenvectors to $(z_0(n)I - A)\mathcal{M}(n)$ (Theorem 1).

**Experiment 1: exact linear rate for the nilpotent Jordan block.** We revisit Example 3 with $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and the disk exhaustion $\Omega_r = \{z : \ |z| < r\}$, $r > \frac{1}{2}$. Writing $\varepsilon = r - \frac{1}{2}$, one has the exact formula $\lambda_{\min}(P_{\Omega_r}(\sigma, A)) = \frac{\varepsilon}{r^2}$, hence linear degeneracy as $\varepsilon \downarrow 0$. Figure 1 compares the computed smallest eigenvalue to the exact expression.

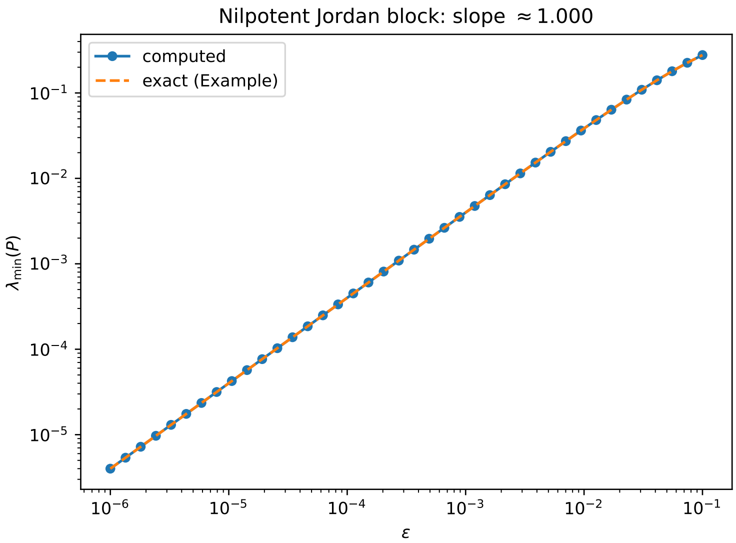


**Figure 1.** Nilpotent Jordan block (Example 3): log–log plot of $\lambda_{\min}$ versus $\varepsilon = r - \frac{1}{2}$, illustrating the predicted linear scaling.

**Experiment 2: generic nonnormal matrix—linear degeneracy and eigenvector convergence.** We generate a fixed random complex matrix $A \in \mathbb{C}^{5\times 5}$ (seeded for reproducibility), fix one direction $n = e^{i\theta}$, and form $\sigma_\varepsilon(n) = z_0(n) + \varepsilon n$ as above. Figure 2 shows $\lambda_{\min}(P_\varepsilon(n))$ against $\varepsilon$ on a log–log scale, together with a reference $\propto \varepsilon$ line; the observed slope is $\approx 1$ on the plotted range. Figure 3 tracks the distance of a min-eigenvector $u_\varepsilon$ of $P_\varepsilon(n)$ to the predicted limiting subspace $(z_0(n)I - A)\mathcal{M}(n)$, quantified by $\|(I - \Pi)\, u_\varepsilon\|$ where $\Pi$ is the orthogonal projector onto $(z_0(n)I - A)\mathcal{M}(n)$ (consistent with Theorem 1).

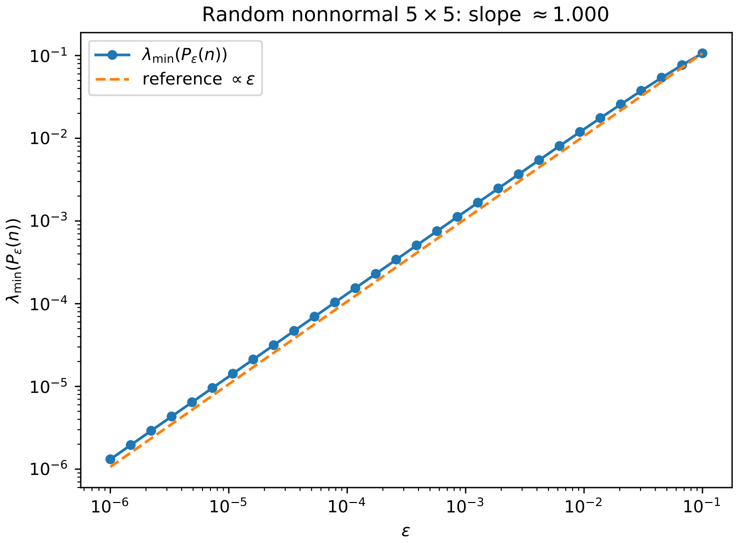


**Figure 2.** Random nonnormal $A \in \mathbb{C}^{5\times 5}$ (fixed seed) and fixed direction $n$: $\lambda_{\min}(P_\varepsilon(n))$ scales linearly with $\varepsilon$ (Corollary 3).

