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From the Pythagorean Dream to the Fermatian Obstruction: Symbolic Representation of $h = \sqrt[3]{(a^3 + b^3)}$ via an Identity Derived from Nicomachus' Cumulative Sum

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From the Pythagorean Dream to the Fermatian Obstruction: Symbolic Representation of $h = \sqrt[3]{a^3 + b^3}$ via an Identity Derived from Nicomachus' Cumulative Sum

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Abstract

This work rigorously explores the conceptual transition between Pythagorean harmony $h^2 = a^2 + b^2$ and Fermatian impossibility $h^3 = a^3 + b^3$, explicitly acknowledging that Fermat's Last Theorem (FLT) prohibits integer solutions for $n = 3$. Starting from Nicomachus' historical formula for the cumulative sum of cubes,

$$S(n) = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2,$$

and applying the first-order retrospective finite difference operator $\nabla S(n) = S(n) - S(n-1)$, we **deduce** the algebraic identity:

$$n^3 = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right].$$

It is crucial to emphasize that Nicomachus (c. 100 CE) **did not explicitly formulate** this identity in terms of symmetric differences; his historical contribution was exclusively limited to the cumulative sum formula. The expression above constitutes a **modern deduction** derived via discrete calculus. Using this deduction, we construct an exact symbolic representation:

$$h = \sqrt[3]{\frac{a^2}{4}\left[(a+1)^2 - (a-1)^2\right] + \frac{b^2}{4}\left[(b+1)^2 - (b-1)^2\right]}.$$

We demonstrate that this expression, while mathematically exact and constructed exclusively through integer operations, does not produce $h \in \mathbb{Z}$ —empirically confirming the arithmetic obstruction of FLT through 2,500 numerical verifications ($1 \leq a, b \leq 50$). We establish the combinatorial uniqueness of exponent $k = 2$ in symmetric differences $(n+1)^k - (n-1)^k$, revealing why the compact representation works exclusively for cubes. We contextualize historically the problem from the Pythagorean school (6th century BCE) to Wiles' proof (1994), highlighting contributions from Nicomachus, Euler, Sophie Germain, and Kummer with historiographical rigor. The genuine value of this proposal resides in its pedagogical capacity to illustrate the fundamental distinction between *internal structure* (local properties of individual cubes) and *additive structure* (relations between distinct cubes), honestly transforming Fermatian impossibility into an opportunity to comprehend the structural limits inherent to mathematics.

Keywords: Pythagorean theorem; cumulative sum of cubes of Nicomachus; finite difference; deduced cubic identity; Fermat's Last Theorem; internal structure; additive structure; combinatorial uniqueness; Sophie Germain; discrete calculus; structural limits; pedagogical value

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1. Introduction: The Seduction of Dimensional Analogy and its Structural Limits

1.1. The Pythagorean legacy and quadratic harmony

The Pythagorean school, founded in Crotona (Magna Graecia) around the 6th century BCE, established one of the first philosophical-mathematical syntheses in Western history: the belief that *"everything is number"* (πάντα ῥεῖ). For the Pythagoreans, numbers were not mere calculation instruments but ontological principles structuring cosmic reality. This worldview found its purest expression in *figurate numbers*: geometric representations of integers through regular point configurations (Heath, 1921).

The theorem bearing Pythagoras' name —though likely known to Mesopotamian and Egyptian cultures earlier— crystallized this harmony between geometry and arithmetic:

$$h^2 = a^2 + b^2, \tag{1}$$

where h represents the hypotenuse of a right triangle and a, b its legs. Euclid, in his *Elements* (Book X, Proposition 29, Lemma 1, c. 300 BCE), provided the complete parametrization of primitive Pythagorean triples:

$$a = m^2 - n^2, \quad b = 2mn, \quad h = m^2 + n^2, \quad m > n > 0,\ \gcd(m, n) = 1,\ m \not\equiv n \pmod 2. \tag{2}$$

This quadratic harmony —the existence of infinitely many integer solutions for a nonlinear Diophantine equation— has captivated generations of mathematicians and has naturally suggested an apparently innocent question: *Can this relationship be generalized to higher powers while preserving integer solutions?*

1.2. Dimensional temptation and structural limits

Dimensional analogy initially appears plausible. If the squares of the legs sum to the square of the hypotenuse in Euclidean plane geometry, could the volumes of "cubic legs" sum to the volume of a "cubic hypotenuse" in three-dimensional space? Formally, does a natural extension of (1) exist to:

$$h^3 = a^3 + b^3 \quad \text{with} \quad h, a, b \in \mathbb{Z}^+? \tag{3}$$

This question, which has haunted mathematicians for centuries, inevitably leads to Fermat's Last Theorem (FLT). The answer, rigorously established by Euler for $n = 3$ (1770) and generalized by Wiles for all $n > 2$ (1994), is negative: **no positive integer solutions exist for (3)**. This impossibility is not a transient computational limitation but an inherent arithmetic obstruction within the structure of integers (Edwards, 1979).

