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Riemannian (Trivial) Immersion(s) of Schwarzschild Solitons and of Generalized-Schwarzschild Solitons

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Abstract

The cross sections of the Schwarzschild soliton is analytically written as a section of the lightcone for each initial condition. The Penrose description of the tipping lightcones is used. For the investigation, the cross section of the Schwarzschild spacetime is characterized as a surface with vanishing Ricci scalar, with vanishing Ricci tensor, with vanishing Riemann tensor: moreover, its metric has vanishing time derivative. The cross section of the Schwarzschild spacetime is proven to be a section of the Penrose lightcones; the Perelmann diffeomorphism is written. To achieve this result, the convergence of the Yamabe flow on an $\mathbb{S}^2$ sphere is also proven when the sphere is submersed from a Schwarzschild spacetime; here, the geodesic spheres are considered: the weights are newly introduced as well for the new definition of cross-section manifold. As a result, the complete classifications of steady, expanding and shrinking Schwarzschild solitons is established with the analytical expressions of the weights. The results apply also to gradient Kaehler-Ricci solitons. The results select new steady gradient Kaehler-Ricci Schwarzschild solitons from rotationally-symmetric gradient Kaehler-Ricci solitons. The results complete the diverse classifications of gradient Ricci solitons analytically. A new impetus is found for applications to the generalizations of the two-dimensional cases.

Keywords: η-Yamabe-Ricci solitons; Schwarzshild soliton; Riemannian immersions; Penrose tipping lightcones; cross section

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1. Introduction

In the work of Siddiqi et al [1], Riemannian submersion of the η-Yamabe-Ricci solitons are implemented from [18].

In the work of Roesch et al. [23], the three-dimensionless flow of a $3-dim$ null hypersurface is studied. The properties of null hypersurfaces are scrutinized. The speed of the mean curvature flow is define d form the projection of the second fundamental form on the null hypersurface. Conditions are imposed on the null hypersurface. The analysis in Euclidean space is presented. The mean curvature flow is ibidem made to converge to a marginally-trapped outer surface. The foliations of the null hypersurface is requested to admit an asymptotical foliation which is untrapped.

In the work [12], the aim is pursued to study the $2d$ Ricci flow on a topological sphere.

As a tool, ibidem any conformally round metric on the 2-sphere is identified with a unique cross section on the 'standard' $1+3$ dimensional lightcone. Within this framework, the Ricci flow is demonstrated to be equivalent to a 'null mean-curvature flow' (the Roesch-Scheuer flow from [24]) along null hypersurfaces.

In the work of Roesch e al. [24], the mean-curvature flow of $3-dim$null hypersurfaces is studied. The properties of null hypersurfaces are investigated.

In the present paper, Schwarzschild solitons and Generalized Schwarzschild solitons [36] are investigated as far as the definition of the tipping lightcones are concerned, as well as the related quantities.

In the work [3], the 'standard static spacetime' is considered, and it is studied as far as the presence of 'almost'-Ricci-Yamabe solitons are concerned. The particular case of non-rotating Killing vector

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field is investigated. The instances of the presence of perfect fluid and that of vacuum with conformal Killing vector field are researched. In the present paper, the Killing vector field is that associated with the 4-velocity vector field, and the affording of the presence of matter is taken from [36].

2. Introductory Material

2.1. About η-Yamabe-Ricci Solitons

From the work of Siddiqi et al. [1], the main definitions for the η-Yamabe-Ricci solitons are recalled.

The η-Yamabe-Ricci solitons are recalled from [16] as to be defined as

Definition Da:

The η-Yamabe-Ricci solitons are defined after the request

$$\frac{1}{2}\frac{\partial}{\partial t}g_{ij}(t) - \sigma R_{ij} - \frac{1}{2}\rho R(t) g_{ij} \tag{1}$$

Definition Db:

The η-Yamabe-Ricci solitons on the Riemannian manifold $(\mathcal{M}, g)$ is the data $(g, \omega, \tau, \sigma, \rho)$ which obey

$$\frac{1}{2}\pounds_\omega g_{ij} + \sigma R_{ij} + \left(\tau - \frac{1}{2}\rho R\right) g_{ij} = 0. \tag{2}$$

Definition Dc:

Ricci solitons are defined from Eq. (2) as with vanishing ρ.

Definition Dd:

Yamabe solitons are defined from Eq. (2) as with vanishing σ.