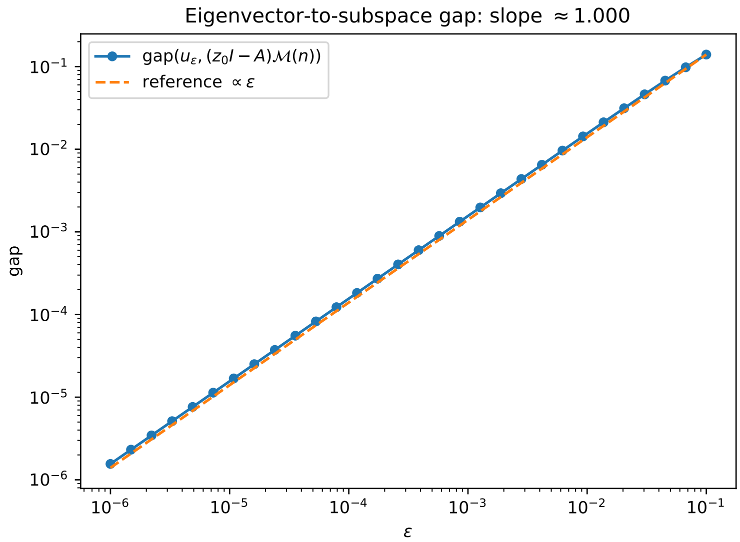


**Figure 3.** Same setup as Figure 2: the direction of a min-eigenvector $u_\varepsilon$ converges to $(z_0(n)I - A)\mathcal{M}(n)$ as $\varepsilon \downarrow 0$ (Theorem 1). The plotted "gap" is $\|(I - \Pi)u_\varepsilon\|$.

**Experiment 2b: local sharpness of norm-based versus refined bounds.** We compare the norm-based bounds from Lemma 3 (c) with the refined bounds of Proposition 5. For the Jordan block, Figure 4 shows that the refined bounds track the exact eigenvalue closely and are asymptotically sharp, while the norm-based upper bound is typically much looser. For the random nonnormal matrix, Figure 5 shows the same qualitative behavior.

doi:10.20944/preprints202602.0563.v2



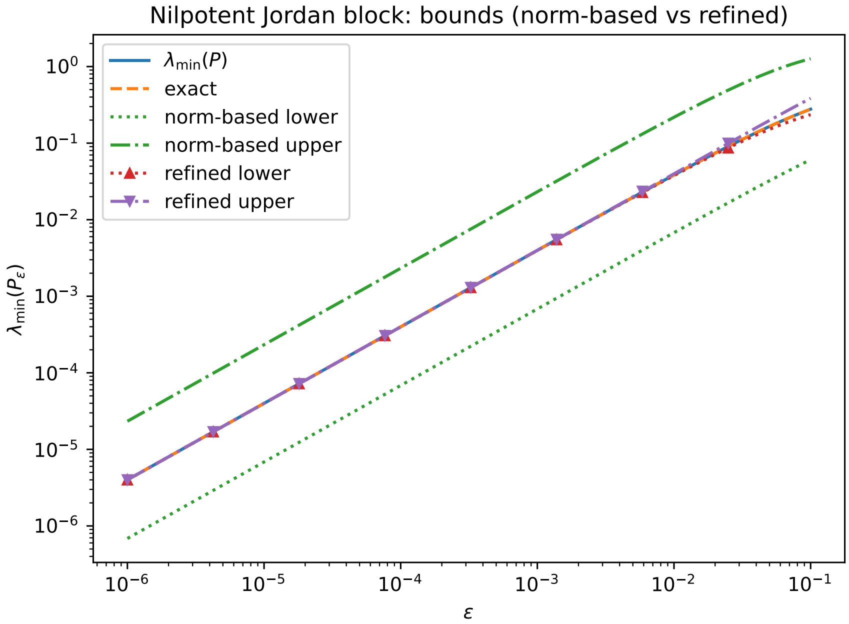


**Figure 4.** Nilpotent Jordan block: comparison of $\lambda_{\min}(P)$ with the norm-based bounds from Lemma 3 (c) and the refined bounds from Proposition 5.

