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Posted Date: 5 February 2026

doi: 10.20944/preprints202602.0367.v1

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From Cumulative Sum to Finite Difference: Nicomachus' Cubic Identity as a Manifestation of Discrete Calculus

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Abstract

This work establishes a rigorous structural connection between Nicomachus' classical formula for the cumulative sum of cubes,

$$S(n) = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4},$$

and the algebraic identity

$$n^3 = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right],$$

through the first-order finite difference operator $\nabla S(n) = S(n) - S(n-1)$. We demonstrate that this identity constitutes the discrete manifestation of the fundamental theorem of calculus applied to the quartic sequence $S(n) \sim n^4/4$, revealing that cubes emerge as "discrete derivatives" of a quartic polynomial function. We establish the combinatorial uniqueness of the case $k = 2$ in the symmetric difference $(n+1)^k - (n-1)^k$, a phenomenon that explains the elegance of the compact representation for cubes. We present exhaustive numerical verifications for $n = 1$ through $n = 25$, analysis of the expression $h = \sqrt[3]{a^3 + b^3}$ for pairs (a, b) with $1 \leq a, b \leq 50$, and historical connections with Pythagorean figurate arithmetic, Boole's umbral calculus, and Faulhaber's theorem. The work highlights the pedagogical value of this perspective for understanding the conceptual transition between classical arithmetic and modern discrete analysis, illustrating the fundamental distinction between *internal* properties (structure of individual powers) and *additive* properties (relations between distinct powers), without misrepresenting the theoretical scope of the presented identity.

Keywords: Nicomachus' theorem; cumulative sum of cubes; finite difference; umbral calculus; internal structure of powers; ∇ operator; discrete derivative; figurate arithmetic

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1. Introduction

1.1. The Legacy of Nicomachus and Figurate Arithmetic

The Pythagorean school (6th century BCE) established a profound connection between geometry and arithmetic through *figurate numbers*: geometric representations of integers via regular point configurations. Triangular numbers $T_n = \frac{n(n+1)}{2}$, square numbers $Q_n = n^2$, pentagonal numbers $P_n = \frac{n(3n-1)}{2}$, and hexagonal numbers $H_n = n(2n-1)$ emerged as pillars of this mathematical conception where the numerical and spatial merged into an ontological unity.

Nicomachus of Gerasa (c. 60–120 CE), Neopythagorean philosopher and mathematician, systematized these ideas in his work *Introduction to Arithmetic* (*Arithmetike eisagoge*), a fundamental text that preserved and expanded classical Greek arithmetic knowledge. Among his most enduring contributions are two theorems about cubic powers that, despite their antiquity, are rarely presented in their mutual structural relationship:

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1. **Theorem of consecutive odd numbers:** Every cube n^3 equals the sum of n consecutive odd numbers centered around n^2:

$$n^3 = (n^2 - n + 1) + (n^2 - n + 3) + \cdots + (n^2 + n - 1).$$

For example:

$$3^3 = 7 + 9 + 11 = 27,$$
$$4^3 = 13 + 15 + 17 + 19 = 64.$$

2. **Theorem of the cumulative sum of cubes:** The sum of the first n cubes equals the square of the triangular sum of the first n integers:

$$S(n) = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}.$$

For example:

$$S(3) = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36 = 6^2 = \left[\frac{3 \cdot 4}{2}\right]^2,$$
$$S(4) = 1 + 8 + 27 + 64 = 100 = 10^2 = \left[\frac{4 \cdot 5}{2}\right]^2.$$

For nearly two millennia, these two results coexisted as isolated arithmetic curiosities. The profound connection between them—that the individual cube n^3 emerges as the first-order finite difference of the cumulative sum $S(n)$—remained concealed until the development of discrete calculus in the 17th–19th centuries. This perspective transforms the algebraic identity

$$n^3 = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right]$$

from a mere coincidence into a structural manifestation of discrete analysis.

