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Multivector Time Generator of Evolution in Phase Plane

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Abstract

The nature of time and its role in physical evolution remain central open questions in theoretical physics, particularly in the presence of irreversibility. In this study, a geometric framework for time evolution is introduced based on a multivector time generator acting on the phase plane. Rather than extending time as a parameter, this approach focuses on the structure of the time derivative and its associated symmetries. Using geometric algebra, the generator decomposes naturally into scalar, bivector, and vector components. The bivector part generates Hamiltonian, symplectic evolution and corresponds to reversible dynamics, while the scalar part produces uniform contraction or expansion and provides a geometric interpretation of irreversibility and entropy production. In addition, vector components generate reversible but anti-symplectic transformations, such as reflections, revealing symmetries that are not captured by the standard Hamiltonian or complex-time approaches. The general solutions of linear systems follow an exponential form, and the reversible generator admits a natural classification into elliptic, hyperbolic, and nilpotent cases, yielding a clear geometric interpretation of oscillatory, overdamped, and critically damped behavior. The framework further clarifies the status of complex time, showing that it arises only as a restricted case when vector components are absent, and the time derivative admits a Wirtinger representation. Outside this regime, the time evolution is a multivector and cannot be described by a single complex parameter. Overall, the proposed framework provides a unified geometric language for analyzing reversible and irreversible dynamics, and highlights the central role of the time generator in shaping temporal evolution.

Keywords: multivector time generator; geometric algebra; symplectic geometry; irreversible dynamics; complex time; time symmetry

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1. Introduction

Time is one of the most fundamental, yet conceptually elusive concepts in physics [1–4]. Historically, it was first introduced through the concept of absolute time, most clearly articulated in classical mechanics, where a single universal time parameter t identically controls dynamical evolution for all observers and remains invariant under Galilean transformations [5,6]. This absolute time enabled a clean formulation of dynamics. It was the discovery of relativity that profoundly altered this picture. In the spacetime formulation introduced by Albert Einstein, time is no longer universal, but becomes a coordinate similar to space, forming a four-dimensional geometric structure (i.e., the spacetime) in which events are specified by relativity equations rather than by their evolution with respect to an external time parameter [7,8]. Within this geometric setting, coordinates serve to label events and worldlines, while "time flow" is not a statement with invariant mathematical content. The conceptual tension about time reappears in quantum mechanics, where time is not treated as an operator (as position or momentum are treated) but as an external parameter governing evolution [9–11]. This becomes acute in quantum gravity. There, the Hamiltonian constraint leads to timeless equations of the form $\hat{H}\psi = 0$, suggesting the absence of a fundamental time parameter [12–15]. This structural mismatch—time as an absolute parameter in classical mechanics, a coordinate in general relativity, an external parameter in quantum mechanics, and effectively absent in quantum

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gravity—constitutes the core of the modern problem of time, extensively discussed in both physics and philosophy-of-physics literature.

A second, closely related difficulty concerns the distinction between the time symmetry of the laws and the observed irreversibility of the physical processes [1,16]. Most equations of motion are invariant under time reversal; for example, in Hamiltonian mechanics, Hamilton's equations are invariant under time reversal $t \to -t$ resulting in $q \to q$ for position and $p \to -p$ for momentum [17,18]. In contrast, macroscopic phenomena, such as friction, viscosity, diffusion, and thermal relaxation, are irreversible at the macroscopic level and define a preferred temporal direction [19,20]. This asymmetry enters physics through thermodynamics, specifically via the Second Law and the concept of entropy (i.e., the total entropy of an isolated system naturally increases or remains the same, never spontaneously decreasing), originally formulated by Ludwig Boltzmann and later developed in statistical and nonequilibrium contexts [21]. The resulting arrow of time is not a property of spacetime geometry itself but an emergent feature arising from the special low-entropy boundary conditions of the universe. Extensive work, from Boltzmann's reversibility debates [1] to the non-equilibrium perspective of Ilya Prigogine [22,23], has emphasized that irreversibility reflects the structure of macroscopic evolution rather than the breakdown of microscopic laws. More recent approaches, such as Carlo Rovelli and Alain Connes's thermal time hypothesis [24–26], further reinforce the view that time's direction and physical relevance emerge from state-dependent generators of evolution rather than from time itself. From this perspective, the arrow of time is not embedded in spacetime but encoded in the dynamical mechanisms—reversible and irreversible—that govern evolution, motivating frameworks that focus on time generators and their algebraic structure rather than on time as a flowing entity.

Given the structural ambiguity of time across physical theories, it is essential to identify the mathematical space in which dynamic evolution can be modeled naturally. In classical mechanics, this role is not played by the configuration space, defined only by generalized coordinates q, which specify positions but do not fully characterize the state of the system, but by the phase space, whose points (q, p) provide a complete instantaneous description through the inclusion of conjugate momenta [27]. The variables q and p correspond to distinct degrees of freedom and can therefore be treated as independent coordinates. However, their independence is specified by a canonical structure: the phase space is endowed with a symplectic form that defines the conjugate relation between q and p and establishes mutually complementary directions, commonly represented as orthogonal in the phase plane [28,29]. Within this framework, dynamical evolution is described as a flow generated in the phase space, with time appearing only as a parameter that characterizes the trajectories of this flow rather than as the source of evolution itself [30]. The symplectic formulation clarifies how conservation laws, reversibility, and stability arise from the geometric structure of the generator, independently of coordinate choices. In this manner, the essential features of evolution, particularly the distinction between reversible and irreversible dynamics, are determined by the properties of the generator acting on the phase space and not by time. Phase space thus provides a minimal and natural arena for analyzing dynamical evolution and examining how modifications of the generator lead to different forms of dynamical behaviors [31].

A closely related and well-established approach to the study of Hamiltonian dynamics is provided by linear algebra, where the evolution of classical systems is represented by linear operators acting on the phase space [32]. In this formulation, Hamiltonian systems are characterized by matrices that preserve the canonical symplectic structure, leading to precise algebraic criteria for reversibility. In particular, for linear systems of the form $\dot{x} = Ax$, the generator A is Hamiltonian if and only if it satisfies the condition $A^T J + JA = 0$, where J denotes the canonical symplectic matrix. This condition guarantees the preservation of the phase-plane area and is equivalent to the requirement that the associated flow must be symplectic and hence reversible. Such criteria are classical results in Hamiltonian matrix theory and underpin much of the modern theories of dynamical systems, stability analysis, and canonical transformations [28,31,33]. Within this linear-algebraic framework, reversibility and irreversibility are already distinguished at the level of the generator itself; deviations from the Hamiltonian condition correspond to non-symplectic evolution and phase-plane contraction or expansion. This perspective highlights that the fundamental distinction between reversible and irreversible dynamics can be expressed algebraically, independent of any particular interpretation of time, and provides a natural bridge between classical symplectic theory and more general formulations of time evolution developed later in this work.

Another aspect of time that has been explored in several studies and should be discussed here is its complexification, that is, whether time can be imaginary or even complex. Imaginary time has generally been regarded as a mathematical trick, i.e., an analytic continuation through Wick rotation. In high-energy and statistical physics, Wick rotation plays a central role by relating the Minkowski-time framework to the Euclidean framework. Replacing $t \mapsto i\tau$ allows path integrals to be reinterpreted as partition functions, thereby establishing a bridge between quantum and statistical mechanics [34,35]. Van Nieuwenhuizen and Waldron [36] showed that the Wick rotation can be understood geometrically as a complex Lorentz boost in a higher-dimensional space, providing a continuous prescription for both bosons and fermions. Their work illustrates that Wick rotation is not just a computational trick, but a well-defined transformation that acts uniformly on fundamental fields. In the context of quantum cosmology and black hole thermodynamics, complex time has also appeared to be more than a formal analytical tool. In a series of influential works, Stephen Hawking and James Hartle [37–39] showed that imaginary time plays a central role in defining well-posed quantum states of spacetime. In particular, the no-boundary formulation that they proposed replaces a singular temporal origin with a smooth Euclidean geometry, obtained via Wick rotation, thereby allowing the quantum state of the universe to be defined without imposing ad hoc initial conditions.

At a more conceptual level, El Naschie [40] was among the first to introduce the idea of conjugate complex time to address the irreversibility problem. By considering both $+it$ and $-it$, he argued that time may emerge as the intersection of forward and backward complex directions. This reconciles the unitary, reversible character of microscopic quantum evolution with the irreversible arrow of time observed macroscopically, in agreement with thermodynamics and the interpretation of quantum mechanics. In this view, complex time provides a natural language for dualities between reversibility and irreversibility, quantum and classical, micro and macro. Recent work in Quantitative Geometrical Thermodynamics (QGT) [19,41–44] has further emphasized the duality between reversible (entropy-free) and irreversible (entropy-producing) dynamics. By associating reversible motion with rotations and dissipative motion with scalar transformations, QGT achieves a geometrical decomposition that closely parallels the roles of the real and imaginary axes of complex time. This suggests that imaginary time could serve as a mathematical encoding of entropy-free evolution, whereas real time captures entropy production, providing a bridge between thermodynamics and geometry.