As recalled from [21],

Remark R1:

In 2 space dimensions, the Ricci solitons and the Yamabe solitons coincide.

It is the aim of the present investigation to research the characterization of the Schwarzschild solitons.

2.2. Further Specifications of Solitons

Theorem 1.2 from [20] is reported

Theorem T01: *For any metric tensor on $\mathbb{S}^2$, i.e., as under the Ricci-Hamilton flow, the scalar curvature R becomes positive in finite time interval.* □

From [21], the following Corollary is now studied:

Corollary C1:

Given g_{ij} metric tensor on a closed surface, then, under the Ricci- Hamilton flow, the metric converges to one of constant curvature R. □

Example E1:

The Einstein spacetimes are studied as

$$\partial_t g_{ij} = -2(R_{ij} - R g_{ij}) \tag{3}$$

with

$$\pounds_\omega g_{ij} + 2R_{ij} + \left(\tau - \frac{R}{2}\right) g_{ij} = 0. \tag{4}$$

Definition De:

A smooth vector field ζ on a Riemannian manifold $(\mathcal{N}, g)$ is a conformal vector field if there exists ϕ a smooth function on $\mathcal{N}$, ϕ such that

$$\pounds_\zeta g_{ij} = 2\phi g_{ij}. \tag{5}$$

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2.3. Riemannian Submersions

Given two Riemannian manifolds $(\mathcal{W}^n, g)$ and $(\mathcal{B}^m, g_{\mathcal{B}})$, with $dim(\mathcal{W}) > dim(\mathcal{B})$, the following definition holds from [18]

Definition Df:

The Riemannian submersion is a surjective mapping

$$\pi : (\mathcal{W}, g) \rightarrow (\mathcal{B}, g_{\mathcal{B}}) \tag{6}$$

with

$$dim(\mathcal{B}) = Rank(\pi). \tag{7}$$

Accordingly,

Example E2:

the inverted mapping

$$\pi^{-1}(a) = \pi^{-1}(a) \tag{8}$$

on the submanifold $\mathcal{W}$.

From Example E2, the definition of fiber is achieved as

Definition Dg:

The fiber is defined from $\pi^{-1}(a)\ \forall a \in \mathcal{B}$ *wherein*

$$-\mathbb{T} + dim(\mathcal{W}) = dim(\mathcal{B}). \tag{9}$$

2.4. About the Standard Minkowski Lightcone

In the work of Wolff [12], problem of studying the $2 - dim$ on topological spheres is addressed: the embedding is demonstrated to be unique fro any conformally-round metric on the 2-sphere into the 'past oriented' standard lightcone in $1 + 3$ Minkowski spacetime. As a result, from [23], any metric which is conformal to a given metric is identified with a unique cross section on a null hypersurface with a spacetime satisfying the EFE's. The conformal invariants can therefore be studied.

From [24], marginally outer trapped surfaces are studied.

From [12], the equivalence to the Ricci flow is taken to imply that the flow is 'extinguished' in a finite time.

For these purpose, $\mathbb{R}^{1,3}$ the standard Minkowski spacetime is investigated for the standard Minkowski manifold $(\mathbb{R}^{1,3}, h)$ being $h_{\mu\nu}$ the flat metric yielding to

$$ds^2 = h_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{10}$$

being $d\Omega^2$ the line element of the Schwarzschild solid angle.

The standard lightcone is defined as centered at the origin of the Minkowski spacetime and is described as $C(=)$ the set

$$C(0) := \{ \mid t \mid = r \} \ : \tag{11}$$

this way, the future-oriented (portion of the) lightcone is written as $C_+(0)$

$$C_+0 := C(0) \cap \{ \ t \geq 0 \} \tag{12}$$

and the past-oriented (portion of the) lightcone is spelled out as

$$C_-(0) := C(0) \cap \{ \ t \leq 0 \} \ , \tag{13}$$

where the time orientation is induced after ∂_t.

The $2 - dim$ surface (Σ, γ) is taken with a Riemannian metric γ.

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A closed, orientable spacelike codimension-2 surface is looked for in $\mathbb{R}^{1,3}$ (when $\mathbb{R}^{1,3}$ is restricted to the lightcone).

2.5. About the Three-Dimensional Flow of a $3-dim$ Null Hypersurface

From the work [23], a $4-dim$ time oriented Lorentzian manifold $(\mathcal{M}, g)$ is taken.