1.2. Discrete Calculus: From Fermat to Boole

The calculus of finite differences has its roots in Pierre de Fermat's work (1636) on power sums and was systematically formalized by Brook Taylor (1715) in his *Methodus Incrementorum Directa et Inversa*. George Boole (1860) consolidated the theoretical framework in his treatise *A Treatise on the Calculus of Finite Differences*, establishing umbral calculus as a fundamental tool for interpolation, numerical series, and difference equations.

The first-order finite difference operator ∇ acts on a sequence $f : \mathbb{N} \to \mathbb{R}$ as:

$$\nabla f(n) := f(n) - f(n-1).$$

Both versions satisfy properties analogous to the continuous derivative: linearity, discrete product rule, and the fundamental theorem of discrete calculus.

The fundamental theorem establishes that the cumulative sum (operator Σ) and the finite difference (∇) are inverse operators:

$$\nabla\left(\sum_{k=1}^{n} f(k)\right) = f(n).$$

This duality is the structural bridge connecting Nicomachus' two theorems.

 doi:10.20944/preprints202602.0367.v1

1.3. Problem Statement and Original Contribution

This work addresses the following structural question: How does the internal decomposition of cubes according to Nicomachus relate to the theoretical framework of modern discrete calculus, and what unique combinatorial properties emerge from this relationship?

Our original contribution consists of:

1. Rigorously establishing that the cubic identity $n^3 = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right]$ is the explicit expression of the retrospective finite difference operator $\nabla S(n) = S(n) - S(n-1)$ applied to Nicomachus' cumulative sum formula.
2. Demonstrating that the denominator 4 in the cubic identity is a structural constant derived directly from $\left[\frac{n(n+1)}{2}\right]^2$, not an arbitrary factor.
3. Revealing the combinatorial uniqueness of exponent $k = 2$ in the symmetric difference $(n+1)^k - (n-1)^k$, explaining why the compact representation works exclusively for cubes and not for higher powers.
4. Presenting exhaustive numerical verifications for $n = 1$ through $n = 25$, including step-by-step breakdowns of the algebraic identity.
5. Numerically analyzing the expression $h = \sqrt[3]{a^3 + b^3}$ for pairs (a, b) with $1 \leq a, b \leq 50$, observing empirical patterns without attributing explanatory significance beyond verified results.
6. Connecting the identity with Faulhaber's theorem on power sums and with polygonal number theory.
7. Proposing genuine pedagogical applications: using this identity to illustrate the conceptual transition between classical figurate arithmetic and modern discrete analysis, highlighting the distinction between internal structures (properties of individual powers) and additive structures (relations between distinct powers).

2. Theoretical Framework

2.1. The Theorem of the Cumulative Sum of Cubes

Theorem 2.1 (Nicomachus' cumulative sum of cubes)**.** For every positive integer $n \in \mathbb{N}$:

$$S(n) = \sum_{k=1}^{n} k^3 = \left[\sum_{k=1}^{n} k\right]^2 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}.$$

Proof. Proof by mathematical induction.

Base case ($n = 1$):

$$S(1) = 1^3 = 1 = \left[\frac{1 \cdot 2}{2}\right]^2 = 1^2 = 1.$$

Inductive hypothesis: Assume valid for n:

$$S(n) = \frac{n^2(n+1)^2}{4}.$$

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Inductive step ($n \to n+1$):

$$\begin{aligned} S(n+1) &= S(n) + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2\left[\frac{n^2}{4} + (n+1)\right] \\ &= (n+1)^2\left[\frac{n^2+4n+4}{4}\right] \\ &= (n+1)^2\left[\frac{(n+2)^2}{4}\right] \\ &= \frac{(n+1)^2(n+2)^2}{4}. \end{aligned}$$

Therefore, the formula holds for all $n \in \mathbb{N}$. □

Observation 2.1. This identity possesses an elegant geometric interpretation: the sum of the first n cubes forms a perfect square whose side equals the n-th triangular number $T_n = \frac{n(n+1)}{2}$. This connection between dimensions (sum of volumes → square area) is unique in figurate arithmetic.

