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*Article*

# Some New Results on N(2,2,0)-Algebras

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**Abstract**

An N(2,2,0)-algebra (abbreviated as NA-algebra) is an algebraic structure equipped with two binary operations, $*$ and $\triangle$, satisfying specific axioms. This paper investigates a special class of NA-algebras where the operation "$*$" exhibits nilpotent properties. We study several fundamental concepts within NA-algebras, including ideals, congruence decomposition, congruence kernels, and multiplicative stabilizers. A notion of NA-morphism is introduced, and a corresponding NA-ismorphism theorem is established. Furthermore, we explore the relationships between NA-algebras and other related logical algebraic structures, such as quantum B-algebras, Q-algebras, CI-algebras, pseudo-BCH-algebras, and RM-algebras. Notably, we prove that any nilpotent NA-algebra forms a quantum B-algebra. These results lay a foundation for further research into the structure and potential applications of NA-algebras.

**Keywords:** NA-algebra; quantum B-algebra; Q-algebra; CI-algebra; RM-algebra

**MSC:** 03G25; 03B52; 06F35; 20M10

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## 1. Introduction

Fuzzy algebras and their axiomatizations have become important topics in theoretical research and in the applications of fuzzy logic [1]. Wu [2] introduced a class of fuzzy implication algebras, FI-algebras for short, in 1990. Recently, researchers such as [3,4] have studied fuzzy implications from different perspectives. Naturally, it is meaningful to investigate the common properties of some important fuzzy implications used in fuzzy logic. Consequently, various interesting properties of FI-algebras, regular FI-algebras, commutative FI-algebras, and some concepts of filter, ideal and fuzzy filter of FI-algebras were proposed, some relations between these FI-algebras and several famous fuzzy algebras were systematically discussed [5].

In the last decades, the development algebraic models for non-commutative multiple-valued logics has become a central topic in the study of fuzzy systems. Various logical algebras have been introduced as the semantical systems of nonclassical logic systems. The notion of NA-algebras was introduced by Deng and Xu in 1996 [7]. We investigate the basic properties of NA-algebra $(S, *, \triangle, 1)$, and showed that if the class of operation $*$ is idempotent, then $(S, *, \triangle, 1)$ is a rewriting systems. Analogously. If the operation $\triangle$ is nilpotent, then $(S, \triangle, 1)$ is an associated BCI-algebra. Furthermove, some relationships between nilpotent NA-algebra and other algebras with the type of (2,0) are obtained [8].

Quantum B-algebras were introduced by Rump and Yang (2014) [19]. A quantum B-algebra is a partially ordered set with two binary operations $\rightarrow$ and $\rightsquigarrow$ satisfying four conditions (see Sect.4). Logically, $\rightarrow$ and $\rightsquigarrow$ represent the left and right implications, while $\leq$ corresponds to the entailment relation. In commutative logic, the two implications $\rightarrow$ and $\rightsquigarrow$ coincide.

NA-algebras are a new kind of algebraic system based on fuzzy implication algebras, and the previous research results show that this algebraic system with a dual semigroup. Under specific

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conditions, it can generate many non-classical logic algebras. Especially, it can constitute quantum B-algebras by introducing a partial order, which further shows that NA- have strong research potential. The present paper is organized as follows: In Section 2, we review some basic concepts, present the main properties of NA-algebras, and show relationship between the NA-algebras and the related fuzzy logic algebras. In Sections 3, we consider subalgebras, ideal, stabilizer and congruence kernels and NA-morphism theorem in NA-algebras. In Sections 4, we investigate the relationships among NA-algebra and other related logical algebraic structures, such as Q-algebras [14], CI-algebras [15] , Quantum B-algebras [19–21], pseudo-BCH-algebras [22], and RM-algebras [23].

## 2. Basis Concepts and Properties of NA-Algebras

The numerical verification in this paper can be referred to the algorithm program in [30]. We recall some basic definitions and results that are necessary for this paper.

Let $S$ be a semigroup. An element $a$ of $S$ is idempotent if $a^2 = a$. The set of idempotents of a subset $A$ of $S$ is denoted by $E(A)$.

An element $z$ of $S$ is a left (respectively, right) zero of $S$ if $zs = z$ (respectively, $sz = z$) for all $s \in S$; $z$ is a zero of $S$ if it is both a left and right zero of $S$. A semigroup all of whose elements are left (respectively, right) zeros is a left (respectively, right) zero semigroup. Elements $a$ and $b$ of $S$ commute if $ab = ba$; $S$ is commutative if any two of its elements commute [6].

**Definition 2.1.** ([2]) A fuzzy implication algebra $X$ with a constant 0 and a binary operation "$\rightarrow$" satisfying the following axioms: for any $x, y, z \in X$,

$(I_1) x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;

$(I_2)(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$;

$(I_3) x \rightarrow x = 1$;

$(I_4) x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$;

$(I_5) 0 \rightarrow x = 1$, where $1 = 0 \rightarrow 0$.

In a fuzzy implication algebra $(X, \rightarrow, 0)$:

(I) The order relation $\leq$ satisfying $x \leq y$ iff $x \rightarrow y = 1$ is a partial order [2];

(II) A multiplication $\circ$ induced by the adjoint relation

$$u \leq a \rightarrow b \Leftrightarrow a \circ u \leq b. \tag{1}$$

$a, b, u \in X$, where $(\circ, \rightarrow)$ is an adjoint pair on $X$ [1].

In [7], we proved that in a fuzzy implication algebra $(X, \rightarrow, 0)$, for all $a, b, u \in X$ if, the following equalities hold:

$$u \rightarrow (a \circ b) = b \rightarrow (u \rightarrow a) \tag{2}$$

$$(a \circ u) \rightarrow b = u \rightarrow (a \rightarrow b), \tag{3}$$

then $(X, \circ)$ is a semigroup [7,8].

By generalizing the expressions (2) and (3), we replace the symbol $\rightarrow$ by the symbol $*$, and the symbol $\circ$ by the symbol $\triangle$, we obtain the basic equations $(N_1)$ and $(N_2)$ of NA-algebra.

**Definition 2.2.** ([7]) A nonempty set $S$ equipped with two binary operations $*$ and $\triangle$ satisfying the axioms

$(N_1) a * (b \triangle c) = c * (a * b)$;

$(N_2)(a \triangle b) * c = b * (a * c)$;

$(N_3) 1 * a = a$,

for all $a, b, c \in S$ is called a **NA-algebra**, where 1 is a constant element on $S$.

In order to distinguish NA-algebras and Fuzzy implication algebras, we have introduced constant 1 in NA-algebra, and axioms $(N_1), (N_2)$ are conditions of the multiplication $*$ forming a semigroup, which induced by the adjointness relation expressions (1) in fuzzy implication algebra $(X, \rightarrow, 1)$.

The derivations of Equations (2) and (3) showing how Axioms $(N_1)$ and $(N_2)$ are derived from the accompanying in fuzzy implication algebra are given in [7,8].

**Example 1.** Let $S = \{1, a, b, c\}$. Define the operations $*$ and $\triangle$ on $S$ by cayley table 1-2 below:

**Table 1.** $*$ table.

| $*$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | $a$ | $b$ | $c$ |
| $b$ | $c$ | $b$ | $b$ | $c$ |
| $c$ | $c$ | $b$ | $b$ | $c$ |

**Table 2.** $\triangle$ table.

| $\triangle$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | 1 | $c$ | $c$ |
| $a$ | $a$ | $a$ | $b$ | $b$ |
| $b$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $c$ |

Based on the Cayley table 1 and table 2, $\forall a, b, c \in S$, we have
$a * (b \triangle c) = a * b = b, c * (a * b) = c * b = b$, so $(N_1)$ holds.
$(a \triangle b) * c = b * c = c, b * (a * c) = b * c = c$, hence $(N_2)$ holds.
Similarly, other cases can be verified and established.
The following propositions gives some basic properties of NA-algebra.

**Theorem 2.1.** Let $(S, *, \triangle, 1)$ be an NA-algebra. Then for all $a, b, c \in S$, the following hold:
$(N_4) a * b = b \triangle a$;
$(N_5) (a * b) * c = a * (b * c)$ ($*$ associative law), $a * (b * c) = b * (a * c)$ ($*$ left commutative law);
$(N_6) (a \triangle b) \triangle c = a \triangle (b \triangle c)$ ($\triangle$ associative law); $a \triangle b) \triangle c = (a \triangle c) \triangle b$ ($\triangle$ right commutative law).

**Proof.** $(N_4)$: $a \triangle b \overset{(N_3)}{=} 1 * (a \triangle b) \overset{(N_1)}{=} b * (1 * a) \overset{(N_3)}{=} b * a$;
$(N_5)$: $(a * b) * c \overset{(N_4)}{=} (b \triangle a) * c \overset{(N_2)}{=} a * (b * c)$; $a * (b * c) \overset{(N_4)}{=} a * (c \triangle b) \overset{(N_1)}{=} b * (a * c)$;
$(N_6)$: $(a \triangle b) \triangle c \overset{(N_4)}{=} c * (a \triangle b) \overset{(N_1)}{=} b * (c * a) \overset{(N_5)}{=} c * (b * a) \overset{(N_5)}{=} (c * b) * a \overset{(N_4)}{=} (b \triangle c) * a \overset{(N_4)}{=} a \triangle (b \triangle c)$;
$(a \triangle b) \triangle c \overset{(N_4)}{=} c * (a \triangle b) \overset{(N_1)}{=} b * (c * a) \overset{(N_4)}{=} (c * a) \triangle b \overset{(N_4)}{=} (a \triangle c) \triangle b$.
This completes the proof.

**Corollary 2.2.** ([7,8]) If $(S, *, \triangle, 1)$ is an NA-algebra, then $(S, *, 1)$ and $(S, \triangle, 1)$ are semigroups.

Let $(S, \circ)$ be an arbitrary semigroup. A semigroup $(S, *)$ is called a dual semigroup to $(S, \circ)$ if $x * y = y \circ x$ for all $x, y \in S$.

A semigroup $(S, \circ)$ is called left commutative (respectively, right commutative) if it satisfies the identity $x \circ (y \circ a) = y \circ (x \circ a)$ (respectively, $(a \circ x) \circ y = (a \circ y) \circ x$) [6].