1.3. Nicomachus and the cumulative sum: Precise historical delimitation

Nicomachus of Gerasa (c. 60–120 CE), Neopythagorean philosopher and mathematician of the late Hellenistic period, systematized in his *Introduction to Arithmetic* (Ἀριθμητικὴ εἰσαγωγή) classical Greek arithmetic knowledge (Nicomachus, c. 100/1926). Among his enduring contributions, Nicomachus rigorously established the following result:

Theorem 1 (Nicomachus' cumulative sum of cubes)**.** *For all $n \in \mathbb{N}$:*

$$S(n) = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}. \tag{4}$$

It is **fundamental** to clarify that Nicomachus **did not formulate nor suggest** the algebraic identity:

$$n^3 = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right], \tag{5}$$

in modern terms of symmetric differences. His work was exclusively limited to expression (4) as a property of cumulative sums. Identity (5) constitutes a **modern deduction** obtained by applying the retrospective finite difference operator $\nabla S(n) = S(n) - S(n-1)$ to the historical formula (4), a procedure that did not exist within the conceptual framework of ancient Greek mathematics (Boole, 1860; Edwards, 1979).

Remark 1 (Rigorous historiographical delimitation). *Precise attribution of mathematical results is an ethical academic imperative. Nicomachus deserves historical recognition for (4), but **not** for (5), which is an algebraic consequence derivable through conceptual tools developed after the 19th century. Any presentation suggesting otherwise would commit an inadmissible historiographical anachronism.*

1.4. Author's contribution: An original pedagogical proposal

Starting from the modern deduction (5) —rigorously derived from Nicomachus' historical formula via discrete calculus— this work proposes an exact symbolic representation of:

$$h = \sqrt[3]{a^3 + b^3}, \tag{6}$$

explicitly accepting that $h \notin \mathbb{Z}$ by FLT for $n = 3$. The original contribution of this work is **exclusively pedagogical and conceptual**: to offer a symbolic representation expressing h through operations exclusively between integers (though the final result is irrational), without falsifying the structural limits imposed by FLT. This proposal does not "solve" FLT (impossible by proven theorem), but transforms Fermatian impossibility into an opportunity to comprehend the conceptual transition between quadratic solubility (Pythagoras) and cubic obstruction (Fermat).

2. Theoretical Framework: Discrete Calculus and Deduction of the Cubic Identity

2.1. Cumulative sum of cubes: Nicomachus' historical theorem

Theorem 2 (Nicomachus' cumulative sum of cubes). *For all $n \in \mathbb{N}$:*

$$S(n) = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}. \tag{7}$$

Proof. By mathematical induction. Base case ($n = 1$): $S(1) = 1^3 = 1 = \frac{1^2 \cdot 2^2}{4} = 1$. Inductive hypothesis: valid for n. Inductive step:

$$\begin{aligned} S(n+1) = S(n) + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2\left[\frac{n^2}{4} + (n+1)\right] = (n+1)^2\left[\frac{n^2+4n+4}{4}\right] \\ &= \frac{(n+1)^2(n+2)^2}{4}. \quad \square \end{aligned}$$

Remark 2 (Geometric interpretation). *Identity (7) possesses an elegant geometric interpretation: the sum of the first n cubes forms a perfect square whose side equals the n-th triangular number $T_n = \frac{n(n+1)}{2}$. This dimensional transformation (sum of volumes → square area) is unique in figurate arithmetic and anticipates profound structures of modern discrete analysis (Conway & Guy, 1996).*

2.2. Finite difference and the fundamental theorem of discrete calculus

The development of finite difference calculus began with Pierre de Fermat's work (1636) on power sums and was systematically formalized by Brook Taylor (1715) in his *Methodus Incrementorum Directa et Inversa*. George Boole (1860) consolidated the theoretical framework in his treatise *A Treatise on the Calculus of Finite Differences*, establishing umbral calculus as a fundamental tool for interpolation, numerical series, and difference equations.

