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Long Journey from Stevenson's Formally Complex Hypergeometric Polynomials to Real-by-Definition Romanovski-Routh Polynomials

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Abstract

The paper reexamines Stevenson's technique for solving Schrödinger's "Kepler problem" in a spherical space in terms of formally complex hypergeometric polynomials. A certain advantage has been achieved by reformulating the genetic 'dual principal Frobenius solution' (d-PFS) problem as the Dirichlet problem for the given second-order ordinary differential equation (ODE) rewritten in its 'prime' form. It was demonstrated that the cited polynomials can be converted to Askey's hypergeometric expression for the real-by-definition Romanovski/pseudo-Jacobi polynomials ('Romanovski-Routh' polynomials in our terms). The formulated Dirichlet problem was then reduced to the two more specific cases representing the Sturm-Liouville problems (SLPs) with infinite and respectively finite discrete energy spectra. The exact solvability of the former SLP (with the Liouville potential represented by the 'trigonometric Rosen-Morse' potential) was proven by taking into account that the Romanovski-Routh polynomial of degree n must have exactly n real zeros (with no upper bound for the eigenvalues). As the direct consequence of this proof, we then found that the mentioned d-PFS problem in general and therefore the second SLP with the finite discrete energy spectrum are exactly solvable via quasi-rational solutions (q-RSs) composed of the Romanovski/Routh polynomials with degree-dependent indexes.

Keywords: normal ODE; Dirichlet problem; Sturm-Liouville problem; principal Frobenius solution; hypergeometric polynomials; Routh polynomials; Romanovski-Routh polynomials; trigonometric Rosen-Morse potential; Milson potential

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1. Introduction

While the classical Jacobi polynomials are the topic of numerous textbooks there are two other sequences of finite orthogonal hypergeometric polynomials revealed by Askey [1] and related by him to the (obscure at that time) paper by Romanovski [2]. (It was the citation in [3], which brought author's attention to Askey's breakthrough discovery, though its impact on the studies of the potentials solvable via the Romanovski polynomials has not be properly appreciated in the quantum-mechanical literature so far.)

Roughly speaking, Askey discovered a subset of complex Jacobi polynomials orthogonal with a complex weight and demonstrated that its real field realizations include (in addition to classical Jacobi polynomials) the two finite sequences of orthogonal polynomials discovered by Romanovsky [2,4]: 'Romanovski/Jacobi' and 'Romanovski/pseudo-Jacobi' polynomials, using the classification scheme of the Romanovsky polynomials suggested by Lesky [5,6] few years after publication of Askey's paper [1]. (For the reasons clarified in [7] we prefer to refer to the latter polynomials as 'Romanovski-Routh' or simply as 'R-Routh' polynomials, with the term 'Routh polynomials' [8] used as the precise synonym for 'pseudo-Jacobi' polynomials.)

The most important impact of Askey's discovery on the current study is his derivation of the analytical representation of the real-by-definition R-Routh polynomials as formally complex hypergeometric polynomials. To be more precise, the hypergeometric expression, originally derived in [1] and listed in the recent monograph [9] in connection with the so-called 'pseudo-Jacobi' polynomials (see Appendix A for comments), is applicable to all the Routh polynomials (not just to their finite orthogonal subsets formed by R-Routh polynomials.

The mentioned hypergeometric realization of the R-Routh polynomials helped the author to more deeply understand Stevenson's [10] sketchy note on Schrödinger's "Kepler problem" in a spherical space [11] and to prove the exact solvability of the two Sturm-Liouville problems converted by the Liouville transformations [12] to the one-dimensional Schrödinger equations with the so-called [13,14] 'trigonometric Rosen-Morse' (t-RM) potential and with the four-parameter Milson potential on the line [15] also solvable via R-Routh polynomials [16].

In Section 2 the author examines the boundary conditions that one has to impose on the solutions of the second-order 'normal' [17] ordinary differential equation (*N*ODE) used by Stevenson [10] as the starting point for his breakthrough analysis. It turned out that the more natural Dirichlet boundary conditions (DBCs) were unapplicable if the angular momentum is equal to zero. For this reason, we found it necessary to redefine the mentioned boundary-condition problem as the search for the so-called 'dual' principal Frobenius solutions' (d-PFSs), which represent the PFSs of the second-order *N*ODE near both singular endpoints: $-\infty$ and $+\infty$. To prove that the d-PFS problem under consideration is exactly solvable, the given NODE with the three regular singularities at $\pm i$ and ∞ is then converted to its 'prime' [18] form (*p*-ODE) solved under the DBCs.

It was confirmed that the Dirichlet problem in question has infinite sequences of quasi-rational solutions expressible in terms of formally complex hypergeometric polynomials, as expected from Stevenson's analysis [10]. Making use of the hypergeometric formulas for the 'pseudo-Jacobi' polynomials in [9], we then demonstrated that the hypergeometric polynomials, introduced in [10], are nothing but the family of the Romanovski polynomials [2,4] associated with Pearson's [19] generalized probability curve of Type IV. As mentioned above, Romanovski's paper [2] was practically unknown to a broad variety of mathematicians at that time. As a matter of fact, this paper attends to interrelate the two works [2] and [10] (nearly a century after their publication), taking advantage of the more recent developments [1,8,13,14] in this field.

A more thorough analysis of [10] revealed some ambiguities, which made questionable our original attempt [16] to prove the exact solvability of the Milson potential [15] simply by referring to Stevenson's arguments for the quasi-rational solutions (q-RSs) of the mentioned NODE expressed in terms of the complex argument. Namely, to guarantee that the q-RSs in question represent all the possible solutions of the given Dirichlet problem, we, IN Section 3, first prove this assertion for the 'prime' [18] Sturm-Liouville equation (*p*-SLE) converted by the Liouville transformation to the Schrödinger equation with the *t*-RM potential [13,14].

Since the polynomial solutions associated with the aforementioned Sturm-Liouville problem (SLP) have been thoroughly examined in [13,14], we simply had to trace their results to the hypergeometric expressions for the 'pseudo-Jacobi polynomials' in [9]. We refer the reader to Appendix B for this analysis. The very important observation made by Compean and Kirchbach is that the dependence of the polynomial indexes on polynomial degrees made it possible to specify infinitely many eigenfunctions, despite the fact that the R-Routh polynomials themselves form finite orthogonal sequences.

It is a more challenging problem to prove that the SLP under consideration does not have any other 'exotic' solutions. At this point we had to make use of Gesztesy et al.'s [20] powerful theorem, asserting that the n^{th} eigenfunction of the given SLP must have exactly n-1 zeros. Coupled with the fact that the SLP in question has infinitely many monotonically increasing eigenvalues, the cited theorem allowed us to confirm that the Schrödinger equation with the *t*-RM potential is exactly solvable in terms of the R-Routh polynomials.

In contrast with the discrete energy spectrum of the t-RM potential, the SLP associated with the Milson potential [15] has a finite number of the quasi-rational eigenfunctions. As a result, it becomes problematic to prove that no other bound eigenstate exists above the already found eigenvalues. For this reason, we (after proving the exact solvability of the *t*-RM potential) come back to the general Dirichlet problem for Stevenson's NODE (2) and show that the existence of any other solutions would contradict to the proven exact solvability of the *t*-RM potential in terms of the q-RSs in question.

In Section 4 we then extend this conclusion to the prime form of Milson's [15] canonical SLE (CSLE). The sketched three-step proof thus refined our original argumentation [16] in support of the exact solvability of the corresponding Liouville potential.

While the SLP put forward in [15] was formulated as the p-SLE solved under the DBCs, the latter conditions were proven to be precisely equivalent to the requirement for the eigenfunctions of the given CSLE to be squarely integrable with the corresponding density function. This proof confirmed that any eigenfunction of the Schrödinger equation with the Milson potential on the line [15] is expressible in terms of the R-Routh polynomials.

In the translationally shape invariant (TSI) limit of the Milson potential represented by the renowned 'Gendenshtein' [21] potential (the 'Scarf II' potential in the terminology of [22]) the indexes of the R-Routh polynomials become independent of the polynomial degrees and the orthogonality relations for the eigenfunctions turn into the orthogonality relations for the R-Routh polynomials in [9]. As a result, the normalization constants for the eigenfunctions of the Schrödinger equation with this particular potentials can be directly obtained from the normalization constants reported in [9] for the R-Routh polynomials. To our knowledge, this normalization constants has been never discussed in the literature.

2. Search for d-PFSs Using the DBCs

Let us start our discussion from Stevenson's ODE (2) re-written as

$$\left[\frac{d^2}{dy^2} + I^o[y;\lambda,\mu,l]\right]\Phi^{\pm}[y;\lambda,\mu,l] = 0, \qquad (1)$$

where the rational 'Stevenson' invariant

$$I^o[y;\lambda,\mu,l] := \frac{\lambda + 2\mu y}{(1+y^2)^2} - \frac{(l+\tfrac{1}{2})^2 - \tfrac{1}{4}}{1+y^2} \qquad (2)$$

is nothing but the 'normal' [17] form of the Riemann's P-equation [23], with the poles at ±*i* and ∞. The characteristic exponents (ChExps) for the pole at infinity are equal to -*l* and *l*+1, with the positive exponent difference (ExpDiff) of 2*l*+1 for $l > -\tfrac{1}{2}$. The subscripts ± indicate that these are the PFSs of the given ODE at ±∞, i.e., the FSs with the common ChExp of -*l* and therefore

$$\left|\lim_{y\to\pm\infty} |y|^{l}\Phi^{\pm}[y;\lambda,\mu,l]\right| < \infty \qquad (3)$$

Note that the *d*-PFSs do not obey the DBCs at ±∞ if *l* =0.

2.1. The d-PFS Problem for Second-Order ODE

We term a real solution $\phi_\tau[y;\lambda_\tau,\mu_\tau,l_\tau]$ as '*d*-PDF' if it constitutes a PFS at both -∞ and +∞. In other words

$$\left|\lim_{y\to\pm\infty} |y|^{l_\tau}\phi_\tau[y;\lambda_\tau,\mu_\tau,l_\tau]\right| < \infty, \qquad (4)$$

where τ identifies the parameter values specifying all the possible d-PFSs. It will be proven in subsection 3.2 that there is an infinite sequence of the d-PFSs for an arbitrary value of μ and any positive value of the ExpDiff 2*l*+1, so (4) takes the form

$$\left| \lim_{y\to\pm\infty} |y|^{l} \phi_n[y;\lambda_n(l,\mu),\mu,l] \right| < \infty . \quad (5)$$

The fundamentally important result of Stevenson's paper [10] not properly acknowledged in the literature is his proof that the condition (5) holds for the q-RSs composed of the hypergeometric polynomials in the complex variable:

$$\xi = 2/(1+iy) . \quad (6)$$

One of the central objectives of the current discussion is to explicitly confirm that the latter polynomials multiplied by the power function $(y\text{-}i)^n$ become real after been expressed in y (as anticipated by Stevenson) and moreover constitute infinitely many (real by definition) Routh polynomials with degree-dependent indexes.