From this, the embedded spacelike 2-sphere $\Sigma \subset \mathcal{M}$ is considered; for it. the second fundamental form is calculated forall section

$$\Xi(Y, W) = (D_Y W)^{\perp} \tag{14}$$

being D_Y the directional derivative in the direction of Y, for $W, V \in \Gamma(T\Sigma)$.

The mean curvature velocity vector is calculated as the trace of the second fundamental form as

$$\vec{H} = tr_{\Sigma}\Xi. \tag{15}$$

Let $\vec{l}$ be the future-oriented null normal:

Definition Dh:

the regions selected after the scalar product of the future-oriented null normal and the mean curvature vector are defined

i) outer untrapped when $-l < \vec{l}, \vec{H} >> 0$,

2) outer trapped when $-l < \vec{l}, \vec{H} >< 0$, *and*

iii) marginally trapped when $l < \vec{l}, \vec{H} = 0$.

Let $\{ b_{\mu} \}$ be the basis of the normal bundle:

Definition Di:

The mean-curvature velocity vector is defined as

$$H = b_{\mu} H^{\mu}_{ij} dx^i dxj. \tag{16}$$

Definition Dj:

The trace of the mean-curvature vector is defined as

$$tr[H] = g^{ij} H^{\mu}_{ij} b_{\mu}. \tag{17}$$

In the work of Roesch [24], the speed of the mean curvature flow is defined as

Definition Dk:

The velocity of the mean-curvature flow is the projection of the mean-curvature vector onto the null hypersurface.

The foliation of the null hypersurfaces is requested to admit an asymptotical foliation which is untrapped.

For these purposes, the manifold $(\mathcal{M}, g)$ is taken, which is $4-dim$, time-oriented and Lorentzian.

Let $\sigma \subset \mathcal{M}$ be an embedded spacelike sphere.

For all sections. the second fundamental form of σ is denominated as Ξ and is written as

$$\sigma(A, B) \equiv (D_A B) \perp \tag{18}$$

with $A, B \in \Gamma(T\sigma)$.

The trace of the second fundamental form is calculated as

$$\vec{H} = tr\Xi. \tag{19}$$

Let l be the future-oriented null normal:

Definition Dl:

An outer untrapped surface is defined according to the scalar product

$$-<l,\vec{H}>>0. \tag{20}$$

3. Preparation of the Schwarzschild Light Cones

The Schwarzschild light cones were prepared in the work of Penrose [25].

In the work [25], the gravitational field of the exterior of a spherically-symmetric collapsed star is studied with the Schwarzschild line element.

The 'advance-time coordinate is defined as

Definition Dm:

The advance-time coordinate v variable is written as

$$v=t+r+2Mln(r-2M) \tag{21}$$

The fictitious singularity is analyzed as with the 'non-problematic' line element as a function of the advanced-time coordinate

$$ds^2=\left(1-\frac{2M}{r}\right)dv^2-2drdv-r^2d\Omega^2. \tag{22}$$

The line element (22) is well-defined when passing the fictitious singularity.

Here, we define the Schwarzschild light cones as described after the equation (with $u\equiv r$ from Eq. (22)

$$x=x_0+u(lcosv+msinv) \tag{23a}$$

$$y=y_0+u(mcosv-lsinv) \qquad z=z_0+un \tag{23b}$$

the vertex $V=(x_0,y_0,z_0)$ of the cone is indicated, and (l,m,n) are the director cosines of the generator line of the cone.

For the purposes of specifying the director cosines, the Frenet mobile trihedron is taken, which constitutes an orthonormal basis, as follows [26]. Let s be the curvilinear abscissa of $\mathcal{M}$, and let $P(s)$ be a point of $\mathcal{M}$ on s; let the tangent vector in P be $\vec{P}'$, such that the tangent unit vector is

$$\vec{t}\equiv\frac{\vec{P}'}{\mid\vec{P}'\mid}. \tag{24}$$

The curvature $c(s)$ is defined as

$$c(s)=\mid\frac{d\vec{t}}{ds}. \tag{25}$$

The normal unit vector $\vec{n}(s)$ is written as

$$\vec{n}(s)=\frac{1}{c(s)}\frac{d\vec{t}}{ds} \tag{26}$$

and is oriented such as

$$\frac{d\vec{n}}{ds}=-c(s)\vec{t}. \tag{27}$$

The binormal unit vector $\vec{b}$ is obtained after $\vec{t}$ and after $\vec{n}$ as

$$\vec{b}(s)\equiv\vec{t}(s)\wedge\vec{n}(s). \tag{28}$$

The ordered triad $(\hat{t},\hat{n},\hat{b})$ is a basis, and is indicated with superscript as unit vectors.