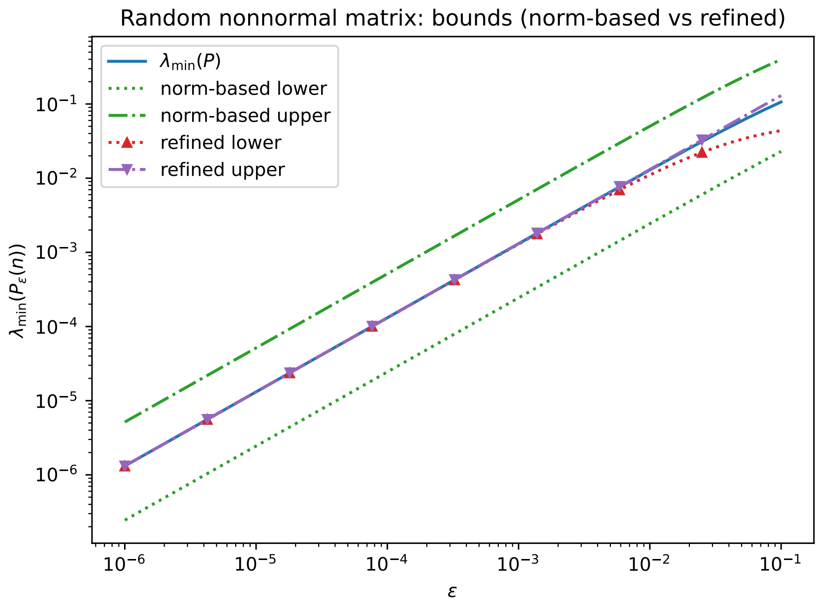


**Figure 5.** Random nonnormal $A \in \mathbb{C}^{5\times5}$, fixed direction $n$: comparison of $\lambda_{\min}(P_\varepsilon(n))$ with the norm-based and refined bounds. The refined bounds recover the correct first-order constant as $\varepsilon \downarrow 0$.

**Experiment 2c: full slope spectrum at a flat face ($m = 2$).** Proposition 7 predicts a *two-dimensional* $O(\varepsilon)$ collapsing cluster when the supporting pencil $H(n)$ has a two-dimensional maximal eigenspace and the chosen support point $\sigma_0$ lies in the interior of the corresponding face. To isolate this mechanism in a fully explicit setting, we take a normal diagonal matrix

$$A = \operatorname{diag}(1,\ 1+2i,\ 0,\ -1), \qquad n = 1,$$

so that $H(n) = \operatorname{Re}(A)$ has $\lambda_{\max} = 1$ with multiplicity $m = 2$ and $\mathcal{M}(n) = \operatorname{span}\{e_1, e_2\}$. With $v = (e_1 + e_2)/\sqrt{2}$, the associated support point is $\sigma_0 = v^*Av = 1 + i \notin \operatorname{spec}(A)$, lying in the relative interior of the face joining 1 and $1+2i$. In this basis, $B_0 = \sigma_0 I - A$ restricts to $\mathrm{diag}(i, -i)$ on $\mathcal{M}(n)$, hence $G = I_2$ and the predicted slope spectrum is $\{1, 1\}$. Numerically we observe $\lambda_1^{\uparrow}(P(\sigma_0 + \varepsilon n, n))/\varepsilon \to 1$ and $\lambda_2^{\uparrow}(P(\sigma_0 + \varepsilon n, n))/\varepsilon \to 1$ as $\varepsilon \downarrow 0$, and the face-detector count in Corollary 5 stabilizes at $m = 2$.

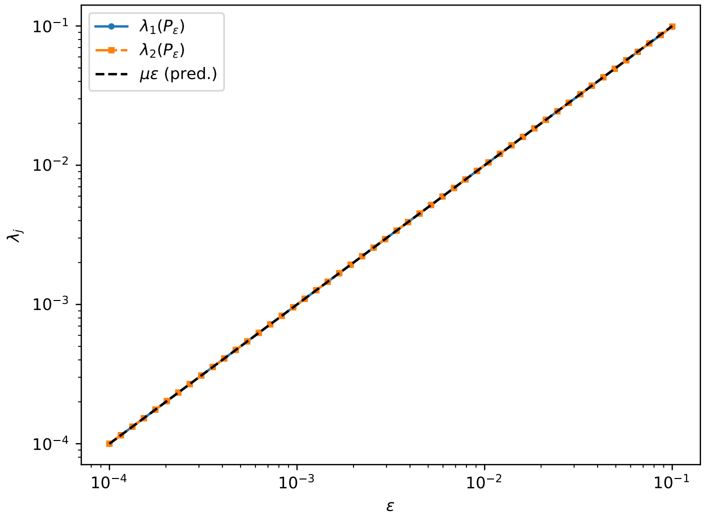


**Figure 6.** Experiment 2c: two collapsing eigenvalues and their rescaled slopes in the $m = 2$ flat-face normal example.

**Experiment 3: approximate global coercivity collapse.** For the same random nonnormal matrix $A$ as in Experiment 2, we approximate the global coercivity constant

$$c(\varepsilon) = \inf_{\sigma \in \partial\Omega_\varepsilon} \lambda_{\min}(P_{\Omega_\varepsilon}(\sigma, A))$$

by sampling a fine grid of directions $\{n_j\}$ and using the offset model $\sigma_\varepsilon(n_j) = z_0(n_j) + \varepsilon n_j$. Figure 7 plots the sampled minimum $\min_j \lambda_{\min}(P_\varepsilon(n_j))$ versus $\varepsilon$, illustrating the collapse asserted by Corollary 2. (Here the offset model has $\Delta(\Omega_\varepsilon) = \varepsilon$, so Corollary 1 also predicts that uniform coercivity cannot persist as $\varepsilon \downarrow 0$.)

