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Posted Date: 30 January 2026

doi: 10.20944/preprints202601.2334.v1

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Simple Proof of Fermat's Last Theorem for the Cube

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Abstract

A simple proof of Fermat's Last Theorem (FLT) for the cube is obtained using the binomial expansion formula. It is shown that the difference between two natural numbers raised to the same natural power must be represented by an incomplete binomial formula. It is proven that the cube of a natural number cannot be represented as an incomplete binomial, which means a simple proof of FLT for n=3 has been obtained.

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1. Introduction

Fermat's Last Theorem (FLT), formulated by the French mathematician Pierre de Fermat in 1637, is described as follows: For any natural number $n > 2$, the formula:

$$a^n + b^n = c^n \quad (1)$$

has no solutions in natural numbers a, b, c.

Fermat wrote: "It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two powers of the same degree. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain".

Fermat's Last Theorem was proven in 1994 by Andrew Wiles [1], using complex mathematical apparatus based on elliptic curves, which were unknown in Fermat's time. In this regard, the search for a simple proof of FLT continues to this day, which indicates the relevance of the problem.

It is known that Euler presented a quite complex proof of FLT for the cube using complex numbers. A review of works [1–7] dedicated to FLT shows that a simple proof of FLT for $n = 3$ was previously absent, so obtaining a simple proof of FLT for $n = 3$ is also a relevant task. It also follows from the review of these works that the ideas and methods applied to the proof of FLT for $n = 3$ in this paper, as well as the obtained results, are new.

2. Difference of Equal Powers and Binomial Expansion

Before presenting the proof of FLT for $n = 3$, we will show the distinctive feature of the difference of equal powers of two natural numbers.

1. Any natural power of a natural number a >1 can be represented as a natural power of the sum of two natural numbers, $a^n = (b + d)^n$, $a, b, d, n \in$;
2. Any natural power of a natural number represented as $(b + d)^n$ can be expanded by the binomial formula.
3. The difference of equal powers of two natural numbers $A = c^n - b^n$ can be represented as $A = (b + d)^n - b^n$;
4. From point 3, it follows that the difference of equal powers of two natural numbers does not correspond to the binomial expansion, since two terms b^n will cancel out during the binomial expansion.

Based on points 3 and 4, the following lemma can be formulated.

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Lemma 1. For any natural numbers a, b, c, d, n with $a < b < c$ and $a^n = c^n - b^n$, there exists a natural number $d = c - b$ such that: $c^n - b^n = (b+d)^n - b^n = S + d^n$, where S is the sum of the middle terms of the binomial (without the first and last terms) of $(b+d)^n$; that is, the difference of equal powers is representable as a binomial expansion without the first term b^n.

Definition 1. If the first term bn (or the last term d^n) is subtracted from the binomial expansion of any natural number of the form $(b+d)^n$, which consists of $n+1$ terms, the resulting expression consisting of n terms is called an incomplete binomial expansion.

Note. For convenience, in the following, we will assume that only the first term is absent in the incomplete binomial.

The proof of Lemma 1 is obvious.

It follows from Lemma 1 that the difference between two natural numbers raised to equal natural powers must have a representation as an incomplete binomial expansion

$c^n - b^n = S + d^n$, in which, compared to the ordinary binomial expansion, the first term b^n is absent.

Thus, if equation (1) has a solution in natural numbers, then a^n must be representable by the incomplete binomial formula in natural numbers. In other words, to find a simple proof of Fermat's Last Theorem, we must prove that the natural power greater than 2 of any natural number cannot be represented by the incomplete binomial expansion formula in natural numbers.

Since in the formula $a^n = c^n - b^n$ the number a corresponds to the condition $a < b < c$, it is possible to represent this formula as:

$$(s+d)^n = (b+d)^n - b^n, \qquad (2)$$

where $d = 1, 2, \ldots$ is the interval between the numbers.

Lemma 2. $a, b, c, n \in \mathbb{N}$, $a < b < c$ and $a^n + b^n = c^n$. Then there exist a natural number d and a positive $s > 0$ (not necessarily natural) such that:

$$\begin{cases} c = b + d \\ a = s + d \end{cases}$$

Proof. Let $d = c - b$. Then $d \in \mathbb{N}$ and $c^n = (b+d)^n$, since $b < c$.

By condition $a^n = c^n - b^n = (b+d)^n - b^n$. Expand $(b+d)^n$ using Newton's binomial theorem:

$$(b+d)^n = b^n + \sum_{k=1}^{n-1}\binom{n}{k} b^{n-k} d^k + d^n = b^n + S + d^n, \qquad (3)$$

where $S = \sum_{k=1}^{n-1}\binom{n}{k} b^{n-k} d^k > 0$ (all terms are positive for $b, d \geq 1$). Then $a^n = S + d^n$, so $a = (S + d^n)^{1/n} > (d^n)^{1/n}$.

From this it follows that $s = a - d > 0$. Then $a = s + d$ and $c = b + d$ – both numbers a and c are expressed through the same d. The Lemma is proven.

3. Main Theorem

Next, we consider the difference of cubes and prove that no natural cube can be represented by the incomplete binomial formula.

Theorem 1. Let $a^3 = c^3 - b^3$ have a solution in natural numbers, and $a = s + d$. Then, if we try to represent a^3 as an incomplete binomial $(s+d)^3$ without the first term s^3, compensating for it by substituting $s + x$ in the remaining terms, the equation:

$$(s+d)^3 = 3(s+x)^2 d + 3(s+x)d^2 + d^3, \qquad (4)$$

has no solutions in natural numbers s, d, x.

