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Article

From the Pythagorean Dream to the Fermatian Obstruction: The Unified Chain

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Abstract

For every prime p and every integer a, the backward finite difference $\delta_p(a) := a^p - (a-1)^p$ equals the cyclotomic binary form $\Phi_p(a, a-1)$ and hence the norm $N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(a - \zeta_p(a-1))$. For $p = 3$ this specialises to $\delta_3(a) = N_{\mathbb{Z}[\omega]}(a - \omega(a-1))$, connecting the individual cubic finite difference—obtained by differencing the classical sum formula of Nicomachus of Gerasa (∼100 CE)—with the Eisenstein norm that appears in Euler's factorisation of $a^3 + b^3$. Starting from the historical identity $S_3(n) = T_n^2$ where $T_n = n(n+1)/2$, and applying the backward finite difference operator $\nabla f(n) := f(n) - f(n-1)$—formalised by Taylor (1715) and systematised by Boole (1860)—the Cubic Identity is derived: $n^3 = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right] = T_n^2 - T_{n-1}^2$. This identity is extended to all $p \geq 1$ via the Universal Faulhaber–Bernoulli Identity (UFBI): $n^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j^+\,\delta_{p+1-j}(n), \qquad \delta_m(n) := n^m - (n-1)^m$. The central contribution of this work is the Unified Chain Formula: $\nabla T_n^2 = \delta_3(a) = N_{\mathbb{Z}[\omega]}(a - \omega(a-1)) = \Phi_3(a, a-1) = N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a - \zeta_3(a-1))$, which connects, in a single proved identity, five centuries of mathematics: Nicomachus (1st century), Boole (19th century), Euler/Eisenstein (18th century), and Gauss/cyclotomic theory (19th–20th centuries). This chain is not present as such in the existing literature; its originality lies in the explicit articulation of these connections, not in the individual equalities, each of which follows from classical results. Beyond the Unified Chain, the following new elements are introduced: (i) the Tower of Norms $a^3 = \sum_{k=1}^{a} N(\alpha_k)$, making explicit how each perfect cube is a stack of hexagonal norms; (ii) the Cyclotomic Compatibility Index ICC(n, p), which quantifies the arithmetic obstruction to $h^p = a^p + b^p$ having integer solutions; (iii) the Window Incompatibility Theorem, formalising why the hexagonal windows $\{a-1, a, a+1\}$ and $\{b-1, b, b+1\}$ can never merge into a single window $\{h-1, h, h+1\}$ in $\mathbb{Z}[\omega]$ for $a, b \geq 2$; (iv) the Order Theorem for $\delta_m(n)$, providing a complete characterisation of prime divisibility of finite differences via multiplicative orders; (v) the Extreme Reduction Theorem (ERT), showing that the Order Filter eliminates every pair (a, b) with $a \geq 2$ from the equation $a^3 + b^3 = c^3$, reducing the problem to the case $a = 1$; (vi) the Fermatian Rigidity Index $R(p)$, a quantitative measure of how far $(a^p + b^p)^{1/p}$ is from an integer. All results are illustrated throughout by the single running example $a = 6$, $b = 10$, and the key number $91 = 7 \times 13$. Verified over 179 700 pairs with 50-digit precision: zero exceptions. This work does not claim to prove Fermat's Last Theorem, definitively established by Wiles [1].

Keywords: Eisenstein integers; cyclotomic binary forms; discrete calculus; centred hexagonal numbers; fermat cubic; unified chain; tower of norms; cyclotomic compatibility index; löschian numbers; lifting-the-exponent lemma; universal faulhaber–bernoulli identity; order theorem; extreme reduction theorem; fermatian rigidity index; structural stratification; Bernoulli numbers; arithmetic obstruction; pedagogical mathematics

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1. Introduction

1.1. Historical Context

The Pythagorean school, founded in Croton (Magna Graecia) around the 6th century BC, established one of the first philosophical-mathematical syntheses in Western history: the belief that "everything is number" ($\pi\alpha\nu\tau\alpha\ \alpha\rho\iota\theta\mu o\varsigma$). For the Pythagoreans, numbers were not mere calculation tools but ontological principles structuring cosmic reality. This worldview found its purest expression in figurate numbers: geometric representations of integers by means of regular point configurations.

The Pythagorean theorem crystallised the harmony between geometry and arithmetic:

$$h^2 = a^2 + b^2,$$

where h denotes the hypotenuse of a right triangle and a, b its legs. Euclid provided the complete parameterisation of primitive Pythagorean triples:

$$a = m^2 - n^2, \quad b = 2mn, \quad h = m^2 + n^2,$$

with $m > n > 0$, $\gcd(m, n) = 1$, $m \not\equiv n \pmod 2$.

Nicomachus of Gerasa (∼60–120 AD) systematised in his *Introductio Arithmetica* the classical Greek arithmetic knowledge. He established the cumulative sum of cubes:

$$S_3(n) = \sum_{k=1}^{n} k^3 = T_n^2, \qquad T_n = \frac{n(n+1)}{2}.$$

It is essential to note that Nicomachus did not formulate nor suggest any identity involving symmetric differences $(n+1)^2 - (n-1)^2$; his contribution was limited exclusively to the cumulative sum formula. Attributing the Cubic Identity (derived in Section 3) to him would be an inadmissible historiographic anachronism.

Johann Faulhaber (1580–1635) computed $S_p(n) = \sum_{k=1}^{n} k^p$ for p up to at least 17 in his *Academia Algebrae* (1631) [14], discovering the polynomial nature of these sums and observing that $S_3(n) = T_n^2$ is the only case with a perfect quadratic factorisation.

Jakob Bernoulli (1654–1705) published posthumously in *Ars Conjectandi* (1713) [8] the general formula for $S_p(n)$ in terms of the coefficients that today bear his name. The Bernoulli numbers B_j vanish for all odd $j \geq 3$; this is a fundamental fact for the Universal Identity.

Pierre de Fermat (1607–1665) wrote in 1637, in the margin of his copy of Diophantus's *Arithmetica*: "*Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.*" Most historians consider that his "marvellous proof" was incomplete.

Leonhard Euler (1707–1783) provided the first proof of FLT for $p = 3$ (∼1770) by infinite descent in the ring $\mathbb{Z}[\omega]$ of Eisenstein integers, where $\omega = e^{2\pi i/3}$. The key factorisation $x^3 + y^3 = (x+y)(x+\omega y)(x+\omega^2 y) = z^3$ requires unique factorisation in $\mathbb{Z}[\omega]$, a property Euler assumed tacitly and that was rigorously established by Gauss and Kummer [2].

Sophie Germain (1776–1831) developed the first systematic framework for attacking FLT via modular congruences, verifying conditions for all primes $p < 100$. She adopted the masculine pseudonym "M. Le Blanc" to correspond with Lagrange and Gauss. Gauss wrote in 1807: "When a person of the sex which, according to our customs and prejudices, must encounter infinitely more difficulties than men to familiarise herself with these thorny researches, succeeds nevertheless in surmounting these obstacles and penetrating their most obscure parts, she must without doubt have the noblest courage, quite extraordinary talents, and superior genius."

Ernst Kummer (1810–1893) introduced the revolutionary concept of ideal numbers and proved FLT for all regular primes.

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Kenneth Ribet proved in 1986 that Frey's elliptic curve (associated with a hypothetical counterexample to FLT) would violate the Taniyama–Shimura–Weil conjecture, reducing FLT to a problem in the theory of modular forms.

Andrew Wiles (1953–) proved FLT in full generality in 1994 (published in 1995) [1] via the Modularity Theorem for elliptic curves. Wiles' proof unifies algebraic geometry, representation theory, complex analysis, and number theory.

Table 1 summarises the chronological development.

Table 1. Historical deductive chain: from Pythagoras to the present work.

Author	Period	Key contribution
Pythagoras et al.	6th c. BC	$h^2 = a^2 + b^2$; figurate numbers
Euclid	∼300 BC	Parameterisation of primitive triples
Nicomachus	∼100 AD	$S_3(n) = T_n^2$
Faulhaber	1631	$S_p(n)$ polynomial for p up to 17
Fermat	1637	Conjecture: $a^p + b^p = c^p$ without solutions, $p \geq 3$
J. Bernoulli	1713	Coefficients B_j^+ in the general formula
Taylor	1715	Operator ∇ formalised
Euler	∼1770	FLT proved for $p = 3$ via $\mathbb{Z}[\omega]$
Gauss	1801	Uniqueness in $\mathbb{Z}[\omega]$; algebraic foundations
Germain	∼1825	Systematic modular framework; FLT for $p < 100$
Kummer	1847	Ideal numbers; FLT for regular primes
Boole	1860	Fundamental Theorem of Discrete Calculus
Ribet	1986	Frey curve ⇒ violation of Taniyama–Shimura
Wiles	1994	Modularity theorem ⇒ FLT in general
This work	2026	UFBI; $C(p)$; Order Theorem; ERT; Unified Chain; Tower of Norms; ICC; Window Theorem

1.2. The Two Traditions and the Bridge

The sum formula of Nicomachus,

$$S_3(n) := \sum_{k=1}^{n} k^3 = T_n^2, \qquad T_n = \frac{n(n+1)}{2},$$

is one of the oldest identities in number theory [3]. Applying the backward-difference operator ∇ yields the individual cubic difference $\delta_3(a) = 3a^2 - 3a + 1$, a formula within reach of any secondary-school student.

More than sixteen centuries later, Euler's factorisation of $a^3 + b^3$ over the ring of Eisenstein integers $\mathbb{Z}[\omega]$ employs the norm of the element $a - \omega(a-1)$ [2]. The central observation of this work is that these two objects are the same:

$$\delta_3(a) = N_{\mathbb{Z}[\omega]}(a - \omega(a-1)).$$

This three-line algebraic identity is the bridge between the two traditions, and is the seed of the Unified Chain developed in Section 6.

1.3. The Gap This Work Covers

The classical approaches to FLT operate at the level of algebraic number theory (Kummer's ideal numbers, 1847) or modular forms (Wiles, 1994). What was missing was an elementary, quantitative, and structural map of the Fermatian obstruction, graded by the exponent p and accessible from first-year modular arithmetic. This work provides exactly that map, and adds a new element: the explicit articulation that the central quantity $\delta_3(a)$ connects four distinct mathematical traditions in a single identity.

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1.4. What Is Genuinely Original

The individual equalities comprising the Unified Chain are consequences of classical results. The originality of this work lies in the following contributions not present in the prior literature:

(i) The explicit articulation of the complete path $\nabla T_n^2 \to \delta_3(a) \to N_{\mathbb{Z}[\omega]} \to \Phi_3 \to N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}$ as a single named, proved, numerically verified identity. No prior work in the known literature presents this chain in this form.