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Definition 1 (Retrospective finite difference)**.** *For a sequence* $f : \mathbb{N} \to \mathbb{R}$*:*

$$\nabla f(n) := f(n) - f(n-1), \quad n \geq 2. \tag{8}$$

Definition 2 (Cumulative sum)**.** *The cumulative sum of a sequence* f *up to* n *is:*

$$\Sigma f(n) := \sum_{k=1}^{n} f(k), \quad n \geq 1. \tag{9}$$

Theorem 3 (Fundamental theorem of discrete calculus)**.** *Let* $S(n) = \Sigma f(n) = \sum_{k=1}^{n} f(k)$ *be the cumulative sum of* f*. Then:*

$$\nabla S(n) = S(n) - S(n-1) = f(n), \quad \forall n \geq 2. \tag{10}$$

Proof. By definition of cumulative sum: $S(n) = f(1) + \cdots + f(n-1) + f(n)$ and $S(n-1) = f(1) + \cdots + f(n-1)$. Subtracting: $S(n) - S(n-1) = f(n)$. □

This theorem is the discrete analog of the fundamental theorem of continuous calculus $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. The finite difference acts as a "discrete derivative" and the cumulative sum as a "discrete integral", establishing a profound structural duality between both operators (Graham, Knuth, & Patashnik, 1994).

2.3. Deduction of the modern cubic identity from Nicomachus' historical formula

Applying the fundamental theorem of discrete calculus to Nicomachus' historical formula (7):

$$\begin{aligned} n^3 &= \nabla S(n) = S(n) - S(n-1) \\ &= \frac{n^2(n+1)^2}{4} - \frac{(n-1)^2 n^2}{4} \\ &= \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right] \\ &= \frac{n^2}{4}\left[(n^2+2n+1) - (n^2-2n+1)\right] \\ &= \frac{n^2}{4}(4n) = n^3. \end{aligned} \tag{11}$$

Remark 3 (Precise conceptual attribution)**.** *The algebraic step* $(n+1)^2 - (n-1)^2 = 4n$ *is an elementary identity derivable from Newton's binomial theorem. However, its interpretation as a finite difference of the cumulative sum is a modern perspective that was not part of Nicomachus' conceptual framework. Expression (11) therefore constitutes a* ***modern deduction attributable to the author of this work*** *within the context of his pedagogical exploration, rigorously derived but not historically attributable to Nicomachus. This distinction is essential for historiographical integrity.*

The denominator 4 in (11) is not arbitrary: it inevitably arises from the quadratic structure $\left[\frac{n(n+1)}{2}\right]^2$ of the cumulative sum. This structural constant is the algebraic trace of the underlying triangular geometry of Pythagorean figurate arithmetic.

3. Symbolic Representation of $h = \sqrt[3]{a^3 + b^3}$ via the Deduced Identity

3.1. Formal construction of the representation

Theorem 4 (Symbolic representation via deduced identity from Nicomachus). *For $a, b \in \mathbb{Z}^+$, define $h = \sqrt[3]{a^3 + b^3}$. Then h admits the exact symbolic representation:*

$$\boxed{h = \sqrt[3]{\frac{a^2}{4}\left[(a+1)^2 - (a-1)^2\right] + \frac{b^2}{4}\left[(b+1)^2 - (b-1)^2\right]}}. \tag{12}$$

Proof. Apply the deduced identity (11) to a and b separately:

$$a^3 = \frac{a^2}{4}\left[(a+1)^2 - (a-1)^2\right], \tag{13}$$

$$b^3 = \frac{b^2}{4}\left[(b+1)^2 - (b-1)^2\right]. \tag{14}$$

Summing (13) and (14):

$$a^3 + b^3 = \frac{a^2}{4}\left[(a+1)^2 - (a-1)^2\right] + \frac{b^2}{4}\left[(b+1)^2 - (b-1)^2\right]. \tag{15}$$

Applying the cubic root to both sides of (15) yields (12). The equality is algebraically exact. □

Remark 4 (Nature and limits of the representation). *Expression (12) is mathematically exact and uses exclusively arithmetic operations between integers in its construction. However, the final result h is generally an irrational algebraic number (root of the irreducible polynomial $x^3 - (a^3 + b^3) = 0$ over $\mathbb{Q}$ when $a^3 + b^3$ is not a perfect cube). This representation does not "solve" FLT; it conceptually illuminates it by revealing the internal structure of each individual term before their additive combination, without falsifying the inherent arithmetic obstruction.*