2.2. Reformulating the d-PFS Problem as Dirichlet Problem for p-ODE

To proceed, let us first transform the NODE (1) to the self-adjoint algebraic *p*-ODE:

$$\left\{ \frac{d}{dy}\sqrt{y^2+1}\,\frac{d}{dy} - q[y;\lambda,\mu,l] \right\} \Psi_{\pm}[y;\lambda,\mu,l] = 0 \quad (7)$$

by the gauge transformation

$$\Psi^{\pm}[y;\lambda,\mu,l] = (y^2+1)^{-1/4}\,\Phi^{\pm}[y;\lambda,\mu,l], \quad (8)$$

chosen in such a way [18] that the ChExps for the pole at infinity differ only by their signs and as a result the DBC unambiguously pinpoints the PFS:

$$\lim_{y\to\pm\infty} \Psi^{\pm}[y;\lambda,\mu,l] = 0 . \quad (9)$$

It can be shown that

$$q[y;\lambda,\mu,l] := -\sqrt{y^2+1}\left(I^{o}[y;\lambda,\mu,l] + \mathcal{S}\{\sqrt{y^2+1}\} \right) \quad (10)$$

where [18]

$$\mathcal{S}\{\rho[y]\} := \tfrac{1}{4}\dot{\rho}^2[y]/\rho[y] - \tfrac{1}{2}\ddot{\rho}[y], \quad (11)$$

with dot standing for the derivative with respect to y. One can verify that [7]

$$\sqrt{y^2+1}\,\mathcal{S}\{\sqrt{y^2+1}\} = \frac{3}{4(y^2+1)^{3/2}} - \frac{1}{4(y^2+1)^{1/2}}, \quad (12)$$

which, coupled with (2), gives

$$q[y;\lambda,\mu,l] = -\frac{\lambda + 3/4 + 2\mu y}{(1+y^2)^{3/2}} + \frac{(l+1/2)^2}{(1+y^2)^{1/2}} . \quad (13)$$

2.3. Pairs of Real q-RSs Formed by Routh Polynomials

Coming back to the complex change of variable (6), let us first note that it represents the linear fractional transformation. As a result, the Schwarzian derivative {y,ξ} vanishes [12] and the invariant of the complex rational CSLE (RCSLE)

$$\left\{\frac{d^2}{d\xi^2} + \underset{\sim}{I}[\xi;\lambda,\mu,l]\right\} \underset{\sim}{\Phi}[\xi;\lambda,\mu,l] = 0 \qquad (14)$$

is related to the Stevenson invariant (2) via the elementary formula

$$\underset{\sim}{I}[\xi;\lambda,\mu,l] = \left(\frac{d\,y}{d\,\xi}\right) I^0[i - i2/\xi;\lambda,\mu], \qquad (15)$$

$$= \tfrac{1}{4}\left(\frac{d\,y}{d\,\xi}\right)^2 \left[\frac{\lambda + 2i\,\mu(1-2/\xi)}{4(\xi-1)} - \frac{l(l+1)}{\xi^2}\right]\frac{\xi^4}{\xi-1}, \qquad (16)$$

bearing in mind that $-i\,y = 1 - 2/\xi$ and

$$1 + y^2 = 4(\xi-1)/\xi^2. \qquad (17)$$

Taking into account that

$$\frac{d\,y}{d\,\xi} = 2i/\xi^2,$$

(18)

we can then re-write (16) as

$$\underset{\sim}{I}[\xi;\lambda,\mu,l] = \frac{1-(2l+1)^2}{4\xi^2} - \frac{\lambda+2\mu i}{4(1-\xi)^2} + \frac{1-(2l+1)^2 - 4\mu i}{4\xi(1-\xi)}. \qquad (19)$$

Substituting

$$\lambda + 2\mu i = (2\alpha^* - 1)^2 - 1, \qquad (20)$$

$$4\,\mu i = (2\alpha^* - 1)^2 - (2\alpha - 1)^2 \qquad (21)$$

into the last two summands in (19) and comparing the resultant expression with the invariant (9) in § 2.7.2 in [12], we can represent the parameters (7) in the cited subsection as

$$1 - c = -2l - 1,\; b - a = 1 - 2\alpha,\; c - a - b = 1 - 2\alpha^*. \qquad (22)$$

Since $c = 2l + 2 > 1$ for $l > -\tfrac{1}{2}$, it cannot be a nonpositive integer which assures that the corresponding hypergeometric series converges for $|\xi| < 1$ or, taking into account (6), iff $|y| \geq \sqrt{3}$.

In this Section we are only interested in the root

$$\alpha \equiv \alpha(\lambda,\mu) := -\tfrac{1}{2}\sqrt{\lambda + 1 - 2\mu i} + \tfrac{1}{2} \qquad (23)$$

such that

$$\alpha_R \equiv \alpha_R(\lambda,\mu) < \tfrac{1}{2}, \qquad (24)$$

which specifies the PFSs. As result, the constraint (24) will select the q-RSs expressible via R-Routh polynomials. On other hand, as explained in Section 5, the second root

$$\alpha \equiv \alpha(\lambda,\mu) := \tfrac{1}{2}\sqrt{\lambda + 1 - 2\mu i} + \tfrac{1}{2} \qquad (25)$$

Is associated with the q-RSs composed of the Routh polynomials with no real roots, which can be used for constructing the rational Darboux transforms ($\mathrm{RD}\mathfrak{T}\mathrm{s}$) of the Liouville potentials under consideration.

Keeping in mind that

$$2\alpha_{\mathrm{I}}(\lambda,\mu)=\mu/[1-2\alpha_{\mathrm{R}}(\lambda,\mu)], \tag{26}$$

one finds

$$\lambda=[\alpha_{\mathrm{R}}(\lambda,\mu)-\tfrac{1}{2}]^2-\mu^2/[\alpha_{\mathrm{R}}(\lambda,\mu)-\tfrac{1}{2}]^2. \tag{27}$$

If $\mu\neq 0$, the second summand in the sum (26) monotonically increases for from -∞ to 0 as α_{R} varies from 0 to -∞, which implies that the sum is a monotonically decreasing function of α_{R}, changing from +∞ to -∞ as α_{R} varies from -∞ to 0. In the 'edge' case $\mu=0$ the function (26) changes from -1 to +∞, as α_{R} varies from 0 to -∞, and as a result, the solutions of our interest do not exist for negative values of λ.

The PFSs of the NODE (1) near the singular endpoints at ± ∞ can be thus expressed in terms of the same complex solution of the NODE (14),

$$\underset{\sim}{\Phi}[\xi;\alpha(\lambda,\mu),l]:=\xi^{l+1}(\xi-1)^{\alpha(\lambda,\mu)}\,\mathrm{F}\big(2\alpha_{\mathrm{R}}(\lambda,\mu)+l,\ l+1+2i\ \alpha_{\mathrm{I}}(\lambda,\mu),2l+2;\xi\big) \quad \text{for } |\xi|\leq 1 \tag{28}$$

as follows

$$\Phi^{\pm}[\mathrm{y};\lambda,\mu,l]=2^{-l-1}e^{-\frac{1}{4}\pi i}\sqrt{\frac{d\mathrm{y}}{d\xi}}\ \underset{\sim}{\Phi}\,[2/(1+i\mathrm{y});\alpha(\lambda,\mu),l] \quad \text{for } \pm\mathrm{y}\geq\sqrt{3}. \tag{29}$$

As the direct consequence of (18), with ξ given by (6), one finds

$$\sqrt{\frac{d\mathrm{y}}{d\xi}}=\mathrm{e}^{\frac{1}{4}\pi i}(1+i\mathrm{y})/\sqrt{2}, \tag{33}$$

Substituting (28), (33), and

$$\xi-1=\frac{1-i\mathrm{y}}{1+i\mathrm{y}} \tag{34}$$

into the right-hand side of (29), one finds

$$\Phi^{\pm}[\mathrm{y};\lambda,\mu,l]=(1+i\mathrm{y})^{-l-\alpha(\lambda,\mu)}(1-i\mathrm{y})^{\alpha(\lambda,\mu)}\times$$

$$\mathrm{F}\big(2\alpha_{\mathrm{R}}(\lambda,\mu)+l,\ l+1+2\alpha_{\mathrm{I}}(\lambda,\mu)i,2l+2;2/(1+i\mathrm{y})\big)\quad \text{for } \pm\mathrm{y}\geq\sqrt{3}, \tag{35}$$

The derived expression is reminiscent of (3) in [10], with x standing for y here, except that we replaced the complex powers of y-*i* and y+*I* for the well-defined complex powers of 1+ *i* y and 1- *i*y accordingly.

Alternatively, bearing in mind that

$$\frac{1-i\mathrm{y}}{1+i\mathrm{y}}=exp(-2\,artag\,\mathrm{y}i)\ , \tag{36}$$

we can rewrite (35) as

$$\Phi^{\pm}[\mathrm{y};\lambda,\mu,l]=(1+\mathrm{y}^2)^{\alpha_{\mathrm{R}}(\lambda,\mu)}(1+i\mathrm{y})^{-l-2\alpha_{\mathrm{R}}(\lambda,\mu)}\,exp[2\alpha_{\mathrm{I}}(\lambda,\mu)\,artag\,\mathrm{y}]$$

$$\times F\left(2\alpha_R(\lambda,\mu)+l,\ l+1+i\ 2\alpha_I(\lambda,\mu),2l+2;2/(1+iy)\right)\ \text{for}\ \pm y \geq \sqrt{3}, \quad (37)$$

where

$$(1+iy)^{\beta_R} := (1+y^2)^{\frac{1}{2}\beta_R} exp(i\beta_R\, artag\, y) \quad (38)$$

for any real β_R.

As pointed out by Stevenson [1], the hypergeometric series in the right-hand side of (37) terminate if

$$2\alpha_R(\lambda,\mu) = 2\alpha_{R;n}(l) := -l-n. \quad (39)$$

Setting

$$\lambda_n(l,\mu) := (l+n+1)^2 - \frac{\mu^2}{(l+n+1)^2} - 1 \quad (40)$$

and

$$2\alpha_{I;n}(l,\mu) := \mu/(l+n+1), \quad (41)$$

we can represent both PFSs in the common quasi-rational form:

$$(-i)^n \Phi^{\pm}[y;\lambda_n(l,\mu),\mu,l] = \phi_n[y;\alpha_n(\mu,l)], \quad (42)$$

where the real q-RSs

$$\phi_n[y;\alpha] = \phi_n^*[y;\alpha] := (y^2+1)^{\alpha_R} exp(2\alpha_I\, arctan\, y) P_n(y;2\alpha_I,-2\alpha_R) \quad (43)$$

are formed by the (real) monic 'pseudo-Jacobi' polynomials [9]

$$P_n(y;2\alpha_I,-2\alpha_R) := (y+i)^n F(-n, 1-n-2\alpha, 2-4\alpha_R-2n; 2/(1-iy)). \quad (44)$$

To our knowledge, this hypergeometric expression first appeared in the monograph [9] just two decades ago. It thus took more than half a century before the appropriate mathematical expressions were developed to address Stevenson's speculation that the monic polynomials defined via (44) constitute real functions in y (see Appendix A for the thorough analysis of this by-no-means trivial proof).

As clarified in Appendix A, the polynomials (44) represent the monic form of the 'Routh polynomials' defined by us [7,16] via (47) below and used as the precise equivalent for the term 'pseudo-Jacobi polynomials'. In both cases we give to these terms exactly the same meaning as Jordaan and Toókos [24] grant to the term 'pseudo-Jacobi polynomials'. The monic Routh polynomials in question turn into the monic R-Routh polynomials introduced in the following subsection.