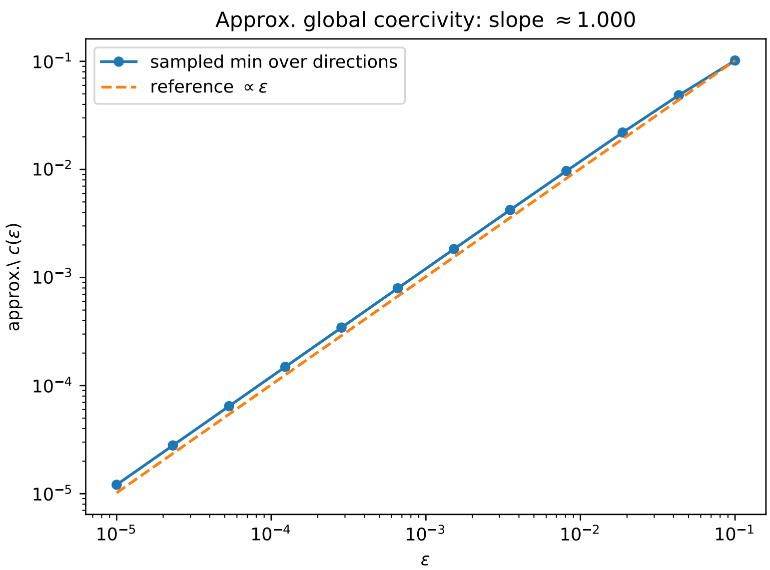


**Figure 7.** Approximate global coercivity constant $c(\varepsilon)$ computed by sampling directions and using the offset model $\sigma_\varepsilon(n) = z_0(n) + \varepsilon n$. The sampled minimum tends to 0 as $\varepsilon \downarrow 0$ (Corollary 2).

*Preprints.org* (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



**Experiment 3b: local slope profile across directions.** For the same random nonnormal matrix and the offset model, Corollary 4 predicts the local first-order constant $\lambda_{\min}(P_\varepsilon(n))/\varepsilon \to 1/\beta_0(n)^2$ as $\varepsilon \downarrow 0$. Figure 8 compares the empirically observed slope $\lambda_{\min}(P_\varepsilon(n))/\varepsilon$ at a small fixed $\varepsilon$ to the prediction $1/\beta_0(n)^2$ across directions $n = e^{i\theta}$, while Figure 9 plots the relative error.

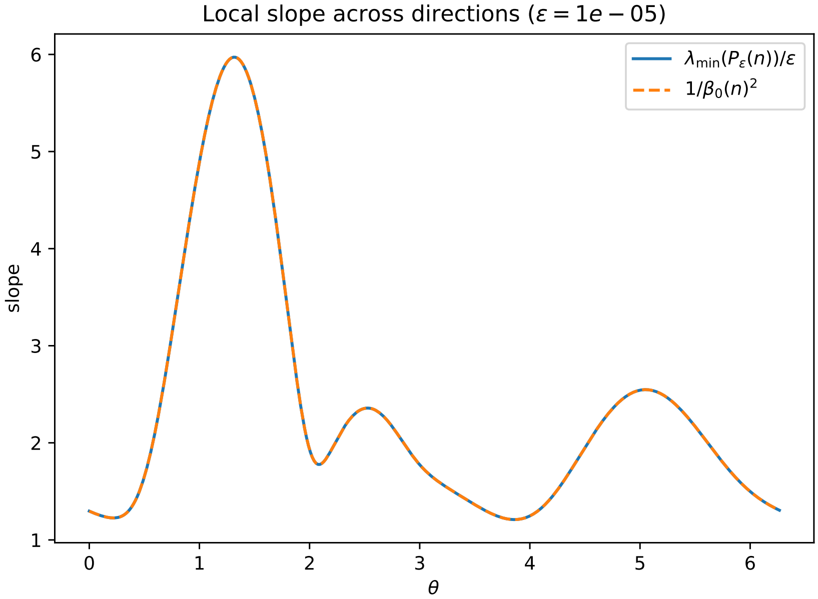


**Figure 8.** Random nonnormal matrix: direction-wise comparison of the empirical slope $\lambda_{\min}(P_\varepsilon(n))/\varepsilon$ with the predicted limit $1/\beta_0(n)^2$ (Corollary 4).