Obviously, the NA-algebra $(S, *, \triangle, 0)$ is an algebra system with a pair of dual semigroups, and $(S, *)$ is a left commutative semigroup, $(S, \triangle)$ is a right commutative semigroup.

Several interesting properties of NA-algebra have been discussed earlier [7,8].

**Theorem 2.3.** Let $(S, *, \triangle, 1)$ be an NA-algebra and for every $x, y \in S$, then the following hold:

(1) If $x * x = 1$, then $x * y = x \triangle y$. In this case, two binary operations $*$ and $\triangle$ coincide, and $(S, *, ^{-1}, 1)$ is an abelian group, where "1" is an unit element, $x^{-1} = x$, for all $x$.

(2) If $x * 1 = 1$, then $x \in E(S)$ and semigroup $(S, *, 1)$ is a right zero semigroups.

**Proof.** (1) By theorem 2.1 $(N_5)$, we have operation" $*$ " is associative.

If $x * x = 1$, then $x * 1 = x * (x * x) \overset{(N_5)}{=} (x * x) * x = 1 * x \overset{(N_3)}{=} x$, we get $x * 1 = 1 * x = x$. Hence, 1 is unique unit element. Since $1' = 1 * 1' = 1' * 1 = 1$, we get 1 is unique unit in $S$.

In this case, $x * y = (x * y) * 1 \overset{(N_5)}{=} x * (y * 1) \overset{(N_5)}{=} y * (x * 1) = y * x = x \triangle y$. $*$ and $\triangle$ coincide, $(S, *, 1)$ is a commutative semigroups.

From that $x * x = 1$, we have $x$ is an inverse $x$, if there exists an inverse element $y$ of $x$ such that $x * y = 1$, then $x = 1 * x = (x * y) * x = x * (y * x) = y * (x * x) = y * 1 = y$. Hence, for any $x \in S$, we have $x$ is itself inverse and unique.

So, for an NA-algebras, if $x * x = 1$, then $(S, *, 1)$ is an Abelian group.

(2) For every $x \in S$, if it satisfies $x * 1 = 1$, then

$$x^2 = x * x = x * (1 * x) = (x * 1) * x = 1 * x = x.$$

We get $x * 1 = 1 \Rightarrow x^2 = x$, i.e., $x \in E(S)$.

Moreover, for any $x, y \in S$, we have $x * y = x * (1 * y) = (x * 1) * y = 1 * y = y$. Hence, $(S, *, 1)$ is a right zero semigroup.

**Theorem 2.4.** Let $(S, *, \triangle, 1)$ be an NA-algebra. For any $x, y \in S$, then $x * y = y * x$ if and only if $x * 1 = x$.

**Proof.** $(\Longleftarrow)$ $x * y = 1 * (x * y) = (x * y) * 1 = x * (y * 1) = y * (x * 1) = y * x$.

$(\Longrightarrow)$ Let us suppose that $x * 1 \neq x$, but $1 * x = x$, by $x * y = y * x$, a contradiction.

It is easy to check that the following results are true.

**Remark 1.** Let $(S, *, \triangle, 1)$ be an NA-algebra. Then, for any $x \in S$, satisfy simultaneously $x * 1 = x$ and $x * 1 = 1$ if and only if $x = 1$. In this case, NA-algebra is trivial, and also it shows the condition $x * x = 1$ are very strong.

**Remark 2.** Let $(S, *, \triangle, 1)$ be an NA-algebra. For all $x \in S$, such that $x * x = 1$ is a sufficient but unnecessary condition semigroup for $(S, *, 1)$ as commutative semigroup.

We illustrate this conclusion with the following example:

**Example 2.** ([7,8]) Let $S = \{1, a, b, c\}$. Define the operations $*$ and $\triangle$ on $S$ by table 3-4 below:

**Table 3.** $*$ table.

| $*$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | $a$ | 1 | $c$ | $b$ |
| $b$ | $b$ | $c$ | 1 | $a$ |
| $c$ | $c$ | $b$ | $a$ | 1 |

**Table 4.** $\triangle$ table.

| $\triangle$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | $a$ | 1 | $c$ | $b$ |
| $b$ | $b$ | $c$ | 1 | $a$ |
| $c$ | $c$ | $b$ | $a$ | 1 |

Where $\forall x \in S$, such that $x * x = 1$, we can verify that $(S, *, \triangle, 1)$ is an NA-algebra, and $(S, *, 1)$ is a commutative semigroup, and $(S, \triangle, 1)$ is also.

**Example 3.** Let $X = \{1, a, b\}$. Define the operations $*$ and $\triangle$ on $X$ by table 5-6 below:

**Table 5.** $*$ table.

| $*$ | 1 | $a$ | $b$ |
|---|---|---|---|
| 1 | 1 | $a$ | $b$ |
| $a$ | $a$ | 1 | $b$ |
| $b$ | $b$ | $b$ | $b$ |

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**Table 6.** $\triangle$ table.

| $\triangle$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ |
| $a$ | $a$ | $1$ | $b$ |
| $b$ | $b$ | $b$ | $b$ |

By table 5 and table 6, we have $a * (b \triangle 1) = a * b = b, 1 * (a * b) = 1 * b = b$, so $(N_1)$ holds. $(a \triangle b) * 1 = b * 1 = b, b * (a * 1) = b * a = b$ , hence $(N_2)$ holds. $1 * 1 = 1, 1 * a = a, 1 * b = b$, So $(N_3)$ hold. Other cases can also be verified $(N_1)$, $(N_2)$ and $(N_3)$ hold. Therefore $(X, *, \triangle, 1)$ is an NA-algebra, where $b = b * b \neq 1$, but $\forall x \in X, x * 1 = x$ , by theorem 2.4, then $(X, *, 1)$ is a commutative semigroup.

So, for all $x \in S$ satisfies $x * x = 1$ is a sufficient but unnecessary condition semigroup for $(S, *, 1)$ as commutative semigroup.

**Remark 3.** If a fuzzy implication algebra $(X, \rightarrow, 1)$ with a partial order $'' \leq ''$, such that any $a, b, u \in X, a \leq b \Leftrightarrow a \rightarrow b = 1, u \leq a \rightarrow b \Leftrightarrow a * u \leq b$, and for all $a, b, u \in X$, the following conditions hold:

$$u \rightarrow (a * b) = b \rightarrow (u \rightarrow a),$$

$$(a * u) \rightarrow b = u \rightarrow (a \rightarrow b),$$

then $(X, \rightarrow, *, 1)$ is an NA-algebra.

**Remark 4.** Let $(S, *, \triangle, 1)$ be an NA-algebra, then semigroups $(S, *, 1)$ and $(S, \triangle, 1)$ are a pair of dual semigroup. A pair of dual operations $(*, \triangle)$ form an adjoint pair $(\rightarrow, *)$, ie., $u \leq a \rightarrow b \Leftrightarrow a * u \leq b \Leftrightarrow u \triangle a \leq b$, for every $a, b, u \in S$.

**Theorem 2.5.** Let $(S, *, \triangle, 1)$ be an NA-algebra and for every $x, y, z$ in $S$, then the following hold:

$$x \triangle (y * z) = y * (x \triangle z), x * (y \triangle z) = y \triangle (x * z).$$

**Proof.** By Theorem 2.1, we have $x \triangle y = y * x$. Hence, $x \triangle (y * z) = (y * z) * x = y * (z * x) = y * (x \triangle z)$. Similarly, $x * (y \triangle z) = y \triangle (x * z)$.

Note that the associativity of $*$ and $\triangle$ of NA-algebra $(S, *, \triangle, 1)$ is equivalent to the equation:

$$x * (y \triangle z) = y \triangle (x * z).$$

**Theorem 2.6.** ([8]) Let $(S, *, \triangle, 1)$ be an NA-algebra. For every $x \in S$ , if the binary operation $*$ satisfies $x * x = 1$, then the following hold:

(1) $x * y \neq 1 \Leftrightarrow x \neq y, \forall x, y \in S$;

(2) $x * y = x \Leftrightarrow y = 1, \forall x, y \in S$;

(3) $x * (x * y) = y, y * (x * y) = x, \forall x, y \in S$.

In this subsection we introduce a new axiomatization of NA-algebras.

**Theorem 2.7.** Let $S$ be a set with a consistent 1 and two binary operations $*, \triangle$, for all $x, y, z \in S$, satisfying

$(I)$ $x * (y \triangle z) = y \triangle (x * z)$,

$(II)$ $x \triangle y = y * x$,

$(III)$ $x * (y * z) = y * (x * z)$, (left commutative)

$(IV)$ $1 * x = x$.

Then $(S, *, \triangle, 1)$ is an NA-algebra. The reverse is also true.

**Proof.** By $(I)$ and $(II)$, for all $x, y, z \in S$, we have

$$x * (z * y) \overset{(II)}{=} x * (y \triangle z) \overset{(I)}{=} y \triangle (x * z) \overset{(II)}{=} (x * z) * y.$$

Thus, for any $x, y, z \in S$, we have $x * (z * y) = (x * z) * y$. Hence the binary operation "$*$" with associative.

Similarly, we can prove $x \triangle (z \triangle y) = (x \triangle z) \triangle y$. That is "$\triangle$" with associative.

Using the hypothesis, we have

$x * (y \triangle z) = x * (z * y) = z * (x * y)$, hence, axiom $(N_1)$ holds.

$(x \triangle y) * z = (y * x) * z = y * (x * z)$, so axiom $(N_2)$ holds.

by hypothesis $(IV)$, we get axiom $(N_3)$ holds. Thus, $(S, *, \triangle, 1)$ is an NA-algebra.

Conversely, every NA-algebra satisfies the condition $(II)$(by$(N_4)$),$(III)$ (by $(N_5)$) and $(IV)$(by $(N_3)$). We only need to verify $(I)$ holds.

Let $(S, *, \triangle, 1)$ be an NA-algebra. By Theorem 2.1, we have

$x * (y \triangle z) \overset{(N_4)}{=} x * (z * y) \overset{(N_5)}{=} (x * z) * y \overset{(N_4)}{=} y \triangle (x * z)$. So, $(I)$ holds.