2.2. The Finite Difference Operator and the Fundamental Theorem

Definition 2.1 (Retrospective finite difference)**.** For a sequence $f : \mathbb{N} \to \mathbb{R}$, the first-order retrospective finite difference is:

$$\nabla f(n) := f(n) - f(n-1), \quad n \geq 2.$$

Definition 2.2 (Cumulative sum)**.** The cumulative sum (or partial sum) of a sequence f up to n is:

$$\Sigma f(n) := \sum_{k=1}^{n} f(k), \quad n \geq 1.$$

Theorem 2.2 (Fundamental theorem of discrete calculus)**.** Let $S(n) = \Sigma f(n) = \sum_{k=1}^{n} f(k)$ be the cumulative sum of a sequence f. Then:

$$\nabla S(n) = S(n) - S(n-1) = f(n), \quad \forall n \geq 2.$$

Proof. By definition of cumulative sum:

$$\begin{aligned} S(n) &= f(1) + f(2) + \cdots + f(n-1) + f(n), \\ S(n-1) &= f(1) + f(2) + \cdots + f(n-1). \end{aligned}$$

Subtracting:

$$S(n) - S(n-1) = f(n).$$

□

This theorem is the discrete analog of the fundamental theorem of continuous calculus $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. The finite difference acts as a "discrete derivative" and the cumulative sum as a "discrete integral".

2.3. Cubes as Discrete Derivatives of the Cumulative Sum

Applying the fundamental theorem to the cumulative sum of cubes $S(n) = \frac{n^2(n+1)^2}{4}$:

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Theorem 2.3 (Cubes as discrete derivatives)**.** The cube n^3 is the first-order retrospective finite difference of the cumulative sum $S(n)$:

$$n^3 = \nabla S(n) = S(n) - S(n-1) = \frac{n^2(n+1)^2}{4} - \frac{(n-1)^2 n^2}{4}.$$

Proof. Algebraic development:

$$\begin{aligned} S(n) - S(n-1) &= \frac{n^2(n+1)^2}{4} - \frac{(n-1)^2 n^2}{4} \\ &= \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right] \\ &= \frac{n^2}{4}\left[(n^2+2n+1) - (n^2-2n+1)\right] \\ &= \frac{n^2}{4}(4n) \\ &= n^3. \end{aligned}$$

□

Observation 2.2. The key step $(n+1)^2 - (n-1)^2 = 4n$ is an elementary algebraic identity derived from Newton's binomial theorem. However, its appearance in this context reveals a profound structural property: the symmetric difference of squares produces a pure linear monomial, a phenomenon unique to exponent $k = 2$ as demonstrated in the following section.

3. Exhaustive Numerical Verifications

We present below a complete table of numerical verifications for $n = 1$ through $n = 25$. Each row shows:

- n: integer value
- n^3: direct cube
- $S(n)$: cumulative sum up to n
- $S(n-1)$: cumulative sum up to $n-1$
- $\frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right]$: evaluation of the algebraic identity
- Verification: equality between n^3 and the identity

Table 1. Exhaustive numerical verifications of the cubic identity for $n = 1$ through $n = 25$. All equalities hold exactly.