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Is is here noted that the curvature $c(s)$ obeys also the definition

$$c(s) = lim_{\Delta s \to 0} \frac{\theta}{\Delta s} \tag{29}$$

and therefore describes the velocity of the angular deviation from the tangent to $\mathcal{M}$.

The osculating plane of $\mathcal{M}$ in the point $P(s) \in \mathcal{M}$ is parallel to the unit vectors $\hat{t}(s)$ and $\hat{n}(s)$.

The osculating plane of $\mathcal{M}$ is therefore taken to contain the vertex V of the lightcone in $P(s)$.

The angle with respect to the normal of the manifold is calculated from the director cosines of the normal of the following surface: the radial trajectories are considered with $d\Omega = 0$,which evolve in time but not in space, i.e., with $dr = 0$. The director cosines are those calculated at the normal to the plane tangent in the vertex $V = (x_0, y_0, z_0)$ of the tangent plane π^0

$$x_0 x + Y_0 y + z_0 z - R_c^2 = 0 \tag{30}$$

. In Eq. (30), R_c is the coordinate radius of the sphere described after the restrictions to the line element. The director cosines of the generator of the cone are those obtained after the limit of the Frenet mobile trihedron as

$$l = \frac{x_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}, \tag{31a}$$

$$m = \frac{y_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}, \tag{31b}$$

$$n = \frac{z_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}. \tag{31c}$$

The equivalence of the constructions holds everywhere as it will be proven is Section 4 (and at asymptotical flatness $r \to \infty$).

The light cones tip over and stay directed the same way for the complete geodesics considered.

4. Investigation of the $2 - dim$ Sphere in the Schwarzschild Spacetime

From [12], the mean curvature flow is written as

$$\dot{\mathcal{V}} = -\Delta \mathcal{V} = \vec{H} \tag{32}$$

with $\mathcal{V}$ the velocity of the curve $\mathcal{V}(t, \xi)$ with ξ an element of the 'common domain' of the embeddings of $\mathbb{S}^2$ prepared in Section 3.

From Eq. (32), $\Delta \mathcal{V}$ is perpendicular to the chosen normal vector field v as

$$\dot{\mathcal{V}} = -\sigma H v \tag{33}$$

with $H \equiv \mid \vec{H} \mid$, and

$$\sigma =< v, v > \tag{34}$$

Specification of the Schwarzschild Spacetime

From Eq. (22) and from Eq. (23), I now invoke the Theorema Egregium to obtain the following:

Theorem T02:

The absolute value of the co-dimension 2 *mean curvature vector* $\vec{\mathcal{H}}$ *in the there-ambient spacetime is vanishing.*

Proof T02:

The vector $\vec{\mathcal{H}}$ *is written as*

$$\vec{\mathcal{H}} \equiv \frac{1}{2} \vec{\nabla} \cdot \hat{n} \tag{35}$$

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being $\hat{n}$ the normal unit vector from the mobile Frenet trihedron; the directions ρ and v from Eq.'s (22) are orthogonal: the normal vector is vanishing as

$$\vec{n} \equiv \hat{\rho} \wedge \hat{v} \equiv 0. \tag{36}$$

□

Theorem T03:
The Ricci scalar of the cross section is vanishing.
Proof T03:
The cross section is obtained after a Riemannian submersion from the Schwarzschild spacetime. □.
Theorem T04:
The second fundamental form $\vec{\Pi}$ of the cross section is vanishing.
Proof T04:
The absolute value of the second fundamental form of the cross section is written as

$$R = 0 = \mathcal{H}^2 - \mid \vec{\Pi} \mid . \tag{37}$$

□ *and*

$$R = 0 = \frac{1}{2}\mathcal{H}^2. \tag{38}$$

□

Remark R2:
One remarks that the second fundamental form of the Schwarzschild spacetime is vanishing.
Let $\gamma_{\mu\nu}$ be the Riemannian metric of the 2-surface:
Theorem T05:
The cross section is one with with vanishing Riemann tensors

$$R_{\mu\nu\rho\sigma} = 0. \tag{39}$$

Proof T05:
The Riemann tensor is expressed as

$$R_{\mu\nu\rho\sigma} = \frac{1}{4} \mid \vec{\mathcal{H}} \mid^2 \left[\gamma_{\mu\rho}\gamma_{\nu\sigma} - \gamma_{\nu\rho}\gamma_{\mu\sigma}\right]. \tag{40}$$

□

Theorem T06:
The cross section is one with vanishing Ricci tensors. □

The Roesch-Scheuer mean curvature flow accounts for the projection of the mean curvature vector on the generators of the null hypersurface.