This completes the proof.

Let $(S, *, \triangle, 1)$ be a NA-algebra. If operations $*$ and $\triangle$ of $(S, *, \triangle, 1)$ coincide, then the NA-algebra becomes a semigroup. Thus, every NA-algebra is a generalization of semigroups.

Let $(L, \odot, 1)$ is a monoid, the operations $\odot, \rightarrow$ and $\rightsquigarrow$ satisfy the so-called right adjointness property:

$$x \odot y \leq z \Leftrightarrow x \leq y \rightarrow z,$$

and the left adjointness property:

$$x \odot y \leq z \Leftrightarrow y \leq x \rightsquigarrow z.$$

**Remark 5.** Let $(S, *, \triangle, 1)$ be an NA-algebra, for any $x, y, z \in S$, by using $x * y = y \triangle x$, and

$$x \leq y \rightarrow z \Leftrightarrow y * x \leq z,$$

$$x \leq y \rightarrow z \Leftrightarrow x \triangle y \leq z$$

we obtain $*$ satisfy the left adjoint property, and $\triangle$ satisfy the right adjoint property.

## 3. Subalgebras, Ideal and Stabilizer of NA-Algebras

### *3.1. Subalgebras, Ideal of NA-Algebras*

In this section, we will introduce ideals and congruences of NA-algebra $(S, *, \triangle, 1)$. We define two quasi-ordering relations and denote them by $\leq_r$ and $\leq_l$ on $E(S)$ as follows, respectively.

$$e \leq_r f \Leftrightarrow f * e = e \Leftrightarrow e \triangle f = e \tag{4}$$

$$e \leq_l f \Leftrightarrow e * f = e \Leftrightarrow f \triangle e = e \tag{5}$$

**Definition 3.1.** Let $A$ be an nonempty subset of NA-algebra $(S, *, \triangle, 1)$ is called a subalgebra of $S$ if $x * y, x \triangle y \in A$ hold for any $x, y \in A$.

**Definition 3.2.** Let $A$ be an nonempty subset of NA-algebra $(S, *, \triangle, 1)$ is called an NA-ideal of $S$, if

(1) $1 \in A$;

(2) for any $x \in A$, and $x * y, x \triangle y \in A \Rightarrow y \in A$.

It is easy to verify that $\{1\}$ and $S$ are two ordinary ideals of NA-algebra $(S, *, \triangle, 1)$.

**Example 4.** Let $S = \{1, a, b\}$. Define the operations $*$ and $\triangle$ on $X$ by table 7-8 below:

**Table 7.** $*$ table.

| $*$ | 1 | $a$ | $b$ |
|---|---|---|---|
| 1 | 1 | $a$ | $b$ |
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $b$ | $a$ | $a$ |

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**Table 8.** $\triangle$ table.

| $\triangle$ | 1 | $a$ | $b$ |
|---|---|---|---|
| 1 | 1 | $a$ | $b$ |
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $b$ | $a$ | $a$ |

Then, it is easy to verity that $(S,*,\triangle,1)$ is an NA-algebra. One can easily check that $A_1 = \{1\}, A_2 = S, A_3 = \{1,a\}, A_4 = \{1,b\}$ are ideals of NA-algebra $(S,*,\triangle,1)$.

**Proposition 3.1.** Let $(S,*,\triangle,1)$ be an NA-algebra. Then we have

$$A(a) = \{x | x, a \in E(S), a \leq_r x\}$$

and

$$B(1) = \{x | x * (x * 1) = 1\}$$

are ideals of $S$.

**Proof.** (1) By equation (4) and $1 * a = a$, we have $a \leq_r 1$. Hence, $1 \in A(a)$. Now, if $x \in A(a)$, then $a \leq_r x$, i.e., $x * a = a$, and $x * y \in A(a)$, so $a \leq_r x * y$, i.e, $(x*y)*a = a$.

By $(x*y)*a = x*(y*a) = y*(x*a) = y*a$, so $y*a = a$, thus $a \leq_r y \Rightarrow y \in A(a)$. Therefore $A(a)$ is an ideal of $S$.

(2) Using $1*(1*1) = 1$, so $1 \in B(1)$. If $x \in B(1)$, and $x*y \in B(1)$, then we have

$(x*y)*((x*y)*1) = 1 \Rightarrow (x*y)*(x*(y*1)) = (x*y)*(y*(x*1)) = 1 \Rightarrow (y*y)*(x*(x*1)) = 1 \Rightarrow y*(y*1) = 1 \Rightarrow y \in B(1)$. Hence, $B(1)$ is an ideal of $S$.

**Example 5.** Let $S = \{1,a,b,c,d,e\}$. Define the operations $*$ and $\triangle$ on $S$ by table 9-10 below:

**Table 9.** $*$ table.

| $*$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
| $a$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
| $b$ | $b$ | $b$ | $b$ | $c$ | $b$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ |
| $d$ | $d$ | $d$ | $b$ | $c$ | $d$ | $d$ |
| $e$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |

**Table 10.** $\triangle$ table.

| $\triangle$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | $b$ | $c$ | $d$ | 1 |
| $a$ | $a$ | $a$ | $b$ | $c$ | $d$ | $a$ |
| $b$ | $b$ | $b$ | $b$ | $c$ | $b$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ |
| $d$ | $d$ | $d$ | $b$ | $c$ | $d$ | $d$ |
| $e$ | $e$ | $e$ | $b$ | $c$ | $d$ | $e$ |

Then, it is easy to check that $(S,*,\triangle,1)$ is an NA-algebra, and

$$A(1) = A(a) = A(e) = \{1,a,e\}, A(c) = S, A(b) = \{1,a,b,d,e\}, A(d) = \{1,a,d,e\}, B(1) = \{1,a,e\}$$

are NA-ideals.

**Proposition 3.2.** Let $(S,*,\triangle,1)$ be an NA-algebra. If $Q$ is an ideal of $S$, and for all $x,y \in Q$, $\exists z \in S$, such that $x*(y*z) = 1$, then $x*z, y*z \in Q$.

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**Proof.** Since $Q$ is an ideal of $S$, we have $1 \in Q$. If $x \in Q, \exists z \in S$, such that $x * (y * z) = 1 \in Q$, then by Definition 3.1, we have $y * z \in Q$. Similarly, $y \in Q, x * (y * z) = y * (x * z) = 1 \in Q$, then we have $y * z \in Q$.

**Proposition 3.3.** Let $(S, *, \triangle, 1)$ be an NA-algebra. If for any $x \in S$, such that $x * 1 = x$, then we have

$$S_g = \{x | g \in E(S), x * g * x = g\} \tag{6}$$

is an ideal of $S$.

**Proof.** (1) By equation(6) and $1 * (g * 1) = g * (1 * 1) = g * 1 = g$, we have $1 \in S_g$ holds.

(2) If $x \in S_g$, then $x * g * x = g$. Let $x * y \in S_g$, then we get $(x * y) * g * (x * y) = g$, it follows that $y * (x * g * x) * y = g \Rightarrow y * g * y = g$. Therefore, $S_g$ is an ideal of $S$.

**Example 6.** Let $S = \{1, a, b, c\}$. Define the operations $*$ and $\triangle$ on $S$ by table 11-12 below:

**Table 11.** $*$ table.

| $*$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | $a$ | $b$ | $c$ |
| $b$ | $c$ | $b$ | $b$ | $c$ |
| $c$ | $c$ | $b$ | $b$ | $c$ |

**Table 12.** $\triangle$ table.

| $\triangle$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | 1 | $c$ | $c$ |
| $a$ | $a$ | $a$ | $b$ | $b$ |
| $b$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $c$ |

Then, it is easy to verity that $(S, *, \triangle, 1)$ is an NA-algebra. One can easily check that

$$S_1 = \{1\}, S_a = \{a\}, S_b = \{a, b\}, S_c = \{1, c\}$$

are ideals of NA-algebra $(S, *, \triangle, 1)$.

An element $a \in S$ is called nilpotent provided that $a^2 = a * a = 1$.

**Proposition 3.4.** Let $(S, *, \triangle, 1)$ be an NA-algebra, $N$ be set of all nilpotent element $S$. Then $N$ is an ideal of $S$.

**Proof.** $1 \in N$ is obvious. For any $x \in N, x * y \in N$, we have $1 = (x * y)^2 = x^2 * y^2 = 1 * y^2 = y^2$, hence $y \in N$. Therefore, $N$ is an ideal of $S$.

From Proposition 3.4 it follows that if $N$ be a set of all nilpotent element on NA-algebra $(S, *, \triangle, 1)$, then $N$ is an ideal of $S$. For any $s, t \in N$, we define

$$s \sim t \Leftrightarrow s * t, s \triangle t \in N,$$

then, the relation $\sim$ in $N$ is an equivalence relation.

In fact, if $x * x = 1 \in N$, we get $x \sim x$, that is $\sim$ satisfying reflexive.

Secondly, from the define of relation $\sim$, for any $x, y \in N, x \sim y \Rightarrow y \sim x$, it follows that we get $\sim$ satisfying symmetric.

Finally, suppose that $x \sim y, y \sim z$, that is , $x * y, y * z \in N, y * x, z * y \in N$, we have

$$(x * y) * (y * z) = x * ((y * y) * z) = x * (1 * z) = x * z,$$

$$(x * z) * (x * z) = ((x * y) * (y * z)) * (x * z) = (x * y) * (x * ((y * z) * z)) = (x * y) * ((x * y) * (z * z)) = 1 * (1 * 1) = 1.$$

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That is, $x * z \in N$. This case of $z * x \in N$ can be proved similarly. Hence, $\sim$ is transitive. Therefore, the relation $\sim$ in $N$ is an equivalence relation.

In what follows, let us check the equivalence classes partition of the set $S$. Call $N_e$ the equivalence class of an element $e$ belong to the $S$. So $N_e$ consists of the elements $t$ such that $e \sim t$:

$$N_e = \{t \in S \mid e \in N, \exists t \in S, s.t., e * t, e \triangle t \in N\}.$$

We denote by $S/N$ the set of equivalence class. Two operations $\circledast, \copyright$ in $S/N$ be defined as follows:

$$N_e \circledast N_f = N_{e*f}, N_e \copyright N_f = N_{e\triangle f}.$$

Then, it is easy to verify that $(S/N, \circledast, \copyright, N_1)$ is an NA-algebras.