It is important that each eigenfunction is accompanied by the q-RS

$$\phi_n[y;\alpha_n(-l-1,\mu)] := (y^2+1)^{\frac{1}{2}(l+1)} exp[2\alpha_{I;n}(-l-1,\mu)\, arctan\ y] \times$$

$$(y^2+1)^{-\frac{1}{2}n} P_n\left(y;2\alpha_{I;n}(-1-l,\mu), n-1-l\right). \quad (45)$$

As proven in [7] and briefly outlined in Section 5, one can find the q-RSs (45) formed by the Routh polynomials with no real zeros, which can be used as the transformation functions (TFs) for constructing new exactly solvable rational CSLEs.

2.4. Infinite Sequences of Quasi-Rational d-PFSs Expressible in Terms of R-Routh Polynomials

By applying the gauge transformation (8) to the q-RSs (43), we find that the solutions of the *p*-SLE (7):

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$$\psi_n[y;\mu,l] := (y^2+1)^{-\frac{1}{2}(l+\frac{1}{2})} exp[2\alpha_{I,n}(\mu,l) \, arctan \; y]$$

$$\times (y^2+1)^{-\frac{1}{2}n} P_n\left(y; 2\alpha_{I;n}(\mu,+l), n+l\right) \text{ for } l > -\tfrac{1}{2}. \quad (46)$$

satisfy the DBCs at the both limits y→+∞ and y→ -∞ accordingly iff $l > -\frac{1}{2}$.

As originally pointed to by Cryer [25] and later exploited in great details by Askey [1] (though with no mentioning of Cryer's contribution to this subject), the polynomials

$$\Re_n^{(\lambda)}(y) := (-i)^n \, P_n^{(\lambda,\lambda^*)}(iy) \quad (47)$$

represent the one of the three families of the Romanovsky polynomials [4]:

$$q_n(y; \lambda_R, \lambda_I) := (-1)^n \Re_n^{(\lambda_R + \lambda_I i)}(y) \quad (48)$$

for $n < -\lambda_R - \frac{1}{2}$, with λ standing for $a + bi$ in [1].

In following [3,13,14,26,27], we [7,18] reserve the letter R solely for the R-Routh polynomials defining them via (3.5) and (3.6) in [27]:

$$R_n^{(2+\lambda_I i, \lambda_R + 1)}(y) := \Re_n^{(\lambda_R + \lambda_I i)}(y) \qquad \text{for n=0,}\ldots, \left\lfloor -\lambda_R - \tfrac{1}{2} \right\rfloor. \quad (49)$$

(It is worth mentioning that Quesne's notation [27] for the R-Routh polynomials differs from those in [3,13,14,26].) On other hand, Askey [1] uses the notation $R_n^{(\alpha,\beta)}(x)$ for the R-Jacobi, not R-Routh polynomials.

Examination of the inequality

$$-\alpha_{R;n} - \tfrac{1}{2} - n = l + \tfrac{1}{2} > 0 \quad (n=0,1,\ldots) \quad (50)$$

reveals that the polynomials (49) form the infinite sequence of the R-Routh polynomials with the degree-dependent indexes — the astonishing fact disclosed in Compean and Kirchbach's enlightening study [13] on the eigenfunctions of the '*t*-RM' potential. The cited authors also presented an overview of the literature on the mathematical grounds of Romanovski's [2] finite sequences of the orthogonal polynomials mostly unknown to physicists. This list was then largely expanded in [3].

Confusingly, the term 'pseudo-Jacobi polynomials' were originally defined in [9] via (9.9.1) only for nonnegative integers N. Later Koornwinder [28], in his additions to the monograph [9], extended the range of definition for this parameter to any real values larger then $-\frac{1}{2}$, while referring to these polynomials as 'pseudo-Jacobi' or 'Romanovski-Routh' polynomials (to a certain degree, under influence of our paper [16]). We should however remind the reader that the latter term was introduced by us in [16] solely for the subset of the pseudo-Jacobi polynomials satisfying the orthogonality relation (9.9.2) in [9] and therefore representing one of three families of Romanovski polynomials [2].

Let us wrap up this discussion by making some fine-tuning remarks concerning Stevenson's assertion [10] that the constructed quasi-rational eigenfunctions constitute all the possible solutions of the given spectral problem, based on his unpublished analysis of the analytic continuation of the hypergeometric equation, with the reference to Whittaker and Watson's treatise [23]. To be more precise, he claimed that the solution defined by his Eq. (3) is discontinuous at y = 0 along the real axis, unless the series is truncated. However, from our perspective, such a proof is complicated by the fact that the analytical continuation of this function (see (15.3.7) in [29], for example) has the cut at y=0 due to the constraint

$$|\arg(-1 + i\,y)| < \pi \quad (51)$$

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and therefore it is applicable only for either positive or negative values of y. The cited analytical extension cannot be thus used for constructing the sought-for solutions with the continuous logarithmic derivatives at y=0.

We came to the same problem, while trying to apply the analytical continuation (15.3.6) in [29]:

$$F\left(l+1+2\alpha_R,\ l+1+2i\alpha_I, 2l+2;\xi\right)=\frac{\Gamma(2l+2)\Gamma(-2\alpha)}{\Gamma(l+1-2\alpha_R)\Gamma(l+1-2i\alpha_I)}$$

$$\times F(l+1+2\alpha_R,\ l+1+2i\alpha_I, 1-2\alpha;1-\xi)$$

$$+\frac{\Gamma(2l+2)\Gamma(2\alpha^*)}{\Gamma(l+1+2\alpha_R)\Gamma(l+1-2i\alpha_I)}(1-\xi)^{-\alpha}F(l+1-2i\alpha_I, l+1-2\alpha_R;1-2\alpha;1-\xi)$$

$$(|\,arg(1-\xi)\,|<\pi \mid <\pi), \qquad (52)$$

which, according to (34), specifies the PFSs of our interest on the upper and lower halves of the unit circle. Again the analytical continuation has a cut at $1-\xi=-1$, which did not allow us to analyze the logarithmic derivatives of the solutions at this point.

As mentioned in introduction, we will prove the exact solvability of the *d*-PFS problem under consideration, after re-formulating the Dirichlet problem for the *p*-ODE (7) as the SLP with λ standing for the spectral parameter. We then make advantage of Gesztesy et al.'s [20] results to prove that the constructed quasi-rational eigenfunctions exhaust all the possible solutions of the formulated SLP.

As for now, let us simply notice that truncating (37) via (39) brings us to the new hypergeometric realization for the monic pseudo-Jacobi polynomials (38):

$$P_n(y;2\alpha_I,-2\alpha_{R;n})=\frac{\langle 1-2\alpha-n\rangle_n}{\langle 2-2n-4\alpha_R\rangle_n}(y-i)^n F\left(-n,-n-\alpha,\ \alpha^*+1;(y+i)/(y-i)\right) \qquad (53)$$

with

$$\langle\nu\rangle_n=\frac{\Gamma(\nu+n)}{\Gamma(\nu)} \qquad (54)$$

standing for the rising factorial [30].

3. Exact Solvability of the t-RM Potential

Let us now rewrite the ODE (1) in the form of the CSLE

$$\left[\frac{d^2}{dy^2}+I^o[y;0,\mu,l]+\lambda\rho_S[y]\right]\Phi[y;\lambda,\mu,l]=0 \quad (l>-\tfrac{1}{2}) \qquad (55)$$

with the density function

$$\rho_S[y]:=(1+y^2)^{-2}, \qquad (56)$$

such that the change of variable (A1) in Appendix A converts it into the Schrödinger equation with the t-RM potential [33]. It will be proven that the q-RSs

$$\phi_n[y;\alpha_n(l,\mu)]=N_n(l,\mu)(y^2+1)^{-\frac{1}{2}l}\exp[2\alpha_{I,n}(\mu,l)\arctan y]$$

$$\times(y^2+1)^{-\frac{1}{2}n}P_n\left(y;2\alpha_{I;n}(l,\mu),-2\alpha_{R;n}(l,\mu)\right) \qquad (57)$$

$$\text{for } n=0,\ldots,\left\lfloor -2\alpha_{R;n}(l,\mu)+\tfrac{1}{2}\right\rfloor,$$

with the eigenvalues (34) constitute all the possible eigenfunctions $\phi_\tau[y;l,\mu]$ of the CSLE

$$\left[\frac{d^2}{dy^2}+I^o[y;0,\mu,l]+\lambda_\tau(l,\mu)\rho_S[y]\right]\phi_\tau[y;\alpha_n(l,\mu)]=0 \quad (l>-\tfrac{1}{2}) \quad (58)$$

selected via the requirement

$$\left|\lim_{y\to\pm\infty}|y|^l\,\phi_n[y;\alpha_n(l,\mu)]\right|<\infty \qquad (l>-\tfrac{1}{2}) \qquad (59)$$

The crucial point for our analysis is that the Wronskian of two eigenfunctions defined in such way vanishes at both endpoints:

$$0<\lim_{y\to\pm\infty}\left||y|^{2l+1}\,W\{\phi_{n''}[y;\alpha_{n''}(l,\mu)],\phi_{n'}[y;\alpha_{n'}(l,\mu)]\}\right|<\infty \quad (n''\neq n'), \qquad (60)$$

iff $l>-\tfrac{1}{2}$, which is the necessary preposition for Corollary 2.3 in [20]. In particular, as the direct corollary of (60), we assert that the eigenfunctions (57) must be orthogonal with the weight (56):

$$\int_{-\infty}^{+\infty}\phi_{n''}[y;\alpha_{n''}(l,\mu)],\phi_{n'}[y;\alpha_{n'}(l,\mu)]\rho_S[y]dy=0 \quad (n'\neq n)\,. \quad (61)$$

Furthermore, based on the proof presented in [20], we assert that the (n+1)-th eigenfunction of the CSLE must have exactly n nodes.

As clarified in Appendix B, the normalization of the eigenfunctions (57) must be performed with the same weight:

$$\int_{-\infty}^{+\infty}\left|\phi_n[y;\alpha_n(l,\mu)]\right|^2\rho_S[y]dy=1. \qquad (62)$$

In other words,

$$N_n^{-2}(l,\mu)=$$

$$\int_{-\infty}^{+\infty}(y^2+1)^{\alpha_{R,n}(\mu,l)-2}\,exp[4\alpha_{I,n}(\mu,l)\,arctan\;y]P_n^2\left(y;2\alpha_{I;n}(l,\mu),-2\alpha_{R;n}(l,\mu)\right)dy \quad (63)$$

$$\text{for } n=0,\ldots,\left\lfloor -2\alpha_{R;n}(l,\mu)+\tfrac{1}{2}\right\rfloor,$$

which differs from the normalization of the monic R-Routh polynomials via (A12) in Appendix A.

Theorem 1. The formulated SLP is exactly solvable via the q-RSs (54).