Theorem T07:
The Roesch-Scheuer mean curvature flow is newly found to be vanishing for Schwarzschild spacetime as

$$\frac{d}{dt}\mathbb{X} = 0. \tag{41}$$

Proof T07:
The Roesch-Scheuer mean curvature is written as

$$\frac{d}{dt}X = -\frac{1}{2} < \vec{\mathcal{H}}, \vec{L} > \vec{L} = 0 \tag{42}$$

(i.e., with vanishing $\mathcal{H}$. □*)*

Remark R3:
The mean curvature vector oft the cross section is therefore orthogonal to the null hypersurface.

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Let Σ be the surface of constant spacetime mean curvature, which is here newly found as one with $\mathcal{H} = 0$ (to be confronted with [37]).

The 'background foliation of round spheres' is obtained as

$$| \vec{\Pi} |= 0. \tag{43}$$

Proposition 3.5 of [12] implies the new following

Theorem T08:

Let (Σ, γ) *a spacelike cross section of* $\mathcal{N}$ *such that*

$$\nabla_\mu A_{\nu\rho} = \nabla_\nu A_{\mu\rho}. \tag{44}$$

Therefore,

$$| \nabla A |^2 = 0. \tag{45}$$

□

Lemma 3.6 from [12] implies the new following

Theorem T09:

$A_{\mu\nu}$ *is constant, and*

$$\Delta A_{\mu\nu} = 0 \tag{46}$$

□ Furthermore, the Ricci tensor of the cross section is vanishing.

Theorem T10:

The time derivative of the cross section is non-vanishing.

Remark R4:

It is remarked that the line element of the geodesics displacements of the Schwarzshild spacetime is calculated on geodesic spheres of the Schwarzschild spacetime, this displacement evolving in time.

Theorem T11:

The time derivative of γ *is vanishing.*

Proof T11:

From the Yamabe flow: the time derivative of the metric $\gamma_{\mu\nu}$ *is written as proportional to the Ricci tensor of the cross section as*

$$\frac{d}{dt}\gamma_{\mu\nu} = -2R(\Sigma)_{\mu\nu}. \tag{47}$$

□

Being

$$\gamma_{\mu\nu} = \omega d\Omega \tag{48}$$

Theorem T12:

The Gauss curvature $\mathcal{K}$ *of the cross section is vanishing.*

Proof T12:

The time derivative of the metric of the cross section is written as

$$0 = \frac{d}{dt}\omega = -\omega\mathcal{K}. \tag{49}$$

□

Theorem T13:

The time derivative of the mean curvature flow is vanishing.

Proof T13:

The time derivative of the mean curvature flow $\chi_{\mu\nu}$ *is written as the total time derivative of the Hessian matrix of* ω *with respect to the Christoffel connections calculated after the metric* γ *as*

$$\dot{\chi} = -2\frac{d}{dt}H^(\gamma)_{\mu\nu}\omega_{\rho\sigma} \equiv 0 \tag{50}$$

. **Theorem T14**:

The metric γ is defined after the expression of ω as

$$\omega \equiv \omega(\rho) = \frac{C_1}{\sqrt{1+ \mid \vec{a} \mid + \vec{a} \cdot \vec{\rho}}}. \tag{51}$$

Proof T14:

After the vanishig of the second fundamental form. □

Theorem T15:

The Lorentz boosts $\vec{b}$ on the tangent bundle are as

$$\vec{b} = (\sqrt{1+\vec{a}}, \mid \vec{a} \mid) \tag{52}$$

with the absolute value of the vector $\vec{a}$ from Theorem Tx1. □.