(ii) The reorientation of the Faulhaber–Bernoulli formula from cumulative sums toward individual powers via $\delta_m(n) = n^m - (n-1)^m$.

(iii) The Tower of Norms representation (Section 7), making explicit how each perfect cube accumulates hexagonal norms layer by layer.

(iv) The Cyclotomic Compatibility Index $\mathrm{ICC}(n, p)$ (Section 8), a new invariant that quantifies whether a given integer can be a cyclotomic norm, and hence whether $h^p = a^p + b^p$ can have integer solutions.

(v) The Window Incompatibility Theorem (Section 9), formalising why hexagonal windows cannot merge.

(vi) The Extreme Reduction Theorem (Section 11), reducing $a^3 + b^3 = c^3$ to the case $a = 1$ by elementary means.

(vii) The Fermatian Rigidity Index $R(p)$ (Section 13.2), a new quantitative measure.

(viii) The structural map of the Fermatian obstruction from elementary discrete calculus, graded by the exponent p and accessible from first-year modular arithmetic.

1.5. Scope and Non-Claims

Explicit declaration. This work does not prove Fermat's Last Theorem, definitively established by Wiles [1]. It does not supersede Euler's classical argument for $p = 3$ [2]: the three-language equivalence (Section 10.1) uses the unique-factorisation-domain (UFD) property of $\mathbb{Z}[\omega]$, which is precisely Euler's key ingredient. The ERT should be read as a structural reduction of the problem to the case $a = 1$, not as a complete proof of FLT.

1.6. Roadmap

Section 2 establishes the theoretical framework. Section 3 derives the Cubic Identity and the Eisenstein norm identity. Section 4 develops the general cyclotomic framework. Section 5 proves the UFBI and the Structural Stratification Theorem. Section 6 states and proves the Unified Chain Formula. Section 7 introduces the Tower of Norms. Section 8 defines and analyses the Cyclotomic Compatibility Index. Section 9 proves the Window Incompatibility Theorem. Section 10 establishes the Three-Language Equivalence, the Order Theorem, and the LTE and Order Filters. Section 11 proves the Extreme Reduction Theorem. Section 12 closes the base case $a = 1$. Section 13 presents the structural comparison between the Pythagorean and Fermat cases. Section 14 derives 3-adic constraints. Section 15 discusses Löschian numbers and arithmetic density. Section 16 presents the universal symbolic representation. Section 17 presents all computational verifications. Section 18 discusses the results. Section 19 lists open questions. Section 20 concludes.

2. Theoretical Framework

2.1. Discrete Calculus: Fundamental Operators

Definition 1 (Fundamental discrete operators)**.** *Let $f : \mathbb{N} \to \mathbb{R}$. Define:*

$$\nabla f(n) := f(n) - f(n-1), \quad n \geq 2 \quad \textit{(backward difference)}, \tag{1}$$

$$\Delta f(n) := f(n+1) - f(n) \quad \textit{(forward difference)}, \tag{2}$$

$$\Sigma f(n) := \sum_{k=1}^{n} f(k) \quad \textit{(cumulative sum)}. \tag{3}$$

Theorem 1 (Fundamental Theorem of Discrete Calculus, Boole 1860 [7]). *If $S(n) = \sum_{k=1}^{n} f(k)$, then $\nabla S(n) = f(n)$ for all $n \geq 2$.*

Proof. $S(n) - S(n-1) = [f(1) + \cdots + f(n)] - [f(1) + \cdots + f(n-1)] = f(n)$. □

Remark 1. *This result is the discrete analogue of the Fundamental Theorem of Calculus $\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$. The cumulative sum Σ acts as a "discrete integral" and ∇ as a "discrete derivative"; they are mutually inverse.*

The analogy between continuous and discrete calculus is best illustrated by the parallel:

$$\underbrace{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}_{\text{Leibniz–Newton (17th c.)}} \quad \longleftrightarrow \quad \underbrace{\nabla S(n) = f(n)}_{\text{Boole (1860)}}$$

2.2. The Individual Finite Difference $\delta_m(n)$

Definition 2 (Individual finite difference). *For integers $m \geq 1$ and $n \geq 1$:*

$$\delta_m(n) := n^m - (n-1)^m.$$

The first values are: $\delta_1(n) = 1$, $\delta_2(n) = 2n - 1$, $\delta_3(n) = 3n^2 - 3n + 1$, $\delta_4(n) = 4n^3 - 6n^2 + 4n - 1$.

Observation 2. *The quantity $\delta_m(n)$ is not the difference of a cumulative sum: it measures the increment of n^m when passing from $n-1$ to n. Reorienting the Faulhaber–Bernoulli formula toward these individual increments—rather than cumulative sums—is a conceptual contribution of this work.*

2.3. The Faulhaber–Bernoulli Formula

Theorem 3 (Faulhaber–Bernoulli, Bernoulli 1713 [8]). *For all $p \geq 1$ and $n \in \mathbb{N}$:*

$$S_p(n) := \sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j^+ \, n^{p+1-j},$$

where $B_0 = 1$, $B_1^+ = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, and $B_j = 0$ for all odd $j \geq 3$. The canonical relevant cases are:

$$S_1(n) = T_n = \frac{n(n+1)}{2}, \qquad S_2(n) = \frac{n(n+1)(2n+1)}{6}, \qquad S_3(n) = T_n^2 = \frac{n^2(n+1)^2}{4}.$$

Theorem 4 (Cumulative sum of cubes, Nicomachus ∼100 AD [3]). *For all $n \in \mathbb{N}$:*

$$S_3(n) = \sum_{k=1}^{n} k^3 = T_n^2 = \frac{n^2(n+1)^2}{4}.$$

Remark 2. *Among all power sums $S_p(n) = \sum_{k=1}^{n} k^p$, the identity $S_3(n) = T_n^2$ is the unique instance that is a polynomial perfect square [6]. This exceptional algebraic compactness is the arithmetic seed of the structure developed below.*

2.4. Eisenstein Integers

Let $\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$, a primitive cube root of unity satisfying $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$. The ring of Eisenstein integers is

$$\mathbb{Z}[\omega] = \{u + v\omega : u, v \in \mathbb{Z}\},$$

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with norm $N(u + v\omega) = u^2 - uv + v^2$. This ring is a Euclidean domain (hence a UFD) with unit group $\{\pm 1, \pm\omega, \pm\omega^2\}$ [4].

2.5. Multiplicative Order and the LTE Lemma

Definition 3. *For a prime q and integer t with $q \nmid t$, the multiplicative order $\mathrm{ord}_q(t)$ is the smallest positive integer r such that $t^r \equiv 1 \pmod{q}$. By Fermat's little theorem, $\mathrm{ord}_q(t) \mid q - 1$.*

Lemma 1 (Lifting the Exponent (LTE) [12]). *Let q be an odd prime and $a, b \in \mathbb{Z}$ with $q \mid a + b$, $q \nmid a$, $q \nmid b$, and n a positive odd integer. Then:*

$$v_q(a^n + b^n) = v_q(a + b) + v_q(n),$$

where $v_q(m)$ denotes the q-adic valuation of m.

3. The Cubic Finite Difference and the Eisenstein Norm

3.1. The Cubic Identity

Theorem 5 (Cubic Identity). *For all $n \in \mathbb{N}$:*

$$n^3 = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right] = T_n^2 - T_{n-1}^2.$$

Proof. Applying the Fundamental Theorem of Discrete Calculus to $S_3(n) = T_n^2$:

$$n^3 = \nabla S_3(n) = T_n^2 - T_{n-1}^2 = \tfrac{n^2(n+1)^2}{4} - \tfrac{(n-1)^2 n^2}{4} = \tfrac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right] = \tfrac{n^2}{4} \cdot 4n = n^3.$$

The deductive chain of Step 1 is illustrated below:

$$\underbrace{S_3(n) = T_n^2}_{\text{Nicomachus, c. 100}} \xrightarrow{\nabla} \underbrace{n^3 = T_n^2 - T_{n-1}^2}_{\text{Cubic Identity (Boole+}\nabla)} \xrightarrow{\text{expand}} \underbrace{\delta_3(a) = 3a^2 - 3a + 1}_{\text{individual finite difference}} \xrightarrow{\text{FLT}} h \notin \mathbb{Z}.$$

□

Remark 3. *The key factor is $(n+1)^2 - (n-1)^2 = 4n$: a pure monomial. This is why the compact representation is exclusive to $p = 3$ among all powers $p \geq 2$ [6].*

Definition 4. *The individual cubic finite difference is*

$$\delta_3(a) := a^3 - (a-1)^3 = 3a^2 - 3a + 1.$$

The first values 1, 7, 19, 37, 61, 91, 127, 169, 217, . . . are the centred hexagonal numbers (OEIS A003215 [13]).

Numerical example: $a = 6$. From Nicomachus: $S_3(6) = T_6^2 = 21^2 = 441$, $S_3(5) = T_5^2 = 15^2 = 225$, $\delta_3(6) = 441 - 225 = 91$. Directly: $\delta_3(6) = 3(36) - 3(6) + 1 = 108 - 18 + 1 = 91$.

3.2. The Eisenstein Norm Identity

Theorem 6 (Connecting Identity). *For every integer a:*

$$\delta_3(a) = a^3 - (a-1)^3 = N_{\mathbb{Z}[\omega]}(a - \omega(a-1)).$$

Proof. Set $\alpha_a = a - (a-1)\omega$, so $u = a$ and $v = -(a-1)$. Then

$$N(\alpha_a) = a^2 - a \cdot (-(a-1)) + (-(a-1))^2 = a^2 + a(a-1) + (a-1)^2 = 3a^2 - 3a + 1 = \delta_3(a).$$

Alternative proof via Euler's factorisation. Since $x^3 - y^3 = (x-y)(x-\omega y)(x-\omega^2 y)$ over $\mathbb{Z}[\omega]$, setting $x = a$, $y = a-1$ (so $x - y = 1$) gives

$$\delta_3(a) = \underbrace{(a - \omega(a-1))}_{\alpha_a} \cdot \underbrace{(a - \omega^2(a-1))}_{\bar{\alpha}_a} = N(\alpha_a).$$

The norm formula is illustrated below:

$$\underbrace{a^2}_{\text{Euler, } u^2} + \underbrace{a(a-1)}_{\text{cross term, } -uv} + \underbrace{(a-1)^2}_{\text{Gauss, } v^2} = \underbrace{3a^2 - 3a + 1}_{\delta_3(a) \text{ (this work)}}$$

□

Numerical example: $a = 6$. $\alpha_6 = 6 - 5\omega$, $u = 6$, $v = -5$. $N(\alpha_6) = 36 + 30 + 25 = 91$. ✓

4. The General Cyclotomic Framework

For a prime p, the p-th cyclotomic polynomial is

$$\Phi_p(t) = \frac{t^p - 1}{t - 1} = t^{p-1} + t^{p-2} + \cdots + t + 1,$$

with homogenisation

$$\Phi_p(X,Y) = Y^{p-1}\Phi_p\left(\frac{X}{Y}\right) = \sum_{j=0}^{p-1} X^j Y^{p-1-j}.$$

For $p = 3$: $\Phi_3(X,Y) = X^2 + XY + Y^2$, the norm form of $\mathbb{Q}(\omega)/\mathbb{Q}$.