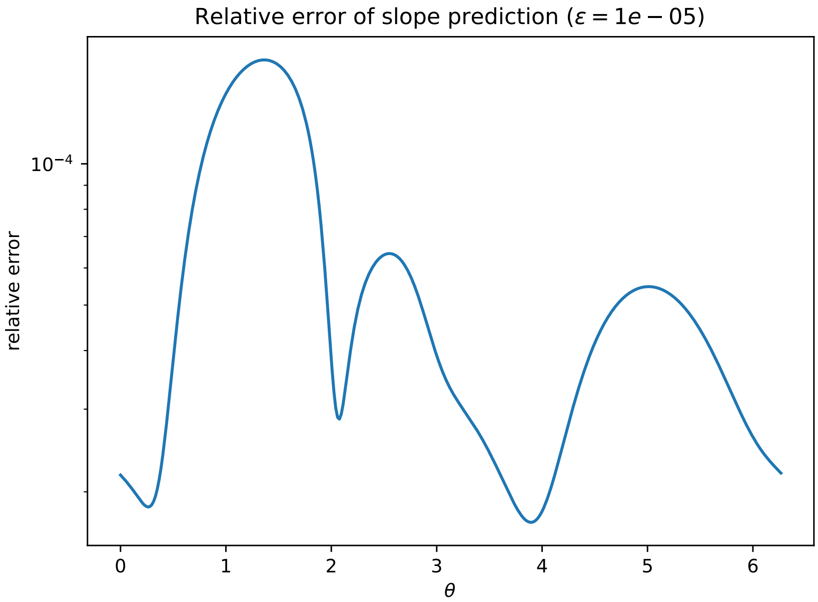


**Figure 9.** Random nonnormal matrix: relative error of the prediction in Figure 8. The error decreases as the test value of $\varepsilon$ is reduced.

**Experiment 3c: global slope constant and the $\beta_0$ profile.** Proposition 6 predicts that the direction-sampled coercivity constant satisfies $\tilde{c}(\varepsilon)/\varepsilon \to 1/\max_\theta \beta_0(\theta)^2$. Figure 10 plots the profile of $\beta_0(\theta)^2$, and Figure 11 compares the sampled ratio $\tilde{c}(\varepsilon)/\varepsilon$ to the predicted limit.

 doi:10.20944/preprints202602.0563.v2



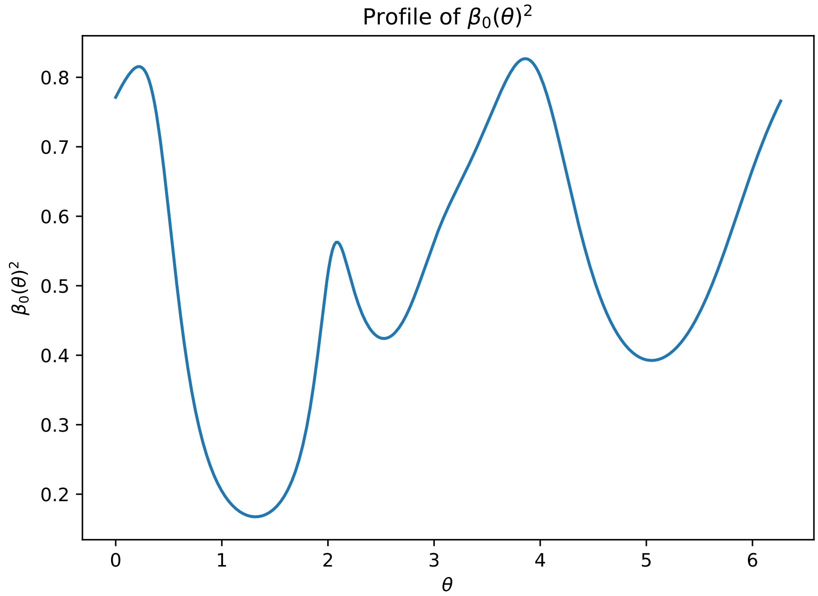


**Figure 10.** Random nonnormal matrix: profile of $\beta_0(\theta)^2 = \|(z_0(\theta)I - A)|_{\mathcal{M}(e^{i\theta})}\|^2$. The global slope constant is set by the maximum of this profile (Proposition 6).