Geometric algebra provides a unified mathematical framework in which scalars, vectors, complex numbers, and higher-grade geometric objects are treated within a single associative algebra [45,46]. Originating in the work of William Kingdon Clifford, geometric algebra was developed for the synthesis of complex numbers, quaternions, and vector algebra and has since been shown to offer compact and geometrically transparent formulations of classical mechanics, electromagnetism, quantum mechanics, and relativity [47,48]. In this approach, the imaginary unit arises naturally as an oriented area element (a bivector) in 2D geometric algebra rather than as an abstract scalar, thereby giving complex structures a direct geometric meaning. This feature makes geometric algebra particularly well suited for describing rotations, boosts, and reflections as intrinsic geometric operations without the need to use matrix representations or ad hoc complexification. In a series of seminal papers [49–54], David Hestenes demonstrated how geometric algebra provides a real, coordinate-free reformulation of quantum mechanics and relativistic physics, clarifying the geometric content of spinors, Dirac theory, and Lorentz transformations. From this perspective, geometric algebra offers a natural setting for extending complex-variable methods and expressing generalized generators of evolution as geometric objects, a viewpoint that will be adopted and developed in the present work.

Based on the above discussion, the present study introduces a geometric formulation of time evolution based on a multivector time generator, providing a unified description of reversible and irreversible dynamics within the phase plane. Rather than treating time as a classical scalar parameter or as a complex/imaginary quantity, the time derivative itself is reinterpreted as a geometric object in 2D geometric algebra, whose algebraic structure encodes distinct contributions to evolution. Within this framework, Hamiltonian (reversible) dynamics arise naturally from bivector components associated with symplectic rotations, whereas entropy-producing (irreversible) processes are associated with scalar components corresponding to contractions in phase space. Interestingly, the multivector formulation reveals additional geometric transformations associated with the vector components of the multivector generator, which are not accessible within standard complex time descriptions. In this sense, commonly used concepts of complex or imaginary time emerge as restricted cases of a more general generator structure. By replacing the matrix representation with a multivector in geometric algebra and embedding time evolution directly in this algebra, the proposed framework clarifies the geometric origin of reversibility/irreversibility and provides a natural extension of the symplectic and linear-algebraic formulations of Hamiltonian systems. Moreover, this work characterizes the symmetries of temporal transformations by identifying the time evolution with the action of a multivector generator in the phase plane. Within this framework, temporal change and the arrow of time acquire a precise meaning through the structure of the reversible and irreversible components of the generator, while the notion of time flow is understood as the geometric effect of these transformations rather than as a primitive property of time itself.

The remainder of this paper is organized as follows: In Section 2, geometric algebra $Cl(2,0)$ is introduced. Section 3 establishes the isomorphism between linear generators and multivector representations. Section 4 examines the dynamics of the phase plane by using both linear and geometric algebra. In Section 5, a multivector time generator is defined, and its basic geometric properties are discussed. Section 6 introduces the decomposition into reversible and irreversible components. Section 7 applies the framework to representative dynamical systems, including the free particle and the harmonic oscillator, highlighting the geometric structure of the solutions. Section 8 discusses the role of symmetries and antisymplectic transformations, along with the interpretation of Poisson brackets using the multivector time generator. Section 9 analyzes the conditions under which complex time emerges as a restricted case of the multivector. Finally, the Discussion and Conclusions sections summarize the main results, address limitations, and outline possible directions for future work.

2. Preliminaries of Geometric Algebra $Cl(2,0)$

The geometric algebra $Cl(2,0)$ is the associative algebra generated by two orthonormal basis vectors e_1 and e_2 satisfying the Euclidean metric [47,49–54]:

- $e_1^2 = +1$
- $e_2^2 = +1$
- $e_1 e_2 = -e_2 e_1$.

From these relations, the algebra contains four basis elements:

- Scalar: 1
- Vectors: e_1, e_2
- Bivector: $J = e_1 e_2$

Every element, i.e., a multivector $M \in Cl(2,0)$ can be written as a linear combination of the four basis elements as follows:

$$M = a1 + be_1 + ce_2 + de_1 e_2, \tag{2.1}$$

where a, b, c, d are real numbers.

Bivector $J = e_1 e_2$ represents an oriented area element in the $e_1 - e_2$ plane. Because the basis vectors anticommute, J obeys:

$$J^2 = (e_1 e_2)^2 = -e_1^2 e_2^2 = -1. \tag{2.2}$$

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Thus, $Cl(2,0)$ naturally contains an element that behaves exactly like the imaginary unit, except that it has a clear geometric meaning: it represents a 90° rotation in the $e_1 - e_2$ plane.

If $v = v_1 e_1 + v_2 e_2$ and $w = w_1 e_1 + w_2 e_2$ are two vectors, then their geometric product decomposes into a symmetric inner product (the dot product) and an antisymmetric wedge product (an oriented area) as follows:

$$vw = v \cdot w + v \wedge w = (v_1 w_1 + v_2 w_2) + (v_1 w_2 - v_2 w_1) e_1 e_2. \quad (2.3)$$

The first term is scalar and the second term is the bivector J. This decomposition is fundamental for interpreting the phase plane geometry presented in this study.

Complex numbers appear naturally as even subalgebra of $Cl(2,0)$ generated by 1 and bivector J as elements of the form:

$$z = a + dJ, \quad (2.4)$$

thus, $\mathbb{C} \cong Span\{1, e_1 e_2\} \subset Cl(2,0)$. Similarly, hyperbolic (split-complex) numbers arise as the subalgebra generated by 1 and the vector e_1, which squares to $+1$ as elements of the form

$$\zeta = a + b e_1. \quad (2.5)$$

In this subalgebra, we have:

$$e_1^2 = +1, \qquad (a + b e_1)(c + d e_1) = (ac + bd) + (ad + bc) e_1, \quad (2.6)$$

which is the standard multiplication rule for hyperbolic numbers. Thus, $\mathbb{H}_{split} \cong Span\{1, e_1\} \subset Cl(2,0)$.

3. Isomorphism Between $Cl(2,0)$ and the Real Matrix Algebra $M_2(\mathbb{R})$

As shown in the previous section, the geometric algebra $Cl(2,0)$ is a four-dimensional real associative algebra. The real matrix algebra $M_2(\mathbb{R})$ is also four dimensional. There exists a well-defined algebraic isomorphism between them, which allows any multivector in $Cl(2,0)$ to be represented uniquely as a 2×2 real matrix [55]. This correspondence allows for the recasting of standard linear phase-plane dynamics in geometric algebra terms.

Let the basis elements be of $Cl(2,0)$ as described in Section 2. A linear map is defined as follows:

$$\Phi: Cl(2,0) \to M_2(\mathbb{R}) \quad (3.1)$$

on the basis by:

$$\Phi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Phi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\Phi(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Phi(J) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.2)$$

These images form a basis in $M_2(\mathbb{R})$, and thus, Φ is a bijection. It is easy to verify that $\Phi(1)$ is actually the identity 2×2 matrix I, whereas $\Phi(J) = \Phi(e_1)\Phi(e_2)$, which ensures that the anticommutation of e_1 and e_2 is preserved in matrix form. $\Phi(J)$ is the standard generator of 90° rotations in the plane, and satisfies $\Phi(J)^2 = -I$, matching $J^2 = -1$.

For a general multivector M as given in Equation (2.1), map Φ yields

$$\Phi(M) = \begin{pmatrix} a + b & c + d \\ c - d & a - b \end{pmatrix}. \quad (3.3)$$

This establishes a one-to-one correspondence between the coefficients a, b, c, d and a unique 2×2 matrix.

It should be emphasized that the mapping Φ is not merely a vector-space isomorphism; it preserves the full algebraic structure $\Phi(MN) = \Phi(M)\Phi(N)$, because Φ contains all multiplication rules of $Cl(2,0)$, that is, $e_1^2 = 1$, $e_2^2 = 1$, and $e_1 e_2 = -e_2 e_1$. Moreover, Φ is defined such that the multiplication of the basis elements corresponds to the matrix multiplication of their images. Because 1, e_1, e_2, and J generate the entire algebra, the preservation of the multiplication on this basis ensures preservation for all multivectors. Thus, Φ is the algebraic isomorphism shown in Equation (3.1).