n	n^3	$S(n)$	$S(n-1)$	$\frac{n^2}{4}\left[(n+1)^2-(n-1)^2\right]$	Verification
1	1	1	0	$\frac{1}{4}[4-0]=1$	✓
2	8	9	1	$\frac{4}{4}[9-1]=8$	✓
3	27	36	9	$\frac{9}{4}[16-4]=27$	✓
4	64	100	36	$\frac{16}{4}[25-9]=64$	✓
5	125	225	100	$\frac{25}{4}[36-16]=125$	✓
6	216	441	225	$\frac{36}{4}[49-25]=216$	✓
7	343	784	441	$\frac{49}{4}[64-36]=343$	✓
8	512	1296	784	$\frac{64}{4}[81-49]=512$	✓
9	729	2025	1296	$\frac{81}{4}[100-64]=729$	✓
10	1000	3025	2025	$\frac{100}{4}[121-81]=1000$	✓
11	1331	4356	3025	$\frac{121}{4}[144-100]=1331$	✓
12	1728	6084	4356	$\frac{144}{4}[169-121]=1728$	✓
13	2197	8281	6084	$\frac{169}{4}[196-144]=2197$	✓
14	2744	11025	8281	$\frac{196}{4}[225-169]=2744$	✓
15	3375	14400	11025	$\frac{225}{4}[256-196]=3375$	✓
16	4096	18496	14400	$\frac{256}{4}[289-225]=4096$	✓
17	4913	23409	18496	$\frac{289}{4}[324-256]=4913$	✓
18	5832	29241	23409	$\frac{324}{4}[361-289]=5832$	✓
19	6859	36100	29241	$\frac{361}{4}[400-324]=6859$	✓
20	8000	44100	36100	$\frac{400}{4}[441-361]=8000$	✓
21	9261	53361	44100	$\frac{441}{4}[484-400]=9261$	✓
22	10648	64009	53361	$\frac{484}{4}[529-441]=10648$	✓
23	12167	76176	64009	$\frac{529}{4}[576-484]=12167$	✓
24	13824	90000	76176	$\frac{576}{4}[625-529]=13824$	✓
25	15625	105625	90000	$\frac{625}{4}[676-576]=15625$	✓

Example 3.1 (Step-by-step breakdown for $n = 7$)**.**

$$\begin{aligned} n &= 7, \\ n^3 &= 343, \\ S(7) &= \frac{7^2 \cdot 8^2}{4} = \frac{49 \cdot 64}{4} = 784, \\ S(6) &= \frac{6^2 \cdot 7^2}{4} = \frac{36 \cdot 49}{4} = 441, \\ S(7) - S(6) &= 784 - 441 = 343, \\ \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right] &= \frac{49}{4}\left[8^2 - 6^2\right] = \frac{49}{4}[64 - 36] = \frac{49}{4} \cdot 28 = 49 \cdot 7 = 343. \end{aligned}$$

4. Numerical Analysis of the Expression $h = \sqrt[3]{a^3 + b^3}$

Applying the cubic identity to two integers $a, b \in \mathbb{Z}^+$, we can express:

$$a^3 = \frac{a^2}{4}\left[(a+1)^2 - (a-1)^2\right], \quad b^3 = \frac{b^2}{4}\left[(b+1)^2 - (b-1)^2\right].$$

The sum of cubes is then written as:

$$a^3 + b^3 = \frac{a^2}{4}\left[(a+1)^2 - (a-1)^2\right] + \frac{b^2}{4}\left[(b+1)^2 - (b-1)^2\right].$$

We define the expression:

$$h = \sqrt[3]{a^3 + b^3}.$$

Below we present a table with computed values for $1 \le a, b \le 15$:

 doi:10.20944/preprints202602.0367.v1

Table 2. Values of $h = \sqrt[3]{a^3 + b^3}$ for $1 \leq a, b \leq 15$

(a, b)	$a^3 + b^3$	h	$h \in \mathbb{Z}$?
(1,1)	2	1.259921049894873	No
(1,2)	9	2.080083823051904	No
(2,2)	16	2.519842099789746	No
(2,3)	35	3.271066310188589	No
(3,3)	54	3.779763149684620	No
(3,4)	91	4.497941445275415	No
(4,4)	128	5.039684199579493	No
(4,5)	189	5.738793548317167	No
(5,5)	250	6.299605249474366	No
(5,6)	341	6.986398597794855	No
(6,8)	728	8.995858485738126	No
(7,14)	3087	14.561585428076244	No
(9,10)	1729	12.002314362764570	No
(12,16)	5824	17.992222257843930	No
(15,20)	11375	22.496531154456260	No
Total pairs evaluated	225	—	—
Pairs with $h \in \mathbb{Z}$	0	—	—

Extending the analysis to $1 \leq a, b \leq 50$ (2,500 pairs), no pair produces $h \in \mathbb{Z}$. This numerical pattern is consistent and reproducible, constituting a solid empirical observation about the behavior of the expression $h = \sqrt[3]{a^3 + b^3}$ in the domain of positive integers.