3.2. Exhaustive numerical verification: Case $a = 3$, $b = 4$

Example 1 (Classical Pythagorean triangle extended to cubes). *For pair $(a, b) = (3, 4)$, classical in the Pythagorean context ($3^2 + 4^2 = 5^2$):*

$$3^3 = \frac{9}{4}\left[4^2 - 2^2\right] = \frac{9}{4}(16 - 4) = \frac{9}{4} \cdot 12 = 27,$$
$$4^3 = \frac{16}{4}\left[5^2 - 3^2\right] = 4(25 - 9) = 4 \cdot 16 = 64,$$
$$h = \sqrt[3]{27 + 64} = \sqrt[3]{91} \approx 4.497941445275415.$$

Representation (12) produces exactly $\sqrt[3]{91}$, an irrational algebraic number. Note that $h \notin \mathbb{Z}$, confirming FLT for $n = 3$. The number 91 is not a perfect cube (since $4^3 = 64 < 91 < 125 = 5^3$), and its prime factorization $91 = 7 \cdot 13$ contains no exponents that are multiples of 3, making it impossible for 91 to be a cube in $\mathbb{Z}$.

3.3. Systematic numerical verifications: 2,500 pairs (a, b)

We exhaustively evaluated all pairs with $1 \le a, b \le 50$ (2,500 combinations). Results are summarized in Table 1.

Corollary 1 (Empirical confirmation of FLT for $n = 3$). *For all $1 \le a, b \le 50$:*

$$h = \sqrt[3]{a^3 + b^3} \notin \mathbb{Z}.$$

Proof. Exhaustive computational evaluation. No pair produces $h \in \mathbb{Z}$. This result is consistent with Euler's theoretical proof for the $n = 3$ case of FLT (Edwards, 1979). □

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Table 1. Selected numerical verifications of $h = \sqrt[3]{a^3 + b^3}$ for $1 \le a, b \le 15$.

(a,b)	$a^3 + b^3$	h	$h \in \mathbb{Z}$?
$(1,1)$	2	1.259921	No
$(1,2)$	9	2.080084	No
$(2,3)$	35	3.271066	No
$(3,4)$	91	4.497941	No
$(4,5)$	189	5.738794	No
$(5,6)$	341	6.986400	No
$(6,8)$	728	8.995858	No
$(9,10)$	1729	12.002314	No
$(12,16)$	5824	17.992222	No
$(15,20)$	11375	22.496531	No
Total evaluated	2,500 pairs	—	**0 integer solutions**

Remark 5 (The taxicab number 1729)**.** *Pair $(9,10)$ yields $a^3 + b^3 = 1729$, Hardy-Ramanujan's famous taxicab number, expressible as a sum of two cubes in two distinct ways ($1729 = 9^3 + 10^3 = 12^3 + 1^3$). Despite this combinatorial richness, $\sqrt[3]{1729} \approx 12.002314 \notin \mathbb{Z}$, illustrating that even numbers with multiple representations as sums of cubes respect the Fermatian obstruction for $n = 3$ (Singh, 1997).*

4. Combinatorial Uniqueness of Exponent $k = 2$ in Symmetric Differences

4.1. Generalized symmetric difference and binomial expansion

Definition 3 (Symmetric difference of order k)**.** *For $k \in \mathbb{N}$ and $n \in \mathbb{N}$:*

$$D_k(n) := (n+1)^k - (n-1)^k. \tag{16}$$

Proposition 1 (Binomial expansion of $D_k(n)$)**.**

$$D_k(n) = 2 \sum_{\substack{j=1 \\ j \text{ odd}}}^{k} \binom{k}{j} n^{k-j}. \tag{17}$$

Proof. By Newton's binomial theorem:

$$(n+1)^k = \sum_{j=0}^{k} \binom{k}{j} n^{k-j},$$
$$(n-1)^k = \sum_{j=0}^{k} \binom{k}{j} n^{k-j} (-1)^j.$$

Subtracting and noting that $1 - (-1)^j = 0$ if j even and 2 if j odd, we obtain (17). □

4.2. Combinatorial classification and uniqueness of the pure monomial

We analyze the number of non-zero terms in $D_k(n)$ for $k = 1$ through $k = 6$:

Theorem 5 (Uniqueness of the pure monomial)**.** *$D_k(n)$ is a pure monomial (single non-constant term) if and only if $k = 2$.*

Proof. According to (17), the number of non-zero terms equals the number of odd integers j in $[1, k]$.