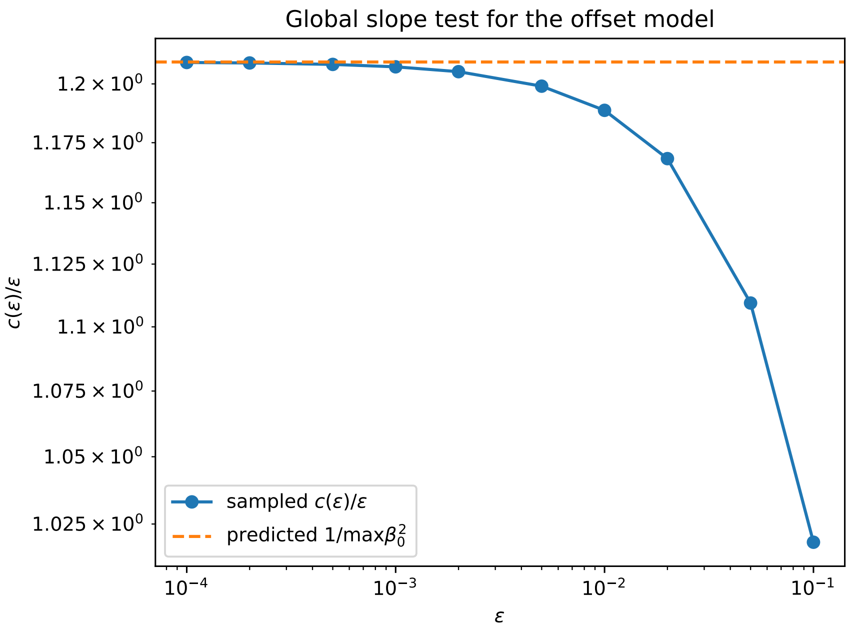


**Figure 11.** Random nonnormal matrix: the sampled ratio $\tilde{c}(\varepsilon)/\varepsilon$ versus $\varepsilon$, together with the predicted limit $1/\max_\theta \beta_0(\theta)^2$ (Proposition 6).

**Experiment 4: normal matrices at spectral support points.** We reproduce the contrasting behavior in Examples 1–2 by evaluating $P(1+\varepsilon, 1) = \mathrm{Re}\big(((1+\varepsilon)I - A)^{-1}\big)$ for two diagonal (hence normal) matrices: $A = \mathrm{diag}(0,1)$ and $A = \mathrm{diag}(1, 1+i)$. Figure 12 shows that the former remains bounded away from 0 as $\varepsilon \downarrow 0$, while the latter degenerates linearly, consistent with Proposition 11.

 doi:10.20944/preprints202602.0563.v2



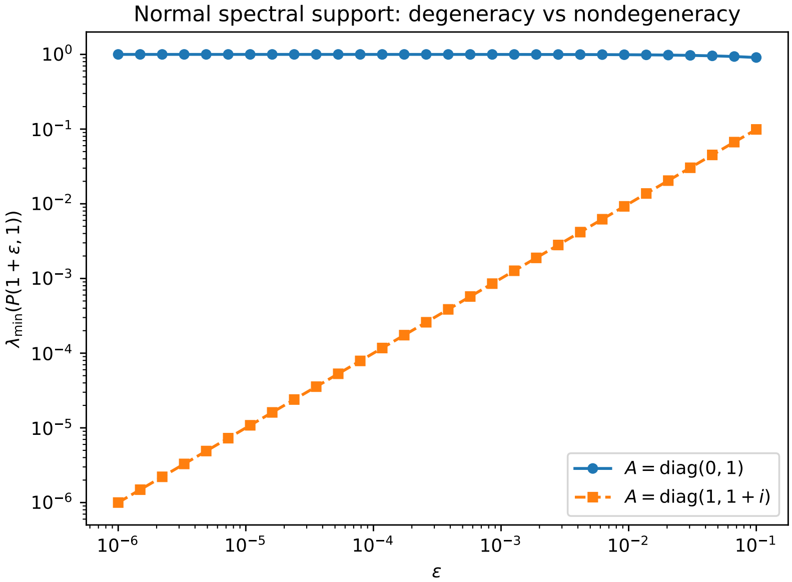


**Figure 12.** Normal matrices at a spectral support point: $A = \mathrm{diag}(0,1)$ remains bounded away from 0, whereas $A = \mathrm{diag}(1,1+i)$ degenerates linearly, consistent with Proposition 11.

**Experiment 4b: three-scale splitting at spectral support points (nonnormal).** Proposition 12 predicts a three-scale structure at $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$ under non-tangential offsets $\sigma_\varepsilon = \sigma_0 + \varepsilon n$: an exact $1/\varepsilon$ blow-up on the geometric eigenspace $E = \mathrm{Ker}(\sigma_0 I - A)$, an $O(\varepsilon)$ collapsing cluster of dimension $m - r$ on $\mathcal{M}(n) \ominus E$, and an $O(1)$ bulk bounded away from 0. We test this on three small nonnormal examples with $n = 1$ and $\sigma_0 = 1$:

(SS1) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, for which $m = r = 1$ (blow-up only; no $O(\varepsilon)$ cluster);

(SS2) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 0 \end{bmatrix}$, for which $m = 2$ and $r = 1$ (one collapsing slope);

(SS3) $A = H + iK$ with $H = \mathrm{diag}(1,1,1,0,0)$ and $K_{24} = K_{42} = 1$, $K_{35} = K_{53} = 2$, for which $m = 3$ and $r = 1$ (two collapsing slopes).

In each case we observe the predicted behavior: the largest eigenvalue scales as $1/\varepsilon$, the smallest $m - r$ eigenvalues scale linearly in $\varepsilon$ with slopes given by $G_F^{-1}$, and the next eigenvalue remains separated from 0 uniformly in $\varepsilon$.