Observation 4.1. The cubic identity allows individual calculation of a^3 and b^3 through finite differences, but the sum $a^3 + b^3$ does not inherit an analogous structure guaranteeing that its cubic root is an integer. This numerical observation reflects a structural difference between *internal* properties of individual powers (captured by the identity) and *additive* properties that emerge when combining distinct powers.

5. Combinatorial Uniqueness of the Case $k = 2$

5.1. Generalized Symmetric Difference

We define the symmetric difference of order k as:

Definition 5.1 (Symmetric difference of order k)**.** For $k \in \mathbb{N}$ and $n \in \mathbb{N}$:

$$D_k(n) := (n+1)^k - (n-1)^k.$$

Applying Newton's binomial theorem:

Proposition 5.1 (Binomial expansion of the symmetric difference)**.**

$$D_k(n) = \sum_{j=0}^{k} \binom{k}{j} n^{k-j} [1^j - (-1)^j] = 2 \sum_{\substack{j=1 \\ j \text{ odd}}}^{k} \binom{k}{j} n^{k-j}.$$

Proof. By Newton's binomial theorem:

$$(n+1)^k = \sum_{j=0}^{k} \binom{k}{j} n^{k-j} 1^j = \sum_{j=0}^{k} \binom{k}{j} n^{k-j},$$

$$(n-1)^k = \sum_{j=0}^{k} \binom{k}{j} n^{k-j} (-1)^j.$$

Subtracting:

$$(n+1)^k - (n-1)^k = \sum_{j=0}^{k} \binom{k}{j} n^{k-j} \left[1 - (-1)^j\right].$$

The factor $\left[1 - (-1)^j\right]$ is:

$$\left[1 - (-1)^j\right] = \begin{cases} 0 & \text{if } j \text{ is even,} \\ 2 & \text{if } j \text{ is odd.} \end{cases}$$

Therefore:

$$D_k(n) = 2 \sum_{\substack{j=1 \\ j \text{ odd}}}^{k} \binom{k}{j} n^{k-j}.$$

□

5.2. Classification by Number of Terms

We analyze the number of non-zero terms in $D_k(n)$ for $k = 1$ through $k = 6$:

Table 3. Classification of $D_k(n)$ by number of terms. Only for $k = 2$ do we obtain a pure linear monomial.

k	$D_k(n) = (n+1)^k - (n-1)^k$	Non-zero terms	Type
1	2	1 (constant)	Constant monomial
2	$4n$	1 (linear)	**Pure monomial**
3	$6n^2 + 2$	2	Binomial
4	$8n^3 + 8n$	2	Binomial
5	$10n^4 + 20n^2 + 2$	3	Trinomial
6	$12n^5 + 40n^3 + 12n$	3	Trinomial

Theorem 5.1 (Uniqueness of the pure monomial). $D_k(n)$ is a pure monomial (a single non-constant term) if and only if $k = 2$.

Proof. According to the previous proposition:

$$D_k(n) = 2 \sum_{\substack{j=1 \\ j \text{ odd}}}^{k} \binom{k}{j} n^{k-j}.$$

The number of non-zero terms equals the number of odd integers j in the interval $[1, k]$.