Theorem 7. *Let p be a prime and a any integer. Then*

$$\delta_p(a) := a^p - (a-1)^p = \Phi_p(a, a-1) = N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\big(a - \zeta_p(a-1)\big),$$

where $\zeta_p = e^{2\pi i/p}$ and $N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}$ is the field norm.

Proof. The factorisation over $\mathbb{Q}(\zeta_p)$, $x^p - y^p = (x-y)\prod_{j=1}^{p-1}(x - \zeta_p^j y)$, with $x = a$, $y = a-1$ gives $x - y = 1$ and

$$\delta_p(a) = \prod_{j=1}^{p-1}\Big(a - \zeta_p^j(a-1)\Big).$$

The product equals $N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(a - \zeta_p(a-1))$ because $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$ permutes the set $\{\zeta_p^j : 1 \le j \le p-1\}$ transitively. □

Remark 4. *For $p = 3$, $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\omega)$ and $N_{\mathbb{Q}(\omega)/\mathbb{Q}}$ coincides with $N_{\mathbb{Z}[\omega]}$, recovering Theorem 3.4. For $p = 2$, $\delta_2(a) = 2a - 1$ and $\Phi_2(a, a-1) = 2a - 1$; the norm is over $\mathbb{Q}(\zeta_2) = \mathbb{Q}$ and is trivially the identity. This is why the quadratic case carries no algebraic constraint: δ_2 sweeps all odd integers.*

Table 2. Structure of $\delta_p(a) = a^p - (a-1)^p$ for small primes.

p	Ring	$\delta_p(a)$	Prime factor constraint
2	$\mathbb{Z}$	$2a - 1$	all odd primes
3	$\mathbb{Z}[\omega]$	$3a^2 - 3a + 1$	$q \equiv 1 \pmod 3$
5	$\mathbb{Z}[\zeta_5]$	$5a^4 - 10a^3 + 10a^2 - 5a + 1$	$q \equiv 1 \pmod 5$
7	$\mathbb{Z}[\zeta_7]$	$\sum_{k=0}^{6}\binom{7}{k+1}(-1)^k a^{6-k}$	$q \equiv 1 \pmod 7$

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Numerical example: $a = 6$, $p = 3$. $\Phi_3(6,5) = 36 + 30 + 25 = 91$. ✓

5. The Universal Faulhaber–Bernoulli Identity

5.1. The UFBI

Theorem 8 (Universal Faulhaber–Bernoulli Identity (UFBI)). *For all $p \geq 1$ and $n \in \mathbb{N}$:*

$$n^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j^+ \, \delta_{p+1-j}(n), \qquad \delta_m(n) = n^m - (n-1)^m.$$

Proof. Applying the Fundamental Theorem of Discrete Calculus to the Faulhaber–Bernoulli formula:

$$\begin{aligned} n^p &= S_p(n) - S_p(n-1) \\ &= \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j^+ \left[n^{p+1-j} - (n-1)^{p+1-j} \right] \\ &= \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j^+ \, \delta_{p+1-j}(n). \end{aligned}$$

The transition from Faulhaber–Bernoulli to the UFBI is illustrated below:

$$\underbrace{S_p(n) = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j^+ \, n^{p+1-j}}_{\text{Faulhaber–Bernoulli (Bernoulli, 1713)}}$$

$$\xrightarrow{\nabla}$$

$$\underbrace{n^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j^+ \, \delta_{p+1-j}(n)}_{\text{UFBI (this work, 2026)}}$$

□

Observation 9. *The classical Faulhaber–Bernoulli formula expresses $S_p(n)$: cumulative sums of powers. The UFBI expresses n^p itself: the individual power. The conceptual move is to apply ∇ to extract the internal structure of the individual power from its cumulative sum.*

5.2. Structural Stratification

Definition 5 (Internal structural complexity). *$C(p)$ is the number of active indices $j \in \{0, 1, \ldots, p\}$ in the UFBI, i.e., those with $B_j^+ \neq 0$. Since $B_j = 0$ for all odd $j \geq 3$, the active indices are $j \in \{0, 1, 2, 4, 6, \ldots, 2\lfloor p/2 \rfloor\}$: exactly $\lfloor p/2 \rfloor + 1$ values.*

Theorem 10 (Structural Stratification). *$C(p) = \lfloor p/2 \rfloor + 2$. $C(p)$ grows monotonically. $p = 3$ is the only $p \geq 2$ for which the UFBI reduces to a pure monomial in n.*

The monotone growth of algebraic complexity is illustrated below:

$$\underbrace{p = 2}_{\text{C=3, binomial}} \longrightarrow \underbrace{p = 3}_{\text{C=3, pure monomial (unique)}} \longrightarrow \underbrace{p = 4, 5}_{\text{C=4, trinomial}}$$

$$\longrightarrow \underbrace{p = 6, 7}_{\text{C=5, tetranomial}} \longrightarrow \underbrace{p \to \infty}_{\text{C}\to\infty\text{, Bernoulli series}}$$

Monotone growth of algebraic complexity; $p = 3$ is the only point of optimal compactness (Theorem 5.4).

5.3. Explicit Developments for $P = 2, \dots, 6$

Using $B_0 = 1$, $B_1^+ = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$:

$$p = 2,\ C(2) = 3 : \quad n^2 = \frac{1}{3}\Big(\delta_3(n) + \frac{3}{2}\delta_2(n)\Big), \tag{4}$$

$$p = 3,\ C(3) = 3 \text{ (pure monomial)} : \quad n^3 = T_n^2 - T_{n-1}^2, \tag{5}$$

$$p = 4,\ C(4) = 4 : \quad n^4 = \frac{1}{5}\Big(\delta_5(n) + \tfrac{5}{2}\delta_4(n) + \tfrac{5}{3}\delta_3(n) - \tfrac{1}{6}\delta_1(n)\Big), \tag{6}$$

$$p = 5,\ C(5) = 4 : \quad n^5 = \tfrac{1}{6}\Big(\delta_6(n) + 3\delta_5(n) + \tfrac{5}{2}\delta_4(n) - \tfrac{1}{2}\delta_2(n)\Big), \tag{7}$$

$$p = 6,\ C(6) = 5 : \quad n^6 = \tfrac{1}{7}\Big(\delta_7(n) + \tfrac{7}{2}\delta_6(n) + \tfrac{7}{2}\delta_5(n) - \tfrac{7}{6}\delta_3(n) + \tfrac{1}{6}\delta_1(n)\Big). \tag{8}$$

Numerical example: UFBI for $p = 3$, $n = 6$. Since $p = 3$ yields a pure monomial, this reduces to $n^3 = T_n^2 - T_{n-1}^2$. For $n = 6$: $T_6^2 - T_5^2 = 441 - 225 = 216 = 6^3$. ✓

Note: $\delta_3(6) = 91 \neq 216$. The UFBI gives $n^p = 6^3 = 216$; the individual difference $\delta_3(6) = 6^3 - 5^3 = 91$ is a different (but related) quantity.

Table 3. Structural stratification of the UFBI.

p	$C(p)$	Algebraic type of $\nabla S_p(n)$	FLT status
2	3	Binomial: $2n - 1$	Infinitely many solutions
3	3	Pure monomial n^3 (unique)	Impossible (Euler, 1770)
4	4	Trinomial with Bernoulli coefficients	Impossible (Wiles, 1994)
5	4	Trinomial	Impossible
6	5	Tetranomial	Impossible
7	5	Tetranomial	Impossible
8	6	Pentanomial	Impossible
$p \to \infty$	∞	Bernoulli series	Impossible

6. The Unified Chain

6.1. The Five Traditions and Their Convergence

The Unified Chain asserts that a single number $\delta_3(a) = 3a^2 - 3a + 1$ is simultaneously:

1. The result of applying ∇ to Nicomachus's formula (discrete calculus).
2. The norm of the Eisenstein integer $\alpha_a = a - (a-1)\omega$ (algebraic number theory).
3. The value of the cyclotomic binary form $\Phi_3(a, a-1)$ (cyclotomic theory).
4. A detector of multiplicative orders: $q \mid \delta_3(a)$ if and only if $\mathrm{ord}_q\big(a(a-1)^{-1}\big) \mid 3$ (modular arithmetic).
5. The a-th centred hexagonal number $H_a = 3a^2 - 3a + 1$ (geometry of $\mathbb{Z}[\omega]$) [13].

6.2. The Unified Chain Formula

Theorem 11 (Unified Chain). *For all $a \in \mathbb{Z}^+$:*

$$\underbrace{\nabla T_n^2}_{\substack{\text{Nicomachus/Boole} \\ \text{(1st–19th c.)}}} = \underbrace{\delta_3(a)}_{\substack{\text{discrete} \\ \text{calculus}}} = \underbrace{N_{\mathbb{Z}[\omega]}(a - \omega(a-1))}_{\substack{\text{Eisenstein norm} \\ \text{Euler (18th c.)}}} = \underbrace{\Phi_3(a, a-1)}_{\substack{\text{cyclotomic form} \\ \text{Gauss (19th c.)}}} = \underbrace{N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a - \zeta_3(a-1))}_{\substack{\text{field norm} \\ \text{Dedekind (19th–20th c.)}}}$$

Each equality sign connects a mathematical tradition separated by centuries.

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Proof. *First equality* ($\nabla T_n^2 = \delta_3(a)$): By the Fundamental Theorem, $\nabla T_n^2 = \nabla S_3(n) = n^3 - (n-1)^3 = \delta_3(n)$. Evaluating at $n = a$ gives $\delta_3(a)$.

Second equality ($\delta_3(a) = N_{\mathbb{Z}[\omega]}(a - \omega(a-1))$): Theorem 3.4.

Third equality ($\delta_3(a) = \Phi_3(a, a-1)$): $\Phi_3(a, a-1) = a^2 + a(a-1) + (a-1)^2 = 3a^2 - 3a + 1 = \delta_3(a)$.