*3.2. Multiplicative Stabilizers in NA-Algebras*

From a logic point of view, stabilizer can be used in studying the consequence connectives in the correspondence logic system, which has a very close relationship with fuzzy set. Since stabilizer was successful in some distinct tasks in various branches of mathematics (Roudabri and Torkzadeh) [11], it has been extended to various logical algebras; for example, Haveshki and Mohamadhasani first introduced the notion of stabilizers in BL-algebras and then, they studied several properties of them [12]. Michiro Kondo considered two types of stabilizers, implicative and multiplicative stabilizers in residuated lattices and proved their fundamental properties [13].

In this section, we will introduce some stabilizers and study related properties of them in NA-algebras.

Let $S$ be an NA-algebras. For an non-empty subset $\emptyset \neq X \subseteq S, a \in S$, the multiplicative stabilizers $X_a$ are defined as follows:

$$X_a = \{x \in S | x * a = a = a \triangle x\} \tag{7}$$

for all $a \in X$.

**Proposition 3.5.** Let $S$ be an NA-algebras. Then, we have the following basic results about multiplicative stabilizers:

(1) $X_1 = E(S)$;

(2) $X_a$ is a subalgebra and an ideal of $S$;

(3) If $x, y \in S$, s.t., $x * y = y \triangle x = 1$, then $x$ and $y$ is a multiplicative stabilizer of the same element.

**Proof.** (1) By Definition 2.2 $(N_3)$, we have $1 * x = x$, hence $1 * 1 = 1$, so we get $x * 1 = 1$. It follows that $1 \in X_1$.

From $X_1 = \{x \in S \mid x * 1 = 1\}$, we have $\forall x \in S, x^2 = x * x = x * (1 * x) = (x * 1) * x = 1 * x = x \Rightarrow x^2 = x$. Therefore, $X_1 = E(S)$.

(2) Let $\forall x, y \in X_a$. Then, $x * a = a = a \triangle x, y * a = a = a \triangle y$. We have $(x * y) * a = x * (y * a) = x * a = a \Rightarrow x * y, y \triangle x \in X_a$. In a similar way, we have $y * x, x \triangle y \in X_a$. Since $x * y = y \triangle x$, we get $(X_a, *, \triangle, 0)$ is a subalgebra of $(S, *, \triangle, 0)$.

It is obvious that $1 \in X_a$, because of $1 * a = a$.

Next, if $x, x * y, y \triangle x \in X_a$, then $x * a = a, (x * y) * a = a, a \triangle x = a, a \triangle (y \triangle x) = a$, we get

$$y * a = y * (x * a) = x * (y * a) = (x * y) * a = a,$$

that is $y \in X_a$. Hence, $X_a$ is an ideal of $S$.

(3) Suppose that $x \in X_a, y \in X_b$. Since $x * y = 1$, it follows that $x * a = a \Rightarrow y * (x * a) = y * a \Rightarrow x * (y * a) = y * a \Rightarrow (x * y) * a = y * a \Rightarrow a = 1 * a = y * a$, we get $y * a = a$, that show $y \in X_a$. $y \in X_b \Rightarrow x \in X_b$ can be proved similarly.

**Proposition 3.6.** Let $S$ be an NA-algebras. We defined two types products of $S$ and $S_a$ as follows:

$$S * S_a = \{s * x \mid s \in S, x \in S_a\},$$

$$S * S_a * S = \{s_1 * x * s_2 \mid s_1, s_2 \in S, x \in S_a\}.$$

Then, the following statements hold.

(1) $(S * S_a, *, \triangle, 1)$ is a subalgebra of NA-algebras $S$. In particular, for any positive integer $n$, we get a NA-algebras sequence as follows:

$$\ldots \leq (S^{n+1}, *, \triangle, 1) \leq (S^n, *, \triangle, 1) \leq (S^{n-1}, *, \triangle, 1) \leq \ldots \leq (S, *, \triangle, 1).$$

(2) $(S * S_a * S, *, \triangle, 1)$ is a subalgebra of NA-algebras $S$. In addition, $(S_a^n, *, \triangle, 0) = (S_a, *, \triangle, 1)$.

(3) For any $a, b \in S$, we have

(A) $(S_1, *, \triangle, 1) \subseteq (S_a \bigcap S_b, *, \triangle, 1) \subseteq (S_a * S_b \bigcap S_b * S_b, *, \triangle, 1) \subseteq (S_{a*b} \bigcap S_{b*a}, *, \triangle, 1)$;

(B) $(S_a, *, \triangle, 1) \subseteq (S_a * S_b \bigcap S_b * S_b, *, \triangle, 1)$;

(C) If $a * b = 1$, then $(S_a, *, \triangle, 1) = (S_b, *, \triangle, 1) = (S_1, *, \triangle, 1)$.

**Proof.** We only show the cases (1) and (C) of (3). Other cases can be proved easily.

(1) It follows from Proposition 3.5 (2) that $X_a$ is a subalgebra of $S$, we have $1 = 1 * 1 \in S * X_a$.

For any $s_1 * x_1, s_2 * x_2 \in S * X_a$, where $s_1, s_2 \in S, x_1, x_2 \in X_a$, we have

$(s_1 * x_1) * (s_2 * x_2) = s_1 * (x_1 * (s_2 * x_2)) = s_1 * (s_2 * (x_1 * x_2)) = (s_1 * s_2) * (x_1 * x_2) \in S * X_a$.

By using the operational duality between $*$ and $\triangle$, we get $(s_1 * x_1) \triangle (s_2 * x_2) \in S \triangle X_a$. Therefore, $(S * S_a, *, \triangle, 0)$ is a subalgebra of NA-algebras $S$.

Obviously, for any positive integer $n$, the subalgebras sequence of NA-algebras $(S, *, \triangle, 1)$

$$\ldots \leq (S^{n+1}, *, \triangle, 1) \leq (S^n, *, \triangle, 1) \leq (S^{n-1}, *, \triangle, 1) \leq \ldots \leq (S, *, \triangle, 1)$$

holds.

(3) We prove that case (C) of (3). Let $a, b \in S$, if $a, b \in S$, satisfies $a * b = 1$, then, for any $x \in S_a$, we have $x * b = x * (1 * b) = x * ((a * b) * b) = (x * a) * (b * b) = a * (b * b) = (a * b) * b = 1 * b = b$, i.e., $x \in S_b$. Therefore, $S_a \subseteq S_b$. In a similar way, we can show that $x \in S_b$ implies $x \in S_a$, i.e., $S_b \subseteq S_a$. Hence, $S_a = S_b$. Moreover, for any $x \in S_b$, $x * 1 = x * (a * b) = a * (x * b) = a * b = 1$ implies $x \in S_1$. It follows from (A) of (3), we get $S_1 \subseteq S_a$. Therefore, we have $(S_a, *, \triangle, 1) = (S_b, *, \triangle, 1) = (S_1, *, \triangle, 1)$.

### *3.3. Congruence Decomposition of NA-Algebras*

We define congruence decomposition of NA-algebras, which plays a central role in the results that follow.

**Definition 3.3.** A congruence on an NA-algebras $(S, *, \triangle, 1)$ is an equivalence relation $\rho$ on $S$ such that $x\rho x'$ and $y\rho y'$, then $(x * y)\rho(x' * y')$ and $(x \triangle y)\rho(x' \triangle y')$.

The condition for an equivalence relation $\rho$ on $S$ to be a congruence ensures that the set $S/\rho$ of equivalence classes under $\rho$ has a well-defined monoid structure inherited from $S$.

**Definition 3.4.** Let $(S, *, \triangle, 1)$ be an NA-algebra. For all $x, y \in S$, if there exists some $u \in S$ with $x * u * y = x * y$, then $u$ is called middle unit of $S$.

The all middle unit set of $S$ will be denoted by $M(S)$. Obviously, $1 \in M(S)$.

**Proposition 3.7.** Let $(S, *, \triangle, 0)$ be an NA-algebras. Then, the following statements hold.

(1) $(M(S), *, \triangle, 1)$ is a subalgebra of $(S, *, \triangle, 1)$.

(2) If $\forall x, y \in S, \exists u \in S$ satisfies $x^2 * u * y^2 = x * y$, then $u \in M(S)$ if and only if $x * y \in E(S)$.

**Proof.** (1) By $1 \in M(S)$, we have $\emptyset \neq M(S) \subseteq S$. $\forall u, v \in M(S), x, y \in S$, using Definition 3.4, this implies that

$$x * u * y = x * y, x * v * y = x * y,$$

$x * (u * v) * y = (x * u) * (v * y) = v * ((x * u) * y) = v * (x * u * y) = v * (x * y) = x * v * y = x * y$, hence, $u * v \in M(S)$. In a similar way, we can show that $v * u \in M(S)$. By using the operational duality between $*$ and $\triangle$, we get $u \triangle v, v \triangle u \in M(S)$. Therefore, $(M(S), *, \triangle, 1)$ is a subalgebra of $(S, *, \triangle, 1)$.

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(2)Suppose that $u \in M(S)$. Then $\forall x, y \in S$, we have $x * u * y = x * y$. On the other hand, from $x^2 * u * y^2 = x * y$, we get $x * y = x^2 * u * y^2 = x * x * u * y * y = x * (x * u * y) * y = x * x * y * y = x * y * x * y = (x * y)^2$, and hence that $x * y \in E(S)$. Conversely, we assume that $x^2 * u * y^2 = x * y$, for all $x, y \in S$, i.e., $u * (x^2 * y^2) = u * (x * y)^2 = x * y$ hold. If $x * y \in E(S)$, then, for any $x, y \in S$, $x * u * y = x * (u * y) = u * (x * y) = u * (x * y)^2 = x * y$. This implies that $u \in M(S)$.

**Proposition 3.8.** Let $(S, *, \triangle, 1)$ be an NA-algebras and $M(S)$ is all middle unit set of $S$. Define the relation $\rho$ as follows: $\forall x, y \in S$, $(x, y) \in \rho \Leftrightarrow$ there exists $u_1, u_2 \in M(S)$ such that $u_1 * x = u_2 * y$. then $\rho$ is a congruence on $S$.