**Reproducibility.** All figures are generated by the accompanying scripts `poisson_utils.py` and `run_numerical_experiments.py`, which require only NumPy and Matplotlib and save PDF figures into a `figs/` folder.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 February 2026 doi:10.20944/preprints202602.0563.v2



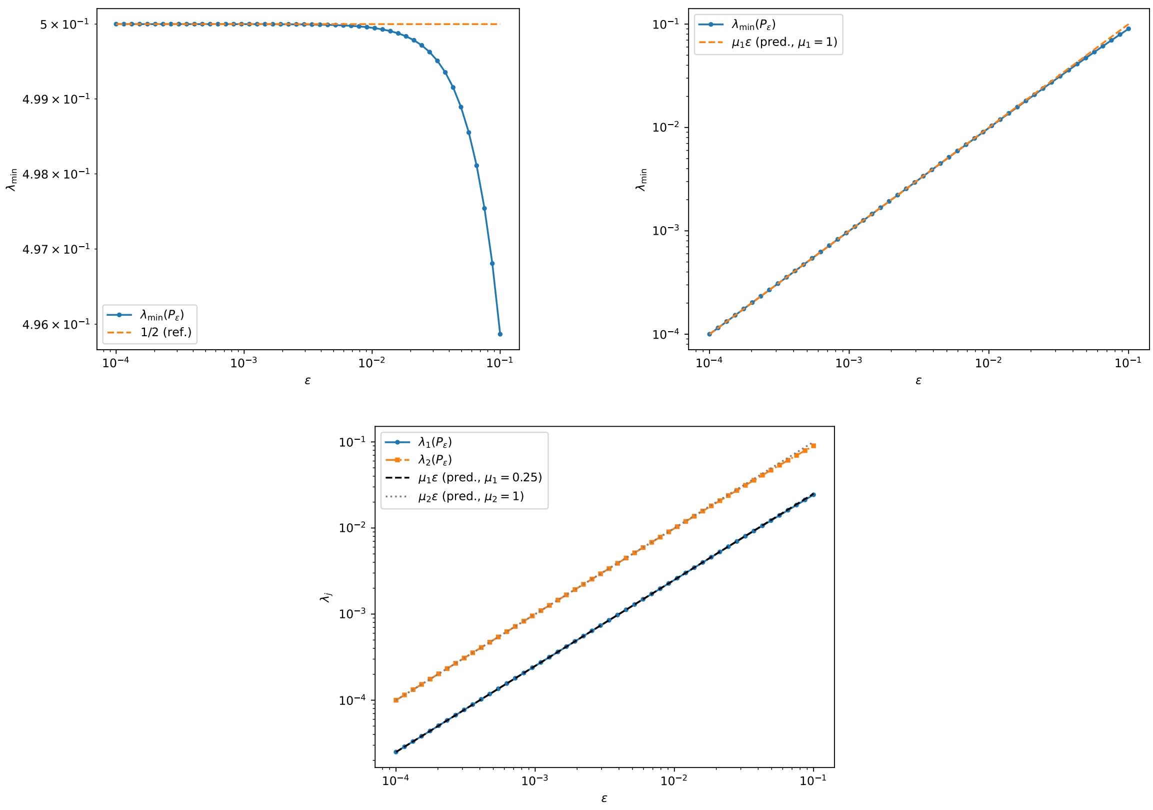

**Figure 13.** Experiment 4b: three-scale splitting at spectral support points (SS0–SS2). Each panel shows the $1/\varepsilon$ blow-up eigenvalue(s), the collapsing $O(\varepsilon)$ cluster, and the $O(1)$ bulk.

### *4.10. A Curvature Surrogate Question at Smooth Exposed Points*

The direction-dependent slope data in Corollary 4 and Proposition 7 suggest that the rescaled Poisson degeneracy rate can be viewed as a quantitative "stiffness" of the numerical range boundary in a given direction.

Let $n = e^{i\theta}$ and consider the support function

$$h(\theta) := \max_{z \in W(A)} \mathrm{Re}(e^{-i\theta} z) = \lambda_{\max}\big(H(e^{i\theta})\big),$$

where $H(e^{i\theta}) = \mathrm{Re}(e^{-i\theta} A)$. At directions where the maximal eigenvalue is simple, choose the corresponding unit eigenvector $v(\theta)$ and set the exposed support point

$$\sigma_0(\theta) := v(\theta)^* A v(\theta) \in \partial W(A).$$

For the non-tangential offset family $\sigma_\varepsilon(\theta) = \sigma_0(\theta) + \varepsilon e^{i\theta}$, define $P_\varepsilon(\theta) = P(\sigma_\varepsilon(\theta), e^{i\theta})$. When $\dim \mathcal{M}(e^{i\theta}) = 1$, Proposition 7 gives the explicit slope

$$s(\theta) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \lambda_{\min}\big(P_\varepsilon(\theta)\big) = \frac{1}{\|(\sigma_0(\theta) I - A) v(\theta)\|^2}. \tag{4.37}$$

A natural question is how $s(\theta)$ compares to geometric curvature data of $\partial W(A)$. In convex geometry, the support function determines the boundary, and for $C^2$ strictly convex curves the radius of curvature can be expressed in terms of $h$ and its second derivative (see, e.g., [12]). For numerical ranges, $\partial W(A)$ is an algebraic curve determined by the Kippenhahn polynomial [14], with corners and flat portions corresponding to eigenvalue multiplicities and supporting-face degeneracies.