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Both complex and hyperbolic numbers can be represented as 2×2 matrices that are easily derived from the general matrix $\Phi(M)$ (see Equation 3.3). Thus, elements of the form of Equation (2.4) are represented as

$$\Phi(z) = \begin{pmatrix} a & d \\ -d & a \end{pmatrix}, \tag{3.4}$$

while elements of the form of Equation (2.5) are represented as:

$$\Phi(\zeta) = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}. \tag{3.5}$$

Both z and ζ are equivalent to the forms seen in Equations (3.4) and (3.5) because matrix representations are defined only up to similarity transformations and reproduce the corresponding standard multiplication rules.

Another aspect of Φ that should be discussed is that the matrices appearing under the latter coincide with the real forms of Pauli matrices. If we define:

- $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $\tilde{\sigma}_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

then the isomorphism identifies:

$$e_1 \leftrightarrow \sigma_z, \qquad e_2 \leftrightarrow \sigma_x, \qquad J = e_1 e_2 \leftrightarrow -\tilde{\sigma}_y. \tag{3.6}$$

Thus, the basis $\{1, e_1, e_2, J\}$ corresponds directly to the familiar set $\{1, \sigma_z, \sigma_x, -\tilde{\sigma}_y\}$ which spans all real 2×2 matrices. In this representation, the bivector J plays the role of a purely real analog of the imaginary Pauli matrix σ_y. This provides a familiar matrix language for the multivector time generator developed in the next section, helping to relate its geometric structure to well-known operators in physics.

4. Classical Phase-Plane Dynamics in Matrix and Geometric Algebra Form

A general two-dimensional system of linear differential equations in the phase plane can be written as follows [33,56]:

$$\frac{dq}{dt} = \alpha q + \beta p, \qquad \frac{dp}{dt} = \gamma q + \delta p, \tag{4.1}$$

where q and p represent the position and momentum (or position–velocity), respectively. This is equivalently expressed in matrix form as

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}, \tag{4.2}$$

or,

$$\frac{d\boldsymbol{X}}{dt} = A\boldsymbol{X}, \tag{4.3}$$

where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is the 2×2 coefficient matrix above and $\boldsymbol{X} = (q, p)$ is the coordinate vector of the two independent variables q and p.

The coordinate vector $\boldsymbol{X}$ can be represented as a geometric vector in $Cl(2,0)$:

$$X(t) = q(t)e_1 + p(t)e_2. \tag{4.4}$$

This method embeds the phase plane into the vector subspace spanned by e_1 and e_2. Because $e_1^2 = e_2^2 = +1$, the Euclidean geometry of the plane is inherited naturally.

$$X^2 = q^2 + p^2. \tag{4.5}$$

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As expected, the bivector $J = e_1 e_2$ is the oriented unit area element of this plane. In differential-geometric terms, thid corresponds to the canonical symplectic form

$$\omega = dq \wedge dp \quad \leftrightarrow \quad J. \tag{4.6}$$

Thus, $Cl(2,0)$ provides a direct algebraic encoding of the symplectic geometry underlying Hamiltonian mechanics. Multiplication by J generates a 90° rotation in the phase plane, mirroring the Hamiltonian flow structure.

Using the algebraic isomorphism $\Phi^{-1}: M_2(\mathbb{R}) \to Cl(2,0)$ as established in Section 3, the matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ corresponds uniquely to the multivector $M = a1 + be_1 + ce_2 + dJ$ with coefficients:

$$a = \frac{\alpha + \delta}{2}, \qquad b = \frac{\alpha - \delta}{2}, \qquad c = \frac{\beta + \gamma}{2}, \qquad d = \frac{\beta - \gamma}{2}. \tag{4.7}$$

Because Φ preserves multiplication, the matrix evolution $\dot{\boldsymbol{X}} = A\boldsymbol{X}$ is equivalent to the geometric-algebra evolution:

$$\dot{X} = MX. \tag{4.8}$$

In this representation, the components of M have clear geometric meaning:

- Scalar term a is isomorphic to the identity matrix, encoding isotropic dilation or decay,
- The bivector term dJ is isomorphic to the rotation matrix encoding the rotation (oscillation).
- The vector terms $be_1 + ce_2$ are isomorphic to the reflection matrices encoding reflections across the q-axis and line $q = p$, respectively.

This decomposition reveals the geometric character of the multivector operator in a manner associated with the matrix form. In the next section, this structure is used to reinterpret the time derivative as a multivector operator, thereby separating its reversible and irreversible components.

5. The Multivector Time Generator

In the previous section, it was established that the matrix evolution law $\dot{\boldsymbol{X}} = A\boldsymbol{X}$ can be rewritten in terms of geometric algebra form as $\dot{X} = MX$ with $M = a + be_1 + ce_2 + dJ$, where the coefficients a, b, c, and d are uniquely determined by the entries of matrix A through Equation (4.7). This shows that the operator associated with the time evolution is not merely a matrix but also a geometric object in $Cl(2,0)$ as well. This also proves that the generator has geometrically meaningful components. Thus, it is proposed that the time-derivative operator itself can be interpreted as a multivector acting on X, i.e.:

$$\frac{d}{dt} \equiv M. \tag{5.1}$$

This interpretation is not literal in the sense of replacing the differential operator, but structural: the time derivative operator acts as if it were generated by the multivector M. Moreover, through the isomorphism of M with A, the time derivative is not simply a scalar operator, but is enriched with specific phase-plane transformations associated with the geometric meaning of the matrices Φ (the bases of matrix A, see Equation 3.2). These transformations are associated with either reversible or irreversible processes depending on the specific conditions discussed below.

5.1. Reversibility Conditions

Classical reversible dynamics preserve the canonical oriented area element J (see Equation 4.6), thereby preserving the oriented area in the phase plane. In this manner, it is compatible with the symplectic structure [28,31–33]. In Hamiltonian matrix theory, a constant matrix A is considered as a Hamiltonian (i.e., it preserves the symplectic structure) if and only if

$$A^T J + JA = 0, \tag{5.2}$$

where A^T is the transpose of matrix A. Using the linear map of J, $\Phi(J)$ (see Equation 3.2), Equation (5.2) can be written in matrix form as follows:

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$$\begin{pmatrix} 0 & \alpha+\delta \\ -(\alpha+\delta) & 0 \end{pmatrix} = 0. \tag{5.3}$$

Thus, the Hamiltonian condition holds if and only if:

$$\alpha + \delta = 0, \tag{5.4}$$

i.e.,

$$\mathrm{tr}A = 0. \tag{5.5}$$

Equation (5.5) characterizes the area-preserving (symplectic) evolution and is a classical criterion for reversibility [33]. It is interesting to show how all these appear naturally and equivalently in the geometric algebraic form of the phase plane dynamics.

As discussed in Section 4, the canonical symplectic form in the phase plane is represented by the bivector $J = e_1e_2$. Thus, a linear flow preserves the oriented area if and only if it preserves this bivector. For the evolution generated by the multivector M, Equation (5.2) is translated into $Cl(2{,}0)$ as

$$\widetilde{M}J + JM = 0, \tag{5.6}$$

where $\widetilde{M} = a + be_1 + ce_2 - dJ$ is the conjugate of M. It is easy to prove that A^T corresponds to $\widetilde{M}$ (see Appendix A). A straightforward computation yields the following:

$$\widetilde{M}J + JM = 2aJ, \tag{5.7}$$

thus,

$$2aJ = 0 \Leftrightarrow a = 0. \tag{5.8}$$

This is exactly the same condition as in Equation (5.4), that $a = \frac{\alpha+\delta}{2}$ (Equation 4.7). Hence, the area preservation in $Cl(2{,}0)$ is identical to the scalar-free condition $a = 0$.

5.2. Geometric Meaning of the Multivector Components

With the reversibility condition established in the previous subsection, each term of the multivector $M = a + be_1 + ce_2 + dJ$ can now be interpreted according to its geometric role and contribution to irreversible (or reversible) processes.

- **Bivector part dJ**

The bivector $J = e_1e_2$ generates a 90° rotation in the phase plane, corresponding to the canonical Hamiltonian structure. It satisfies:

$$JJ = -1. \tag{5.9}$$

Considering that $\tilde{J} = -J$, it is easy to prove that Equation (5.6) holds for dJ as

$$\widetilde{(dJ)}J + J(dJ) = 0. \tag{5.10}$$

Thus, this component always preserves the area and corresponds to a reversible oscillatory motion, such as an undamped harmonic oscillator (see Section 7.3).

- **Vector parts be_1, ce_2**

The vectors e_1 and e_2 correspond (using the linear map Φ) to the matrices $\Phi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\Phi(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (see Equation 3.2). Both are reflections; $\Phi(e_1)$ corresponds to the reflection along the q-axis, whereas $\Phi(e_2)$ corresponds to the reflection along the line $q = p$ (flipping q and p axes). Reflections are isometries, and although they are not generated by rotors in the GA, they are still reversible linear transformations; each reflection is its own inverse. Considering that $\widetilde{e_i} = e_i$, $i \in \{1, 2\}$, it is easy to prove that

$$\widetilde{e_i}J + Je_i = 0, \qquad i \in \{1, 2\}. \tag{5.11}$$

Hence, Equation (5.6) also holds for the vector parts of the multivector M, be_1 and ce_2. Thus, the vector terms of M are reversible.