- $k = 1$: $j = 1$ (one term) $\Rightarrow D_1(n) = 2$ (constant).
- $k = 2$: $j = 1$ (one term) $\Rightarrow D_2(n) = 2\binom{2}{1}n = 4n$ (pure linear monomial).
- $k = 3$: $j = 1, 3$ (two terms) $\Rightarrow D_3(n) = 2\left[\binom{3}{1}n^2 + \binom{3}{3}\right] = 6n^2 + 2$.

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- $k \geq 4$: number of odds in $[1, k]$ is $\lceil k/2 \rceil \geq 2 \Rightarrow$ at least two non-zero terms.

Table 2. Classification of $D_k(n)$ by number of non-zero terms.

k	$D_k(n) = (n+1)^k - (n-1)^k$	Non-zero terms
1	2	1 (constant)
2	$4n$	1 (pure monomial)
3	$6n^2 + 2$	2 (binomial)
4	$8n^3 + 8n$	2 (binomial)
5	$10n^4 + 20n^2 + 2$	3 (trinomial)
6	$12n^5 + 40n^3 + 12n$	3 (trinomial)

Therefore, only for $k = 2$ do we obtain a non-constant pure monomial. □

Remark 6 (Consequence for the deduced cubic identity)**.** *This combinatorial uniqueness explains why the compact representation (11) is possible for cubes but not for higher powers. For n^4, for example:*

$$n^4 \neq \frac{n^3}{c}\left[(n+1)^3 - (n-1)^3\right] \quad \text{for any constant } c \in \mathbb{Q},$$

since $D_3(n) = 6n^2 + 2$ contains two terms and cannot be factored as $c \cdot n$. The algebraic elegance of (11) is therefore an exclusive structural property of the cubic exponent, derived from the combinatorial uniqueness of $k = 2$ in symmetric differences.

5. Historical Context I: Pythagoras and the Pythagorean School

Historical Note 1 (Pythagoras of Samos (c. 570–495 BCE))**.** *Pythagoras was born on the island of Samos (modern Greece) during the Archaic Greek period. After traveling through Egypt and possibly Babylon, he founded in Crotona (southern Italy) a philosophical-religious community with strict ascetic rules. The Pythagoreans believed numbers were the ultimate essence of reality: "numbers constitute the essence of all things" (Aristotle, Metaphysics A, 5). This numerical worldview manifested in fundamental mathematical discoveries: Pythagorean theorem, classification of even/odd numbers, perfect numbers, and figurate numbers (triangular, square, pentagonal) (Heath, 1921).*

Pythagorean geometry was deeply intertwined with arithmetic. A right triangle was not merely a geometric figure but a visible manifestation of harmonious numerical relationships. The discovery of Pythagorean triples such as $(3, 4, 5)$ or $(5, 12, 13)$ reinforced the belief in a perfect correspondence between the sensible world (geometry) and the intelligible world (arithmetic).

Historical Note 2 (The crisis of incommensurables)**.** *Pythagorean harmony suffered an existential crisis with the discovery of incommensurable magnitudes (likely $\sqrt{2}$ as the diagonal of the unit square). This finding, attributed to Hippasus of Metapontum (5th century BCE), contradicted the fundamental belief that every geometric relationship could be expressed as a ratio of integers. Legend claims Hippasus was thrown into the sea for revealing this secret, though historicity is doubtful. This crisis led to the development of Eudoxus' theory of proportions (c. 370 BCE), precursor to the modern concept of real numbers (Heath, 1921).*

6. Historical Context II: Nicomachus of Gerasa and Figurate Arithmetic

Historical Note 3 (Nicomachus of Gerasa (c. 60–120 CE))**.** *Nicomachus was born in Gerasa (modern Jerash, Jordan), a Hellenistic city of the Roman Decapolis. His principal work, Introduction to Arithmetic (Ἀριθμητικὴ εἰσαγωγή), written in Greek koiné, became the standard arithmetic text in the Greco-Roman world and later in medieval Europe (translated into Latin by Boethius, c. 500 CE). Unlike Euclid, Nicomachus prioritized intuitive and philosophical understanding over axiomatic rigor, emphasizing the mystical and cosmological properties of numbers (Nicomachus, c. 100/1926).*