- If $k = 1$: $j = 1$ (one term) $\Rightarrow D_1(n) = 2$ (constant).
- If $k = 2$: $j = 1$ (one term) $\Rightarrow D_2(n) = 2\binom{2}{1}n^1 = 4n$ (pure linear monomial).
- If $k = 3$: $j = 1, 3$ (two terms) $\Rightarrow D_3(n) = 2\left[\binom{3}{1}n^2 + \binom{3}{3}n^0\right] = 6n^2 + 2$.
- If $k \geq 4$: the number of odd integers in $[1, k]$ is $\lceil k/2 \rceil \geq 2$, therefore $D_k(n)$ has at least two non-zero terms.

Therefore, only for $k = 2$ do we obtain a non-constant pure monomial. □

Observation 5.1. This combinatorial uniqueness explains why the compact representation

$$n^3 = \frac{n^2}{4} D_2(n) = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right]$$

is possible for cubes but not for higher powers. For n^4, for example:

$$n^4 \neq \frac{n^3}{c}\left[(n+1)^3 - (n-1)^3\right] \quad \text{for any constant } c,$$

since $D_3(n) = 6n^2 + 2$ contains two terms and cannot be factored as $c \cdot n$.

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6. Historical and Theoretical Connections

6.1. Faulhaber's Theorem and Power Sums

Johann Faulhaber (1580–1635) discovered polynomial formulas for power sums:

$$S_p(n) = \sum_{k=1}^{n} k^p.$$

For $p = 1, 2, 3$:

$$S_1(n) = \frac{n(n+1)}{2},$$
$$S_2(n) = \frac{n(n+1)(2n+1)}{6},$$
$$S_3(n) = \left[\frac{n(n+1)}{2}\right]^2.$$

The case $p = 3$ is exceptional: $S_3(n)$ is a perfect square polynomial. This unique property enables the connection with finite differences that we have explored. For $p \geq 4$, $S_p(n)$ is a polynomial of degree $p+1$ without simple quadratic factorization, which prevents an analogous compact representation for n^p.

6.2. Polygonal Numbers and Figurate Arithmetic

Polygonal numbers of order m are defined as:

$$P_n^{(m)} = \frac{n\left[(m-2)n - (m-4)\right]}{2}.$$

For $m = 3$ (triangular), $m = 4$ (square), $m = 5$ (pentagonal):

$$T_n = \frac{n(n+1)}{2},$$
$$Q_n = n^2,$$
$$P_n = \frac{n(3n-1)}{2}.$$

Nicomachus' identity connects cubes with triangular numbers:

$$n^3 = T_n^2 - T_{n-1}^2 = \left(T_n - T_{n-1}\right)\left(T_n + T_{n-1}\right) = n \cdot n^2 = n^3,$$

since $T_n - T_{n-1} = n$ and $T_n + T_{n-1} = n^2$. This is another manifestation of the internal structure of cubes in terms of classical figurate arithmetic.

7. Conceptual Discussion: Internal Structures vs. Additive Structures

A crucial aspect for rigorous mathematical understanding is distinguishing between two levels of structural analysis:

- **Internal structure (local property):** How an individual power n^3 decomposes through algebraic operations or finite differences. Nicomachus' identity describes *exclusively* this internal structure: it reveals that each individual cube can be expressed as a weighted symmetric difference of adjacent squares.
- **Additive structure (relational property):** Relations between distinct powers through arithmetic operations such as addition. The expression $a^3 + b^3$ represents an additive structure that combines two individual cubes.

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Nicomachus' cubic identity is mathematically correct, elegantly formulated, and reveals a genuine local property of individual cubes. However, it **does not imply any restriction** on the behavior of additive expressions like $a^3 + b^3$. Confusing a local property with a restriction on additive structures would constitute a categorical error that must be avoided in rigorous mathematical discourse.

The genuine value of this identity resides in:

1. Connecting ancient figurate arithmetic with modern umbral calculus, illustrating the historical continuity of mathematical thought.
2. Pedagogically illustrating the fundamental theorem of calculus in its discrete version, showing how the "derivative" of a cumulative sum recovers the original term.
3. Serving as a paradigmatic example for teaching the distinction between local analysis (properties of individual objects) and relational analysis (combinations of distinct objects)—a fundamental distinction across all branches of mathematics.
4. Revealing a genuine combinatorial uniqueness (the case $k = 2$ in symmetric differences) that explains why certain algebraic representations are possible for cubes but not for higher powers.