- **Scalar part $a1$**

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The scalar part of the multivector M corresponds to the matrix $\Phi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, that is, the 2×2 identity matrix, which produces a uniform expansion or contraction of the phase-plane area. One can immediately see that by applying Equation (5.6) to $a1$, the following is derived.

$$\tilde{\alpha}J + J\alpha = 2aJ \neq 0, \tag{5.12}$$

which leads to the main condition required for Equation (5.6) to hold, that is, $a = 0$ (Equation 5.8). Thus, a non-zero a violates Equations (5.6) and (5.8) and therefore generates irreversible, non-symplectic evolution. It is concluded that *the scalar part is the unique source of irreversibility in the multivector generator* M.

6. General Solution and Geometric-Algebra Interpretation

Once the time evolution has been expressed in geometric-algebraic form as:

$$\dot{X} = MX, \tag{6.1}$$

where $M = a + be_1 + ce_2 + dJ$ is the multivector time generator, which follows immediately by analogy to the matrix case. For constant coefficients a, b, c, and d in M, the solution is given by the exponential operator

$$X(t) = e^{Mt}X_0, \tag{6.2}$$

where $X_0 = X(0)$ denotes the initial phase-plane vector. Through algebraic isomorphism $\Phi: Cl(2,0) \to M_2(\mathbb{R})$, this solution is equivalent to the classical matrix solution:

$$X(t) = e^{At}X_0. \tag{6.3}$$

However, the geometric-algebra formulation provides a direct interpretation of the exponential in terms of the transformations (rotations, reflections, and dilations) acting on the phase plane.

Let the multivector time generator be decomposed as:

$$M = a1 + M_{rev}, \qquad M_{rev} = be_1 + ce_2 + dJ, \tag{6.4}$$

where, as shown in Section 5, the scalar part a generates irreversible dilation or decay, whereas M_{rev} is the reversible (Hamiltonian) part satisfying Equation (5.6) as $\widetilde{M_{rev}}J + JM_{rev} = 0$. Because the scalar commutes with all elements of the algebra, the exponential separates as:

$$e^{Mt} = e^{at}e^{M_{rev}t}, \tag{6.5}$$

Thus, the general solution in Equation (6.2) takes the form:

$$X(t) = e^{at}e^{M_{rev}t}X_0. \tag{6.6}$$

This factorization clearly separates irreversible amplitude modulation from reversible geometric evolution.

6.1. Reversible Evolution Operator

The reversible time generator M_{rev} is a linear combination of elements whose squares are ± 1 and are mutually anticommute in pairs. Consequently, its square is reduced to a scalar.

$${M_{rev}}^2 = b^2 + c^2 - d^2 \equiv \kappa^2. \tag{6.7}$$

Depending on the sign of κ^2, exponential $e^{M_{rev}t}$ takes one of the standard closed forms.

- *Elliptic (oscillatory) case* $(\kappa^2 < 0)$:

$$e^{M_{rev}t} = \cos(\omega_o t) + \frac{M_{rev}}{\omega_o}\sin(\omega_o t), \quad \omega_o = \sqrt{-\kappa^2}. \tag{6.8}$$

- *Hyperbolic case* $(\kappa^2 > 0)$:

$$e^{M_{rev}t} = \cosh(\kappa t) + \frac{M_{rev}}{\kappa}\sinh(\kappa t). \tag{6.9}$$

- *Nilpotent case* $(\kappa^2 = 0)$:

$$e^{M_{rev}t} = 1 + M_{rev}t. \tag{6.10}$$

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In all the cases, $e^{M_{rev}t}$ represents a reversible area-preserving linear transformation of the phase plane.

Equation (6.6) admits a clear geometric interpretation; the factor e^{at} produces uniform dilation or decay of phase plane trajectories and is responsible for irreversibility; the operator $e^{M_{rev}t}$ produces pure geometric motion: rotations – via the bivector J, reflections – via the vectors e_1 and e_2, or combinations thereof. Thus, the multivector exponential acts as a generalized propagator that combines oscillatory, parity-changing, and dissipative effects in a single compact form. Therefore, the solution $X(t) = e^{Mt}X_0$ may be viewed as a time-ordered geometric action on the initial state X_0 generated by the multivector M. The algebraic structure of M directly determines whether the evolution is reversible or irreversible, whether trajectories rotate, reflect, or exponentially contract/expand, and how the classical Hamiltonian behavior is modified by additional geometric contributions. This interpretation will be made explicit in the next section through concrete examples, including the free particle and the harmonic oscillator.

7. Illustrative Examples

In this section, the multivector time-generator framework developed in Sections 5 and 6 is applied to two representative systems: the free particle and the harmonic oscillator with and without damping. These examples illustrate how the geometric-algebra formulation naturally separates the reversible and irreversible components of the dynamics and provides a compact interpretation of the time evolution operator.

7.1. Free Particle

Consider a free particle in one dimension without the influence of an external force governed by

$$\dot{q} = \frac{p}{m}, \quad \dot{p} = 0, \tag{7.1}$$

where m denotes the particle mass. In matrix form, the system is

$$\dot{X} = AX, \quad A = \begin{pmatrix} 0 & 1/m \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} q \\ p \end{pmatrix}. \tag{7.2}$$

Using the isomorphism $\Phi^{-1}: M_2(\mathbb{R}) \to Cl(2,0)$ (Equation 4.7), the corresponding multivector generator is:

$$M = \frac{1}{2m}e_2 + \frac{1}{2m}J. \tag{7.3}$$

Because $a = 0$, this generator consists only of reversible parts, that is $M = M_{rev}$. Moreover, it satisfies ${M_{rev}}^2 = 0$, and therefore represents a nilpotent case. Hence, the general solution follows from Section 6.

$$X(t) = e^{Mt}X_0 = (1 + Mt)X_0, \tag{7.4}$$

where $X_0 = q_0e_1 + p_0e_2$ is the initial phase-plane vector with $q_0 = q(0)$ and $p_0 = p(0)$ denoting the initial conditions for position and momentum, respectively. Equation (7.4) can be explicitly written as

$$X(t) = \begin{pmatrix} q_0 + p_0t \\ p_0 \end{pmatrix}. \tag{7.5}$$

Equation (7.5) represents the standard free-particle solution.

Several key points emerge immediately. As mentioned above, the scalar part of M vanishes ($a = 0$), so the evolution, as expected, is reversible. In addition, the generator $M = \frac{1}{2m}e_2 + \frac{1}{2m}J$ satisfies Equation (5.6), preserving the area of the phase plane. Finally, although M is not an even multivector (i.e., not a rotor consisting only of a bivector part), it still generates reversible dynamics. Thus, the free particle provides a clear example showing that reversibility is not tied to the even subalgebra but to the symplectic condition, that is, Equation (5.6).

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7.2. Free Particle with Linear Friction

Here, a minimal extension of the free particle discussed in the previous subsection was considered by introducing a constant linear friction force proportional to the velocity. Other external driving forces were ignored. The equation of motion is written as

$$m\ddot{q} = -\mu\dot{q}, \tag{7.6}$$

where $\mu > 0$ denotes the constant friction coefficient. Using the definition of momentum $p = m\dot{q}$, Equation (7.6) is equivalent to the first-order system:

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ 0 & -\mu/m \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}. \tag{7.7}$$

By comparing Equation (7.8) with the general linear system of Equation (4.3), the corresponding matrix generator $A = \begin{pmatrix} 0 & 1/m \\ 0 & -\mu/m \end{pmatrix}$ is immediately identified. Using the algebraic isomorphism $\Phi^{-1}: M_2(\mathbb{R}) \to Cl(2,0)$ (Equation 4.7), the associated multivector time generator M is obtained as

$$M = -\frac{\mu}{2m} + \frac{\mu}{2m}e_1 + \frac{1}{2m}e_2 + \frac{1}{2m}J. \tag{7.8}$$

Using the definition of the reversible time generator M_{rev} (Equation 6.7), M in Equation (7.8) can be written as:

$$M = -\frac{\mu}{2m} + M_{rev}, \qquad M_{rev} = \frac{\mu}{2m}e_1 + \frac{1}{2m}e_2 + \frac{1}{2m}J. \tag{7.9}$$

This decomposition has a direct geometric interpretation. The reversible part M_{rev} is identical to that of the multivector generator of the free particle derived in Equation (7.3), except for the additional vector component $\frac{\mu}{2m}e_1$. As expected, all the components of M_{rev} satisfy the symplectic condition (Equation 5.6), corresponding to reflections and rotations in the phase plane. The scalar term $-\frac{\mu}{2m}$ represents the only contribution that breaks the symplectic condition and introduces phase-space contraction. The additional vector component $\frac{\mu}{2m}e_1$ in M_{rev} is also associated with the geometrical meaning of friction, which is a direct consequence of the anisotropic nature of the latter, breaking the symmetry between the position and momentum directions in the phase plane. The scalar part alone is responsible for phase-plane contraction and irreversibility, whereas the vector part encodes the directional character of the dissipative mechanism. Further discussion considering the vector parts of the multivector time generator can be found in Section 10.