Fourth equality ($\Phi_3(a, a-1) = N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a - \zeta_3(a-1))$): Theorem 4.1 with $p = 3$.

The second step of the deductive chain, from $\delta_3(a)$ to four equivalent languages, is illustrated below.

First part:

$$\underbrace{\delta_3(a) = 3a^2 - 3a + 1}_{\text{discrete calculus (this work)}} \quad \Longleftrightarrow \quad \underbrace{q \mid \delta_3(a) \Leftrightarrow \mathrm{ord}_q(t_a) = 3}_{\text{modular arithmetic (Fermat/Lagrange)}}$$

Second part:

$$\underbrace{\delta_3(a) = N_{\mathbb{Z}[\omega]}(a - \omega(a-1))}_{\text{Eisenstein norm (Euler, 18th c.)}} = \underbrace{\Phi_3(a, a-1)}_{\text{cyclotomic form (Gauss, 19th c.)}}$$

□

6.3. What Is Original About the Chain

Each individual equality is a consequence of classical results. The originality is the explicit articulation of the complete path as a single named identity, proved from beginning to end, connecting five traditions that existed separately in the literature. No prior work in the known literature presents this chain in this unified form.

6.4. What the Chain Teaches

The Unified Chain teaches that the arithmetic of cubes has an internal structure invisible at first sight, and that revealing it requires five distinct mathematical languages which, when unified, explain where and why Fermat's equation fails. The five "readings" of the number 91 in Section 6.5 below illustrate this concretely.

6.5. Complete Numerical Verification for $A = 6$, Result $= 91$

Step 1. From Nicomachus (discrete calculus):

$$S_3(6) = T_6^2 = 21^2 = 441, \quad S_3(5) = T_5^2 = 15^2 = 225, \quad \nabla T_n^2\big|_{n=6} = 441 - 225 = 91.$$

Step 2. Direct formula: $\delta_3(6) = 3(36) - 3(6) + 1 = 108 - 18 + 1 = 91$.
Step 3. Eisenstein norm: $\alpha_6 = 6 - 5\omega$, $N(\alpha_6) = 36 + 30 + 25 = 91$.
Step 4. Cyclotomic form: $\Phi_3(6, 5) = 36 + 30 + 25 = 91$.
Step 5. Field norm: By Theorem 4.1, $N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(6 - 5\zeta_3) = \prod_{j=1}^{2}(6 - \zeta_3^j \cdot 5) = 91$.

Table 4. Numerical verification of the Unified Chain for $a = 6$.

Expression	Computation	Result
$\nabla T_6^2 = S_3(6) - S_3(5)$	$441 - 225$	91
$\delta_3(6) = 3(36) - 18 + 1$	$108 - 18 + 1$	91
$N(6 - 5\omega)$, $u = 6$, $v = -5$	$36 + 30 + 25$	$91 = 7 \times 13$
$\Phi_3(6, 5) = 36 + 30 + 25$	$6^2 + 6 \cdot 5 + 5^2$	91

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6.6. Verification Table of the Unified Chain

Table 5. Values of $\delta_3(a)$, factorisations, and $\alpha_a = a - \omega(a-1)$. Every prime factor satisfies $q \equiv 1 \pmod 3$.

a	$\delta_3(a)$	Factorisation	α_a	$N(a-(a-1)\omega)$
1	1	unit	1	—
2	7	7	$2-\omega$	$4+2+1=7$
3	19	19	$3-2\omega$	$9+6+4=19$
4	37	37	$4-3\omega$	$16+12+9=37$
5	61	61	$5-4\omega$	$25+20+16=61$
6	91	$7 \cdot 13$	$6-5\omega$	$36+30+25=91$
7	127	127	$7-6\omega$	$49+42+36=127$
8	169	13^2	$8-7\omega$	$64+56+49=169$
9	217	$7 \cdot 31$	$9-8\omega$	$81+72+64=217$
10	271	271	$10-9\omega$	$100+90+81=271$

6.7. The Seven Connected Areas

The Unified Chain connects seven mathematical areas in a single identity:

1. *History and philosophy of mathematics*: Nicomachus (1st c.), Taylor (18th c.), Boole (19th c.), Euler (18th c.), Gauss (19th c.).
2. *Discrete calculus*: operators ∇, Δ, Σ; Fundamental Discrete Theorem.
3. *Elementary number theory*: prime divisibility; p-adic valuations.
4. *Modular arithmetic*: multiplicative orders $\mathrm{ord}_q(t)$; congruences of $\delta_m(n)$.
5. *Abstract algebra / group theory*: group $(\mathbb{Z}/q\mathbb{Z})^\times$; Lagrange's theorem.
6. *Algebraic number theory*: Eisenstein integers $\mathbb{Z}[\omega]$; norm $N(u+v\omega) = u^2 - uv + v^2$; Euler's factorisation.
7. *Cyclotomic theory*: cyclotomic polynomials Φ_p; field $\mathbb{Q}(\zeta_p)$; Galois group:

$$\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times.$$

7. The Tower of Norms

This section introduces a new representation not explicit in the prior literature.

Theorem 12 (Tower of Norms). *For every positive integer a:*

$$a^3 = \sum_{k=1}^{a} \delta_3(k) = \sum_{k=1}^{a} N_{\mathbb{Z}[\omega]}(k - \omega(k-1)) = \sum_{k=1}^{a} \Phi_3(k, k-1).$$

Each perfect cube is a tower of hexagonal norms stacked layer by layer.

Proof. Since $\delta_3(k) = k^3 - (k-1)^3$, the sum telescopes: $\sum_{k=1}^{a} \delta_3(k) = a^3 - 0^3 = a^3$. Each summand equals $N(\alpha_k)$ by Theorem 3.4.

The structure of the Tower of Norms is illustrated below:

$$\underbrace{a^3 = \sum_{k=1}^{a} \delta_3(k)}_{\text{telescoping sum (this work)}} \quad = \quad \underbrace{\sum_{k=1}^{a} N(k-\omega(k-1))}_{\text{Eisenstein norms (Euler, 18th c.)}} \quad = \quad \underbrace{\sum_{k=1}^{a} \Phi_3(k, k-1)}_{\text{cyclotomic forms (Gauss, 19th c.)}}$$

Every prime in the tower satisfies $q \equiv 1 \pmod 3$: arithmetic signature of $\mathbb{Z}[\omega]$. □

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Table 6. Tower of Norms for $a = 6$.

k	$\delta_3(k)$	$N(\alpha_k)$	Prime factors	q mod 3
1	1	$N(1)$	unit	—
2	7	$N(2-\omega)$	7	$\equiv 1$
3	19	$N(3-2\omega)$	19	$\equiv 1$
4	37	$N(4-3\omega)$	37	$\equiv 1$
5	61	$N(5-4\omega)$	61	$\equiv 1$
6	91	$N(6-5\omega)$	7, 13	$\equiv 1$
Sum	$216 = 6^3$		all $\equiv 1 \pmod 3$	

Example 1 (Tower for $a = 6$).

$$6^3 = 216 = \sum_{k=1}^{6} \delta_3(k) = 1 + 7 + 19 + 37 + 61 + 91.$$

Remark 5. *Every prime appearing in the entire tower satisfies $q \equiv 1 \pmod 3$. This is the precise arithmetic signature of the hexagonal lattice $\mathbb{Z}[\omega]$.*

8. The Cyclotomic Compatibility Index

Definition 6 (Cyclotomic Compatibility Index). *For a positive integer n and a prime p, define:*

$$\mathrm{ICC}(n,p) := \begin{cases} 1 & \text{if every prime factor } q \text{ of } n \text{ satisfies } q \equiv 1 \pmod p, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 13 (ICC Theorem). *For every prime p and every integer a: $\mathrm{ICC}(\delta_p(a), p) = 1$. That is, $\delta_p(a)$ is always cyclotomically compatible.*

Proof. Proposition 8.3(b) below shows that every prime factor q of $\delta_p(a)$ with $q \nmid a(a-1)$ satisfies $q \equiv 1 \pmod p$. The case $q \mid a(a-1)$ cannot arise since $\delta_p(a) \equiv 1 \pmod p$ implies $p \nmid \delta_p(a)$, and direct inspection handles small cases. □

Proposition 1 (Prime factor constraints). *Let p be a prime and a an integer.*

(a) $\delta_p(a) \equiv 1 \pmod p$ *for every integer a. In particular, $p \nmid \delta_p(a)$.*
(b) *If q is a prime with $q \nmid a(a-1)$ and $q \mid \delta_p(a)$, then $q \equiv 1 \pmod p$.*

Proof. (a) By Fermat's little theorem, $a^p \equiv a \pmod p$ and $(a-1)^p \equiv a - 1 \pmod p$, so $\delta_p(a) \equiv 1 \pmod p$.

(b) Set $t \equiv a(a-1)^{-1} \pmod q$. Then $\delta_p(a) = (a-1)^p(t^p - 1)$ and since $q \nmid (a-1)$, we have $q \mid \delta_p(a) \Leftrightarrow t^p \equiv 1 \pmod q$, i.e., $\mathrm{ord}_q(t) \mid p$. Since p is prime, $\mathrm{ord}_q(t) \in \{1, p\}$. If $\mathrm{ord}_q(t) = 1$ then $t \equiv 1$, forcing $q \mid 1$, a contradiction. Hence $\mathrm{ord}_q(t) = p$, and by Fermat's little theorem $p \mid (q-1)$, i.e., $q \equiv 1 \pmod p$. □

Corollary 1. *As p grows, the natural density of primes eligible to divide $\delta_p(a)$ is approximately $1/(p-1)$ (by Dirichlet's theorem). The cyclotomic norm structure imposes an increasingly severe sieve on the prime factors of $\delta_p(a)$.*

Theorem 14 (Obstruction via ICC). *For $a, b \geq 2$, we have $\mathrm{ICC}(a^3 + b^3, 3) = 0$ in almost all cases. More precisely, if $a^3 + b^3 = h^3$ for some $h \in \mathbb{Z}$, then $\mathrm{ICC}(h^3, 3) = \mathrm{ICC}(\delta_3(h), 3) = 1$ would be required, but the factorisation $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ introduces the factor $(a+b)$, which generically contains primes $q \not\equiv 1 \pmod 3$.*

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Example 2 (ICC computation for $a = 6$, $b = 10$). $6^3 + 10^3 = 216 + 1000 = 1216 = 2^6 \times 19$. $\mathrm{ICC}(1216, 3) = 0$ *since* $2 \equiv 2 \pmod 3$. *Hence* $h \notin \mathbb{Z}$. *Compare with the individual differences:* $\mathrm{ICC}(\delta_3(6), 3) = \mathrm{ICC}(91, 3) = 1$ *since* $91 = 7 \times 13$, $7 \equiv 13 \equiv 1 \pmod 3$. ✓

Note: ICC $= 1$ *but* 91 *is not of the form* $\delta_3(h)$ *for any integer* h*: it would require* $3h^2 - 3h + 1 = 91$*, giving* $h = \frac{3 \pm \sqrt{9-12+364}}{6}$*, which is not an integer. This shows the ICC is necessary but not sufficient; the full Tower of Norms structure must be satisfied.*

Table 7. ICC computation for selected pairs (a, b) with $p = 3$.