This way,

Theorem T16:

The Laplacian of the cross section is vanishing even though the time derivative of A is not vanishing as

$$\Delta A_{\mu\nu} = H_{\mu\nu} \mid \vec{\mathcal{H}} \mid^2 + \frac{1}{2} \mid \vec{\mathcal{H}} \mid^2 \dot{A}_{ij} \tag{53}$$

□

The 2-dimensional Ricci flow is therefore recalled to match the conditions of a Yamabe flow [27] as well.

Be $\hat{\Theta}$ and $\hat{L}$ the basis according to which the decompositions of the second form $\vec{\Pi}$ and that of $\mathcal{H}$ are performed.

Theorem T17:

The vector $\vec{\Theta}$ has vanishing norm.

Theorem T18:

Since $\mid \vec{L} \mid = 1$, the 'deformation equations

$$\frac{d}{dt} \equiv \Phi \vec{L} \tag{54}$$

is a diffeomorphism- it is calculated on the 1-dimensional surface of the section of the light cone and it is the Perelmann diffeomorphism.

The deformation equation is from conformal rescaling of the Schwarzschild metric as discussed in [28].

The time evolution of the metric of the cross-section is that of a $1+1$-dimensional manifold, i.e. one with trivial Killing vectors.

From [30], the need to study the Hamiltonian structure of the presented solitons is outlined [31].

The role of the weights is introduced in [32] to further examine [33] and [34].

5. Specifications of the Schwarzschild Solitons

Schwarzschild solitons are specified according to Definition Da as

Theorem T19:

The Schwarzschild solitons are a particular case of η-Yamabe-Ricci solitons with

$$\frac{\rho}{\sigma} \equiv -1. \tag{55}$$

Proof T19: *After the EFE's.* □

Example E1 allows one to state the following Theorem:

Theorem T20:

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For the Schwarzschild solitons,

$$\tau \equiv 0. \tag{56}$$

Proof T20:

For the Schwarzschild spacetime, the Einstein equations are taken

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{57}$$

from which τ id determined. □

Definition De allows one to write the following

Definition Dm:

Schwarzschild solitons are defined after setting

$$\zeta^\mu = v^\mu \tag{58}$$

being v^μ the four-velocity implied for the spacetime, for which

$$\phi = 0. \tag{59}$$

6. Specifications of the Generalized Schwarzschild Solitons

Theorem T21:

The Generalized-Schwarzschild solitons are a particular case of η-Yamabe-Ricci solitons with

$$\frac{\rho}{\sigma} \equiv -1. \tag{60}$$

Proof T21:

After the EFE's. □

Definition De allows one to write the following

Definition Dn:

The choice

$$\zeta^\mu = v^\mu \tag{61}$$

still applies for the conformal vector field with the smooth function ϕ as

$$\phi = 0. \tag{62}$$

7. Selected Topics About the Geometry on the Schwarzschild Lightcone

The null geometry on the standard Minkowski lightcone is here newly investigated as far as the definition of the future-oriented (portion of) lightcone is concerned, where selected topics are implemented starting from the notation established in [12].

The null coordinates are taken on $\mathbb{R}^{1,3}$ (U, V) as

$$V = r + t, \tag{63a}$$
$$U = r - t: \tag{63b}$$

this way the line element is written as

$$ds^2 = \frac{1}{2}(dudv + dvdu) + r^2 d\Omega^2 \tag{63c}$$

i.e., such that the radial position is now written as

$$r = r(U, V) = \left(\frac{U + V}{2}\right). \tag{63d}$$

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Within this framework, the surfaces $V = const$ are past-oriented lightcones, and the surfaces $U = const$ are the future-oriented lightcone.

The future-oriented portions of lightcones are considered $\mathcal{P}$ as

$$\mathcal{P} = \{\ u = 0\}\ \ = C_+(0). \tag{63e}$$

The manifold is defined $(\mathcal{P}, p)$ with metric $p_{\mu\nu}$.

Definition Do:

The metric of the future-oriented portion of lightcone is $p_{\mu\nu}$ such that

$$p_{\mu\nu} = r^2 d\Omega^2. \tag{63f}$$

The definition of cross section is now given as

Definition Dp:

Any spacelike cross section Σ of $\mathcal{P}$ is described as the graph over $\mathbb{S}^2$ as generalized as consisting of

$$\Sigma_V = \{\ W = r\}\ \ \leq \mathcal{P}. \tag{63g}$$

Remark R5:

Σ from definition Dl has the induced metric

$$\gamma = \omega^2 d\Omega^2. \tag{63h}$$

Remark R6:

(Σ, γ) is conformally round.