**Question.** At smooth exposed boundary points where $\dim \mathcal{M}(e^{i\theta}) = 1$, is the slope $s(\theta)$ in (4.37) comparable (in an appropriate quantitative sense) to the curvature (or radius of curvature) of $\partial W(A)$

at $\sigma_0(\theta)$? Numerically, plots of $s(\theta)$ (Experiment 3b) show sharp spikes near directions associated with nearly-flat faces, suggesting that $s(\theta)$ may serve as an easily computable curvature surrogate.

### 4.11. Discussion and Remaining Open Problems

**Remark 9** (Beyond the geometric three-scale picture)**.** *The bounded-resolvent regime $\sigma_0 \notin \operatorname{spec}(A)$ is treated in Theorem 1 and quantified by the slope spectra in Section 4.4. At spectral support points $\sigma_0 \in \operatorname{spec}(A) \cap \partial W(A)$, Proposition 11 and Corollary 7 give a complete normal-matrix description, and Proposition 12 provides a three-scale splitting for general matrices under a gap hypothesis for the supporting pencil $H(n)$.*

*Several aspects of the nonnormal spectral-support regime remain open:*

*(i)* ***Defective eigenvalues.*** *When $\sigma_0$ has nontrivial Jordan chains, generalized eigenvectors may exhibit higher-order resolvent blow-up (typically $1/\varepsilon^p$ for a length-$p$ Jordan block). A systematic description of how Jordan structure and the support pencil interact to determine the full singular-value/eigenvalue profile of $P(\sigma_0 + \varepsilon n, n)$ is closely related to pseudospectral growth; see [13].*

*(ii)* ***Tangential approach geometry.*** *Our slope spectra are derived for non-tangential offsets and for general $C^1$ exhaustions after normalization by the scalar support gap $\delta$. Understanding when the $O(\delta)$ scaling persists under more tangential approach, or when higher-order scalings appear, is largely open.*

*(iii)* ***Loss of spectral isolation.*** *The explicit slope spectra rely on a gap separating $\lambda_{\max}(H(n))$ from the rest of the spectrum. When this gap closes, the associated projectors become ill-conditioned and new multi-scale behaviors may appear.*

## References


1. O. Toeplitz, *Das algebraische Analogon zu einem Satze von Fejér*, Math. Z. **2** (1918), no. 1–2, 187–197. doi:10.1007/BF01212904
2. F. Hausdorff, *Der Wertevorrat einer Bilinearform*, Math. Z. **3** (1919), no. 1, 314–316. doi:10.1007/BF01292610
3. M. Crouzeix, *Bounds for analytical functions of matrices*, Integr. Equ. Oper. Theory **48** (2004), 461–477. doi:10.1007/s00020-002-1188-6
4. M. Crouzeix, *Numerical range and functional calculus in Hilbert space*, J. Funct. Anal. **244** (2007), no. 2, 668–690. doi:10.1016/j.jfa.2006.10.013
5. M. Crouzeix and C. Palencia, *The numerical range is a $(1+\sqrt{2})$-spectral set*, SIAM J. Matrix Anal. Appl. **38** (2017), no. 2, 649–655. doi:10.1137/17M1116672
6. B. Delyon and F. Delyon, *Generalization of von Neumann's spectral sets and integral representation of operators*, Bull. Soc. Math. France **127** (1999), no. 1, 25–41. doi:10.24033/bsmf.2340
7. C. Badea, M. Crouzeix, and B. Delyon, *Convex domains and K-spectral sets*, Math. Z. **252** (2006), no. 2, 345–365. doi:10.1007/s00209-005-0857-y
8. F. L. Schwenninger and J. de Vries, *The double-layer potential for spectral constants revisited*, Integr. Equ. Oper. Theory **97** (2025), 13. doi:10.1007/s00020-025-02800-2
9. M. Crouzeix and A. Greenbaum, *Spectral Sets: Numerical Range and Beyond*, SIAM J. Matrix Anal. Appl. **40** (2019), no. 3, 1087–1101. doi:10.1137/18M1198417
10. T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Classics in Mathematics, Springer, 1995. doi:10.1007/978-3-642-66282-9
11. C. Davis and W. M. Kahan, *The rotation of eigenvectors by a perturbation. III*, SIAM J. Numer. Anal. **7** (1970), 1–46. doi:10.1137/0707001
12. R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, 2nd ed., Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2014. doi:10.1017/cbo9781139003858
13. L. N. Trefethen and M. Embree, *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*, Princeton University Press, 2005. doi:10.1515/9780691213101
14. R. Kippenhahn, *Über den Wertevorrat einer Matrix*, Math. Nachr. **6** (1951), 193–228. doi:10.1002/mana.19510060306