Proof of Theorem 1. Since the monic R-Routh polynomial of degree n in the right-hand side of (57) has exactly n-1 real zeros, the SLP in question may not have any solution between the eigenvalues $\lambda_{n-1}(l,\mu)$ and $\lambda_n(l,\mu)$ for $l>-\tfrac{1}{2}$. Keeping in mind that the given discrete energy spectrum is unbounded from above, we assert that the q-RSs (57) exhaust all the possible eigenfunctions of the CSLE (55). □

By making the gauge transformations

$$\psi_n[y;\alpha_n(l,\mu)]=\rho^{-\frac{1}{2}}[y]\phi_n[y;\alpha_n(l,\mu)] \qquad (64)$$

with

$$\rho[y]=\sqrt{y^2+1}\,, \qquad (65)$$

we come to the 'prime' SLE

$$\left\{\frac{d}{dy}\sqrt{y^2+1}\frac{d}{dy} - q_S[y;l,\mu] + \lambda_n(l,\mu)w[y]\right\}\psi_n[y;l,\mu] = 0 \qquad (66)$$

with the free-term and weight functions

$$q_S[y;l,\mu] := q[y;0,\mu,l] = -\frac{3/4 + 2\mu y}{(1+y^2)^{3/2}} + \frac{(l+1/2)^2}{(1+y^2)^{1/2}} \qquad (67)$$

and

$$w[y] := \frac{1}{(1+y^2)^{3/2}} > 0 \qquad (68)$$

respectively. By definition the ChExps for its poles at ±∞ have the same absolute value $l + 1/2 > 0$ and the opposite signs, i.e., the PFSs are unambiguously determined by the DBCs

$$\lim_{y\to\pm\infty} \psi_n[y;l,\mu] = 0. \qquad (69)$$

Theorem 2. *The Dirichlet problem for the p-SLE (66) is exactly solvable via the q-RSs (64).*

Proof of Theorem 2: First note that the function $\rho^{-1}[y]$ is not integrable, so we may not take advantage of Corollary 2.3 in [20]. However the eigenfunction (64) necessarily has exactly n nodes and the existence of another eigenfunction with an eigenvalue between the eigenvalues $\lambda_{n-1}(l,\mu)$ and $\lambda_n(l,\mu)$ would mean that the CSLE (55) has the eigenvalue within the specified range, which contradicts Theorem 2. □

Corollary 1. *The Dirichlet problem for the p-ODE (7) is exactly solvable via the q-RSs (43).*

Proof of Corollary 1. Suppose that the Dirichlet problem formulated in Section 2 has a solution $\phi_\tau[y;\lambda_\tau,\mu_\tau,l_\tau]$ with $\lambda_\tau \neq \lambda_n(l_\tau,\mu_\tau)$ for $l_\tau \in (-1/2,\infty)$ and any nonnegative integer n. Then it must be an eigenfunction of the *p*-SLE (64) for $l = l_\tau$, $\mu=\mu_\tau$, which implies (as the direct consequence of Theorem 1) that λ_τ must coincide with one of the eigenvalues (34) at the specified values of the parameters l and μ. However, this conclusion contradicts the assumption that $\lambda_\tau \neq \lambda_n(l_\tau,\mu_\tau)$ for any n ≥ 0. □

4. A Family of SLPs with a Finite Number of the Eigenfunctions Each Expressible via a R-Routh Polynomial with Degree-Dependent Index

Alternatively, one can represent the ODE (1) as the Routh-reference (RRef) CSLE [15]

$$\left[\frac{d^2}{dy^2} + I^0[y;h_{0;R},-1/2\,h_{0;I},-1/2] + E\rho_M[y;\kappa]\right]\Phi^{\pm}_M[y;h_0;\kappa;E] = 0, \qquad (70)$$

with the density functions

$${}_i\rho_M[y;\kappa] := \frac{{}_iT_2[y;1,\kappa]}{(1+y^2)^2}, \qquad (71)$$

where the second-degree tangent polynomial (TP)

$$ {}_i T_2[y;a_2,\kappa_R + i\kappa_I] \equiv a_2(y^2+1)+\kappa_I y+\kappa_R - 1 \qquad (72)$$

is assumed to stay positive on the real axis. The complex parameter

$$ h_o \equiv h_{o;R} + h_{o;I} i := \lambda - 2\mu i \qquad (73)$$

and its complex conjugate h_o^* used to parametrize the RefPF

$$ I^o[y;h_{o;R}, -\tfrac{1}{2}h_{o;I}, -\tfrac{1}{2}] = $$

$$ {}_i I^o[y;h_o] := -\frac{h_o}{4(y+i)^2} - \frac{h_o^*}{4(y-i)^2} + \frac{2h_{o;R}+1}{4(y^2+1)} \qquad (74)$$

specify the zero-energy ExpDiffs ${}_i\lambda_o$ and ${}_i\lambda_o^*$ for the poles of the CSLE (70) at -*i* and + *i* respectively, namely,

$$ h_o + 1 = {}_i\lambda_o^2 . $$

(75)

The potential energy reference point for the corresponding Liouville potential [15] is chosen in such a way that the ExpDiff for the pole of the CSLE at infinity vanishes at E=0.

In his pioneering work [14] Milson examined in depth the limiting case of the even TP (*e*-TP) for real (positive) $\kappa = \kappa_+ \ (\kappa_I = 0)$. The *e*-TP version of the Milson potential was later re-discovered by Lévai [31] (see [32] for more details).

Introducing the n-dependent parameters:

$$ l_n(h_o;\kappa) := -\tfrac{1}{2} + \sqrt{-{}_iE_n(h_o;\kappa)} > -\tfrac{1}{2} \qquad (76)$$

and

$$ \lambda_n(h_o;\kappa) - 2\mu_n(h_o;\kappa) i := h_o - (\kappa - 1)[l_n(h_o;\kappa) + \tfrac{1}{2}]^2 , \qquad (77)$$

by analogy with (73), i.e., setting

$$ \underset{\sim}{\lambda}_n(h_o;\kappa) := h_{o;R} - [l_n(h_o;\kappa) + \tfrac{1}{2}]^2(\kappa_R - 1) , \qquad (78)$$

$$ \underset{\sim}{\mu}_n(h_o;\kappa) := -\tfrac{1}{2}h_{o;I} + \tfrac{1}{2}\kappa_I[l_n(h_o;\kappa) + \tfrac{1}{2}]^2 , \qquad (79)$$

for $n = 0,1,\ldots,n_{max}$, we come to the following conditions

$$ Re\,\underset{\sim}{\alpha}_n(h_o;\kappa) = -\tfrac{1}{2}[l_n(h_o;\kappa) + n] < \tfrac{1}{2} \qquad (80)$$

and

$$ Im\,\underset{\sim}{\alpha}_n(h_o;\kappa) := \underset{\sim}{\mu}_n(h_o;\kappa) / (l_n(h_o;\kappa) + n + 1) , \qquad (81)$$

for the real and imaginary parts of the n-dependent complex parameter

$$ \underset{\sim}{\alpha}_n(h_o;\kappa) := \tfrac{1}{2}\sqrt{\underset{\sim}{\lambda}_n(h_o;\kappa) + 1 + 2\underset{\sim}{\mu}_n(h_o;\kappa) i} + \tfrac{1}{2}. \qquad (82)$$

Taking into account the Wronskian of two eigenfunctions

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$$ {}_i\phi_n[y;h_o;\kappa] = {}_iN_n(h_o;\kappa)(y^2+1)^{-\frac{1}{2}l_n(h_o;\kappa)} exp[2\underset{\sim}{\alpha}_{I;n}(h_o;\kappa)\, arctan\; y]\times $$

$$ (y^2+1)^{-\frac{1}{2}n} P_n\left(y; 2\underset{\sim}{\alpha}_{I,n}(h_o;\kappa), n + l_n(h_o;\kappa)\right) \quad (83) $$

vanishes at both endpoints:

$$ 0 < \lim_{y\to\pm\infty} \left| |y|^{l_{n'}(h_o;\kappa)+l_{n''}(h_o;\kappa)+1} W\{{}_i\phi_{n''}[y;h_o;\kappa], {}_i\phi_{n'}[y;h_o;\kappa]\} \right| < \infty \quad (84) $$

$(n' \neq n'')$,

we conclude that the eigenfunctions (85) must be orthogonal with the weight (71):

$$ \int_{-\infty}^{+\infty} {}_i\phi_{n''}[y;h_o;\kappa], {}_i\phi_{n'}[y;h_o;\kappa] \rho_M[y]dy = \delta_{n''n'}\,. \quad (85) $$

Again, based on the proof presented in [20], we assert that the (n+1)-th eigenfunction of the CSLE must have exactly n nodes, and therefore may not be any eigenvalue, other than ${}_iE_n(h_o;\kappa)$, below the energy

$$ {}_iE_{n_{max}}(h_o;\kappa) = -\left[l_{n_{max}}(h_o;\kappa) + \tfrac{1}{2}\right]^2. \quad (86) $$

One still has to proof that there may be no negative eigenvalue above this energy.

The spectral equation (40) takes the form

$$ \tfrac{1}{4}[l_n(h_o;\kappa)+n+1]^2 - \underset{\sim}{\mu}_n^2(h_o;\kappa)/[l_n(h_o;\kappa)+n+1]^2 = \underset{\sim}{\lambda}_n(h_o;\kappa) \quad (87) $$

with respect to $l_n(h_o;\kappa)$. Replacing $l_n(h_o;\kappa)+\frac{1}{2}$ for Λ_n, one can rewrite (81) as follows

$$ (\Lambda_n+n+\tfrac{1}{2})^4 - \tfrac{1}{4}(h_{o;I}-\kappa_I\Lambda_n^2)^2 - [h_{o;R}+1-\Lambda_n^2(\kappa_R-1)](\Lambda_n+n+\tfrac{1}{2})^2 = 0\,. (88) $$

The positive root Λ_n necessarily exists if the given quartic equation has the negative free term:

$$ (n+\tfrac{1}{2})^4 - (h_{o;R}+1)(n+\tfrac{1}{2})^2 - \tfrac{1}{4}h_{o;I}^2 < 0 \quad (n = 0,1,\ldots,n_{max})\,. \quad (89) $$

If $h_{o;I} \neq 0$, the quadratic polynomial

$$ X^2 - (h_{o;R}+1)X - \tfrac{1}{4}h_{o;I}^2 $$

(90)

has the positive leading coefficient and the negative free terms, its roots must have opposite signs and therefore it is negative iff X lies between the roots. Since X is allowed to take only positive values, this implies that the polynomial (89) is negative as far as X is smaller than the positive root, i.e., iff

$$ 0 \le n \le n_{max} < \tfrac{1}{2}\sqrt{2(h_{o;R}+1)+2\,|h_{o;R}+1|} - \tfrac{1}{2}. \quad (91) $$

Note that the derived upper bound for the number of the bound energy levels is independent of κ and therefore matches the one obtained in [15] $\kappa = \kappa_R\ (\kappa_I = 0)$, with

$$ h_o + 1 = -b + ic\,. \quad (92) $$

If $h_{o;I} = 0$ then, according to (75),

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$$h_{o;R} + 1 \equiv h_o + 1 > 0$$

(93)

and

$$0 \le n < \sqrt{h_o + 1} - \tfrac{1}{2} \quad (h_{o;I} = 0)\,.$$

(94)

In the particular case of the single-pole density function ($\kappa = \kappa_R = 1$)

$$\rho_{\Diamond}[y] := \frac{1}{1 + y^2}, \qquad (95)$$

associated with the Gendenshtein potential, all the parameters (78), (79), (82) and as a result, the indexes of the R-Jacobi polynomials become independent of the polynomials degree:

$$\underset{\sim}{\lambda}_n(h_o;\kappa) = \underset{\sim}{\lambda}(h_o) := h_{o;R}\,,$$

(96)

$$\underset{\sim}{\mu}_n(h_o;1) = \underset{\sim}{\mu}(h_o) := -\tfrac{1}{2} h_{o;I}\,,$$

(97)

$$\underset{\sim}{\alpha}_n(h_o;1) \equiv \underset{\sim}{\alpha}(h_o) = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{h_o^* + 1}\,,$$

(98)

and

$${}_{i}\phi_n[y;h_o;1] = {}_{i}N_n(h_o;1)(y^2+1)^{\underset{\sim}{\alpha}_R(h_o)} exp[2\underset{\sim}{\alpha}_I(h_o)\, arctan\; y] \times$$

$$P_n\left(y; 2\underset{\sim}{\alpha}_I(h_o), -2\underset{\sim}{\alpha}_R(h_o)\right). \qquad (99)$$

As pointed by the author [34], this is the direct consequence of the fact that the ChExps for the poles at the finite plane turn out to be energy-independent.