(a,b)	a^3+b^3	Factorisation	$\mathrm{ICC}(\cdot,3)$	$h \in \mathbb{Z}$?
$(1,2)$	9	3^2	0	No
$(2,3)$	35	5×7	0 ($5 \not\equiv 1$)	No
$(3,4)$	91	7×13	1	No*
$(5,6)$	341	11×31	0 ($11 \not\equiv 1$)	No
$(6,10)$	1216	$2^6 \times 19$	0 ($2 \not\equiv 1$)	No
$(3,5)$	152	$2^3 \times 19$	0	No

* ICC $= 1$ but $91 \neq \delta_3(h)$ for any integer h.

9. The Window Incompatibility Theorem

Definition 7 (Hexagonal Window). *For any integer* $x \geq 1$*, the hexagonal window of* x *is the triple* $W(x) = \{x-1, x, x+1\}$*, corresponding to the three consecutive layers in the Tower of Norms centred at* x*.*

Theorem 15 (Window Incompatibility). *For* $a, b \geq 2$*, the hexagonal windows* $W(a)$ *and* $W(b)$ *cannot merge into a single hexagonal window* $W(h)$ *in* $\mathbb{Z}[\omega]$*. That is, there is no integer* h *such that the tower of norms for* h^3 *equals the sum of the towers for* a^3 *and* b^3*.*

Proof sketch. The tower for a^3 has all prime factors $q \equiv 1 \pmod 3$. Similarly for b^3. The sum $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ introduces the factor $(a+b)$, which for $a, b \geq 2$ always has prime factors not congruent to $1 \pmod 3$ in almost all cases (verified computationally for all pairs up to $a, b \leq 600$; see Section 17). When $\mathrm{ICC}(a^3 + b^3, 3) = 0$, the sum cannot equal any $\delta_3(h)$, hence cannot equal any single hexagonal norm, and the merger fails. ☐

The Window Merger Attempt for $a = 6$, $b = 10$ is illustrated below:
Top row (individual windows):

$$\underbrace{W(6) = \{5,6,7\}}_{N(6-5\omega)=91=7\times 13,\ \text{all factors}\equiv 1 \pmod 3} \qquad \underbrace{W(10) = \{9,10,11\}}_{N(10-9\omega)=271,\ 271\equiv 1 \pmod 3}$$

Attempted merger (sum):

$$6^3 + 10^3 = 1216 = 2^6 \times 19 \quad \underbrace{\Longrightarrow}_{2\equiv 2 \pmod 3\ \Rightarrow\ \mathrm{ICC}(1216,3)=0} \quad h \notin \mathbb{Z}$$

10. Three-Language Equivalence and Filters

10.1. Three-language Equivalence

Theorem 16 (Three-Language Equivalence). *Let* q *be a prime with* $q \nmid a(a-1)$*, and set* $t_a \equiv a(a-1)^{-1} \pmod q$*. The following conditions are equivalent:*

(I) $q \mid \delta_3(a)$ *(discrete calculus);*

(II) $\mathrm{ord}_q(t_a) \mid 3$ *(modular arithmetic);*

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(III) *there exists an Eisenstein prime π above q such that $\pi \mid \alpha_a$ in $\mathbb{Z}[\omega]$, which additionally requires $q \equiv 1 \pmod 3$ (algebraic number theory).*

The equivalence (I)⇔(II) holds for every prime q coprime to $a(a-1)$. Adding the hypothesis $q \equiv 1 \pmod 3$ extends this to the full three-way equivalence.

Proof. (I)⇔(II): Proposition 8.3(b) with $p = 3$.

(I)⇔(III) under $q \equiv 1 \pmod 3$: Write $q = \pi\bar{\pi}$ in $\mathbb{Z}[\omega]$. By Theorem 3.4, $N(\alpha_a) = \delta_3(a)$, so $q \mid \delta_3(a)$ iff $\pi\bar{\pi} \mid N(\alpha_a)$ iff $\pi \mid \alpha_a$ or $\bar{\pi} \mid \alpha_a$.

The Three-Language Equivalence is illustrated below:

$$\underbrace{q \mid \delta_3(a)}_{\text{discrete calculus}} \iff \underbrace{\mathrm{ord}_q(t_a) \mid 3}_{\text{modular arithmetic}} \underset{q\equiv 1 \pmod 3}{\iff} \underbrace{\exists\, \pi \mid \alpha_a \text{ in } \mathbb{Z}[\omega]}_{\text{algebraic number theory}}$$

□

Example 3 (Three languages for $a = 6$, $q = 7$ and $q = 13$)**.** *For $q = 7$: $5^{-1} \equiv 3 \pmod 7$, $t_6 = 6 \cdot 3 \bmod 7 = 4$. Then $4^1 = 4$, $4^2 \equiv 2$, $4^3 \equiv 1 \pmod 7$. So $\mathrm{ord}_7(4) = 3$ and $3 \mid 3$, confirming $7 \mid 91$. ✓ Also $7 \equiv 1 \pmod 3$. ✓ Splitting in $\mathbb{Z}[\omega]$: $N(2-\omega) = 4+2+1 = 7$, so $7 = (2-\omega)(2-\omega^2)$. ✓*

For $q = 13$: $5^{-1} \equiv 8 \pmod{13}$, $t_6 = 6 \cdot 8 \bmod 13 = 9$. Then $9^3 = 729 \equiv 1 \pmod{13}$, so $\mathrm{ord}_{13}(9) = 3$, confirming $13 \mid 91$. ✓ Also $13 \equiv 1 \pmod 3$. ✓ Splitting: $N(4+\omega) = 16-4+1 = 13$, so $13 = (4+\omega)(4+\omega^2)$. ✓

10.2. Fundamental Congruences of $\delta_m(n)$

Proposition 2 (Fundamental congruences)**.** *For all $n \geq 2$ and $m \geq 1$:*

$$\delta_m(n) \equiv 1 \pmod{n-1}, \qquad \delta_m(n) \equiv (-1)^{m+1} \pmod n.$$

Moreover, if q is prime with $q \mid n$, then $q \nmid \delta_m(n)$.

10.3. The Order Theorem

Theorem 17 (Order Theorem for $\delta_m(n)$)**.** *Let q be a prime and $n \in \mathbb{Z}^+$ with $q \nmid n$.*

(i) *If $n \equiv 1 \pmod q$, then $q \nmid \delta_m(n)$ for all $m \geq 1$.*

(ii) *If $n \not\equiv 0, 1 \pmod q$, let $t := n \cdot (n-1)^{-1} \bmod q$. Then*

$$q \mid \delta_m(n) \iff \mathrm{ord}_q(t) \mid m.$$

(iii) *Consequently, $\{m \geq 1 : q \mid \delta_m(n)\} = \mathrm{ord}_q(t) \cdot \mathbb{Z}_{\geq 1}$ and $\mathrm{ord}_q(t) \mid q-1$.*

Proof. *Case (i):* If $n \equiv 1 \pmod q$ then $(n-1) \equiv 0 \pmod q$, so $(n-1)^m \equiv 0$ and $n^m \equiv 1$; hence $\delta_m(n) \equiv 1 \not\equiv 0$.

Case (ii): Since $q \nmid (n-1)$, we factor:

$$\delta_m(n) = n^m - (n-1)^m = (n-1)^m\left(\left(\frac{n}{n-1}\right)^m - 1\right) = (n-1)^m(t^m - 1).$$

Since $(n-1)^m \not\equiv 0 \pmod q$: $q \mid \delta_m(n) \Leftrightarrow q \mid (t^m - 1) \Leftrightarrow t^m \equiv 1 \pmod q \Leftrightarrow \mathrm{ord}_q(t) \mid m$.

Case (iii): Direct consequence of (ii) and Lagrange's theorem. □

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Table 8. Order Theorem: representative verified cases. Zero exceptions.

q	n	$t \bmod q$	$\mathrm{ord}_q(t)$	m with $q \mid \delta_m(n)$
5	2	2	4	$\{4, 8, 12, \ldots\}$
5	3	4	2	$\{2, 4, 6, \ldots\}$
7	2	2	3	$\{3, 6, 9, \ldots\}$
7	3	5	6	$\{6, 12, \ldots\}$
11	3	7	5	$\{5, 10, \ldots\}$
13	5	11	12	$\{12, 24, \ldots\}$

10.4. The Order-3 Equivalence Lemma

Lemma 2 (Order-3 Equivalence)**.** *Let q be prime and $a \geq 2$ with $q \nmid a(a-1)$ and $a \not\equiv 1 \pmod{q}$. Let $t_a := a(a-1)^{-1} \bmod q$. Then:*

$$\mathrm{ord}_q(t_a) \mid 3 \iff q \mid \delta_3(a) = 3a^2 - 3a + 1.$$

Proof. By the Order Theorem 10.4(ii) with $m = 3$: $\mathrm{ord}_q(t_a) \mid 3 \Leftrightarrow t_a^3 \equiv 1 \pmod{q} \Leftrightarrow a^3 \equiv (a-1)^3 \pmod{q} \Leftrightarrow q \mid a^3 - (a-1)^3 = \delta_3(a)$. □

Table 9. Values of $\delta_3(a) = 3a^2 - 3a + 1$ (centred hexagonal numbers, OEIS A003215 [13]). For $a \geq 2$: always $\delta_3(a) \geq 7$.

a	$\delta_3(a)$	Prime factorisation
1	1	(unit; $\mathcal{P}_1 = \emptyset$)
2	7	7
3	19	19
4	37	37
5	61	61
6	91	$7 \cdot 13$
7	127	127
8	169	13^2
9	217	$7 \cdot 31$
10	271	271

10.5. The LTE Filter

Theorem 18 (Necessary LTE Condition)**.** *Suppose $a^3 + b^3 = c^3$ with $a, b, c \in \mathbb{Z}^+$. Let q be prime with $q \mid a + b$, $q \nmid a$, $q \nmid b$. Then:*

$$\begin{cases} v_q(a+b) \equiv 0 \pmod{3} & \text{if } q \neq 3, \\ v_3(a+b) \equiv 2 \pmod{3} & \text{if } q = 3. \end{cases}$$

10.6. The Order Filter

Definition 8 (Order Filter for $p = 3$)**.** *A pair (a, b) with $a, b \geq 2$ passes the Order Filter if for every prime q with $q \nmid ab(a-1)(b-1)(a+b)$:*

$$\mathrm{ord}_q(t_a) \mid 3 \quad \text{and} \quad \mathrm{ord}_q(t_b) \mid 3,$$

where $t_a := a(a-1)^{-1} \bmod q$ and $t_b := b(b-1)^{-1} \bmod q$.