Considering the square of the reversible time generator ${M_{rev}}^2$, it is calculated as:

$${M_{rev}}^2 = \left(\frac{\mu}{2m}\right)^2 \tag{7.10}$$

which is a positive scalar. Therefore, according to the classification introduced in Section 6.1, the reversible evolution belongs to the hyperbolic case. Consequently, the exponential $e^{M_{rev}t}$ is expressed as:

$$e^{M_{rev}t} = \cosh\left(\frac{\mu t}{2m}\right) + \frac{2m}{\mu}M_{rev}\sinh\left(\frac{\mu t}{2m}\right). \tag{7.11}$$

Because the scalar part of M commutes with all multivector components, the evolution operator factorizes as

$$e^{Mt} = e^{-\frac{\mu t}{2m}}e^{M_{rev}t} = e^{-\frac{\mu t}{2m}}\left[\cosh\left(\frac{\mu t}{2m}\right) + \frac{2m}{\mu}M_{rev}\sinh\left(\frac{\mu t}{2m}\right)\right]. \tag{7.12}$$

At first glance, the appearance of hyperbolic functions may seem to complicate the solution. However, when the scalar decay factor is combined with the hyperbolic form of Equation (7.11), e^{Mt}

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is combined at a single exponential rate. Using the standard identities for hyperbolic functions as follows:

$$e^{-x}\cosh x = \frac{1}{2}(1+e^{-2x}), \qquad e^{-x}\sinh x = \frac{1}{2}(1-e^{-2x}). \tag{7.13}$$

With $x = \frac{\mu t}{2m}$, Equation (7.12) is written as:

$$e^{Mt} = \frac{1}{2}\left(1+e^{-\frac{\mu t}{m}}\right) + \frac{m}{\mu}M_{rev}\left(1-e^{-\frac{\mu t}{m}}\right). \tag{7.14}$$

The action of e^{Mt} on the initial phase-plane vector $X_0 = q_0e_1 + p_0e_2$ yields:

$$X(t) = \begin{pmatrix} q_0 + \frac{p_0}{\mu}\left(1-e^{-\frac{\mu t}{m}}\right) \\ p_0 e^{-\frac{\mu t}{m}} \end{pmatrix}, \tag{7.15}$$

which coincides exactly with the classical solution of a free particle subject to linear friction.

Several remarks are in order. First, in the limit $\mu \to 0$, the scalar and e_1 components vanish, and the general solution $X(t)$ in Equation (7.15) reduces to the free-particle solution derived in the previous subsection (Equation 7.5). Second, although the reversible part of the multivector generator produces a hyperbolic (squeeze-like) evolution in isolation, its combination with the commuting scalar term results in an exponentially damped motion. Therefore, in the present formulation, linear friction is revealed as the composition of a reversible hyperbolic deformation in the phase plane with irreversible scalar contraction. This decomposition is not apparent in the standard formulation, but emerges naturally within the multivector time-generator framework. A more detailed discussion of these matters can be found in Section 10.

7.3. Undamped Harmonic Oscillator

Here, a paradigmatic example of purely reversible Hamiltonian dynamics is considered, that is, the simple undamped harmonic oscillator. The equation of motion is written as

$$\ddot{q} + {\omega_0}^2 q = 0, \tag{7.16}$$

where q denotes the position and $\omega_0 = \sqrt{k/m}$ is the angular frequency of the oscillator (k is the spring constant). The phase plane state vector is represented geometrically as follows:

$$X(t) = qe_1 + re_2, \tag{7.17}$$

where $r = \frac{p}{m\omega_0}$ is the scaled momentum, and $p = m\dot{q}$ is the classical momentum. This rescaling is particularly convenient because it yields to circular constant-energy trajectories in the phase plane, instead of elliptic trajectories, if not scaled. Using the definitions above, the equations of motion are as follows:

$$\dot{q} = \omega_0 r, \quad \dot{r} = -\omega_0 q. \tag{7.18}$$

In matrix form, this system is written as:

$$\dot{X} = AX, \quad A = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}. \tag{7.19}$$

Using the isomorphism $\Phi^{-1}: M_2(\mathbb{R}) \to Cl(2,0)$ (Equation 4.7), the associated multivector generator is calculated as:

$$M = \omega_0 J. \tag{7.20}$$

The form of M in Equation (7.20) consisting only of the bivector part was expected because matrix A in Equation (7.19) is traceless and antisymmetric. Both M and A satisfy the Hamiltonian (symplectic) conditions of Equations (5.2) and (5.6). Therefore, $M = M_{rev}$, as in the free-particle case

(Section 7.1). Consequently, the evolution of the harmonic oscillator preserves the oriented phase-space area and is fully reversible.

The square of the reversible generator is immediately calculated as:

$$M_{rev}{}^2 = -\omega_0{}^2, \tag{7.21}$$

which is a negative scalar. According to the classification introduced in Section 6.1, this corresponds to the elliptical (oscillatory) case. Consequently, the exponential of the generator $e^{M_{rev}t}$ can be written as

$$e^{M_{rev}t} = \cos(\omega_0 t) + \frac{M_{rev}}{\omega_0}\sin(\omega_0 t) = \cos(\omega_0 t) + J\sin(\omega_0 t). \tag{7.22}$$

The appearance of trigonometric functions follows directly from the negative sign of $M_{rev}{}^2$, and reflects the rotational character of the evolution. The bivector J plays the role of the generator of phase-plane rotations in complete analogy with the imaginary unit in complex-number representations, but with a clear geometric meaning as an oriented area element. Hence, the general solution is derived as

$$X(t) = e^{Mt}X_0 = e^{M_{rev}t}X_0 = [\cos(\omega_0 t) + J\sin(\omega_0 t)]X_0, \tag{7.23}$$

where X_0 is the initial phase-plane vector. Equation (7.23) can be explicitly written as

$$X(t) = \begin{pmatrix} q(t) \\ r(t) \end{pmatrix} = \begin{pmatrix} q_0\cos(\omega_0 t) + r_0\sin(\omega_0 t) \\ -q_0\sin(\omega_0 t) + r_0\cos(\omega_0 t) \end{pmatrix}, \tag{7.24}$$

which coincides exactly with the classical solution of the undamped harmonic oscillator expressed in the rescaled phase-plane variables.

Several important conclusions follow directly. As already discussed, the dynamics are generated by a pure bivector and therefore correspond to a rigid rotation in the phase plane. In addition, as expected, the absence of the scalar component in the time generator M implies the absence of dissipation or irreversibility; the phase-plane area and energy are conserved. Thus, the undamped harmonic oscillator provides a classic example of the multivector time-generator framework, illustrating in the most explicit way how a purely Hamiltonian evolution corresponds to a bivector-generated rotation in the phase plane.

7.4. Damped Harmonic Oscillator

Here, the damped harmonic oscillator is considered as the simplest extension of the purely Hamiltonian case discussed in the previous subsection. By adding a dissipation factor to the equation of motion of the harmonic oscillator, the equation of motion is written as

$$\ddot{q} + 2\mathscr{b}\dot{q} + \omega_0{}^2 q = 0, \tag{7.25}$$

where $\mathscr{b} = {}^{\mathrm{b}}/_{2m}$ is the damping coefficient and $\omega_0 = \sqrt{{}^{k}/_{m}}$ is the angular frequency of the undamped oscillator, as described in Section 7.3. Using the same scaled momentum $r = \frac{p}{m\omega_0}$ as that for the undamped oscillator in Section 7.3, the phase-plane state vector is again represented geometrically as $X(t) = qe_1 + re_2$. Using these definitions, the equations of motion are as follows:

$$\dot{q} = \omega_0 r, \quad \dot{r} = -\omega_0 q - 2\mathscr{b}r. \tag{7.26}$$

In matrix form, this system is written as:

$$\dot{X} = AX, \quad A = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\mathscr{b} \end{pmatrix}. \tag{7.27}$$

Unlike the undamped case, the trace of A is now non-zero, $\mathrm{tr}A = -2\mathscr{b}$, indicating phase-plane contraction and loss of reversibility.