**Proof.** It is easy to verify that reflexive and symmetric are satisfied. We only show the proof of transitivity.

From the relation $\rho$ on $S$, it is defined by the following rule:

$(x, y) \in \rho$ if and only if there exists $u_1, u_2 \in M(S)$ such that $u_1 * x = u_2 * y$, for any $x, y \in S$.

In fact, if $\forall x, y, z \in S, (x, y) \in \rho, (y, z) \in \rho$, then there exists $u_1, u_2, u_3, u_4 \in S$, such that

$$u_1 x = u_2 y, u_3 y = u_4 z.$$

It follows that $(u_1 * u_3) * x = u_3 * (u_1 * x) = u_3 * (u_2 * y) = u_2 * (u_3 * y) = u_2 * (u_4 * z)$.

From proposition 3.7(1), $(M(S), *, \triangle, 1)$ is a subalgebra of $(S, *, \triangle, 0)$, we get $u_1 * u_3, u_2 * u_4 \in M(S)$, Thus $(x, z) \in \rho$. Therefore, $\rho$ is an equivalence relation on $S$.

Next, we show that $\rho$ is a congruence on $S$. Suppose that $x_1, x_2, y_1, y_2 \in S, (x_1, y_1) \in \rho, (x_2, y_2) \in \rho$, i.e., $a, b, c, d \in M(S)$, such that

$$a * x_1 = b * y_1, c * x_2 = d * y_2,$$

we have $(a * c) * (x_1 * x_2) = (a * x_1) * (c * x_2) = (b * y_1) * (d * y_2) = (b * d) * (y_1 * y_2)$. Hence, $\rho$ is a congruence on $S$.

We denote equivalence classes belonging to $x$ by $[x]$ . $S/\rho = \{[x] | x \in S\}$. For any $[x], [y] \in S/\rho$, we define $[x] * [y] = [x * y], [x] \triangle [y] = [x \triangle y]$, then, $(S/\rho, *, \triangle, [1])$ form an NA-algebras, which is a quotient algebra of $(S, *, \triangle, 1)$.

Suppose that $a$ is a fixed element in $S$. Now, we define another congruence relation on $S$ by using $S_a$ as follow.

**Proposition 3.9.** Let $(S, *, \triangle, 1)$ be an NA-algebras. $\forall x, y \in S, (x, y) \in \theta \Leftrightarrow \exists h_1, h_2 \in S_a$, such that $x * h_1 = y * h_2$. Then, $\theta$ is a congruence relation on $S$.

**Proof.** It is easy to verify that reflexive and symmetric are satisfied. We only show the proof of transitivity.

In fact, let $\forall x, y, z \in S, (x, y) \in \theta, (y, z) \in \theta$, then there exists $h_1, h_2, h_3, h_4 \in S_a$, such that

$$x * h_1 = y * h_2, y * h_3 = z * h_4.$$

It follows that:

$x * (h_1 * h_3) = (x * h_1) * h_3 = (y * h_2) * h_3 = y * (h_2 * h_3) = h_2 * (y * h_3) = h_2 * (z * h_4) = z * (h_2 * h_4)$.

From Proposition 3.5(2), we get $h_1 * h_3, h_2 * h_4 \in S_a$. Thus $(x, z) \in \theta$. Therefore, $\theta$ is a equivalence relation on $S$.

Let now $x_1, x_2, y_1, y_2 \in S, x_1 \theta y_1, x_2 \theta y_2$, then there exists $h_1, h_2, h_3, h_4 \in S_a$, such that $x_1 * h_1 = y_1 * h_2, x_2 * h_3 = y_2 * h_4$. Thus, we have

$(x_1 * x_2) * (h_1 * h_3) = x_1 * (x_2 * (h_1 * h_3)) = x_1 * (h_1 * (x_2 * h_3)) = (x_1 * h_1) * (x_2 * h_3)$
$= (y_1 * h_2) * (y_2 * h_4) = y_1 * (h_2 * (y_2 * h_4)) = y_1 * (y_2 * (h_2 * h_4)) = (y_1 * y_2) * (h_2 * h_4)$.

Hence, $\theta$ is a congruence relation on $S$.

Similar to Proposition 3.8, we denote equivalence classes belong to $x$ by $[x]$ . $S/\theta = \{[x] | x \in S\}$. For any $[x], [y] \in S/\theta$, we define

$$[x] * [y] = [x * y], [x] \triangle [y] = [x \triangle y],$$

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Then, $(S/\theta, *, \triangle, [1])$ forms an NA-algebra, which is a quotient algebra of $(S/\theta, *, \triangle, [1])$.

### *3.4. Congruence Kernels of NA-Algebras*

In this section, we give a characterization of congruence kernels in an NA-algebra. Let $\theta$ be a binary relation on an NA-algebra $(S, *, \triangle, 1)$. We denote $\{x \in S \mid (x, 1) \in \theta\}$, by $[1]_\theta$. If $\theta$ is a congruence relation on $S$, then $[1]_\theta$ is called a congruence kernel.

**Proposition 3.10.** Let $(S, *, \triangle, 1)$ be an NA-algebras. We denote natural homomorphism $f : S \to S/\theta, x \longmapsto [x]$. Then, we have

$$kerf = \{x \mid x \in S, f(x) = [1]\}.$$

is a subalgebra and an ideal of $(S, *, \triangle, 1)$.

**Proof.** In fact, $kerf = \{x \mid x \in S, [x] = [1]\} = \{x \mid x \in S, x\theta 1\} = \{x \mid x \in S, \exists h_1, h_2 \in S_a, x * h_1 = 1 * h_2\}$.

Obviously, $1 \in kerf$. For all $x, y \in kerf$, there exists $h_1, h_2, h_3, h_4 \in S_a$, such that

$$x * h_1 = 1 * h_2 = h_2, y * h_3 = 1 * h_4 = h_4.$$

This means that $(x * y) * (h_1 * h_3) = x * (y * (h_1 * h_3)) = x * (h_1 * (y * h_3)) = (x * h_1) * (y * h_3) = h_2 * h_4 = 1 * (h_2 * h_4)$. Hence $x * y \in kerf$.

By using the operational duality between $*$ and $\triangle$, we can show that $x\Delta y \in kerf$. Therefore, $kerf$ is a subalgebra of $(S, *, \triangle, 1)$.

Next, suppose $x, x * y \in kerf$, then there exists $h_1, h_2, h_3, h_4 \in S_a$, such that

$$x * h_1 = 1 * h_2 = h_2, (x * y) * h_3 = 1 * h_4 = h_4.$$

Hence $y * (h_2 * h_3) = y * ((x * h_1) * h_3) = (x * h_1) * (y * h_3) = x * (h_1 * (y * h_3)) = x * (y * (h_1 * h_3)) = (x * y) * (h_1 * h_3)) = h_1 * ((x * y) * h_3)) = h_1 * h_4 = 1 * (h_1 * h_4)$.

This means that $y \in kerf$.

Therefore $kerf$ is an ideal of $(S, *, \triangle, 1)$.

### *3.5. NA-Morphisms*

Let $X := (X, *, \triangle, 1)$ and $Y := (Y, \overline{*}, \overline{\triangle}, \overline{1})$ be two NA-algebras unless otherwise specified.

**Definition 3.5.** A mapping $\varrho : X \to Y$ is called an NA-morphism if it satisfies: for all $x, y \in X$,

$$\varrho(x * y) = \varrho(x)\overline{*}\varrho(y), \varrho(x \triangle y) = \varrho(x)\overline{\triangle}\varrho(y).$$

**Example 7.** Let $X = \{1, a, b, c\}$, define the operations $*$ and $\triangle$ with Cayley Table 13-14:

**Table 13.** $*$ table.

| $*$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | $a$ | $b$ | $c$ |
| $b$ | $c$ | $b$ | $b$ | $c$ |
| $c$ | $c$ | $b$ | $b$ | $c$ |

**Table 14.** $\triangle$ table.

| $\triangle$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | 1 | $c$ | $c$ |
| $a$ | $a$ | $a$ | $b$ | $b$ |
| $b$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $c$ |

Let $Y = \{\overline{1}, w, s, t\}$, define the operations $\overline{*}$ and $\overline{\triangle}$ with Cayley Table15-16:

**Table 15.** $*$ table.

| $\overline{*}$ | $\overline{1}$ | $w$ | $s$ | $t$ |
|---|---|---|---|---|
| $\overline{1}$ | $\overline{1}$ | $w$ | $s$ | $t$ |
| $w$ | $\overline{1}$ | $w$ | $s$ | $t$ |
| $s$ | $t$ | $s$ | $s$ | $t$ |
| $t$ | $t$ | $s$ | $s$ | $t$ |

**Table 16.** $\triangle$ table.

| $\overline{\triangle}$ | $\overline{1}$ | $w$ | $s$ | $t$ |
|---|---|---|---|---|
| $\overline{1}$ | $\overline{1}$ | $\overline{1}$ | $t$ | $t$ |
| $w$ | $w$ | $w$ | $s$ | $s$ |
| $s$ | $s$ | $s$ | $s$ | $s$ |
| $t$ | $t$ | $t$ | $t$ | $t$ |

It is routine to verify that $(X; *, \triangle, 1)$ and $(Y; \overline{*}, \overline{\triangle}, \overline{1})$ both are NA-algebras.
Now, let $\varrho : X \to Y$ be a mapping defined by

$$\varrho(x) = \begin{cases} \overline{1}, & x \in \{1, a, b\} \\ w, & x \in \{c\} \end{cases} \tag{8}$$

It is routine to verify that $\varrho$ is an NA-morphism [30].

**Proposition 3.11.** If $\delta : X \to Y$ is an NA-morphism, then $\delta(1) = 1$ and $\delta(x) \leq \delta(y)$ for all $x, y \in X$ with $x \leq y$.

**Proof.** We have $\delta(1) = \delta(1 * 1) = \delta(1) * \delta(1) = 1$. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so $\delta(x) * \delta(y) = \delta(x * y) = \delta(1) = 1$. This shows that $\delta(x) \leq \delta(y)$.