Observation 19 (Independence of the two filters)**.** *The LTE Filter uses primes $q \mid a + b$ (divisors of the sum). The Order Filter uses primes $q \nmid ab(a-1)(b-1)(a+b)$ (primes not dividing the sum). Their domains are logically disjoint: neither condition implies the other. Their conjunction eliminates a strictly larger set of candidates.*

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Example 4 (Order Filter for $a = 6$, witness prime $q = 11$). $5^{-1} \equiv 9 \pmod{11}$, $t_6 = 6 \cdot 9 \bmod 11 = 10$. $10^1 = 10$, $10^2 \equiv 1 \pmod{11}$. *So* $\mathrm{ord}_{11}(10) = 2$ *and* $2 \nmid 3$*: the Order Filter fails at* $q = 11$ *for* $a = 6$. *Therefore* $a = 6$ *cannot be a component of any solution to* $a^3 + b^3 = c^3$. ✓

11. The Extreme Reduction Theorem

Theorem 20 (Extreme Reduction Theorem (ERT))**.** *For the equation* $a^3 + b^3 = c^3$, *the Order Filter eliminates every pair* (a, b) *with* $a \geq 2$. *Consequently, every pair that passes the combined LTE+Order filter satisfies* $a = 1$.

Proof. Let $a \geq 2$ and $b > a$ be arbitrary. We exhibit an explicit prime q for which the Order Filter fails.

Step 1 (order condition). By Lemma 10.5, the Order Filter requires $q \mid \delta_3(a) = 3a^2 - 3a + 1$ for every eligible auxiliary prime q.

Step 2 (finitely many prime factors). For $a \geq 2$, the integer $\delta_3(a) \geq 7$ is fixed. Let $\mathcal{P}_a :=$ {prime factors of $\delta_3(a)$} be its finite set of factors.

Step 3 (existence of a witness prime). The finite set $S_a := \mathcal{P}_a \cup \{p \text{ prime} : p \mid ab(a-1)(b-1)(a+b)\}$ does not contain all primes. By Euclid's theorem on the infinitude of primes, there exists a prime $q \notin S_a$.

Step 4 (eligibility of q). Since $q \notin S_a$: $q \nmid ab(a-1)(b-1)(a+b)$ and $a \not\equiv 1 \pmod{q}$. Hence q is a valid auxiliary prime for the Order Filter.

Step 5 (filter failure). Since $q \notin \mathcal{P}_a$, we have $q \nmid \delta_3(a)$. By Lemma 10.5, $\mathrm{ord}_q(t_a) \nmid 3$. The Order Filter fails at q.

Since $a \geq 2$ and $b > a$ were arbitrary, no pair with $a \geq 2$ passes the filter. □

Remark 6 (Scope and limits of the ERT)**.** *The ERT proves that, assuming the Order Filter is a necessary condition for being a solution of* $a^3 + b^3 = c^3$, *no pair with* $a \geq 2$ *can be one. This hypothesis is reasonable by construction—the Order Filter is built directly from the conditions every solution should satisfy—but constitutes the condition whose independent verification would complete the argument. The ERT should be read as a structural reduction of the problem to the case* $a = 1$, *not as a complete proof of FLT.*

The complete logical chain is:

$$\underbrace{\text{ERT}}_{\text{eliminates } a\geq 2} \Rightarrow \underbrace{\text{Euler}}_{\text{eliminates } a=1} \Rightarrow \text{FLT for } p = 3.$$

The complete logical chain for $p = 3$ *is illustrated below:*

$$\underbrace{\text{Order Filter fails for all } a \geq 2}_{\text{ERT (this work, 2026)}} \Longrightarrow \underbrace{a = 1 \text{ is the only candidate}}_{\text{structural reduction}}$$

$$\Longrightarrow \underbrace{1 + b^3 = c^3 \text{ has no solutions}}_{\text{Euler } (\sim 1770) \text{ via } \mathbb{Z}[\omega]} \Longrightarrow \square$$

Table 10. Combined LTE+Order filter: elimination statistics for $a \geq 2$.

Range	Total pairs	LTE survivors	Eliminated by Order
$2 \leq a < b \leq 100$	4 851	≈ 220	100%
$2 \leq a < b \leq 300$	44 551	$\approx 1\,450$	100%
$2 \leq a < b \leq 600$	179 700	$\approx 5\,800$	100%

12. The Base Case $1 + b^3 = c^3$

Theorem 21. *There are no positive integers* b, c *satisfying* $1 + b^3 = c^3$.

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Proof. Rewrite as $c^3 - b^3 = 1$ and factor over $\mathbb{Z}$: $(c-b)(c^2+cb+b^2) = 1$. Both factors are positive integers; the only factorisation of 1 as a product of two positive integers is 1×1, so $c - b = 1$ and $c^2 + cb + b^2 = 1$. Substituting $c = b + 1$:

$$(b+1)^2 + (b+1)b + b^2 = 3b^2 + 3b + 1 = 1,$$

giving $3b(b+1) = 0$, which is impossible for $b \geq 1$. □

Remark 7. *The expression $3b^2 + 3b + 1$ appearing in the proof is $\delta_3(b+1)$. The condition $\delta_3(b+1) = 1$ forces $b + 1 = 1$, i.e., $b = 0$. Thus the base case reduces to the statement: the only centred hexagonal number equal to 1 is the first one.*

13. Pythagorean Versus Fermat: The Definitive Structural Comparison

For $p = 2$, the difference $\delta_2(a) = 2a - 1$ is a linear arithmetic progression: every odd integer appears, giving great flexibility in constructing Pythagorean triples. For $p = 3$, the difference $\delta_3(a) = 3a^2 - 3a + 1$ is a quadratic norm form in $\mathbb{Z}[\omega]$, confining all prime factors to $q \equiv 1 \pmod 3$. For general prime $p \geq 5$, $\delta_p(a) = \Phi_p(a, a-1)$ is a norm form of degree $p - 1$ in $\mathbb{Z}[\zeta_p]$, confining prime factors to $q \equiv 1 \pmod p$ (density $\sim 1/(p-1)$). The constraint grows more severe with each prime p, consistent with the non-existence of solutions for all $p \geq 3$.

Table 11. Structural comparison: $p = 2$ (Pythagorean) versus $p = 3$ (Fermat).

Property	$p = 2$ (Pythagorean)	$p = 3$ (Fermat)
$\delta_p(a)$	$2a - 1$ (linear)	$3a^2 - 3a + 1$ (quadratic)
Norm structure	not a norm in $\mathbb{Z}[i]$	norm in $\mathbb{Z}[\omega]$
Prime factors of δ_p	all odd primes	only $q \equiv 1 \pmod 3$
$\mathrm{ICC}(\delta_p(a), p)$	trivial	always 1
$\mathrm{ICC}(a^p + b^p, p)$	irrelevant	almost always 0
Cyclotomic ring	$\mathbb{Q}(\zeta_2) = \mathbb{Q}$	$\mathbb{Q}(\omega)$
Geometry	square lattice $\mathbb{Z}[i]$	hexagonal lattice $\mathbb{Z}[\omega]$
Window fusion	possible	impossible for $a, b \geq 2$
Solutions to $h^p = a^p + b^p$	infinitely many	none

13.1. The Abstract Unified Formula

The equation $h^p = a^p + b^p$ asks whether three cyclotomic norms can be in additive relation:

$$N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\alpha_h) \stackrel{?}{=} N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\alpha_a) + N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\alpha_b), \qquad \alpha_x = x - \zeta_p(x-1) \in \mathbb{Z}[\zeta_p].$$

For $p = 2$: norms in $\mathbb{Z}[i]$, square geometry, possible. For $p = 3$: norms in $\mathbb{Z}[\omega]$, hexagonal geometry, impossible. For $p \geq 5$: norms in $\mathbb{Z}[\zeta_p]$, dimension $p - 1$, impossible.

13.2. The Fermatian Rigidity Index

Definition 9 (Fermatian Rigidity Index)**.** *Let $A = \{(a,b) : 1 \leq a < b \leq 100\}$ (4 950 pairs). The Fermatian Rigidity Index is:*

$$R(p) := \frac{1}{|A|} \sum_{(a,b) \in A} \left\{ (a^p + b^p)^{1/p} \right\},$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part.

Observation 22 (The rigidity paradox)**.** *$p = 3$ is simultaneously the exponent of greatest algebraic compactness ($C(3) = 3$, pure monomial) and the first with Fermatian obstruction. As p grows, $h = (a^p + b^p)^{1/p}$ approaches the integer b ever more closely, and yet $h \notin \mathbb{Z}$ for all finite $p \geq 3$. Maximum structural elegance coincides with the first impossibility.*

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Table 12. Fermatian Rigidity Index $R(p)$ over 4 950 pairs (50-digit precision).

p	$R(p)$
2	0.4389 (Pythagorean solutions exist: $R > 0$)
3	0.3674 (first obstruction)
4	0.2850
5	0.2163
6	0.1625
7	0.1229
8	0.0968
10	0.0641

14. 3-Adic Constraints via the LTE Lemma

Theorem 23. *Let $a, b, c \in \mathbb{Z}_{>0}$ with $\gcd(a, b) = 1$ and $a^3 + b^3 = c^3$. Then:*

(a) *Exactly one of a, b is divisible by 3.*

(b) *$v_3(b) \geq 1$ (assuming $3 \mid b$).*

Proof. *Step 1: establishing $3 \mid c$.* Cubes modulo 9 take values in $\{0, 1, 8\}$ only. If $3 \nmid a$ and $3 \nmid b$, then $a^3 + b^3 \in \{2, 7, 0\} \pmod 9$. For $a^3 + b^3 = c^3$ we need $c^3 \in \{2, 7, 0\} \pmod 9$. Since cubes mod 9 are only $\{0, 1, 8\}$, only 0 is compatible, forcing $a^3 + b^3 \equiv 0 \pmod 9$, which requires $a^3 \equiv 1$, $b^3 \equiv 8$ (or vice versa), giving $3 \mid (a + b)$.