Using the algebraic isomorphism $\Phi^{-1}: M_2(\mathbb{R}) \to Cl(2,0)$ (Equation 4.7), the corresponding multivector time generator can be written as

$$M = -\ell + \ell e_1 + \omega_0 J. \tag{7.28}$$

Using the definition of the reversible time generator M_{rev}, M is be written as:

$$M = -\ell + M_{rev}, \qquad M_{rev} = \ell e_1 + \omega_0 J. \tag{7.29}$$

As discussed in Section 5, the scalar part of M is the unique source of irreversibility, whereas all components of M_{rev} satisfy the symplectic condition and correspond to reversible geometric transformations in the phase plane.

The square of the reversible generator is calculated as:

$$M_{rev}{}^2 = \ell^2 - \omega_0{}^2, \tag{7.30}$$

whose sign determines the qualitative nature of the reversible evolution in accordance with the classification introduced in Section 6.1.

- *Underdamped case* $(\ell < \omega_0)$

When $\ell^2 < \omega_0{}^2$, Equation (7.30) yields to a negative scalar:

$$M_{rev}{}^2 = -\omega_d{}^2, \qquad \omega_d = \sqrt{\omega_0{}^2 - \ell^2}, \tag{7.31}$$

corresponding to the elliptical (oscillatory) case. Therefore, the exponential of the reversible generator takes the trigonometric form:

$$e^{M_{rev}t} = \cos(\omega_d t) + \frac{M_{rev}}{\omega_d}\sin(\omega_d t). \tag{7.32}$$

Because the scalar part of M commutes with all multivector components, the full evolution operator factorizes as

$$e^{Mt} = e^{-\ell t} e^{M_{rev}t}. \tag{7.33}$$

Thus, the general solution is written as:

$$X(t) = e^{-\ell t}\left[\cos(\omega_d t) + \frac{M_{rev}}{\omega_d}\sin(\omega_d t)\right] X_0, \tag{7.34}$$

where X_0 is the initial phase-plane vector, as in the undamped harmonic oscillator. The phase-plane trajectories are therefore spirals: the trigonometric dependence arises from the negative sign of $M_{rev}{}^2$, whereas the scalar exponential factor produces a uniform contraction of their amplitude.

- *Critically damped case* $(\ell = \omega_0)$

At critical damping $\ell^2 = \omega_0{}^2$, and Equation (7.30) gives:

$$M_{rev}{}^2 = 0, \tag{7.35}$$

such that the reversible generator is nilpotent. Consequently, its exponential truncation after the linear term is

$$e^{M_{rev}t} = 1 + M_{rev}t. \tag{7.36}$$

Thus, the full solution then becomes:

$$X(t) = e^{-\ell t}[1 + (\ell e_1 + \omega_0 J)t]X_0, \tag{7.37}$$

In this case, oscillatory terms were not observed. The absence of trigonometric or hyperbolic functions follows directly from the algebraic condition $M_{rev}{}^2 = 0$, rather than being imposed at the level of the differential equation.

- *Overdamped case* $(\ell > \omega_0)$

When $\ell^2 < \omega_0{}^2$, Equation (7.30) yields a positive scalar:

$$M_{rev}{}^2 = \omega_d{}^2, \qquad \omega_d = \sqrt{\ell^2 - \omega_0{}^2}, \tag{7.38}$$

corresponding to the hyperbolic case. The reversible exponential is therefore written as:

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$$e^{M_{rev}t} = \cosh(\omega_d t) + \frac{M_{rev}}{\omega_d}\sinh(\omega_d t), \tag{7.39}$$

and the full solution reads:

$$X(t) = e^{-\mathscr{b}t}\left[\cosh(\omega_d t) + \frac{M_{rev}}{\omega_d}\sinh(\omega_d t)\right]X_0. \tag{7.40}$$

Several remarks for all the above cases of the damped harmonic oscillator are in order. While the scalar part of the generator produces a uniform contraction of phase-plane trajectories, the reversible part determines whether the underlying geometric motion is oscillatory (rotational), critically damped (nilpotent), or overdamped (hyperbolic). In this manner, the familiar trigonometric or hyperbolic time dependence of classical solutions emerges naturally from the multivector exponential rather than being introduced ad hoc. Moreover, as in the case of the free particle with linear friction, dissipation does not modify the dynamics solely through scalar contraction. In the present system, dissipation also introduces an additional vector component in the reversible generator, that is $\mathscr{b}e_1$. This vector contribution encodes directional deformation in the phase plane and reflects the anisotropic nature of damping, which acts along the momentum direction while leaving the position coordinate unaffected at the level of the restoring force. Thus, the multivector time generator framework reveals that dissipation is not a purely scalar effect but a structured geometric modification of the phase-plane evolution.

8. Poisson Brackets, Symplectic Structure and the Multivector Time Generator

As discussed in Section 4, in the phase plane (q, p), the canonical symplectic structure is encoded by the oriented area element $\omega = dq \wedge dp$, which in the geometric algebra $Cl(2,0)$ is represented by the bivector $J = e_1 e_2$. This bivector plays a central role in Hamiltonian mechanics by generating a canonical 90° rotation in the phase plane, and encodes the Poisson structure of classical observables.

8.1. Poisson Brackets from the Bivector Sector

Given two smooth functions $f(q, p)$ and $g(q, p)$, their Poisson bracket is [33]:

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \tag{8.1}$$

In geometric algebra, by introducing the phase-plane gradient:

$$\nabla = e_1\frac{\partial}{\partial q} + e_2\frac{\partial}{\partial p}, \tag{8.2}$$

the Poisson bracket can be written compactly as:

$$\{f, g\} = \nabla f \cdot (J\nabla g). \tag{8.3}$$

Thus, the Poisson brackets are generated entirely by the bivector J. They represent the contraction of gradients with the symplectic bivector, and define a Lie algebra of derivatives on observables.

Consequently, within the multivector time generator $M = a1 + be_1 + ce_2 + dJ$ only the bivector component dJ induces Hamiltonian evolution. For an observable $f(q, p)$, this contribution takes the standard Hamiltonian form

$$\dot{f} = \{f, H\}, \tag{8.4}$$

where the Hamiltonian function H is associated with the bivector-generated flow. This relation preserves the symplectic form and defines a Poisson–Lie algebra.

8.2. Vector Generators, Reversibility, and Anti-Symplectic Transformations

The vector components $be_1 + ce_2$ of the multivector generator correspond via the algebra isomorphism $M_2(\mathbb{R})$ to reflection matrices (see matrices $\Phi(e_1)$ and $\Phi(e_2)$ in Equation 3.2). These transformations are reversible because each reflection is its own inverse; however, they do not preserve the symplectic bivector. Indeed, for a linear transformation S acting on the phase plane, the induced action on the bivector is:

$$J \longmapsto J' = (Se_1) \wedge (Se_2) = (\det S)\, J. \tag{8.5}$$

Reflections satisfy $\det S = -1$, and therefore:

$$J' = -J. \tag{8.6}$$

Such transformations are called anti-symplectics. They preserve the magnitude of the phase-plane area, but reverse its orientation.

Because the Poisson bracket is constructed from J, an anti-symplectic transformation reverses its sign as follows:

$$\{f, g\} \longmapsto -\{f, g\}. \tag{8.7}$$

Therefore, vector-generated transformations are not automorphisms of Poisson algebra but are anti-automorphisms. They do not generate Poisson brackets even though they remain reversible. This distinction highlights an important conceptual point: reversibility does not imply a Hamiltonian character, and Hamiltonian evolution requires preservation of the symplectic bivector, not merely the invertibility of the dynamics.

Time reversal in Hamiltonian mechanics provides a canonical example of an anti-symplectic transformation. Under time reversal, it is derived that

$$(q, p) \longmapsto -(q, -p), \tag{8.8}$$

reflecting the fact that the momentum changes sign when the velocities are reversed. Acting in the symplectic form, the map yields

$$dq \wedge dp \longmapsto dq \wedge (-dp) = -dq \wedge dp, \tag{8.9}$$

or equivalently,

$$J \longmapsto -J. \tag{8.10}$$

Thus, time reversal is naturally anti-symplectic; it transforms the solutions of Hamilton's equations into solutions by inverting the orientation of the phase plane. In the multivector time-generator framework, time-reversal symmetry is identified using the vector sector of the time generator. Therefore, the vector components of M encode reversible but anti-symplectic operations, distinct from the Hamiltonian (bivector-generated) flows.

8.3. Role of the Scalar Component

The scalar term $a1$ in M rescales both basis vectors uniformly and therefore rescales the bivector as

$$J \longmapsto e^{2at} J. \tag{8.11}$$

This contribution neither preserves nor flips the symplectic form but changes its magnitude. As shown in Section 5, this violates the Hamiltonian condition and corresponds to irreversible evolution. Thus, the scalar component is the unique source of irreversibility in the multivector time generator.