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Nicomachus systematized the theory of figurate numbers inherited from the Pythagoreans:

- **Triangular**: $T_n = \frac{n(n+1)}{2}$ (sum of first n integers)
- **Square**: $Q_n = n^2$
- **Pentagonal**: $P_n = \frac{n(3n-1)}{2}$
- **Hexagonal**: $H_n = n(2n-1)$

His theorem on the sum of cubes (4) profoundly connects these categories: the sum of the first n cubes equals the square of the n-th triangular number. This bidimensional relationship (sum of volumes $\rightarrow$ square area) is unique in figurate arithmetic and anticipates structures of modern combinatorial analysis.

Historical Note 4 (Precise delimitation of Nicomachus' contribution)**.** *It is historiographically imperative to clarify that Nicomachus* ***did not formulate*** *identity (5) in terms of symmetric differences $(n+1)^2 - (n-1)^2$. His contribution was exclusively limited to the cumulative sum formula (4). Expression (5) is an algebraic consequence derivable through the finite difference operator ∇, a concept developed in the 17th–19th centuries with the advent of discrete calculus (Boole, 1860). Attributing (5) directly to Nicomachus would constitute an inadmissible anachronism falsifying the history of mathematics.*

7. Historical Context III: Faulhaber's Theorem and Power Sums

Johann Faulhaber (1580–1635), German mathematician, discovered polynomial formulas for power sums (Faulhaber, 1631):

$$S_p(n) = \sum_{k=1}^{n} k^p. \tag{18}$$

For $p = 1, 2, 3$:

$$S_1(n) = \frac{n(n+1)}{2}, \tag{19}$$

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}, \tag{20}$$

$$S_3(n) = \left[\frac{n(n+1)}{2}\right]^2. \tag{21}$$

The case $p = 3$ is exceptional: $S_3(n)$ is a perfect square polynomial. This unique property enables the connection with finite differences that we have explored. For $p \geq 4$, $S_p(n)$ is a polynomial of degree $p+1$ without simple quadratic factorization, preventing an analogous compact representation for n^p (Conway & Guy, 1996).

The exceptionality of $p = 3$ in Faulhaber's theorem is not coincidental: it is intimately linked to the combinatorial uniqueness of $k = 2$ in symmetric differences. Only for $p = 3$ does the cumulative sum $S_p(n)$ possess a quadratic structure that collapses elegantly under the ∇ operator.

8. Historical Context IV: Fermat, Euler, and the Path to FLT

Historical Note 5 (Pierre de Fermat (1607–1665))**.** *Fermat, French jurist and amateur mathematician, made fundamental contributions to number theory, probability, and analytic geometry. In 1637, in the margin of his copy of Diophantus' Arithmetica (Bachet edition, 1621), he wrote:*

> *"Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet."*

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("It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general any power higher than the second into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.") (Singh, 1997).

Most historians agree that Fermat likely possessed an incomplete or erroneous proof, possibly based on the method of infinite descent that he correctly applied to particular cases such as $x^4 + y^4 = z^2$. The conjecture remained unproven for more than three centuries, becoming the most famous mathematical problem in history.

Historical Note 6 (Leonhard Euler (1707–1783) and the $n = 3$ case)**.** *Euler provided the first partial proof of FLT for $n = 3$ around 1770, using infinite descent in the ring $\mathbb{Z}[\omega]$ of Eisenstein integers, where $\omega = e^{2\pi i/3}$ is a primitive cubic root of unity. The key factorization:*

$$x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y) = z^3$$

requires unique factorization in $\mathbb{Z}[\omega]$. Euler tacitly assumed this property, which was not rigorously established until Gauss and Kummer's work in the 19th century. Despite this historical gap, Euler's strategy laid the foundations for the modern algebraic approach to FLT (Edwards, 1979).

The $n = 3$ case proved by Euler directly implies that equation (3) has no solutions in positive integers. This impossibility is not a conjecture but a rigorously established mathematical theorem.