Numerical observations about $h = \sqrt[3]{a^3 + b^3}$ constitute interesting empirical data worthy of study, but must be interpreted within their appropriate context: as observable patterns without *a priori* attribution of explanatory significance beyond experimentally verified results.

8. Pedagogical Applications

8.1. Teaching Discrete Calculus in Secondary Education

Nicomachus' identity offers an accessible entry point to discrete calculus without requiring limits or continuous derivatives. Students can:

- Numerically verify the identity for small values of n (as in Table 1).
- Experimentally discover that $(n+1)^2 - (n-1)^2 = 4n$ through direct calculation.
- Generalize to higher-order symmetric differences and observe the loss of simplicity (Table 3).
- Connect with the formula for the sum of the first n integers $T_n = \frac{n(n+1)}{2}$.

This approach builds intuition for advanced concepts (finite differences, interpolation, series) starting from elementary arithmetic.

8.2. Bridge Between Arithmetic and Algebra

The transition from:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

to:

$$n^3 = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right]$$

illustrates how algebra enables "decomposing" cumulative relations into instantaneous structures. This is a first conceptual step toward differential calculus, where the derivative decomposes a cumulative integral.

9. Conclusions

1. We have rigorously demonstrated that the cubic identity $n^3 = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right]$ is the explicit manifestation of the retrospective finite difference operator $\nabla S(n) = S(n) - S(n-1)$ applied to the cumulative sum of cubes $S(n) = \frac{n^2(n+1)^2}{4}$. This is the discrete expression of the fundamental theorem of calculus applied to a quartic function.
2. We have established that the denominator 4 in the cubic identity is a structural constant derived directly from $\left[\frac{n(n+1)}{2}\right]^2$, not an arbitrarily introduced factor.

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3. We have demonstrated that the symmetric difference $(n+1)^k - (n-1)^k$ produces a pure monomial if and only if $k = 2$, explaining the unique, non-generalizable elegance of the compact representation for cubes. This combinatorial uniqueness is a genuine mathematical fact.
4. We have presented exhaustive numerical verifications for $n = 1$ through $n = 25$ (Table 1), confirming the algebraic validity of the identity across a broad range of values.
5. We have numerically analyzed the expression $h = \sqrt[3]{a^3 + b^3}$ for 2,500 pairs (a, b) with $1 \leq a, b \leq 50$, observing that $h \notin \mathbb{Z}$ in all evaluated cases. This pattern constitutes a solid empirical observation about the behavior of this expression in the domain of positive integers.
6. We have connected the identity with Faulhaber's theorem on power sums and with classical polygonal number theory, situating it within a broad historical and theoretical framework.
7. We have proposed genuine pedagogical applications: using this identity to understand:
   - Discrete calculus and its conceptual relationship with continuous calculus.
   - The fundamental distinction between internal structures (properties of individual powers) and additive structures (relations between distinct powers).
   - How discrete algebraic symmetries anticipate differential structures without determining deep arithmetic properties.
8. Nicomachus' cubic identity, correctly interpreted as the finite difference of the cumulative sum, constitutes a valuable bridge between classical figurate arithmetic and contemporary discrete analysis, **without misrepresenting its theoretical scope or suggesting unverified implications**.

This work does not claim to contribute new theorems to advanced number theory. Its contribution lies in **conceptual clarity**, **historical synthesis**, and **pedagogical value**: transforming an apparently curious algebraic identity into a window toward profound mathematical structures, while always respecting the rigorous limits of what this identity can and cannot explain.

Acknowledgments: The author thanks the Laboratory of the Department of Industrial Design (LIDDI) of the National University of La Plata for institutional support and space for developing interdisciplinary research between mathematics, design, and education.

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