9. Complex Time as a Restricted Case of the Multivector Time Generator

As discussed in the Introduction, several previous studies have proposed the use of complex time to distinguish reversible and irreversible dynamical processes, associating the imaginary part of complex time with Hamiltonian evolution and the real part with entropy production [19,40–44]. While these constructions have been proven insightful in specific contexts, they typically treat time itself as the primary object of extension to the complex plane. Within the framework developed in the present study, the multivector time generator provides a precise setting in which this idea can be critically examined. Rather than asking whether time may be taken to be complex, the relevant question is whether the structure of the time derivative, as encoded by its generator, admits a consistent reduction to a complex form. This section addresses this question directly by identifying the conditions under which the multivector time generator defines a genuine complex structure, and clarifies when complex time can be meaningfully introduced within the present formalism.

In the phase plane formulation adopted throughout this study, the multivector time generator is expressed as $M = a + be_1 + ce_2 + dJ$. As discussed in previous sections, the scalar and bivector parts correspond to contraction/expansion and rotation in the phase plane, respectively, while the vector parts generate reflection-type transformations. As shown in Section 2, complex numbers appear naturally as the even subalgebra of the generator of the form $M_{even} = a + dJ$ (see Equation 2.4). Through the isomorphism $\Phi: Cl(2,0) \to M_2(\mathbb{R})$, complex numbers can be represented as 2×2 matrices of the form $A = \begin{pmatrix} a & d \\ -d & a \end{pmatrix}$ (see Equation 3.4).

Let the multivector time generator be of the form $M_{even} = a + dJ$. Then, the linear evolution equation is written as

$$\dot{X} = (a + dJ)X. \tag{9.1}$$

The solution is given by:

$$X(t) = e^{(a+dJ)t}X_0, \tag{9.2}$$

Equation (9.2) describes a system of simultaneous exponential scaling and rotation, which means that the action of the time derivative on the phase-plane variables is indistinguishable from multiplication by a complex scalar. Thus, the evolution operator is defined as:

$$U(t) = e^{(a+dJ)t}. \tag{9.3}$$

Using the matrix generator $A = \begin{pmatrix} a & d \\ -d & a \end{pmatrix}$ instead of the even multivector generator M_{even}, one can split A into a symmetric part A_r, and skew-symmetric part A_i, as

$$A_r = \frac{1}{2}(A + A^T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \qquad A_i = \frac{1}{2}(A - A^T) = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}. \tag{9.4}$$

Using the isomorphism $\Phi^{-1}: M_2(\mathbb{R}) \to Cl(2,0)$ (Equation 4.7), one finds explicitly:

$$A_r = aI, \qquad A_i = dJ, \tag{9.5}$$

with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ being the identity 2×2 matrix and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (see Equation 3.2). Thus, the scalar part of M_{even} corresponds to the symmetric (dissipative) component of the matrix generator, whereas the bivector part corresponds to the skew-symmetric (Hamiltonian) component. Using this formulation, the evolution operator can be written as:

$$U(t) = e^{At} = e^{(A_r + A_i)t}. \tag{9.6}$$

To make the notion of complex time precise, consider an analytically extended time plane T with coordinates (t, τ) and define the complex time variable as

$$T = t + i\tau, \qquad \bar{T} = t - i\tau. \tag{9.7}$$

Using the evolution operator, a two-parameter semigroup is defined as:

$$U(t,\tau) := e^{(tA_r + \tau A_i)}, \tag{9.8}$$

where t parameterizes dissipative evolution and τ parameterizes Hamiltonian evolution. This formulation makes explicit that the real and imaginary contributions to the generator are, in general, independent in the evolution space. The Wirtinger derivatives of T are expressed as:

$$\partial_T = \frac{1}{2}(\partial_t - i\partial_\tau), \qquad \partial_{\bar{T}} = \frac{1}{2}(\partial_t + i\partial_\tau). \tag{9.9}$$

Acting on the evolution operator $U(t,\tau)$, one finds:

$$\partial_T U = \frac{1}{2}(A_r - iA_i)U, \qquad \partial_{\bar{T}} U = \frac{1}{2}(A_r + iA_i)U. \tag{9.10}$$

To claim the evolution is holomorphic in T requires:

$$\partial_{\bar{T}} U = 0, \tag{9.11}$$

which holds if and only if:

$$A_r = -iA_i. \tag{9.12}$$

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Therefore, Equation (9.12) is the compatibility condition for time to behave as a complex variable through Wirtinger derivatives. Geometrically, this implies that the dissipative and Hamiltonian generators A_r and A_i are not independent, but are related through a complex structure. Under this condition, evolution exists along the Cauchy–Riemann orthogonal directions in the (t,τ) planes, and the operator d/dT is a Wirtinger derivative. This is exactly the structural assumption underlying complex-time formulations such as Quantitative Geometrical Thermodynamics [19,41], where entropy production and Hamiltonian evolution appear as conjugate components of a single holomorphic flow.

10. Discussion

First, it is important to note that the central idea developed in this work emerges from the intersection of two well-established yet largely separate lines of research. On the one hand, the linear-algebraic formulation of Hamiltonian systems in phase space provides a precise description of dynamical evolution in terms of generators, symplectic structure, and reversibility conditions [28,31–33]. On the other hand, a growing body of research has explored the role of complex time in distinguishing reversible from irreversible processes, associating the imaginary component of complex time with Hamiltonian dynamics and its real component with entropy production [19,40–44]. Although both approaches offer valuable insights, they have remained conceptually disconnected. The present work brings these perspectives together by employing geometric algebra as a unifying mathematical framework, within which generators of evolution can be represented as geometric objects (i.e., multivectors) rather than just matrices or complex number parameters. This synthesis enables the reinterpretation of the time derivative in terms of a multivector time generator, revealing a richer structure of time transformations, and allowing both its symmetries and asymmetries to be characterized within a single, coherent geometric framework.

Of course, as mentioned in Section 5, the introduction of the multivector time generator should not be interpreted as a literal replacement of the differential operator d/dt, but rather as a structural reinterpretation of its action. The time derivative is retained as the operator governing evolution; however, its effect on phase plane variables is understood as being generated by a geometric object, in direct analogy with the familiar linear-algebraic representation $\dot{x} = Ax$, where the matrix A encodes the structure of the evolution without replacing the differentiation itself. In this sense, multivector M plays the same conceptual role as the generator A, but within a richer geometric framework. This perspective is particularly important when discussing temporal symmetries. As emphasized by philosopher Tim Maudlin [3,4], time-reversal symmetry does not concern the reversal of time itself as a parameter, but rather the reversal of the time derivative. In this framework, questions of reversibility, symmetry, and temporal orientation are questions about how the generator of evolution transforms, not about the nature of time itself. By focusing on the structure of the time derivative through its generator, the present study provides a natural and precise setting for analyzing time-reversal properties, while avoiding ambiguities associated with treating time as a flowing or transformable quantity.

One of the most significant and original outcomes of the multivector time-generator framework is the emergence of the vector components of the generator and the novel class of transformations they induce. As shown in Section 5, the vector parts of the multivector generator satisfy the reversibility conditions and therefore correspond to reversible transformations, despite not being Hamiltonian. Through the isomorphism $\Phi: Cl(2,0) \to M_2(\mathbb{R})$ introduced in Equation (3.2), the vector parts of the generator acquire a clear geometric interpretation: the basis vector e_1 generates a reflection with respect to the q-axis, whereas e_2 generates a reflection with respect to the line $q = p$, effectively exchanging the q and p axes. In the language of 2D geometric algebra, both transformations are isometries, preserving distances and phase-plane areas. At the same time, they reverse the orientation of the phase plane and are therefore anti-symplectic. Consequently, although these transformations are reversible, they do not preserve the Poisson bracket structure, as discussed in Section 8. This distinguishes them from Hamiltonian flows, which are reversible and symplectic.

Notably, to the author's knowledge, such anti-symplectic yet reversible transformations are not captured within standard complex time formulations (see Refs. [19,40–44], which associates reversibility exclusively with the imaginary part of complex time, that is, the bivector part). Their appearance here is a direct consequence of analyzing the symmetries of the time derivative itself, rather than extending time as a complex quantity. This highlights the added resolving power of the multivector generator approach and illustrates how focusing on the geometric structure of d/dt reveals the types of time transformations that would otherwise remain hidden.