Let $f : X \to Y$ be an NA-morphism. For any subset $B$ of $Y$ , the set

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}$$

is called the preimage of $B$ under $f$ . The preimage of $f^{-1}(1) := \{x \mid f(x) = 1\}$ under $f$ is called the kernel of $f$ and is denoted by $f^{-1}(1)$ or $ker(f)$, and so

$$f^{-1}(1) = ker(f) = \{x \in X | f(x) = 1\}.$$

**Proposition 3.12.** The preimage of an NA-ideal under an NA-morphism is an NA-ideal.

**Proof.** Let $\phi : X \to Y$ be an NA-morphism and let $G$ be an NA-ideal of $Y$. It is clear that $1 \in \phi^{-1}(G)$. Let $x, y \in X$ be such that $x * y \in \phi^{-1}(G)$ and $x \in \phi^{-1}(G)$. Then $\phi(x) \in G$ and $\phi(x) * \phi(y) = \phi(x * y) \in G$. It follows from Definition 3.2 that $\phi(y) \in G$. Hence $y \in \phi^{-1}(G)$, and therefore $\phi^{-1}(G)$ is an NA-ideal of $X$. Now, let $G$ be an NA-ideal of $Y$.

Given a constant element $c$ and an non-empty subset $H$ of an NA-algebra $S$, we consider a special set:

$$cH := \{x \in S \mid c * x \in H, c \in S\}.$$

**Theorem 3.13.** If $H$ is an NA-Ideal of $S$ and $S/H$ is the quotient NA-algebra induced by $H$, then the map $\varphi : S \to S/H, a \mapsto aH$ is an NA-morphism.

**Proof.** Assume that $H$ is an NA-Ideal of $S$. We define a map $\varphi : S \to S/H, a \mapsto aH$. This implies that

$\forall a, b \in S$, we have

$$a \mapsto aH = \{x \in S \mid a * x \in H\},$$

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$$b \mapsto bH = \{y \in S \mid b * y \in H\},$$

$$a * b \mapsto a * bH = \{x \in S \mid (a * b) * x \in H\}.$$

Obvious that

$$aH * bH = a * bH.$$

We say that $\varphi$ is the natural NA-morphism.

In the theory of semigroups, the isomorphism theorem is very similar to the first isomorphism theorem in group theory. The core idea is: the structure of original semigroup is "compressed" into the image via the quotient structure. With such a thought, we establish the isomorphism theorem of NA-algebra.

**Theorem 3.14.(NA-ismorphism theorem)** Let $(S; *, \triangle, 1)$ and $(T; \overline{*}, \overline{\triangle}, \overline{1})$ be arbitrary NA-algebras. If $f : S \to T$ is an onto NA-morphism, that is for all $a, b \in S$ such that $f(a * b) = f(a)\overline{*}f(b)$, and $f(a \triangle b) = f(a)\overline{\triangle}f(b)$, then we define the relation $'' \sim ''$ : $a \sim b \Leftrightarrow f(a) = f(b)$, then

(1) $\sim$ is a congruence relation on S.

(2) Quotient NA-algebra $S/\sim \cong Imf$.

**Proof.** (1) By the definition of relation $'' \sim ''$ : $a \sim b \Leftrightarrow f(a) = f(b)$, and f is a mapping , we have $'' \sim ''$ is naturally an equivalence relation (reflexive, symmetric, transitive). We only need to verify that it satisfies the congruence, i.e.: if $a \sim \overline{a}$, and $b \sim \overline{b}$, then $a * b \sim \overline{a * b}$, and $a \triangle b \sim \overline{a \triangle b}$.

In fact, if $a \sim \overline{a}$, $b \sim \overline{b}$, By the definition of relation $'' \sim ''$, we have $f(a) = f(\overline{a}), f(b) = f(\overline{ab})$. Since $f$ is a homomorphism, then

$$f(a * b) = f(a)\overline{*}f(b) = f(\overline{a})\overline{*}f(\overline{b}) = f(\overline{a} * \overline{b}),$$

$$f(a \triangle b) = f(a)\overline{\triangle}f(b) = f(\overline{a})\overline{\triangle}f(\overline{b}) = f(\overline{a} \triangle \overline{b}).$$

Hence, $a * b \sim \overline{a * b}$ and $a \triangle b \sim \overline{a \triangle b}$. Therefore, $\sim$ is a congruence relation on S.

(2) Step 1: We construct quotient NA-algebra. Since $\sim$ is a congruence relation on S, we define the quotient set $S/\sim$, where each element is an equivalence class of the form $[a] = \{x \in S | x \sim a\}$.

Step 2: Construct the quotient NA-algebra. We define the operation as:

$$[a] * [b] = [a * b], [a] \triangle [b] = [a \triangle b].$$

Below, we verify that the operation is well-defined, that is, if $[a] = [a']$ and $[b] = [b']$, then $[a * b] = [a' * b']$, and $[a \triangle b] = [a' \triangle b']$. By using the congruence relation, the well-definedness of this operation is obviously established. Therefore, $S/\sim$ is a quotient NA-algebra.

Step 3: Construct the natural mapping $\phi$ and prove it is an isomorphism. Define the mapping $\phi : S/\sim \to Imf$ as:

$$\phi([a]) = f(a).$$

We verify the well-definedness of $\phi$: if $[a] = [a']$, then $a \sim a'$, that is $f(a) = f(a')$. Therefore, $\phi([a]) = f(a) = f(a') = \phi([a'])$. So $\phi$ is a well-defined mapping.

$$\phi[a] * [b] = \phi([a * b]) = f(a * b) = f(a)f(b) = \phi([a])\phi([b]).$$

We now show that $\phi$ is a bijection.

∘ Injection: if $\phi([a]) = \phi([b])$, then $f(a) = f(b)$, that is $a \sim b$. So, $[a] = [b]$. Therefore, $\phi$ is a injection.

∘ Surjection: $\forall y \in Imf, \exists a \in S$ such that $f(a) = y$, i.e., $\phi([a]) = y$. So, $\phi$ is a surjection.

To sum up, we have mapping $\phi$ is an isomorphism. It follows that $S/\sim \cong Imf$. This completes the proof.

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## 4. Relation Between NA-Algebras and Other Related Logical Algebras

This section aims to systematically position NA-algebras in the spectrum of non-classical logical algebras.

An algebra is called a bicommutative algebra [9], if it satisfies the following identities

$$x(yz) = y(xz), (xy)z = (xz)y.$$

One-sided commutative algebras first appeared in the paperby Cayley in 1857 [10] . The variety of right commutative algebras is defined by the following identity:

$$(xy)z = (xz)y.$$

Similarly, the variety of left commutative algebras is defined by the following identity:

$$x(yz) = y(xz).$$

Thus, for a given NA-algebra $(S, *, \triangle, 1)$, $(S, *, 1)$ is a left commutative algebras, and $(S, \triangle, 1)$ is a right commutative algebra. If two binary operations $*$ and $\triangle$ coincide, then NA-algebra $(S, *, \triangle, 1)$ is a bicommutative algebra.

Meanwhile, NA-algebra $(S, *, \triangle, 1)$ has close connections with several important non-classical logical algebraic structures:

**Relations with CI-algebras and Q-algebras:** When operations in a NA-algebra satisfy the condition of power-zero (i.e. $x * x = 1, \forall x \in S$), its algebraic structure $(S, *, 1)$ forms a CI-algebra, and its dual structure $(S, \triangle, 1)$ forms a Q-algebra. This shows that a NA-algebra which satisfies certain conditions be embedded into the category of CI/Q-algebras.

**Relation with quantum B-algebras:** This is a central conclusion of the paper. Any power-zero NA-algebra which satisfies $x * x = 1$ forms a quantum B-algebra. This makes NA-algebras a subclass of the wide quantum B-algebraic framework (which contains-effect algebras, residuated lattices, etc.).

**Relation with pseudo-BCH-algebras:** In an NA-algebra, let $x * x = 1$ and use its properties, one can derive a pseudo-BCH-algebra. The converse is not true, and there are examples which are pseudo-BCH-algebras but not-algebras, showing that pseudo-BCH-algebras are a broader class than NA-algebras.

**Relation with RM-algebras:** Also, an NA-algebra $(S, *, 1)$ which satisfies $x * x = 1$ forms an RM-algebra. However, a general RM-algebra is not necessarily a NA-algebra. Therefore a NA-algebra which satisfies the power-zero condition forms a proper subclass of RM-algebras.

Below we discuss the relationship between NA-algebras and CI/Q-algebras, quantum B-algebras, pseudo BCH-algebras, RM-algebras, etc.

### *4.1. Relations with CI-Algebras and Q-Algebras*

J.Neggers, S.S.Ahn and H.S. Kim introduced a new notion, called an Q-algebra, which is a generalization of the idea of BCH/BCI/BCK-algebras and they generalized some theorems discussed in BCI-algebras [14]. CI-algebras as a generalization of BE-algebras and BCK/BCI/BCH-algebras was initiated by B.L.Meng in 2009, and studied its important properties and relations with BE-algebras [15]. Arsham Borumand Saeid proved that CI-algebra is equivalent to dual Q-algebra [16].

In this section, we investigate the relation between NA-algebras and CI-algebras and Q-algebras.

**Definition 4.1.** ([15]) An algebra $(X; *, 1)$ of type (2, 0) is called a CI-algebra if it satisfies the following axioms: for all $x, y, z \in X$

$(CI_1) x * x = 1$ for all $x \in X$;

$(CI_2) 1 * x = x$ for all $x \in X$;

$(CI_3) x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

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A CI–algebra $X$ satisfying the condition $x * 1 = 1$ is called a BE-algebra. In any CI-algebra $X$ one can define a binary relation $'' \leq ''$ by $x \leq y$ if and only if $x * y = 1$.

A CI-algebra $X$ has the following properties:

$(i)$ $y * ((y * x) * x) = 1$;

$(ii)$ $(x * 1) * (y * 1) = (x * y) * 1$;

$(iii)$ if $1 \leq x$, then $x = 1$, for all $x, y \in X$.