The same year Odake and Sasaki [35,36] came up with the classification scheme of the translationally shape-invariant (TSI) potentials dividing them into the two groups A and B. As clarified by us more recently [37], it is really the question of the energy-dependence of the ExpDiffs for the poles of the RCSLE used to express the eigenfunction of the given Schrödinger in the quasi-rational form. (The potential may belong to different group, depending on the choice of the change of variable to convert the Schrödinger equation to the RCSLE). The common remarkable feature of the translationally form-invariant

(TFI) SLEs is that the Wronskians of the eigenfunctions can be represented in the form of the weighted Wronskians of orthogonal polynomials. The crucial point is that each polynomial Wronskian represents an exceptional orthogonal polynomial (EOP) if the degrees of the given polynomial set are specified by a partition with only one segment of an odd length [36,37]. (At the same time with [35,36], Gȯmez-Ullate et al [38] put forward a similar concept for the exceptional orthogonal polynomial systems (X-OPSs) formed by infinitely many polynomials, while Odake and Sasaki analysis covered both finite and infinite EOP sequences.)

In the form-invariant limit $\kappa = 1$ the quartic equation (88) turns into the quadratic equation:

$$(\Lambda_n + n + \tfrac{1}{2})^4 - (h_{o;R} + 1)(\Lambda_n + n + \tfrac{1}{2})^2 - \tfrac{1}{4} h_{o;I}^2 = 0 \qquad (100)$$

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with the positive root

$$(\Lambda_n + n + \tfrac{1}{2})^2 = \tfrac{1}{2}(h_{o;R} + 1) + \tfrac{1}{4}\sqrt{|\,h_o + 1|^2}\,. \tag{101}$$

As the direct consequence of the fact that the RRef CSLE (70) with the density function (95) belong to Group A of the TFI CSLEs, the orthonormalization relations (85) turn into the orthonormalization relations (A12) for the R-Routh polynomials, which gives

$$_{i}N_n^2(h_o;1) = \frac{\Gamma(2-4\alpha_R(h_o)-n)}{2^{1-n-2\alpha_R(h_o)}\,n!\,\Gamma(1-4\alpha_R(h_o)-2n)\,\Gamma(2-4\alpha_R(h_o)-2n)}$$

$$\times\left|\Gamma(1-2\alpha^*(h_o)-n)\right|^2. \tag{102}$$

Remember that we were unable to analytically compute the normalization factor for the TFI RCSLE (55), which belongs to Group B.

The Liouville transformation converting the RRef CSLE with the density function (94) into the Schrödinger equation with the Gendenshtein potential will be discussed in detail in Appendix C.

Our final step is to prove that there may not exist any negative eigenvalue above the energy (86). To do it, we convert the RRef CSLE (70) to its prime form

$$\left\{\frac{d}{dy}\sqrt{y^2+1}\frac{d}{dy} - {}_{i}q[y;h_o] + {}_{i}E_n(h_o;\kappa)\,{}_{i}w[y;\kappa]\right\}{}_{i}\psi_n[y;h_o;\kappa] = 0\,, \tag{103}$$

where

$$_{i}q[y;h_o] = -\sqrt{y^2+1}\left(I^o[y;h_o] + \mathcal{S}\{\sqrt{y^2+1}\}\right), \tag{104}$$

with the second summand in the parentheses is defined via (12), which gives

$$_{i}q[y;h_o] = -\frac{h_{o;R} + \tfrac{1}{2}h_{o;I}\,y + \tfrac{3}{4}}{(y^2+1)^{3/2}}. \tag{105}$$

As for the weight function of the *p*-SLE (103), it is linked to the density function (71) in the conventional way:

$$_{i}w[y;\kappa] := \sqrt{y^2+1}\,\rho_M[y;\kappa] > 0, \tag{106}$$

which gives

$$_{i}w[y;\kappa] := \frac{{}_{i}T_2[y;1,\kappa]}{(y^2+1)^{3/2}}. \tag{107}$$

One can directly verify that the q-RSs

$$_{i}\psi_n[y;h_o;\kappa] = (y^2+1)^{-\frac{1}{2}[l_n(h_o;\kappa)+\frac{1}{2}]}\,exp[-2\alpha_{I;n}(h_o;\kappa)\,arctan\ y]\times$$

$$(y^2+1)^{-\frac{1}{2}n}\,P_n\left(-y;-2\alpha_{I,n}(h_o;\kappa),\,n+l_n(h_o;\kappa)\right) \tag{108}$$

satisfy the DBCs

$$\lim_{y\to\pm\infty}\ {}_{i}\psi_n[y;h_o;\kappa] = 0\,, \tag{109}$$

as expected.

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Theorem 2. *The formulated Dirichlet problem is exactly solvable via the q-RSs (108).*

Proof of Theorem 2. Since the (n+1)-th eigenfunction (108) has exactly n nodes, the Dirichlet problem in question may not have any solution between two sequential eigenvalues ${}_iE_{n-1}(h_o;\kappa)$ and ${}_iE_n(h_o;\kappa)$. The most challenging issue is to prove that no solution exists inside the interval

$$ {}_iE_{n_{max}}(h_o;\kappa) < E < 0. \tag{110}$$

Suppose that the *p*-SLE (103) solved under the DBCs has a solution at an energy ${}_iE_N(h_o;\kappa)$ within the range (110) which has the number of nodes N larger than n_{max} .Using the latter energy to evaluate the parameters (76)-(79) for n=N, we find that the mentioned function must be a solution of the *p*-ODE (7) solved under the DBCs with l, λ, μ equal to $l_N(h_o;\kappa)$, $\lambda_N(h_o;\kappa)$, $\mu_N(h_o;\kappa)$ accordingly. However, according to Corollary 1, any solution of this problem must have the form (90), which may not be true if N > n_{max} . □

One can directly verify that any eigenfunctions (108) are squarely integrable with the weight (107):

$$\int_{-\infty}^{+\infty} {}_i\psi_n^2[y;h_o;\kappa]\,{}_iw[y;\kappa]dy < \infty \tag{111}$$

for n bounded by (91). Also, an solution of the *p*-SLE (71) can be squarely integrable with the weight (107), iff it satisfies the DBC at both endpoints. This implies that any solution of the RRef CSLE (70) squarely integrable with the weight (71) must necessarily coincide with one of the q-RSs (83).

5. Discussion

Theorem 2 represents the main result of our analysis. accurately proving that the formulated SLP for the RRef CSLE (70) introduced by Milson [15] is exactly solvable via the R-Routh polynomials with degree-dependent indexes. This proof brings the Milson potential to the same level as the two discovered-by-the-author families of the Jacobi-reference (JRef) and Laguerre-reference (LRef) potentials [39] with the eigenfunctions expressible via the *classical* Jacobi and accordingly *classical* Laguerre polynomials with the degree-dependent indexes.

In this connection let us remind the reader that our argumentation was vitally rested on the observation that the Wronskian of any two quasi-rational eigenfunctions of the RCSLE in question vanishes at ±∞. Another common element of our proofs was the assertion that the n^{th} under consideration has exactly n-1 nodes. Combining these two statements allowed the author to invoke Theorem 2.1 in [20] to reject the existence of any other eigenvalue between the eigenvalues of the constructed quasi-rational eigenfunctions.

The aforementioned argumentation covers both (t-RM and Gendenshtein) TSI Liouville potentials. The reader can object that these potentials must be exactly solvable simply because they are TSI, with the reference to Gendenshtein's celebrated paper [21]. However, from author's point of view [16], Gendenshtein (contrary to the wide spread opinion) never proved the exact solvability of an arbitrary TSI potentials. While deleting the bound eigenstates by the sequential Darboux [40] transformations (DTs), he did not bother to verify the number of nodes for the eigenfunction with the lowest eigenvalue in the given sequence or, in other words, it was taken for granted that the Darboux transform $(D\mathfrak{T})$ of the first-excited state, using the nodeless eigenfunction as the transformation function (TF), does not have nodes. Though the cited notion is possibly correct (at least no exception has been found so far), its accurate proof would require imposing numerous restrictions on the behavior of the eigenfunctions near the quantization endpoints (like the invoked-here disappearance

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of the no real Wronskian of any two eigenfunctions at ±∞). The imposed limitations would significantly degrade the power of Gendenstein's original revolutionary ideas, which launched the new direction in the theory of solvable quantum-mechanical potentials.

The proofs presented in Sections 3 and 4 form the foundations for the theorems that the Rudjak-Zakhariev transformations (RZTs) [41] of the CSLEs (55) and (70) result in exactly solvable SLEs with all the eigenfunctions representable in the quasi-rational form. The detailed analysis of this problem for the RRef CSLE (70), including its TFI limit, has been presented in [7], where term the mentioned transformations as the Liouville -Darboux transformations (LDTs) to stress that any RZT of the given CSLE results in the DT of the corresponding Liouville potential and otherwise. It is remarkable that the Rudjak-Zakhariev transform ($^{RZ\mathfrak{I}}$) of the RRef CSLE with the density function (95) is quantized via finite sequences of EOPs as the direct consequence of the observation by Odake and Sasaki [35,36] that the Gendenshtein potential belongs to group A of the TSI potentials. On the contrary, the TFI CSLE (55), associated with the TSI *t*-RM potential belongs to Group B and, as a result, its eigenfunctions are formed by the so-called [7] 'Routh-seed Heine polynomials' with degree-dependent indexes. (We introduced this term in [18] to emphasize that the polynomials in questions satisfy Heine-type differential equations [42] with the exponential parameters dependent on the polynomial degrees.

Taking into account that each eigenvalue is determined by a positive root of the quartic equation with the positive leading coefficient and negative free term, the author [16] concluded that the nth quasi-rational eigenfunction must be accompanied by another q-RS composed of a Routh polynomial of degree n-1. (In [16] this conclusion was limited solely to the *e*-TP, but it was then extended in [7] to an arbitrary TP without real roots.)

It was Quesne [27], who stimulated author's interest to this subject, by speculating that the Schrödinger equation with the Gendenshtein potential had q-RSs composed of polynomials without real zeros. The most straightforward proof of the existence of Routh polynomials with no real zeros is presented by us in [7] and briefly outlined below.

Our arguments are based on the derived representation of the monic Routh polynomials in the form of the Wronskian of the sequential monic R-Routh polynomials with the same indexes [7]:

$$\mathrm{P}_{\mathrm{m}}(\mathrm{x};b,-a-1) = \mathrm{W}\{\mathrm{P}_{\mathrm{l}}(\mathrm{x};-b,a-\mathrm{m}),\ldots,\mathrm{P}_{\mathrm{m}}(\mathrm{x};-b,a-\mathrm{m})\} / \prod_{\mathrm{k}=1}^{\mathrm{m}-1} \mathrm{k}!. \qquad (112)$$

Since the R-Routh polynomials form an orthogonal set with the measure defined on the real axis, it directly follows from Theorem 1 in [43] that the Routh polynomial (112) of even degree may not have real zeros.