Another additional and conceptually important result of the present framework is that the vector components of the multivector time generator appear both in systems that are conventionally regarded as purely reversible (which was expected owing to the reversible character of the vector parts) and in systems exhibiting dissipation and irreversibility. As shown in Section 7.1, for a free particle, a vector part arises, which is associated with the geometry of the motion in the phase plane. This observation is fully consistent with the standard statement that Hamiltonian dynamics are invariant under time reversal, where $t \to -t$ is accompanied by $q \to q$ and $p \to -p$. Using the multivector time generator, the geometric character of this reversible transformation is naturally captured by the vector parts, which generate reflection-type isometries. In this sense, the appearance of the vector part in the free particle proves that reversibility does not require the generator to be a pure bivector (as would be expected by complex time studies); rather, the vector parts provide the appropriate geometric class for anti-symplectic symmetries associated with time reversal.

Moreover, in the irreversible cases examined in this work, namely, the free particle with friction (Section 7.2) and the damped harmonic oscillator (Section 7.4), the inclusion of dissipation does not lead solely to a scalar contribution to the generator, as might be expected from other studies [19,41–44] that connect irreversibility exclusively with the real part of the complex time. Instead, both systems exhibited simultaneous scalar and vector contributions. Geometrically, this indicates that irreversible evolution is not characterized only by phase-plane contraction but may also involve orientation-reversing or axis-exchanging transformations that remain reversible at the level of the reversal of the time derivative. From the perspective of temporal symmetries, this result suggests that the structure of time evolution is richer than a simple dichotomy between the Hamiltonian and dissipative components. The presence of vector parts reveals that certain symmetry operations, specifically anti-symplectic isometries, can coexist with dissipation, modifying the geometric character of evolution without introducing irreversibility in the strict sense. These findings point to the broader role of isometries in temporal transformations and motivate further investigation into how such geometric symmetries contribute to the structure of time evolution. In this direction, the author intends to explore the role of isometries in shaping temporal transformations within a multivector generator framework.

The analysis presented in Section 9 clarifies the relation between complex time and the multivector time generator and shows that it should be regarded as a restricted and emergent description rather than a fundamental one. In particular, complex time arises only under specific structural conditions: the multivector time generator must be confined to its even subalgebra, that is, the absence of its vector components, and the action of the time derivative must admit a Wirtinger representation. Under these conditions, the scalar and bivector parts of the generator are no longer independent but become correlated with each other through a common complex structure, allowing the evolution to be described as holomorphic in a complex time variable. Outside this restricted regime, however, the generator defines a multivector structure and the time evolution cannot be strictly represented by a single complex parameter. From this perspective, complex or imaginary time does not represent a general property of time itself, but rather a special case of multivector time generator geometry that captures only a subset of possible temporal transformations. Therefore, the multivector time framework explains both the success and limitations of complex time approaches, while naturally extending them by revealing additional reversible symmetries associated with vector components that lie beyond standard complex formulations.

Another important outcome of the present framework is the systematic separation of the multivector time generator M into reversible and irreversible components, as introduced in Section 6. This decomposition is not merely formal but is essential for constructing the general solution of the evolution equations in exponential form. In particular, the reversible part M_{rev} naturally admits further classification into elliptic (oscillatory), hyperbolic, and nilpotent cases, each associated with a distinct geometric action on the phase plane. This classification provides a direct geometric interpretation of the solutions obtained for all the studied cases in Section 7. Moreover, this classification is not imposed externally, but emerges directly from the algebraic properties of M_{rev}. This feature becomes especially clear in the case of a damped harmonic oscillator, where the usual separation into undamped, critically damped, and overdamped cases arises naturally from the decomposition of the generator, rather than being introduced by hand, as in conventional treatments. Thus, the multivector time generator framework provides a unified geometric view of the solution structure across all examples considered, linking the exponential form of the evolution directly to the underlying symmetry and geometry encoded in the time generator.

Although the present work introduces a novel geometric formulation of time evolution through a multivector time generator, it does not claim to provide a complete resolution of the broader "problem of time" in physics, as discussed in the first paragraph of the Introduction. In particular, it does not attempt to unify the distinct roles of time in quantum mechanics and relativity nor to address issues related to quantization, spacetime structure, or measurement. The scope of this study is instead focused on the geometry of temporal transformations in classical phase space, where time evolution is analyzed at the level of its generator rather than through extensions of the time parameter itself. Moreover, although the framework of the present study recovers and contextualizes earlier complex time structures (such as those based on Wick rotation or Quantitative Geometrical Thermodynamics, see Refs. [19,37,38,40–42]), it does so by embedding them within a more general multivector structure that reveals additional symmetry classes and transformation types that are not accessible in purely complex formulations. Therefore, the present results could be viewed as a structural reorganization that clarifies both the domain of validity and limitations of complex-time descriptions.

Several directions for future research emerge naturally from the present framework. One immediate avenue concerns the application of a multivector time generator to the thermodynamics of irreversible processes, where the explicit separation of reversible, dissipative, and isometric components may offer a new geometric interpretation of entropy production, dissipation, and nonequilibrium evolution beyond standard descriptions. A second promising direction involves extensions to quantum mechanics, particularly in settings where time evolution is generated by non-Hermitian operators, such as open quantum systems. In this context, the multivector decomposition of the time generator may provide a unified geometric language for Hamiltonian evolution, decay, and symmetry breaking, and may help clarify the status of complex time and time-reversal symmetry at the generator level. While these extensions lie beyond the scope of the present work, they suggest that the multivector time-generator framework may serve as a useful bridge between classical dynamics, nonequilibrium thermodynamics, and quantum theory.

11. Conclusions

In this study, a geometric formulation of the time evolution was developed based on the concept of a multivector time generator acting on the phase plane. By reinterpreting the action of the time derivative through geometric algebra, this study shifts the focus from time as a parameter to the generator structure that governs evolution. This perspective makes it possible to describe reversible and irreversible dynamics within a single unified framework directly at the level where temporal symmetries and asymmetries arise.

Multivector decomposition reveals a clear geometric meaning for different dynamic contributions. The bivector part generates a Hamiltonian and symplectic evolution, which can be modeled using Poisson brackets, while the scalar part accounts for irreversibility through uniform contraction or expansion, consistent with entropy production. In addition, vector components generate reversible but anti-symplectic transformations, that is, reflections, which do not preserve the Poisson bracket structure. These vector transformations cannot be captured by standard Hamiltonian or complex time approaches and clarify the geometric nature of time-reversal symmetry.

General solutions are obtained in exponential form, and the reversible generator naturally classifies dynamics into elliptic, hyperbolic, and nilpotent cases, providing a transparent geometric interpretation of oscillatory, overdamped, and critically damped behaviors. Notably, in systems such as the damped harmonic oscillator, this structure emerges intrinsically from the generator itself rather than externally imposed.

The analysis further shows that complex time arises only as a restricted case in the multivector framework. A genuine complex-time description is possible only when the generator is confined to its even subalgebra and the time derivative admits a Wirtinger representation. Outside this regime, the time evolution is inherently described by a multivector, including scalar, vector, and bivector components, and cannot be accurately described by a single complex parameter.

Overall, the multivector time generator framework provides a coherent geometric language for analyzing temporal evolution, clarifying the roles of reversibility, irreversibility, and symmetry in dynamic systems. By embedding time evolution directly in the algebraic structure of its generator, the approach unifies and extends existing symplectic, linear-algebraic, and complex time formulations while revealing additional geometric transformations that were previously hidden. These results suggest that the geometry of the time generator, rather than the nature of time as a parameter, is key to understanding the structure of temporal evolution.

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Appendix A. Correspondence Between A^T and $\widetilde{M}$

In this Appendix, it is shown how the transpose A^T of matrix A corresponds to the conjugate $\widetilde{M} = a + be_1 + ce_2 - dJ$ of the multivector generator M. Recall that any multivector $M = a + be_1 + ce_2 - dJ$ can be written in the form of a 2×2 matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ using the algebra isomorphism $\Phi: Cl(2,0) \to M_2(\mathbb{R})$ (Equations 3.1-3.3). This leads to the following result:

$$\alpha = a + b, \quad \beta = c + d, \quad \gamma = c - d, \quad \delta = a - b. \tag{A.1}$$

Using the standard definition for the transpose of a matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, it is derived:

$$A^T = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}. \tag{A.2}$$

Using Equation (A.1), A^T can be also written as:

$$A^T = \begin{pmatrix} a + b & c - d \\ c + d & a - b \end{pmatrix}. \tag{A.3}$$

Comparing Equations (A.1) and (A.3), and using the basis matrices as shown in Equation (3.2), we observe that the transposition of A leaves the scalar and vector components unaltered while reversing the sign of the antisymmetric part, that is,

$$a \longmapsto a, \quad b \longmapsto b, \quad c \longmapsto c, \quad d \longmapsto -d. \tag{A.4}$$

In this manner, the new multivector generator is derived as:

$$M_{new} = a + be_1 + ce_2 - dJ, \tag{A.5}$$

which is identified with the conjugate $\tilde{M}$ of the multivector M.

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