**Definition 4.2.** ([14]) An algebra $(X; *, 1)$ of type (2, 0) is called a Q-algebra if it satisfies the following axioms: for all $x, y, z \in X$

$(Q_1) x \triangle x = 1$ for all $x \in X$;

$(Q_2) x \triangle 1 = x$ for all $x \in X$;

$(Q_3) (x \triangle y) \triangle z = (x \triangle z) \triangle y$ for all $x, y, z \in X$.

**Definition 4.3.** ([14,18]) Let $(X, \triangle, 1)$ be a $Q$-algebra and binary operation $'' * ''$ on $X$ is defined as follows:

$$x \triangle y = y * x.$$

Then $(X, *, 1)$ is called dual $Q$-algebra. In fact, the axioms of that are as follows:

$(DQ_1) x * x = 1$;

$(DQ_2) 1 * x = x$;

$(DQ_3) x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

**Example 8.** Let $X = \{1, a, b, c\}$. We define binary operations $\triangle$ on $X$ with the following table 17-18:

**Table 17.** $\triangle$ table.

| $\triangle$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | $a$ | 1 | $c$ | $b$ |
| $b$ | $b$ | $c$ | 1 | $a$ |
| $c$ | $c$ | $b$ | $a$ | 1 |

It is not complicated to check that $(X; \triangle, 1)$ is an Q-algebra, this is also a Klein four-group. By Definition 4.3, we can immediately write down the dual algebra of the Q-algebra $(X; *, 1)$ with the following table 18:

**Table 18.** $*$ table.

| $*$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | $a$ | 1 | $c$ | $b$ |
| $b$ | $b$ | $c$ | 1 | $a$ |
| $c$ | $c$ | $b$ | $a$ | 1 |

Obviously, here $\forall a, b \in X, a \triangle b = a * b$. Therefore $(X; *, 1)$ is a self-dual Q-algebras.

**Theorem 4.1.** ([16,17]) Any CI-algebra is equivalent to the dual Q-algebra.

Applying Theorem 2.1, based on Definition 4.1-4.3, for all $x, y, z \in S$, the following hold,

**Theorem 4.2.** Let $(S, *, \triangle, 1)$ be an NA-algebra and for all $x \in S$, it holds that $x * x = 1$, then

(1) $(S, *, 1)$ is a CI-algebra;

(2) $(S, \triangle, 1)$ is a Q-algebra.

**Proof.** The proof is straightforward.

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**Definition 4.4.**([29]) An algebra$(X;*,\triangle,1)$ of type (2, 2, 0) is called a pseudo-CI algebra if, for all $x,y,z\in X$, it satisfies the following axioms:
$(psCI_1)x*x=x\triangle x=1$,
$(psCI_2)1*x=1\triangle x=x$,
$(psCI_3)x*(y\triangle z)=y\triangle(x*z)$,
$(psCI_4)x*y=1\Leftrightarrow x\triangle y=1$.
It follows from Definition 2.2 and Theorem 2.1 that the following conclusion is obtained easily:
**Theorem 4.3.** An NA-algebra $(S,*,\triangle,1)$ if, for all $x\in S$, it satisfies $x*x=1$, then $(S,*,\triangle,1)$ is a pseudo-CI algebra, i.e, every pseudo-CI algebra is contained in the class of NA-algebras.

*4.2. Relation with Quantum B-Algebras*

Rump and Yang introduced the concept of quantum B-algebras [19], and proved that the quantum B-algebras can provide a unified semantic for non-commutative algebraic logic. Almost all implicational algebras studied before - pseudo-effect algebras, residuated lattices, pseudo MV/BL/MTL-algebras, bounded non-commutative R-monoids, pseudo-hoops, pseudo BCK/BCI-algebras are quantum B-algebras. Now, let us proceed by stating the definitions, some of them being well known.

**Definition 4.5.** ([19]) A quantum B-algebra is a partially ordered set $X$ with two binary operations $\rightarrow$ and $\rightsquigarrow$ such that the following are satisfied for $x,y,z\in X$ :

$$y\le z\Rightarrow x\rightarrow y\le x\rightarrow z \tag{9}$$

$$x\le y\rightarrow z\Leftrightarrow y\le x\rightsquigarrow z \tag{10}$$

$$x\rightarrow(y\rightsquigarrow z)=y\rightsquigarrow(x\rightarrow z). \tag{11}$$

Both $\rightarrow$ and $\rightsquigarrow$ stand for logical implication, a left one and a right one, which have to be distinguished in an non-commutative framework. The partial order relation $\le$ stands for entailment.
**Proposition 4.4.** ([19**?** ]) Let $(X,\le,\rightarrow,\rightsquigarrow)$ be a quantum B-algebra. The following hold, for all $x,y,z\in X$,
$(1^\circ)$ $x\le(x\rightarrow y)\rightsquigarrow y, x\le(x\rightsquigarrow y)\rightarrow y$;
$(2^\circ)$ $y\le z$ implies $x\rightsquigarrow y\le x\rightsquigarrow z$;
$(3^\circ)$ $x\le y$ implies $y\rightarrow z\le x\rightarrow z, y\rightsquigarrow z\le x\rightsquigarrow z$;
$(4^\circ)$ $x\rightarrow y=((x\rightarrow y)\rightsquigarrow y)\rightarrow y, x\rightsquigarrow y=((x\rightsquigarrow y)\rightarrow y)\rightsquigarrow y$;
$(5^\circ)$ $x\rightarrow y\le(y\rightarrow z)\rightsquigarrow(x\rightarrow z), x\rightsquigarrow y\le(y\rightsquigarrow z)\rightarrow(x\rightsquigarrow z)$.
Now, we study the relationships between NA-algebra and quantum B-algebras as follows.
**Theorem 4.5.** Let $(X,*,\triangle,1)$ be an NA-algebra which satisfies $x*x=1$ for any $x\in X$, we have
$(I_1)$ the order is given by $x\le y\Leftrightarrow x*y=1$(or $y\triangle x=1$) is a partial order.
$(I_2)$ $(X,*,\triangle,1)$ is a trivial quantum $B$-algebra, the element $x$ is group-like element.
**Proof.** $(I_1)$ 1)By $x*x=1$, we have $x\le x$;
2) Assume that $x*y=1=y*x$ holds for given $x,y\in X$. Then it implies that $x*(y*x)=(x*y)*x=1*x=x\Rightarrow x*1=x\Rightarrow y=y*1=y*(x*x)=x*(y*x)=x*1=x$. Hence $x\le y, y\le x\Rightarrow x=y$.
3) Assume that $x,y\in X$ satisfies $x*y=1, y*z=1\Rightarrow 1=1*1=(x*y)*(y*z)=x*((y*y)*z)=x*(1*z)=x*z$.
Hence $x\le y,y\le z\Rightarrow x\le z$. So the order relation "$\le$" is a partial order in $X$.
$(I_2)$ In the semigroup $(X,*,1)$ of NA-algebra $(X,*,\triangle,1)$, for all $x,y\in X$, we have $x*y=y\triangle x$ holds, hence for all $x,y,z\in X$,

$$x*(y\triangle z)=x*(z*y)=(x*z)*y=y\triangle(x*z) \tag{12}$$

Using equality (12), by $*$ instead of $\rightarrow$, and use $\triangle$ instead of $\rightsquigarrow$, we obtain equality (11) is verified.
From that $y\le z\Leftrightarrow y*z=1\Rightarrow(x\rightarrow y)\rightarrow(x\rightarrow z)=(x*y)*(x*z)=x*((x*y)*z)=(x*x)*(y*z)=1*1=1$. It follows that (9) $y\le z\Rightarrow x\rightarrow y\le x\rightarrow z$.

By $x * x = 1 \Rightarrow x * 1 = x \to (x * x) = (x * x) * x = 1 * x = x \Rightarrow x * y = x * (y * 1) = y * (x * 1) = y * x = x \triangle y$, i.e., $x * y = y * x = x \triangle y)$. In this case, $x \to y = x \rightsquigarrow y$. and if $x \to (y \to z) = 1$, then $y \to (x \rightsquigarrow z) = 1$, the converse is also true. Thus, we obtain $x \leq y \to z \Leftrightarrow y \leq x \rightsquigarrow z$. So, the inequality (10) is verified.

This proves $(X, *, \triangle, 1)$ is a quantum $B$-algebra.

Furthermore, we can be obtain:

$$(x * y) \triangle y = y * (x * y) = x * (y * y) = x * 1 = x,$$

$$(x \triangle y) * y = (y * x) * y = y * (x * y) = x * (y * y) = x * 1 = x.$$

Therefore, $(X, *, \triangle, 1)$ is a commutative quantum $B$-algebra, and the element $x$ is group-like element.

**Remark 6.** In an NA-algebra $(S, *, \triangle, 1)$, a order relation is defined by $'' \leq ''$ is defined by $x \leq y \Leftrightarrow x * y = 1$. We can prove that only if condition $x * x = 1$ holds, this order relation is a partial order. Therefore, to construct a quantum B-algebra using NA-algebra, an additional condition $''x * x = 1''$ must be imposed to ensure reflexivity.

The following example shows that a nilpotent NA-algenra is a quantum B-algebra.

Let $S = \{a, 1\}$. Define the operations $*$ and $\triangle$ on $S$ by cayley tables below 19-20:

**Table 19.** $*$ table.

| $*$ | $a$ | $1$ |
|---|---|---|
| $a$ | $1$ | $a$ |
| $1$ | $a$ | $1$ |

**Table 20.** $*$ table.

| $\triangle$ | a | 1 |
|---|---|---|
| $a$ | $1$ | $a$ |
| $1$ | $a$ | $1$ |

Here $(S, *, \triangle, 1)$ is a NA-algebra. We putting $* = \to$, $\triangle = \rightsquigarrow$, and $a = 1$, then $(S, \to, \rightsquigarrow, 1)$ is also a quantum B-algebra.

In [20], Shengwei Han and Xiaoting Xu use the identity (11) to propose the concept of C-algebras and show that a C-algebra is a group if and only if each of its elements is dualizing and by dualizing elements of a C-algebra $X$ define different binary operations on $X$ such that $X$ is a moniod. In this section, we will to define C-algebras and investigate the relation between C-algebras and NA-algebras.