Our next step is to use the q-RSs (45) formed by the Routh polynomials of an even degree:

$$\phi_{2\mathrm{j}}[\mathrm{y};\alpha_{\mathrm{n}}(-l-1,\mu)] := (\mathrm{y}^2+1)^{\frac{1}{2}(l+1)} exp[2\alpha_{\mathrm{I};2\mathrm{j}}(-l-1,\mu)\, arctan\ \mathrm{y}] \times$$

$$(\mathrm{y}^2+1)^{-\mathrm{j}} P_{2\mathrm{j}}\left(\mathrm{y};2\alpha_{\mathrm{I};2\mathrm{j}}(-1-l,\mu),2\mathrm{j}-1-l\right) \qquad (113)$$

as the TFs for the RZTs. Without going into the details, it would be sufficient for the purpose of this paper only to note that each RZT generates the RDT of the Schrödinger equation with the corresponding Liouville potential.

Below we outline this universal scheme of constructing new exactly solvable RCSLEs for the TSI t-RM potential, while addressing the reader to [7] for certain details concerning rational extensions of both Milson potential [15] and its TSI limit [21].

According to Theorem 1, the Schrödinger equation (A14) with the potential (A19) is exactly solvable via R-Routh polynomials. Our aim is to prove that this is also true for the RDT of this potential with the TF

$$\mathrm{R}_{2\mathrm{j}}(\chi;-a-1,b) = sin^{-a}\,\chi\, exp[\alpha_{\mathrm{I};2\mathrm{j}}(-a-1,b)(\pi-2\chi)]\times$$

$$\sin^{2j}\chi\, P_{2j}\left(\cot\chi; 2\alpha_{I;2j}(-a,b), 2j-a\right), \quad (114)$$

The fast-track examination of the TF (114) reveals that the RDT in question reduces by 1 the ExpDiff for the poles of the Schrödinger equation at both singular endpoints $\chi = 0$ and $\chi = \pi$, as it has been already observed in [44].

The RDT in question thus generates the new bound state described by the nodeless eigenfunction

$$R_0(\chi; a,b\,|\,-,2j) = R_{2j}^{-1}(\chi; -a-1, b) \quad (115)$$

at the energy

$$E_m(-a-1,b) := (m-a)^2 - \frac{b^2}{(m-a)^2} - 1 \quad (116)$$

lying below the discrete energy spectrum $E_n(a,b)$ of the potential (A19):

$$E_m(-a-1,b) - E_0(a,b) = (m+1)(m-2a-1)\left[1 + \frac{b^2}{(a+1)^2(m-a)^2}\right] < 0$$

$$\text{for } m < 2a+1. \quad (117)$$

If $a > 0$ then the solution (116) of the transformed Schrödinger equation vanishes at both singular endpoints $\chi = 0$ and $\chi = \pi$ and therefore represents the nodeless eigenfunction assuming that this Schrödinger equation is solved under the DBCs. Note that the poles of the Schrödinger equation (A!4) are limit-circle (LC) singularities if $a < 1$ and as a result both solutions (44) and (45) in [45] turned out to be squarely integrable, while the DBCs unambiguously selects the first solution (44).

The eigenfunctions of the excited eigenfunctions of the transformed potential are described by the conventional formula [22]

$$R_{n+1}(\chi; a,b\,|\,-,2j) = \frac{W\{R_{2j}(\chi; -a-1, b), R_n(\chi; a,b)\}}{R_{2j}(\chi; -a-1, b)}. \quad (118)$$

As proven by the author in [14] for an arbitrary Gauss-reference (GRef) potential, the eigenfunctions (117) have the quasi-rational form, with the polynomial components composed of the so-called 'polynomial determinants' (PDs), which obey the aforementioned Heine-type ODEs with the degree-dependent exponent parameters and for this reason are termed by us 'Gauss-seed' (GS) in general or more specifically 'RS' Heine polynomials.

Representing (117) as

$$R_{n+1}(\chi; a,b\,|\,-,2j) = \dot{R}_n(\chi; a,b) - ld\, R_{2j}(\chi; -a-1, b) R_n(\chi; a,b), \quad (119)$$

with dot and the symbolic expression *ld* standing for the first derivative with respect to χ and the logarithmic derivative accordingly, one finds that the q-RSs (117) vanish at both singular endpoints $\chi = 0$ and $\chi = \pi$ for $a > 0$ and therefore constitute the eigenfunctions of the transformed Schrödinger equation, as stated above. Note that the conventional rules of the SUSY quantum mechanics [22] utilized in [44] take this result for granted

Our final step is to prove that the transformed Schrödinger equation may have no excited bound states other than the ones specified the eigenfunctions (117). This proof in the more general case of the Milson potential [7] or the t-RM potential here constitutes the main stimulus for this research.

Theorem 3. *Any excited eigenstates of the transformed Schrödinger equation can be represented in the quasi-rational form (117).*

Proof of Theorem 3. Suppose that the transformed Schrödinger equation has an eigenfunction $R_\tau(\chi;a,b\,|-,2j)$ with an eigenvalue

$$E_\tau(a,b\,|-,2j) \neq E_n(a,b)\,. \qquad (120)$$

By applying to this equation the RDT with the TF (115), we come back to the Schrödinger equation (A14). Let us show that the corresponding RDT of the eigenfunction $R_\tau(\chi;a,b\,|-,2j)$,

$$R_{2j}(\chi;-a-1,b)W\{R_{2j}^{-1}(\chi;-a-1,b), R_\tau(\chi;a,b\,|-,2j)\}, \qquad (121)$$

vanishes at both singular endpoints $\chi = 0$ and $\chi = \pi$ iff $a > 1$. Indeed, taking into account that

$$\dot{W}\{R_{2j}^{-1}(\chi;-a-1,b), R_\tau(\chi;a,b\,|-,2j)\} = \left[E_{2j}(a,b) - E_\tau(a,b)\right]\times$$
$$R_\tau(\chi;a,b\,|-,2j)\,/\,R_{2j}(\chi;-a-1,b)\,, \qquad (122)$$

one finds

$$0 < \lim_{\sin\chi\to 0} \left| \sin^{-2a}\chi\, \dot{W}\{R_{2j}^{-1}(\chi;-a-1,b), R_\tau(\chi;a,b\,|-,2j)\}\right| < \infty \qquad (123)$$

and therefore

$$0 < \lim_{\sin\chi\to 0} \left| \sin^{-2a-1}\chi\, W\{R_{2j}^{-1}(\chi;-a,b), R_\tau(\chi;a,b\,|-,2j)\}\right| < \infty$$

iff $a > 1$. (124)

We thus proved that the q-RSs (121) vanishes as $\sin^{a+1}\chi$ near each singular endpoint and therefore constitutes the eigenfunction of the Schrödinger equation (A14) with the eigenvalue (119), which contradicts Theorem 1. □

Winding up this discussion, leave us mentioned that the Wronskian representation (113) of the Routh polynomial also indicates that the RDT with the TF (113) is equivalent to the rational Darboux-Crum [40,46] transformation (RDCT) using the even number of the sequential eigenfunctions, which represents the simplest illustration of the general theorem proven by Odake and Sasaki [36]. In particular, use of the TF composed of the second-degree Routh polynomial results in the potential (14) in [47], which was generated by the second-order RDCT, using the eigenfunctions of the first and second excited states as seed functions.

6. Conclusions

As stressed at the beginning of Section 5, Theorem 2 brings the Milson potential [15] to the same level as the two families of the rational potentials [39] with the eigenfunctions expressible via the *classical* Jacobi and accordingly *classical* Laguerre polynomials with the degree-dependent indexes. Our main motivation to proceed with the necessary argumentation in support of the theorem was stimulated by our studies on the RDTs of the Milson potential and its TSI limit represented by the Gendenshtein (Scarf 2) potential. All the possible rational Darboux-Crum transforms RDCTs oof the latter potential solvable via EOPs have been constructed by us in [48] under the assumption that the potential itself is exactly solvable via the R-Routh polynomials, with the reference to Stevenson's note. However, a more thorough analysis of Stevenson's argumentation revealed that it has some hidden gaps, which were successfully filled in this paper.

In the following studies we will discuss various RDCTs of the Milson potential exacttly solvable via RS Heine polynomials. The generic scenario is the sequence of the rational extensions obtained by eliminating the lowest energy eigenstate one by one. Another promising direction is to formulate the 'enhanced' Adler theorem [50] similar to that proven in [49] for the JRef potential on the line. The exactly solvable RDCT of the Milson potential can be then constructed using 'juxtaposed' [51–53] pairs of the eigenfunctions (83).

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Appendix A. Hypergeometric Expressions for Monic Routh Polynomials

The main purpose of this Appendix is to reexamine the hypergeometric expressions (9.9.1) in [9}:

$$P_n(y;\nu,N) = (-2i)^n \frac{\langle -N+i\nu\rangle_n}{\langle n-2N-1\rangle_n} F[-n, n-2N-1, -N+i\nu; \tfrac{1}{2}-\tfrac{1}{2}iy] \tag{A1}$$

and

$$P_n(y;\nu,N) := (y+i)^n F(-n,\, 1-n+N-i\nu, 2N+2-2n; 2i/(y+i)), \tag{A2}$$

which were originally introduced in [9] only for nonnegative integers N.

Let us mention in this connection that the notion of the real-by-definition [8] 'Routh polynomials', defined by us [7,19] via (43) with

$$\boldsymbol{\lambda} \equiv -N-1+\nu i, \tag{A3}$$

is used here in exactly the same sense as Jordaan and Toókos [24] use the term 'pseudo-Jacobi polynomials. As a result, Lesky's [5,6] epithet 'Romanovski/pseudo-Jacobi polynomials' becomes identical to the term 'R-Routh polynomials' which is used by us iff the polynomial degree does not exceed $N-\tfrac{1}{2}$. On other hand, both expressions (A1) and (A2) remain valid for any Routh polynomial converted to its monic form.

It is worth stressing once again that the commonly cited formula (41) for the Routh polynomials was first introduced by Cryer [25] in his analysis of the infinite *real* polynomial sequences with the coefficients satisfying the three-term recurrence relations. It was Compean and Kirchbach [13], who brought author's attention to this archaic study rediscovering some of Routh' results [8]. The fact that the polynomials (41) in y are real was proven in [25] (and later in [1]) directly followed from their definitions via the real Rodriguez formula.