9. Historical Context V: Sophie Germain and Gender Barriers

Historical Note 7 (Sophie Germain (1776–1831))**.** *Germain, French self-taught mathematician, faced insurmountable barriers to participating in the formal mathematical community due to her gender. She adopted the male pseudonym "M. Le Blanc" to correspond with Lagrange and Gauss. After revealing her true identity, Gauss wrote in 1807:*

> *"When a person of the sex which, according to our customs and prejudices, encounters infinitely more obstacles than men to familiarize herself with these thorny investigations, succeeds nevertheless in surmounting these obstacles and penetrating the most obscure parts of them, then without doubt she must have the noblest courage, quite extraordinary talents and superior genius."*

(Laubenbacher & Pengelley, 1999).

Germain developed the first systematic framework for attacking FLT through her theorem (c. 1825), establishing conditions under which any hypothetical solution $x^p + y^p = z^p$ must have at least one of x, y, z divisible by p^2. She manually verified the existence of auxiliary primes for all primes $p < 100$, proving FLT for these cases under additional conditions. Her strategy —using modular congruences to restrict possible solutions— anticipated fundamental techniques of modern algebraic number theory.

Full recognition of her contribution arrived only in the 20th century, when historians like Laubenbacher and Pengelley (1999) rescued her original manuscripts from oblivion. Germain represents not only a mathematical milestone but also a testament to the systematic exclusion of women from historical science.

10. Historical Context VI: Ernst Kummer and Ideal Numbers

Ernst Kummer (1810–1893) discovered that many rings of algebraic integers lack unique factorization. For example, in $\mathbb{Z}[\sqrt{-5}]$:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}),$$

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two distinct factorizations into irreducibles (Edwards, 1979). To overcome this limitation, Kummer introduced the revolutionary concept of *ideal numbers* (precursors of modern ideals), allowing restoration of a form of unique factorization at the ideal level. He proved FLT for all regular primes (those not dividing the numerator of any Bernoulli number B_k with $k < p - 1$). This conceptual advance —algebraic abstraction as a tool for solving concrete arithmetic problems— permanently transformed number theory.

11. Historical Context VII: Wiles and the Modern Culmination

Historical Note 8 (Andrew Wiles (1953–))**.** *Wiles did not directly attack FLT. He proved a special case of the Taniyama-Shimura conjecture (modularity theorem), establishing a profound correspondence between elliptic curves and modular forms. Gerhard Frey observed in 1984 that a hypothetical solution $a^p + b^p = c^p$ would generate the "non-modular" elliptic curve (Frey curve):*

$$y^2 = x(x - a^p)(x + b^p).$$

Ken Ribet proved in 1986 that such a curve would violate Taniyama-Shimura. Therefore, Wiles' proof (1994, with error correction in collaboration with Richard Taylor) automatically implied FLT (Wiles, 1995; Singh, 1997).

Wiles' proof, over 100 pages long, unified seemingly disjoint mathematical fields: algebraic geometry, representation theory, complex analysis, and number theory. It did not solve an isolated problem but revealed profound underlying structures of modern mathematics. FLT ceased to be an arithmetic curiosity and became a corollary of a much broader structural theory.

12. Conceptual Discussion: Internal Structure versus Additive Structure

12.1. Definition of the two structural levels

Definition 4 (Internal structure)**.** *Local property of an individual power n^k expressed through algebraic operations or finite differences. Example: the deduced identity (11) describes the internal structure of n^3 through adjacent symmetry $(n-1, n, n+1)$.*

Definition 5 (Additive structure)**.** *Relationship between distinct powers through arithmetic operations such as addition. Example: $a^3 + b^3$ represents an additive structure combining two individual cubes where, generally, $|a - b| \geq 1$.*

12.2. Proof of adjacent symmetry breakdown

Theorem 6 (Symmetry breakdown when transitioning from internal to additive structure)**.** *The adjacent symmetry $(n-1, n, n+1)$ supporting the deduced identity (11) collapses when considering $a^3 + b^3$ with $|a - b| \geq 1$.*

Proof. For a^3: symmetry around a via $(a-1, a, a+1)$.

For b^3: symmetry around b via $(b-1, b, b+1)$.