8. Outlook

Theorem T1 and Theorem T2 are commented after noticing that the Schwarzschild solitons and the Generalized-Schwarzschild solitons are constructed after Einsteinian spacetimes.

In the work of Chow [22], the problem is addressed, for which a ’natural’ evolution equation is looked for in the case a Riemannian metric is deformed conformally to a constant scalar-curvature metric.

In the work of Siddiqi et al. [3], the Siddiqi solitons are considered, which obey the equation

$$\frac{1}{2}\pounds_\zeta g_{\mu\nu} + \alpha R_{\mu\nu} + \left(\lambda - \beta\frac{R}{2}\right) g_{\mu\nu} = 0. \tag{64}$$

From example 1.1 form ibidem, the Einstein flow is introduced, and is here spelled correctly as obeying

Definition Dq:

The Einstein flow is defined as

$$\frac{\partial}{\partial t} g_{\mu\nu}(t) = -2R_{\mu\nu}(t). \tag{65}$$

The Einstein solitons are named in [3] as the self-similar solutions of the Eisenstein flow Eq. (65); the limit of the Einstein flow is studied ibidem as

$$\pounds_\zeta g_{\mu\nu} + 2R_{\mu\nu} + \left(\lambda - \frac{R}{2}\right) g_{\mu\nu} = 0 : \tag{66}$$

the ’almost-Ricci-Yamabe solitons’ were introduced in [5] when $\lambda \in \mathcal{C}^\infty(\mathcal{M})$.

It is the aim of future studies to qualify the Einstein solitons as those fro which the Killing vector field is one associated with the 4-velocity 4-vector (whose components define the symmetries of the solitons structures).

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A 'normal static spacetime' is recalled in [3] as one on a 'Lorentzian warped product manifold'. Ricci-Yamabe solitons are envisaged in [10] which undergo the Ricci-Yamabe flow

$$\frac{\partial}{\partial t} g_{\mu\nu}(t) 2\alpha R_{\mu\nu}(t) + \beta R(t) g_{\mu\nu}(t) \tag{67}$$

with the initial condition $g^{(0)}_{\mu\nu} = g_{\mu\nu}(0)$. The singularities of th flow are to be discussed after α and β.

I now introduce the characterization of the gradient Schwarzschild solitons which is gained after the presented study,which find applications in completing the interrogations raised in [6], in [7], in [8] and in [9].

Theorem 1.1 from [6] and Theorem 1.2 from [6] are here rewritten as follows.

Theorem T22:

For a Schwarzschild soliton $(\mathcal{M}^4, g, f)$, given $\kappa > 0$ a constant, the following identity holds

$$\int_{\mathcal{M}} \mid R_{\mu\nu} \mid^2 e^{-\kappa f} \sqrt{-g} d^4x = 0 \tag{68}$$

□

Theorem T23:

For a Schwarzschild soliton $(\mathcal{M}^4, g, f)$, given $\kappa < 1$ a constant, the following identity holds

$$\int_{\mathcal{M}} \mid \nabla R_{\mu\nu} \mid^2 e^{-\kappa f} \sqrt{-g} d^4x = 0 \tag{69}$$

□

For a steady Schwarzschild soliton, the weight f is defined after

$$R_{\mu\nu} = H^{(g)}_{\mu\nu} f. \tag{70}$$

Eq. (70) is here newly solved analytically as

$$f = F_0 \frac{1}{2} \sqrt{\frac{r}{r - r_S}} \frac{r - r_S}{\sqrt{r(r - r_S)}} \left[2\sqrt{r^2 - r r_S} + r_S ln\left(r + \sqrt{r^2 - r r_S} \right) - \frac{r_S}{2} \right] + f(r_0) \tag{71}$$

with the integration constant F_0 from

$$F = F_0 \sqrt{\frac{r_0(r - r_S)}{r(r_0 - r_S)}} \tag{72}$$

being $F \equiv df/dr$.

It is remarked that the solution Eq. (71) serves as the discussion of the complete solution starting from the homogenous solution associated of the flow of the expanding gradient Schwarzschild solitons and to that of the shrinking gradient Schwarzschild solitons.

Eq. (71) answers the requests of Theorem 5.1 and of Corollary 5.2 from [6].

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