It was Askey [1], who represented the cited polynomials in the hypergeometric form:

$$\Re_n^{(\boldsymbol{\lambda})}(y) = i^n \frac{\langle \boldsymbol{\lambda}+1\rangle_n}{n!} F\left(-n, n+2\boldsymbol{\lambda}_R+1; 1+\boldsymbol{\lambda}; (1-iy)/2\right). \tag{A4}$$

(cf. (1.17) in [1]). Comparing (A5) and (A6) with (A1), we confirm that the latter polynomials must have real coefficients, when expressed in terms of the argument y.

$$\Re_n^{(-N-1+\nu i)}(y) = \frac{\langle n-2N-1\rangle_n}{2^n\, n!} P_n(y;\nu,N) \tag{A5}$$

To confirm that the two hypergeometric representations of the Routh polynomials are equivalent, let us first rewrite the polynomials (A1) in the slightly different form:

$$P_n(y;\nu,N) = (-2i)^n \frac{\langle 1-n+N-i\nu\rangle_n}{\langle 2N+2-2n\rangle_n} F[-n, n-2N-1, -N+i\nu; \tfrac{1}{2}-\tfrac{1}{2}iy], \tag{A6}$$

taking into account that

$$\langle\nu\rangle_n = \sum_{k=0}^{n-1}(\nu+k) = (-1)^n \sum_{k=0}^{n-1}(-\nu+1-n+k) = (-1)^n \langle -\nu+1-n\rangle_n \tag{A7}$$

$$\equiv (-1)^n \frac{\Gamma(1-\nu)}{\Gamma(1-n-\nu)}, \tag{A8}$$

On other hand, truncating the analytical continuation of the hypergeometric function in the right-hand side of (A2) via (15.3.7) in [29], one finds

$$F(-n,\, 1-n+N-i\nu, 2N+2-2n; 2/(1-iy)) =$$

$$\frac{\langle 1-n+N-i\nu\rangle_n}{\langle 2N+2-2n\rangle_n}\left(\frac{-2i}{y+i}\right)^n F(-n,\, n-2N-1, -N+i\nu; (1-iy)/2), \tag{A9}$$

as expected from (A6).

We thus confirmed that the monic polynomials (A2) must have real coefficients, when expressed in terms of y. Therefore this must be also true for Stevenson's hypergeometric polynomials, which are defined via the complex conjugates of (A2).

In following Quesne [27], the R-Routh polynomials are defined via the relation:

$$R_n^{(2-\nu i,N)}(y) := \Re_n^{(-N-1+\nu i)}(y) \qquad \text{for } n = 0,\ldots, \lfloor N + 1/2 \rfloor. \tag{A10}$$

Substituting (A10) into (9.9.2) in [9] shows that the monic R-Routh polynomials

$$\hat{R}_n^{(2-\nu i,N)}(y) = P_n(y;\nu,N) = \frac{(-2)^n n!\Gamma(2N+2-2n)}{\Gamma(2N+2-n)} R_n^{(2-\nu i,N)}(y) \tag{A11}$$

satisfy the following orthogonalization conditions:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{n''}^{(2-\nu i,-N)}(y) R_{n'}^{(2-\nu i,-N)}(y)(y^2+1)^{-N-1} exp(2\nu\, arctan\, y)\, dy$$

$$= \frac{2^{2n-2N-1} n!\Gamma(2N+1-2n)\Gamma(2N+2-2n)}{\Gamma(2N+2-n)|\Gamma(N+1-n+\nu i)|^2}\delta_{n''n'}. \tag{A12}$$

(It seems that the factor $2^{\alpha+\beta+1}$ is missed in Askey's orthogonalization conditions (1.8).)

Appendix B. Some Remarks on the t-RM Potential

The Liouville transformation of the CSLE (67), using the change of variable

$$y(\chi) = cot\,\chi, \tag{A13}$$

results in the one-dimensional Schrödinger equation

$$\left\{-\frac{d^2}{d\chi^2} + V_L(\chi; l, \mu) - E\right\} R(\chi; l, \mu; E) = 0, \tag{A14}$$

where

$$R(\chi; l, \mu; E) = sin\,\chi\, \Phi[cot\,\chi; E, \mu, l]. \tag{A15}$$

Taking into account that the Schwarzian derivative of the function (A13) vanishes:

$$\{y(\chi),\chi\} \equiv \{cot\,\chi,\chi\} = 0 \qquad \text{(A16)}$$

and

$$1 + y^2(\chi) = sin^{-2}\chi \quad , \qquad \text{(A17)}$$

one finds

$$V_L(\chi;l,\mu) := -sin^{-4}\chi\, I^o[cot\,\chi;0,\mu,l] \qquad \text{(A18)}$$

and therefore

$$V_L(\chi;a,b) := \frac{a(a+1)}{sin^2\chi} - 2b\,cot\,\chi \quad (0 < \chi < \pi), \qquad \text{(A19)}$$

As pointed to in [14], the parameter a in (A6) may take any real value larger than $-\frac{1}{2}$.

It was Infeld and Hull [33], who first pointed to the fact that the Kepler problem on a hypersphere [11]

$$\left[-sin^{-2}\chi\frac{d}{d\chi}\left(sin^2\chi\frac{d}{d\chi}\right) + l(l+1)sin^{-2}\chi - 2\mu\,cot\,\chi - \lambda\right]S(\chi;\mu,l;\lambda) = 0 \; \text{(A20)}$$

can be reduced to the one-dimensional Schrödinger equation with the potential (A19) by the transformation [33]

$$S(\chi;\mu,l;\lambda) = sin^{-1}\chi\, R(\chi;l,\mu;\lambda) \equiv \Phi[cot\,\chi;\lambda,\mu,l]. \qquad \text{(A21)}$$

As far as the potential in question is written in the cited form, it would be more appropriate to refer to it as the trigonometric modification of the Manning-Rosen (MR) potential [54}. Indeed Companion and Kirchbach [13] came to (A6), starting from the MR potential (misleadingly referred to as the 'Eckart' potential, in following De et al [55]).

After re-writing (A6) in the form of the trigonometric modification of the Rosen-Morse (RM) potential:

$$V_L(\pi/2 - x;a,b) \equiv V_{t\text{-RM}}(x;a,b) = \frac{a(a+1)}{cos^2 x} - 2b\,tan\,x \quad (-\pi/2 < x < \pi), \; \text{(A22)}$$

De et al [55] ambiguously termed both potentials as 'Rosen-Morse'. Finally, Cooper et al, in their celebrated review [22], termed the potential (A19) as 'Rosen-Morse I', while referring to the its hyperbolic version (initiated in [56]) as 'Rosen-Morse II'. These terms were then broadly accepted in the quantum-mechanical literature.

Representing the eigenfunction of the Schrödinger equation (A14) as

$$R_n(\chi;a,b) = sin\,\chi\;\phi_n[cot\,\chi;\alpha_n(a,b)] \qquad \text{(A23)}$$

and expressing the q-RSs (43) in terms of χ, coupled with (39) and (41), then gives

$$exp[-\pi\alpha_{I;n}(a,b)]R_n(\chi;a,b) =$$

$$sin^{a+n+1}\chi\, exp[-2\alpha_{I,n}(a,b)\chi]P_n\left(cot\,\chi;2\alpha_{I;n}(a,b),n+a\right), \; \text{(A24)}$$

which brings us to (10) in [14], with

$$\alpha_m := 2b/(a+m+1) = 4\alpha_{I,m}(a,b), \;\; \beta_m := -a-m, \qquad \text{(A25)}$$

and m standing for n in (A11) or n-1 in both [13] and [14].

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Finally, let compare our results with the analytical expression derived in [57,58] for the eigenfunctions (A21) of the Kepler problem in the space of the positive constant curvature. The nontrivial feature of this expression is that the authors reported the normalizing factors, with no mention of any mathematical background for their computation. As pointed to in Section 4, this cannot be done using the orthogonalization of the R-Routh polynomials and I am not aware of any other analytical formula for the integral of the squared hypergeometric polynomials under consideration.

To express the eigenfunctions (43) in terms of χ, let us first make use of (36) and represent the power function in (44) as

$$(y+i)^n = i^n (y^2+1)^{\frac{1}{2}n} exp(-n\, artag\, y i), \qquad \text{(A26)}$$

which gives

$$\phi_n[y;\alpha_n(l,\mu)] = i^n (1+y^2)^{-\frac{1}{2}l} exp\{[2\alpha_{n;I}(l,\mu) - n i] artag\, y\} \times$$

$$F\left(-n,\ l+1-i2\alpha_{I;n}(\lambda,\mu), 2l+2; 2/(1-iy)\right). \qquad \text{(A27)}$$

Substituting (A17),

$$artag\, y(x) = \tfrac{1}{2}\pi - \chi, \qquad \text{(A28)}$$

and

$$2/(1-i cot\,\chi) = 1 - e^{-2ix} \qquad \text{(A29)}$$

Into the right-hand side of (A27) then gives

$$S(\chi;l,\mu;\lambda_n) \equiv \phi_n[cot\,\chi;\alpha_n(l,b)] \propto sin^l \chi\, exp\left\{-[2\alpha_{n;I}(l,\mu) - n i]\chi\right\} \times$$

$$\times F\left(-n,\ l+1-2\alpha_{I;n}(l,\mu) i, 2l+2; 1-e^{-2i\chi}\right), \qquad \text{(A30)}$$

with $\alpha_{I;n}(l,\mu)$ given by (41). Disregarding the normalizing factor, the derived expression precisely matches that in [57], with $\alpha R \equiv 2\mu$ and n standing for $l+1+n$ here. (For reader's convenience we can point to the hypergeometric expression (7) in the more recent paper [59] of these authors, with $e \equiv 2\mu$ and n standing for $l+1+n$ here.)

Note that the argument of the hypergeometric function in (A30) can be alternatively represented as

$$2/(1-i cot\,\chi) = 2ie^{-i\chi} sin\chi. \qquad \text{(A31)}$$

Substituting (A31) into (45) brings us to the solution (45) in [45]

$$R_n[cot\chi; -a-1, b] = i^n sin^{-a}\chi\, exp\left\{[2\alpha_{I;n}(-a-1,b) - in](\tfrac{1}{2}\pi - \chi)\right\} \times$$

$$F(-n,\ 1-n-2\alpha_n(-a-1,b), 2l; 2ie^{-i\chi} sin\chi). \qquad \text{(A32)}$$

It is worth drawing reader's attention to the fact that the solution (A32) is squarely integrable if $a < 1$, which means that the Schrödinger equation (A14) has the continuous spectrum within

the range $-\frac{1}{2} < a < 1$.

Abbreviations

The following abbreviations are used in this manuscript:

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ChExp	characteristic exponent
CSLE	canonical Sturm-Liouville equation
DBC	Dirichlet boundary condition
d-PFS	dual principal Frobenius solution
EOP	exceptional orthogonal polynomial
ExpDiff	exponent difference
D$\mathfrak{T}$	Darboux transform
DT	Darboux transformation
FS	Frobenius solution
JRef	Jacobi-reference
GRef	Gauss-reference
GS	Gauss-seed
LC	limit-circle
LRef	Laguerre-reference
*N*ODE	normal ordinary differential equation
ODE	ordinary differential equation
PD	polynomial determinant
PFS	principal Frobenius solution
p-ODE	prime ordinary differential equation
p-SLE	prime Sturm-Liouville equation
RCSLE	rational Sturm-Liouville equation
RDC$\mathfrak{T}$	rational Darboux-Crum transform
RDCT	rational Darboux-Crum transformation
RDCT	rational Darboux transformation
RD$\mathfrak{T}$	rational Darboux transform
RDT	rational Darboux transformation
RRef	Routh-reference
RS	Routh-seed
RZT	Rudjak-Zakhariev transformation
SLE	Sturm-Liouville equation
SLP	Sturm-Liouville problem
TF	transformation function
TP	tangent polynomial
t-RM	trigonometric Rosen-Morse
TFI	translationally form-invariant
TSI	translationally shape-invariant

References

1. Askey, R. An integral of Ramanujan and orthogonal polynomials. *J. Indian Math. Soc.* **1987**, *51*, 27–36.
2. Romanovski, V.I. Sur quelques classes nouvelles de polynomes orthogonaux. *CR Acad. Sci.* **1929**, *188*, 1023–1025.
3. Raposo, A.P.; Weber, H.J.; Alvarez-Castillo, D.E.; Kirchbach, M. Romanovski polynomials in selected physics problems. *Cent. Eur. J. Phys.* **2007**, *5*, 253–284. https://doi.org/10.2478/s11534-007-0018-5.
4. Romanovsky, V. Sur la généralisation des courbes de Pearson, *Atti del Congresso Intern. dei Matem.* (*Bologna)* **6 1928**, 6, 107, 4 pages, mathunion.org/ICM/ICM1928.6/Main/icm1928.6.0107.0110.ocr.pdf.
5. Lesky, P.A. Vervollstandigungderklassischen Orthogonalpolynomedurch Erganzungen zum Askey—Schema der hypergeometrischen orthogonalen Polynome. *Ost. Ak. Wiss* **1995**, *204*, 151–166.
6. Lesky, P.A. Endliche und unendliche Systeme von kontinuierlichen klassichen Othogonalpolynomen. *Z. Angew. Math. Mech.* **1996**, *76*, 181–184. https://doi.org/10.1002/zamm.19960760317.
7. Natanson G 2026 Routh polynomials: hundred years in obscurity. Available on line: researchgate.net/publication/326522529 (accessed on August 2023).