If $|a - b| \geq 1$, no integer c exists such that simultaneously:

$$\{a-1, a, a+1\} \cup \{b-1, b, b+1\} \subseteq \{c-1, c, c+1\}.$$

Therefore, the adjacent symmetry required by (11) cannot extend to the sum $a^3 + b^3$. This structural breakdown is the origin of the arithmetic obstruction preventing $h \in \mathbb{Z}$. □

Remark 7 (Fundamental pedagogical consequence)**.** *This breakdown explains* ***why*** *Pythagorean harmony $(n = 2)$ does not extend to $n = 3$: the combinatorial uniqueness of exponent $k = 2$ in symmetric differences (pure monomial only for $k = 2$) is not preserved when transitioning to additive structures. Representation (12) visually illuminates this conceptual transition without falsifying its structural limits.*

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13. Original Pedagogical Value of the Proposal

The genuine contribution of this proposal resides in its capacity to honestly transform Fermatian impossibility into a pedagogical opportunity to comprehend the structural limits of mathematics. Specifically:

1. **Rigorous historical-conceptual bridge**: Connects ancient figurate arithmetic (Nicomachus, 1st century CE) with modern discrete calculus (Boole, 19th century) and contemporary number theory (Wiles, 20th century), illustrating the historical continuity of mathematical thought **without falsifying historical attributions**.
2. **Fundamental structural distinction**: Rigorously teaches the difference between internal structure (local properties of individual objects) and additive structure (relationships between distinct objects), a crucial distinction across all branches of mathematics (algebra, analysis, topology).
3. **Visualization of the obstruction**: Representation (12) makes tangible the breakdown of adjacent symmetry when transitioning from individual n^3 to $a^3 + b^3$, allowing students to *conceptually see* why FLT prohibits integer solutions for $n = 3$.
4. **Epistemological honesty**: Explicitly acknowledges the limits of the deduced identity without falsifying them, modeling rigorous scientific attitude toward the temptations of superficial analogy.
5. **Connection with continuous calculus**: Illustrates the fundamental theorem of calculus in its discrete version, preparing conceptual ground for advanced mathematical analysis without requiring limit or continuity notions.

This pedagogical approach does not "solve" FLT (impossible), but transforms its impossibility into a profound lesson about the nature of mathematical structures: true structures possess precise limits defining their domain of validity, and recognizing these limits constitutes the supreme act of mathematical comprehension.

14. Conclusions

1. We have rigorously established that identity $n^3 = \frac{n^2}{4}[(n+1)^2 - (n-1)^2]$ is a **modern deduction** obtained by applying the finite difference operator ∇ to Nicomachus' historical formula $S(n) = [n(n+1)/2]^2$, **not** an explicit statement by Nicomachus in his original work.
2. We have constructed the exact symbolic representation (12) for $h = \sqrt[3]{a^3 + b^3}$, built exclusively through integer operations though the final result is irrational.
3. We have demonstrated the fundamental conceptual distinction between *internal structure* (local property of individual cubes through adjacent symmetry) and *additive structure* (relationship between distinct cubes where such symmetry collapses).
4. We have proven the combinatorial uniqueness of exponent $k = 2$ in symmetric differences $(n+1)^k - (n-1)^k$, explaining why the compact representation is exclusive to cubes.
5. We have presented exhaustive numerical verifications for 2,500 pairs (a, b) with $1 \leq a, b \leq 50$, empirically confirming the absence of integer solutions for h —illustrating the $n = 3$ case of FLT theoretically proved by Euler.
6. We have historically contextualized the problem from the Pythagorean school (6th century BCE) to Wiles (1994), highlighting contributions from Nicomachus, Faulhaber, Euler, Sophie Germain, Kummer and others with historiographical rigor and precise attribution.
7. We have explicitly emphasized that Nicomachus **never formulated** identity (5) in terms of symmetric differences; his contribution was limited to the cumulative sum (4).
8. We have proposed genuine pedagogical value: honestly transforming Fermatian impossibility into an opportunity to comprehend the structural limits inherent to mathematics, without falsifying such limits or historical attributions.

This work does not aim to prove FLT (established by Wiles, 1994) nor to offer a prohibited "arithmetic solution." Its contribution is conceptual and pedagogical: transforming the identity deduced

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from Nicomachus' cumulative sum, correctly interpreted via discrete calculus, into a bridge for comprehending the structural frontier between quadratic solubility (Pythagoras) and cubic impossibility (Fermat).

The temptation to force the deduced identity toward a "cubic Pythagorean theorem" fails not due to human limitation, but due to an inherent arithmetic obstruction within the structure of integers. This failure is not a weakness but a manifestation of mathematics' internal coherence. Recognizing these limits —as Germain, Euler, and Wiles did in their respective contexts— constitutes the deepest act of mathematical comprehension: knowing where each domain of validity ends, and finding in those very limits the source of new structures yet to be discovered.

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