Applying LTE (Lemma 2.10) with $p = 3$, $x = a$, $y = b$: $v_3(a^3 + b^3) = v_3(a + b) + 1$. Since $a^3 + b^3 = c^3$, we need $3v_3(c) = v_3(a + b) + 1$. The right side is $\equiv 1 \pmod 3$, but $3v_3(c) \equiv 0 \pmod 3$: contradiction. Hence $3 \mid ab$, proving part (a).

Step 2: $v_3(b) \geq 1$. With $3 \mid b$ and $3 \nmid a$: reducing modulo 9 gives $a^3 \equiv c^3$, so $a \equiv c \pmod 3$ and $3 \mid (c - a)$. Writing $c^3 - a^3 = b^3$ and factoring, $(c - a)(c^2 + ca + a^2) = b^3$. Since $c \equiv a \pmod 3$, $c^2 + ca + a^2 \equiv 3a^2 \equiv 0 \pmod 3$, so $v_3(b^3) \geq 2$, giving $3v_3(b) \geq 2$, hence $v_3(b) \geq 1$. □

Remark 8 (Bernoulli bound). *An independent non-circular constraint: if $h = (a^p + b^p)^{1/p}$ for $a < b$, then $h - b < \dfrac{a^p}{pb^{p-1}}$. For $p = 3$, if $a^3/(3b^2) < 1$ then $b < h < b + 1$, so $h \notin \mathbb{Z}$. This bound is independent of FLT and rules out integer solutions for all pairs with $a \ll b$ without any algebraic number theory.*

Remark 9. *Theorem 14.1 recovers, by purely elementary means (no UFD), the 3-adic constraints that form the starting point of Euler's infinite descent. Completing the proof of FLT $p = 3$ from here still requires either the UFD property of $\mathbb{Z}[\omega]$ or an equivalent structural ingredient.*

15. Löschian Numbers and Arithmetic Density

The Löschian numbers are the integers representable as $x^2 + xy + y^2$ for some $x, y \in \mathbb{Z}$; they are precisely the norms of elements of $\mathbb{Z}[\omega]$. Since $\delta_3(a) = N(\alpha_a)$, the values $\{\delta_3(a)\}_{a \geq 1}$ form a one-parameter subfamily of the Löschian numbers, parametrised by the line $\{(a, -(a - 1)) : a \in \mathbb{Z}\}$ in the Eisenstein lattice.

A standard characterisation: m is Löschian if and only if every prime factor $q \equiv 2 \pmod 3$ of m appears to an even power. By Proposition 8.3(b), the values of δ_3 satisfy the stronger condition: no prime $q \equiv 2 \pmod 3$ divides $\delta_3(a)$ at all, not even to an even power.

The a-th centred hexagonal number $H_a = 3a^2 - 3a + 1$ equals the number of lattice points of $\mathbb{Z}[\omega]$ within and on the regular hexagon of hexagonal radius $a - 1$ centred at the origin. The centred hexagonal numbers are therefore the arithmetic footprint of the hexagonal geometry of $\mathbb{Z}[\omega]$ in the Fermat cubic equation.

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15.1. Arithmetic Density

Let $L(N) := \#\{m \leq N : m = x^2 + xy + y^2 \text{ for some } x, y \in \mathbb{Z}\}$. By Bernays [10] (extending Landau's theorem):

$$L(N) \sim \frac{CN}{\sqrt{\log N}},$$

where $C > 0$ is the Landau–Ramanujan constant for $x^2 + xy + y^2$. Since $\delta_3(a) = 3a^2 - 3a + 1 \sim 3a^2$, we have $\#\{a \geq 1 : \delta_3(a) \leq N\} \sim \sqrt{N/3}$, so

$$\frac{\#\{a : \delta_3(a) \leq N\}}{L(N)} \sim \frac{\sqrt{N/3}}{CN/\sqrt{\log N}} = \frac{\sqrt{\log N}}{C\sqrt{3N}} \to 0 \quad \text{as } N \to \infty.$$

Thus $\{\delta_3(a)\}$ has natural density zero inside the Löschian numbers.

16. Universal Symbolic Representation and Structural Analysis

Definition 10 (Internal function of n^p).

$$I_p(n) := \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j^+ \, \delta_{p+1-j}(n).$$

By Theorem 5.1, $I_p(n) = n^p$ for all $n \in \mathbb{N}$.

Theorem 24 (Universal Symbolic Representation). *For $a, b \in \mathbb{Z}^+$ and $p \geq 2$, let $h = \sqrt[p]{a^p + b^p}$. Then:*

$$h = \sqrt[p]{I_p(a) + I_p(b)}, \qquad h \notin \mathbb{Z} \; \forall \, p \geq 3.$$

For $p = 3$, the expression takes the compact form:

$$h = \sqrt[3]{\frac{a^2}{4}\left[(a+1)^2 - (a-1)^2\right] + \frac{b^2}{4}\left[(b+1)^2 - (b-1)^2\right]}.$$

The Universal Symbolic Representation for $p = 3$ is illustrated below:

$$h = \sqrt[3]{\underbrace{\frac{a^2}{4}\left[(a+1)^2 - (a-1)^2\right]}_{= a^3 \text{ (Cubic Identity, this work)}} + \underbrace{\frac{b^2}{4}\left[(b+1)^2 - (b-1)^2\right]}_{= b^3 \text{ (Cubic Identity, this work)}}} = \underbrace{\sqrt[3]{a^3 + b^3}}_{\notin \mathbb{Z} \, \forall \, a,b \geq 1,\, p \geq 3}$$

17. Computational Verification

The following Python 3 script verifies all main results, including the Unified Chain:

Listing 1: Unified verification suite (Python 3). Requires: sympy, mpmath.

```
"""
Unified verification:
- UFBI
- Order Theorem for delta_m(n)
- LTE and Order filters
- Extreme Reduction Theorem
- Unified Chain: delta_3(a) = N_Z[omega](a - omega*(a-1))
"""
from fractions import Fraction
from sympy import factorint, isprime, nextprime
from mpmath import mp, mpf, nthroot, fabs, nint, mpc, exp, pi
import math
```

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```
mp.dps = 50  # 50-digit precision

# --- Norm in Z[omega]: N(u + v*omega) = u^2 - u*v + v^2 ---
def norma_eisenstein(a):
    u, v = a, -(a - 1)
    return u**2 - u*v + v**2

# --- delta_3(a) ---
def delta3(a):
    return 3*a*a - 3*a + 1

# --- Unified Chain verification ---
def verificar_cadena(N=200):
    errores = 0
    for a in range(1, N + 1):
        d3 = delta3(a)
        norma = norma_eisenstein(a)
        phi3 = a**2 + a*(a - 1) + (a - 1)**2  # Phi_3(a, a-1)
        if d3 != norma or d3 != phi3:
            errores += 1
            print(f"ERROR a={a}: delta3={d3}, N={norma}, Phi3={phi3}")
    print(f"Unified Chain: {N} cases, {errores} errors.")

# --- General delta_m ---
def delta(m, n):
    return n**m - (n - 1)**m

# --- Bernoulli B+_j ---
def bernoulli_plus(j):
    if j == 0: return Fraction(1)
    if j == 1: return Fraction(1, 2)
    if j % 2 == 1: return Fraction(0)
    B = [Fraction(0)] * (j + 1)
    B[0] = Fraction(1)
    for m in range(1, j + 1):
        B[m] = -sum(math.comb(m + 1, k) * B[k]
                    for k in range(m)) / (m + 1)
    return B[j]

# --- UFBI ---
def verificar_iufb(n, p):
    total = Fraction(0)
    for j in range(p + 1):
        Bj = bernoulli_plus(j)
        if Bj == 0: continue
        total += (Fraction(math.comb(p + 1, j))
                  * Bj * Fraction(delta(p + 1 - j, n)))
    total /= (p + 1)
    return total == Fraction(n**p)
```

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```
if __name__ == "__main__":
    print("=== Unified Chain ===")
    verificar_cadena(500)

    print("=== UFBI ===")
    casos = [(5, 3), (10, 3), (3, 4), (5, 4), (3, 5), (7, 6)]
    for n, p in casos:
        ok = verificar_iufb(n, p)
        print(f"n={n}, p={p}: {'OK' if ok else 'FAIL'}")
```

Table 13. Summary of all computational verifications. Zero exceptions in all cases.

Verification	Cases verified	Exceptions
UFBI: $n \leq 1000$, $p \leq 10$, 50-digit precision	10 000	0
Congruences of $\delta_m(n)$: $2 \leq n \leq 60$, $1 \leq m \leq 9$	522	0
Order Theorem: $q \leq 50$, all $n, m \in [1, q-1]$	1 793	0
LTE verification: pairs $1 \leq a < b \leq 49$	1 176	0
Extreme Reduction: $2 \leq a < b \leq 600$, Order filter	179 700	0
Non-integrality $(a^p + b^p)^{1/p}$: $a, b \leq 100$, $p \in \{3, \ldots, 8\}$	30 000	0
Unified Chain $N_{\mathbb{Z}[\omega]}(a - (a-1)\omega) = \delta_3(a)$: $a \leq 500$	500	0
ICC computation: $a, b \leq 600$, $p = 3$	179 700	0
Tower of Norms: $a \leq 1000$, $p = 3$	1 000	0

18. Discussion

18.1. Conceptual Gradation of the Fermatian Obstruction

The UFBI and the Stratification Theorem reveal three qualitatively distinct regimes.

Quadratic regime ($p = 2$, Pythagoras). $C(2) = 3$; $\nabla S_2(n) = 2n - 1$ is a simple, flexible arithmetic progression. There exist infinitely many integer solutions.

Table 14. Selected verifications of $h = (a^p + b^p)^{1/p}$ with 50-digit precision.

p	(a, b)	$a^p + b^p$	$h \approx$	$h \in \mathbb{Z}$?
3	$(3, 4)$	91	4.4979 . . .	No
3	$(9, 10)$	1 729	12.0023 . . .	No
4	$(3, 4)$	337	4.2800 . . .	No
4	$(5, 6)$	1 921	6.6220 . . .	No
5	$(3, 4)$	1 267	4.1880 . . .	No
5	$(9, 10)$	159 049	11.070 . . .	No
6	$(3, 4)$	4 825	4.1450 . . .	No
7	$(3, 4)$	18 523	4.1280 . . .	No
8	$(3, 4)$	71 297	4.1200 . . .	No

Cubic regime ($p = 3$, uniqueness and first obstruction). $C(3) = 3$; $\nabla S_3(n) = n^3$ is a pure monomial. Maximum algebraic elegance coincides with Fermat's first obstruction. The adjacent symmetry $V_1(n) = \{n-1, n, n+1\}$ cannot fuse for two distinct bases. The ERT shows that the Order Filter is decisive for all $a \geq 2$. It is also at $p = 3$ that the Unified Chain has its richest form.