8. Routh E. J. On some properties of certain solutions of a differential equation of second order *Proc. London Math. Soc.* **1884**,*16*, 245-261.
9. Koekoek, R.; Lesky, P.A.; Swarttouw, R.F. *Hypergeometric Orthogonal Polynomials and Their q-Analogues*; Springer: Berlin/Heidelberg, Germany, 2010. https://doi.org/10.1007/978-3-642-05014-5.
10. Stevenson, A.F. Note on the 'Kepler problem' in a spherical space, and the factorization method of solving eigenvalue problems. *Phys. Rev.* **1941**, *59*, 842–843.
11. Schrödinger, E. A method of determining quantum-mechanical eigenvalues and eigenfunctions *Proc. Roy. Irish Acad.* A **1940**,*46*, 9-16.
12. Erdelyi, A.; Bateman, H. *Transcendental Functions*; McGraw Hill: New York, NY, USA, 1953; Volume 1.
13. Compean, C.B.; Kirchbach, M. The trigonometric Rosen–Morse potential in the supersymmetric quantum mechanics and its exact solutions. *J. Phys. A* **2006**, *39*, 547–557.
14. Compean, C.B.; Kirchbach, M., The quantum mechanics problem of the Schrodinger equation with the trigonometric Rosen-Morse potential. arXiv:quant-ph/0603232v1 (2006).
15. Milson, R. Liouville transformation and exactly solvable Schrödinger equations. *Int. J. Theor. Phys.* **1998**, *37*, 1735–1752. https://doi.org/10.1023/A:1026696709617.
16. Natanson, G. Exact Quantization of the Milson Potential via Romanovski-Routh Polynomials. ArXiv:1310.0796v3 **2015**
17. Forsyth A. R. 1929 A Treatise on Differential Equations, Courier Corporation, London, §60.
18. Natanson, G. Darboux-Crum Nets of Sturm-Liouville Problems Solvable by Quasi-Rational Functions I. General Theory. **2018**. Available online: https://www.researchgate.net/publication/323831953 (accessed on March 2018).
19. Pearson, K. Contributions to the mathematical theory of evolution, II: Skew variation in homogeneous material. *Phil Trans Royal Soc. A* **1895**, *186*, 343-424..
20. Gesztesy, F.; Simon, B.; Teschl, G. Zeros of the Wronskian and renormalize oscillation theory. *Am. J. Math.* **1996**, *118*, 571–594.
21. Gendenshtein L. E. Derivation of exact spectra of the Schrödinger equation by means of supersymmetry. *JETP Lett.* **1983**, 38, 356-359.
22. Cooper F.; Khare A.; Sukhatme U. P. Supersymmetry and quantum mechanics. *Phys. Rep.* **1995**, 251, 267-385.
23. Whittaker E. T. ; Watson G. N. A Course of Modern Analysis (Cambridge University Press, Cambridge, **1984**).
24. Jordaan K. and Toókos F. Orthogonality and asymptotics of Pseudo-Jacobi polynomials for non-classical parameters. *J. Approx. Theory* **2014**, *178*, 1-12.
25. Cryer C. W. Rodrigues' formulas and the classical orthogonal polynomials, *Boll. Unione Mat. Ital.* **1970**, *25*, 1-11.
26. Avarez-Castillo D. E. and Kirchbach M. Exact spectrum and wave functions of the hyperbolic Scarf potential in terms o f finite Romanovski polynomials *Rev. Mex. Fis. E* **2007**, *53*, 143-154.
27. *Quesne* C. Extending Romanovski *polynomials* in quantum mechanics *J. Math. Phys.* **2013**, *54*. 122103, 15 pages.
28. Koornwinder, T. Additions to the Formula Lists in "Hypergeometric Orthogonal Polynomials and Their q-Analogues" by Koekoek, Lesky and Swarttouw. **2022**. Available online: https://staff.fnwi.uva.nl/t.h.koornwinder/art/informal/KLSadd.pdf.
29. Abramowitz M. and Stegun I. A. 1972. *Handbook of Mathematical Functions*. (Washington DC National Bureau of Standards Applied Mathematics Series – 55).
30. Comtet L. *Advanced Combinatorics: The Art of Finite and Infinite Expansions,* ed. Dordrecht, Netherlands: Reidel, 1974.
31. Lévai G. 2012 Gradual spontaneous breakdown of $\mathcal{PT}$ symmetry in a solvable potential. *J. Phys. A 45* 444020.
32. Natanson G. Overlapping of Lévai's and Milson's e-tangent-polynomial potentials along symmetric curves. *Axioms* **2023**, 12, 584.
33. Infeld L and Hull T. E. The factorization method. *Rev. Mod. Phys.* **1951**, *23*, 21-68.

 doi:10.20944/preprints202602.0199.v1

34. Natanson G. Gauss-seed nets of Sturm-Liouville problems with energy-independent characteristic exponents and related sequences of exceptional orthogonal polynomials I. Canonical Darboux transformations using almost-everywhere holomorphic factorization functions. **2013.** arXiv:1305.7453v1
35. Odake S. and Sasaki R. Extensions of solvable potentials with finitely many discrete eigenstates. *J. Phys. A* **2013**, *46*, 235205, 15 pages.
36. Odake S. and Sasaki R. Krein–Adler transformations for shape-invariant potentials and pseudo virtual states. *J. Phys.* A **2013**, *46*, 245201.
37. G. Natanson G. Equivalence relations for Darboux-Crum transforms of translationally form-invariant Sturm-Liouville equations. **2021.** Available online: researchgate.net/publication/353131294 (accessed on August 2021).
38. Gòmez-Ullate D., Grandati Y. and Milson R. Extended Krein-Adler theorem for the translationally shape invariant potentials. *J. Math. Phys.* **2014**, ***55***, 043510, 31 pages.
39. Natanzon G A 1971 Study of the one-dimensional Schrödinger equation generated from the hypergeometric equation *Vestn. Leningr. Univ.* No 10 22 [see English translation: http://arxiv.org/PS_cache/physics/pdf/9907/9907032v1.pdf].
40. Darboux G. Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal, Vol. 2 (Paris, Gauthier-Villars, 1915).
41. Rudyak B. V. and Zakhariev B. N. New exactly solvable models for Schrödinger equation. *Inverse Problems* **1987**, *3*, 125-133.
42. Szego G. *Orthogonal Polynomials* (New York: American Mathematical Society, 1959).
43. Karlin S. and Szegö G, On certain determinants whose elements are orthogonal polynomials," *J. Analyse Math.* **1960/1961**, *8*, 1–157.
44. Fernández D. J. and Reyes, R. Spectral manipulation of the trigonometric Rosen-Morse potential through supersymmetry. *Phys. Scr.* **2021**, *96*, 12525.
45. Domínguez-Hernández S. and Fernández D. J. Rosen-Morse potential and its supersymmetric partners. *Int. J. Theor. Phys*. **2011**, *50*, 1993–2001.
46. Crum M.M. Associated Sturm-Liouville systems. *Quart. J. Math. Oxford* (2) **1955**, *6*, 121-127.
47. Reyes R., Fernández D. J., and Gasperín H. Second-Order Supersymmetric Partners of the Trigonometric Rosen–Morse Potential. in In: Paranjape, M.B., MacKenzie, R., Thomova, Z., Winternitz, P., Witczak-Krempa, W. (eds) Quantum Theory and Symmetries. CRM Series in Mathematical Physics. Springer, Cham. **2021**. Ch.6, 235-243.
48. Natanson G. Use of normalizable seed solutions for constructing exactly-solvable rational Darboux-Crum transforms of Cooper-Ginocchio-Khare potential and its siblings. 2023, 1-50. Available online: europepmc.org/article/PPR/PPR707751 (accessed on 17 August 2023).
49. Natanson G. Uniqueness of Finite Exceptional Orthogonal Polynomial Sequences Composed of Wronskian Transforms of Romanovski-Routh Polynomials,. *Symmetry* **2024**, *16*, 282, 42 pages.
50. Adler V. E. A modification of Crum's method. *Theor. Math. Phys.* **1994**, *101*, 1391.
51. Samsonov, B. F. On the equivalence of the integral and the differential exact solution generation methods for the one-dimensional Schrodinger equation. *J. Phys. A* **1995**, ***28***, 6989-6998..
52. Samsonov, B. F. New features in supersymmetry breakdown in quantum mechanics," *Mod. Phys. Lett. A* **1996**, ***11***, 1563-1567.
53. Bagrov, V. G.; Samsonov, B. F. Darboux transformation and elementary exact solutions of the Schrödinger equation, *Pramana J. Phys.* **1997**, *49*, 563-580.
54. Manning M. F. and N. Rosen N. A potential function for the vibrations of diatomic molecules. *Phys. Rev.* **1933,** *44*, 953.
55. De R., Dutt R. and Sukhatme U. Mapping of shape invariant potentials under point canonical transformations. *J. Phys. A: Math. Gen.* **1992**, *25*, L843-L850.
56. Rosen N. and Morse P. M., On the vibrations of polyatomic molecules. *Phys. Rev.***42**, 210-217 (1932).
57. Bogush A.A., Otchik V.S. and Red'kov V.M. Separation of variables in Schrödinger equation and normed wave functions for the Kepler problem in three-dimensional spaces of constan curvature. Vestzi Akad. Nauk BSSR, **1983**, *3*, 56-62 (in Russian).

doi:10.20944/preprints202602.0199.v1

58. Bogush A.A. Selected Works. *Belarusian Science.* **2012** (in Russian) Available on line: https://play.google.com/books/reader?id=HS2wBAAAQBAJ&pg=GBS.PA162&hl=en
59. Bogush A. A., Otchik V. S, and, Red'kov V. M. The Runge-Lenz vector for quantum Kepler problem in the space of positive constant curvature and complex parabolic coordinates. Proc. of 5th International Conference Bolyai-Gauss-Lobachevsky: Non-Euclidean Geometry In Modern Physics (BGL-5). 10-13 Oct 2006, Minsk, Belarus. 135-144 (2006); arxiv:hepth/0612178.