**Definition 4.6.** ([20]) A set $X$ with two binary operations $\to$ and $\rightsquigarrow$ is called a C-algebra if $\to$ and $\rightsquigarrow$ satisfy the condition (11).

**Theorem 4.6.** Every NA-algebra $(S, *, \triangle, 1)$ is a C-algebra.

**Proof.** By Theorem 2.1, we putting $* = \to$, $\triangle = \rightsquigarrow$, for all $x, y, z \in S$, we have

$$x \rightsquigarrow (y \to z) = x \triangle (y * z) = x \triangle (z \triangle y),$$

$$y \to (x \rightsquigarrow z) = y * (x \triangle z) = (x \triangle z) \triangle y) = x \triangle (z \triangle y),$$

Hence, $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$. So, axiom (11) hold. The proof is complete.

### *4.3. Relation with Pseudo-BCH-Algebras*

In order to characterize fuzziness, non-commutability or uncertainty in quantum logic, it is necessary to break through the rigid structures of traditional Boolean algebra, such as excluded

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neutrality and commutative law. Pseudo-prefix algebra, such as pseudo-Boolean algebra and pseudo-BCI algebra, has become an effective mathematical tool for describing complex logical relationships by weakening axiomatic conditions such as non-commutativity and non-associativity. 2015, Andrzej Walendziak introduced the concept of pseudo-BCH-algebras, which is a theoretical exploration in the evolution of non-commutative logic algebra. Its core value lies in extending the BC/BCI framework to adapt to non-commutative logic scenarios and filling the theoretical gap between strict Boolean algebra and completely free algebra [22]. **Definition 4.7.** ([22]) A pseudo-BCH-algebra is an algebra $X = (X; *, \diamond, 1)$ of type (2, 2, 0) satisfying the axioms:

(pBCH-1) $x * x = x \diamond x = 1$;

(pBCH-2) $(x * y) \diamond z = (x \diamond z) * y$;

(pBCH-3) $x * y = y \diamond x = 1 \Rightarrow x = y$;

(pBCH-4) $x * y = 1 \Leftrightarrow x \diamond y = 1$.

Replacing $\diamond$ by $\triangle$ in Definition 4.7 we obtain

**Remark 7.** Observe that if $(S; *, \triangle, 1)$ is a NA-algebra, letting $x * x := 1$. By Theorem 2.5 and Theorem 2.6, then we have $x * y = y * x = x \triangle y$, and produces a pseudo-BCH-algebra $(S; *, \triangle, 1)$. Its inverse is not generally true as shown in the following example.

**Example 9.** Let $X = \{1, a, b, c\}$. Define the operations $*$ and $\diamond$ on $X$ by table 21-22 below:

**Table 21.** $*$ table.

| $*$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| $a$ | $a$ | 1 | $a$ | 1 |
| $b$ | $b$ | $b$ | 1 | 1 |
| $c$ | $c$ | $b$ | $c$ | 1 |

**Table 22.** $\diamond$ table.

| $\diamond$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| $a$ | $a$ | 1 | $a$ | 1 |
| $b$ | $b$ | $b$ | 1 | 1 |
| $c$ | $c$ | $c$ | $a$ | 1 |

Then, it is easy to verity that $(X; *, \diamond, 1)$ is a pseudo-BCH-algebra, which is not a NA-algebras, since $b * c = 1 \neq a = c \diamond b$.

### *4.4. Relation with RM-Algebras*

In 2019, Walendziak, A. introduced the notion of RM algebras and investigated its elementary properties [23]. The class of all RM algebras contains BCK, BCI, BCH, BZ, RME, pre-BZ algebras and many others.

In this section, for all $x, y, z \in X$, we apply the identity

(I) $x * (y * z) = (z * x) * y$

to NA-algebras $(S, *, \triangle, 1)$ and investigate some relations between this condition with other axioms in some algebras of logic and some examples are given to illustrate them. The relations between NA-algebras and RM-algebras are given.

**Definition 4.8.** ([23])An RM-algebra is an algebra $(X, \rightarrow, 1)$ of type (2,0) satisfying the following axioms: for all $x, y \in X$

(R) $x \rightarrow x = 1$,

(M) $1 \rightarrow x = x$.

**Example 10.** Let $X = \{1, a, b\}$. Define the operations $\rightarrow$ on $X$ with Cayley Table 23:

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**Table 23.** $\rightarrow$ table.

| $\rightarrow$ | 1 | $a$ | $b$ |
|---|---|---|---|
| 1 | 1 | $a$ | $b$ |
| $a$ | $a$ | 1 | $b$ |
| $b$ | $b$ | $b$ | 1 |

then $(X, \rightarrow, 1)$ is an RM-algebra and satisfies condition (I) [24], but $b * (a * b) = b * b = b * b = 1 \neq a * (b * b) = a * 1 = a$, hence it does not satisfy

(E) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

So, RM-algebra is not NA-algebra.

**Theorem 4.7.** If $(S, *, \triangle, 1)$ is a NA-algebra with $x * x = 1$, then $(S, *, 1)$ is an RM-algebra.

**Proof.** By $(N_3)$, we have (M) holds. Based on the assumption $x * x = 1$, we obtain $(S, *, 1)$ is an RM-algebra.

**Theorem 4.8.** Let $(S, *, \triangle, 1)$ be an NA-algebra. If it satisfies condition (I), for all $x, y, z \in S$, we have:

$$(z * x) * y = (z * y) * x.$$

**Proof.** Let $x, y, z \in S$. By using $x * (y * z) = y * (x * z)$, if $S$ satisfies condition (I), we have

$$(z * x) * y = x * (y * z) = y * (x * z) = (z * y) * x.$$

Therefore, $(z * x) * y = (z * y) * x$.

**Proposition 4.8.** If NA-algebra $(S, *, \triangle, 1)$ satisfies $x * x = 1$ and $x \rightarrow 1 = 1$, then $S = \{1\}$.

**Proof.** Let $x \in S$. suppose $x \rightarrow x = 1$, then $x = 1 \rightarrow x = (x \rightarrow x) \rightarrow x = x \rightarrow (x \rightarrow x) = x \rightarrow 1$. If $S$ satisfies $x \rightarrow 1 = 1$, we have $x = 1$, and $S = \{1\}$.

**Proposition 4.9.** Let $(S, *, \triangle, 1)$ be an NA-algebra, we have

(1) if it satisfies condition (I), then $x \rightarrow 1 = x$, for any $x \in S$.

(2) if for all $x, y, z \in S$, then $(z * x) * y = (z * y) * x$ iff $x * y = y * x$.

**Proof.** (1) For any $x \in S$, we apply identity (I) to NA-algebra $(S, *, \triangle, 1)$, we obtain

$$x * 1 = 1 * (x * 1) = (1 * 1) * = 1 * x = x.$$

(2) Put $z = 1$ with identity $(z * x) * y = (z * y) * x$, we get $(1 * x) * y = (1 * y) * x \Leftrightarrow x = y$. Conversely, if all $x, y \in S$, have $x * y = y * x$, then $(z * x) * y = y * (z * x) = z * (y * x) = (z * y) * x$. The proof is complete.

**Proposition 4.10.** ([24]) If $(X, \rightarrow, 1)$ is an RM-algebra satisfying (I), then (E) holds.

From theorem 4.7, Proposition 4.9 and Proposition 4.10, we have: The RM-algebras are a special subclass of the NA-algebras with $x * x = 1$ for all $x \in S$.

According to the above analysis, the following diagram of inclusion relation can be outlined (where "Nilpotent NA" denotes a nilpotent NA-algebra satisfying $x * x = 1$):

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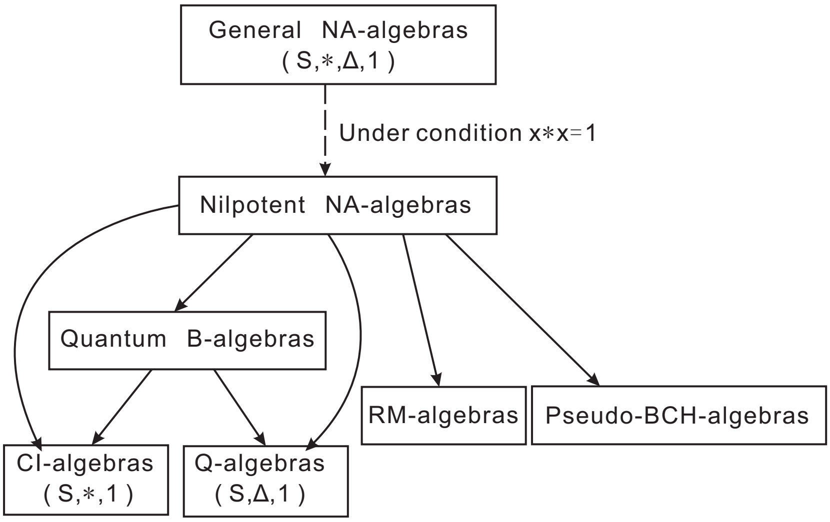


**Figure 1.** The relationships between NA-algebras and other algebras.

## 5. Conclusions and Future Work

In this paper, we have introduced the concept of NA-algebras and investigated some of their useful properties. It is well known that the ideals with special properties play an important role in the logic system. The aim of this article is to investigate ideals and stabilizers in NA-algebra, we establish the NA-morphism theorem. We characterize congruence kernels in a NA-algebra. Finally, the relationships between NA-algebras and other algebras, like Q-algebras, CI-algebras, quantum B-algebras, pseudo-BCH-algebras and RM-algebras, and are also investigated. As a direction of research, one can investigate and extend these results to the other algebraic structures.

We believe that these results are very useful in developing algebraic structures also these definitions and main results can be similarly extended to some other algebraic systems such as lattices and Lie algebras etc. We hope this work would serve as a foundation for further studies on the structure of NA-algebras like fuzzy NA-algebras, soft NA-algebras and hyper NA-algebras.

NA-algebras have applications in the dialgebra theory. It is also closely connected with dimonoids and Digroups introduced by Loday [25,26,28]. Digroups which form a subclass of dimonoids, play an important role in the theory of Leibniz algebras [28]. Next, we will embed the NA-algebra into a vector space, deeply further study its relationship with other algebraic systems, and expand the field of NA-algebra.

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