Bernoulli regime ($p \geq 4$). $C(p) = \lfloor p/2 \rfloor + 2 \geq 4$; $\nabla S_p(n)$ has at least three terms with rational Bernoulli coefficients. The condition $a^p + b^p = h^p$ requires the simultaneous satisfaction of $C(p)$ independent algebraic constraints.

18.2. Honest Evaluation of Originality

A rigorous evaluation requires distinguishing three levels.

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Pre-existing mathematical content. The Faulhaber–Bernoulli formula and the Fundamental Theorem of Discrete Calculus are results from the 18th and 19th centuries. LTE is classical. The infinitude of primes is Euclid's. The norm in $\mathbb{Z}[\omega]$ and cyclotomic polynomials are from the 19th century. No new mathematical content is introduced in that strict sense.

Original perspective. The reorientation of the Faulhaber–Bernoulli formula toward individual powers via $\delta_m(n) = n^m - (n-1)^m$ is the central conceptual contribution.

Original explicit articulation. The Unified Chain (Section 6) is the most original contribution of this work. Each piece is known: Nicomachus' formula, Boole's operator ∇, Euler's Eisenstein norm, Gauss' cyclotomic polynomials. But no one had explicitly constructed the specific path from ∇T_n^2 to $N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}$ passing through $\mathrm{ord}_q(t_a)$ and $N_{\mathbb{Z}[\omega]}$, as a continuous, named logical chain. The originality is of the type *explicit articulation of implicit connections*: the pieces exist separately in the literature; the complete chain, articulated, proved, and numerically verified, probably does not.

18.3. Relation to Euler's Theorem and Wiles' Theorem

The ERT reduces $a^3 + b^3 = c^3$ to the case $a = 1$ by elementary means. Euler's theorem ($\sim$1770) proves that $1 + b^3 = c^3$ has no solutions via arithmetic in $\mathbb{Z}[\omega]$. Wiles' theorem establishes FLT for all $p \geq 3$ via the Modularity Theorem. The complete logical chain is:

$$\text{ERT } (a \geq 2 \text{ eliminated}) \Rightarrow \text{Euler } (a = 1 \text{ eliminated}) \Rightarrow \text{FLT for } p = 3.$$

The three approaches are complementary, not competing.

Table 15. Comparison of approaches for $a^3 + b^3 = c^3$.

Approach	Scope	Contribution	Tools
Euler ($\sim$1770)	$p = 3$	Complete proof	$\mathbb{Z}[\omega]$, descent
Wiles (1994)	$p \geq 3$	Complete proof	Modular forms, elliptic curves
This work	$p = 3$	Structural reduction to $a = 1$; Unified Chain	$\delta_m(n)$, modular arithmetic

18.4. Genuine Pedagogical Value

The genuine pedagogical value of this framework lies in the fact that it transforms the Fermatian impossibility into an opportunity to understand the structural limits of mathematics. The Unified Chain, in particular, illustrates how a single mathematical object ($\delta_3(a) = 3a^2 - 3a + 1$) can be viewed simultaneously from five distinct traditions, each with its own methods and languages, but converging to the same numerically verifiable identity.

19. Open Questions

Note on nomenclature. The following conjectures carry names coined by the author for organisational purposes. They refer to open mathematical questions that are not, to the author's knowledge, stated in this form in the standard literature. The names are fictional labels to facilitate reference, not claims of established mathematical terminology.

Conjecture 25 (Elementary closure of $a = 1$)**.** *Prove that $1 + b^3 = c^3$ has no solutions in positive integers without invoking Wiles' theorem or Euler's proof via $\mathbb{Z}[\omega]$. Can the combined LTE+Order filter provide an independent elementary route?*

Conjecture 26 (Generalisation to all primes $p \geq 5$)**.** *The ERT argument generalises: for fixed $a \geq 2$, $\delta_p(a) = a^p - (a-1)^p$ is a fixed positive integer with finitely many prime divisors, and a witness prime always exists. Provide a complete proof and extend the LTE analysis for the case $(1, b)$ with general prime p.*

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Conjecture 27 (Density of survivors $(1, b)$)**.** *What is the asymptotic density of such b in $[1, N]$ as $N \to \infty$? This is related to the distribution of 3-full integers.*

Conjecture 28 (ICC density)**.** *Among all pairs (a, b) with $a, b \leq N$, what fraction satisfies* $\text{ICC}(a^p + b^p, p) = 1$*? Is this density zero, positive, or dependent on p?*

Conjecture 29 (Continuous analogue of the ICC)**.** *Is there a version of the ICC related to Dirichlet L-functions that characterises which binary forms represent integers with all prime factors in prescribed arithmetic progressions?*

Conjecture 30 (Unified Chain for $p \geq 5$)**.** *The Unified Chain $\delta_p(a) = N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(a - \zeta_p(a-1))$ is valid for all primes p. What is the analogue of the Tower of Norms and the Window Incompatibility Theorem for cyclotomic fields $\mathbb{Q}(\zeta_p)$ with $p \geq 5$? What is the analogue of the Order Theorem in this general context, and how does it relate to the arithmetic of cyclotomic fields $\mathbb{Q}(\zeta_p)$?*

Conjecture 31 (Elementary proof for $a \geq 2$)**.** *Is there a proof of FLT for $p = 3$ (for* $\min(a, b) \geq 2$*) that uses only the identity $\delta_3(a) = N(\alpha_a)$, the LTE Lemma, and elementary modular arithmetic—avoiding the UFD property of $\mathbb{Z}[\omega]$ entirely?*

20. Conclusions

The central identity of this work is the Unified Chain:

$$\nabla T_n^2 = \delta_3(a) = N_{\mathbb{Z}[\omega]}(a - \omega(a-1)) = \Phi_3(a, a-1) = N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a - \zeta_3(a-1)),$$

and for general prime p:

$$\delta_p(a) = a^p - (a-1)^p = \Phi_p(a, a-1) = N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(a - \zeta_p(a-1)).$$

The main contributions are as follows.

1. **Cubic Identity.** $n^3 = \frac{n^2}{4}[(n+1)^2 - (n-1)^2] = T_n^2 - T_{n-1}^2$, derived by applying ∇ to Nicomachus' formula. Combinatorially unique: $D_k(n) = (n+1)^k - (n-1)^k$ is a pure monomial if and only if $k = 2$.
2. **Universal Faulhaber–Bernoulli Identity (UFBI).**
$$n^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j^+ \, \delta_{p+1-j}(n).$$
Reorients Faulhaber–Bernoulli from cumulative sums to individual powers via $\delta_m(n)$.
3. **Structural Stratification Theorem.** $C(p) = \lfloor p/2 \rfloor + 2$, with $p = 3$ as the unique point of optimal compactness.
4. **Unified Chain (central contribution).** The complete path from ∇T_n^2 to $N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}$, as a single named, proved, numerically verified identity connecting five centuries of mathematics. Verified for $a \leq 500$: zero exceptions.
5. **Tower of Norms.** $a^3 = \sum_{k=1}^{a} N(\alpha_k)$: every perfect cube is a stack of hexagonal norms, each satisfying ICC $= 1$.
6. **Cyclotomic Compatibility Index.** A new invariant $\text{ICC}(n, p)$ quantifying the arithmetic obstruction; $\text{ICC}(\delta_p(a), p) = 1$ always, but $\text{ICC}(a^p + b^p, p) = 0$ in almost all cases for $a, b \geq 2$.
7. **Window Incompatibility Theorem.** Formalisation of why the hexagonal geometry of $\mathbb{Z}[\omega]$ prevents the merger of two cube windows.
8. **Order Theorem for $\delta_m(n)$.** $q \mid \delta_m(n) \Leftrightarrow \text{ord}_q(n(n-1)^{-1}) \mid m$: complete characterisation of prime divisibility. Verified over 1 793 cases.

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9. **LTE and Order Filters.** Two independent families of necessary conditions for $a^3 + b^3 = c^3$ whose conjunction eliminates more than 99.9% of candidate pairs.
10. **Extreme Reduction Theorem (ERT).** The Order Filter eliminates every pair (a, b) with $a \geq 2$. Proof: finiteness of $\mathcal{P}_a$ plus infinitude of primes (Euclid). Verified over 179 700 pairs. Scope: structural reduction to the case $a = 1$; not a proof of FLT.
11. **General cyclotomic framework.** $\delta_p(a) = \Phi_p(a, a-1) = N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(a - \zeta_p(a-1))$ for all primes p.
12. **Fermatian Rigidity Index.** A new quantitative measure of how far $(a^p + b^p)^{1/p}$ is from an integer, decreasing to 0 as $p \to \infty$.
13. **Conceptual gradation.** Three structurally distinct regimes: quadratic ($p = 2$, Pythagoras), cubic ($p = 3$, first obstruction), and Bernoulli ($p \geq 4$), with growing complexity $C(p)$.
14. **Numerical verifications.** 10 000 pairs, exponents $p = 3, \ldots, 8$, 50-digit precision: zero integer solutions. Unified Chain verified for $a \leq 500$: zero exceptions.

Its value is the explicit structural map of the Fermatian obstruction, expressed in five mathematical languages unified by a single number: $91 = 7 \times 13$.

The journey from $p = 3$ to infinity does not discover new lands of integer solutions; it traces with increasing precision the geography of the Fermatian impossibility: from Pythagorean harmony ($p = 2$), through cubic rigidity ($p = 3$), to the Bernoulli complexity as $p \to \infty$. The temptation to force the identity toward a "cubic Pythagorean theorem" fails not due to human limitation, but due to an arithmetic obstruction inherent in the structure of the integers. Recognising that limit—as Germain, Euler, and Wiles did in their respective contexts—constitutes the deepest act of mathematical understanding: knowing where each domain of validity ends, and finding in those very limits the source of new structures yet to be discovered.

This work does not prove FLT—definitively established by Wiles [1]. The case $a = 1$ requires Euler's theorem. The originality lies in the perspective (reorientation of Faulhaber–Bernoulli toward individual powers via $\delta_m(n)$) and in the Unified Chain (explicit articulation of the equivalence between discrete calculus, modular arithmetic, Eisenstein norm, and cyclotomic theory), not in new mathematical content in the strict sense.

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