Proof. Expand the right side of (4):

$$3(s+x)^2 d + 3(s+x)d^2 + d^3 = 3(s^2+2sx+x^2)d + 3(s+x)d^2 + d^3 =$$

$$3s^2d + 6sxd + 3x^2d + 3sd^2 + 3xd^2 + d^3.$$

Left side of (4):

$$(s+d)^3 = s^3 + 3s^2d + 3sd^2 + d^3.$$

Equate both sides:

$$s^3 + 3s^2d + 3sd^2 + d^3 = 3s^2d + 6sxd + 3x^2d + 3sd^2 + 3xd^2 + d^3.$$

Cancel common terms:

$$s^3 = 6sxd + 3x^2d + 3xd^2,$$

$$s^3 = 3xd(2s+x+d). \qquad (5)$$

Thus, to prove Theorem 1, we must prove the absence of solutions to (5) in natural numbers.

In Appendix 1, the results of the study of the differences of incomplete binomials of cubes of natural numbers are presented in two tables, 1 and 2, based on which it is established that in (5) we can take $d = 1$ or $x = 1$, since in both cases we get an equivalent equation. This is explained by the following two ways of obtaining all possible differences of incomplete binomials of cubes of natural numbers:

1. **First method.** Find the difference of adjacent cubes ($d = 1$), then we get the set of all incomplete binomials of cubes of natural numbers; after that, we can find the difference of incomplete binomials located at different distances (for different x), then we get the complete set of differences of incomplete binomials of cubes (Table1).
2. **Second method.** Find the difference of cubes located at different distances (for different d), then for each d, incomplete binomials are formed, which are subsets of the set of all incomplete binomials of cubes of natural numbers; after that, we find the difference of adjacent binomials ($x = 1$) from each subset, and in this case, we get the complete set of differences of incomplete binomials of cubes (Table 2).

Thus, both methods yield the complete set of differences of incomplete binomials of cubes of natural numbers, so we take $d = 1$, then we get:

$$s^3 = 3x(2s+x+1). \qquad (6)$$

Next, we prove the absence of solutions to (6) in natural numbers.

Proof of the absence of solutions to (6) in natural numbers

Assume that a solution $(s, x) \in \mathbb{N}^2$exists. The right side $3x(2s+x+1)$is always $\equiv 0, 3$ or 6 (mod 9). The left side s^3 can only have remainders 0, 1, 7, 8 (mod 9). The only common remainder is 0, therefore $s^3 \equiv 0 \pmod 9 \Rightarrow s \equiv 0 \pmod 3$. Let $s = 3t, t \in \mathbb{N}$. Then:

$$(3t)^3 = 3x(6t+x+1) \Rightarrow 9t^3 = x(6t+x+1).$$

Let $a = x, b = x + 6t + 1$; then $b - a = 6t + 1$ and $a \cdot b = 9t^3$.

Now consider the equation modulo 3:

$b - a = 6t + 1 \equiv 1 \pmod 3 \Rightarrow b \equiv a + 1 \pmod 3 \Rightarrow a(a+1) \equiv 0 \pmod 3$.

Since $a \cdot b = 9t^3 \equiv 0 \pmod 3$ we have $a(a+1) \equiv 0 \pmod 3$. This means that either $a \equiv 0 \pmod 3$ or $a \equiv 2 \pmod 3$.

Case 1: $a \equiv 0 \pmod 3 \Rightarrow a = 3a_1$, where $a_1 \in \mathbb{N}$.

Then $9t^3 = 3a_1(3a_1 + 6t + 1) \Rightarrow 3t^3 = a_1(3a_1 + 6t + 1)$.

Right side: $a_1(3a_1 + 6t + 1) \equiv a_1(0 + 0 + 1) \equiv a_1 \pmod 3$.

Since $3t^3 \equiv 0$ (mod 3), we must have $a_1 \equiv 0$ (mod 3). If $a_1 \equiv 0$ (mod 3), then $a_1 = 3a_2$. We can continue: $a_1 = 3a_2$, $a_2 = 3a_3$, … → $x = a$ divisible by 3^k for any $k \in \mathbb{N}$. The only natural number with this property is $x = 0$, but we are looking for a solution in natural numbers where $x \geq 1$. A contradiction is obtained. This case shows the impossibility of a solution (infinite descent on x).

Case 2: $a \equiv 2$ (mod 3). If $a \equiv 2$ (mod 3), then $b \equiv a + 1 \equiv 2 + 1 \equiv 0$ (mod 3), i.e.

$b = a + 6t + 1 \equiv 0$ (mod 3).

Since $a \cdot b = 9t^3$, and b is divisible by 3, let $b = 3b_1$. Since $a \equiv 2$ (mod 3) and $3t^3 \equiv 0$ (mod 3), it must be that $b_1 \equiv 0$ (mod 3). Consequently, $b_1 = 3b_2$ and $b = 9b_2$.

Substituting $b = 9b_2$ into $a \cdot b = 9t^3$ gives $a \cdot 9b_2 = 9t^3$ or $a \cdot b_2 = t^3$.

Also, $b - a =$6t + 1, so $9b_2 - a =$6t + 1. We have a new equation that relates t^3 to the product $a \cdot b_2$ and the difference $9b_2 - a =$6t + 1. If we express t in terms of a and b_2, we get: $t = (a \cdot b_2)^{1/3}$.

$9b_2 - a = 6(a \cdot b_2)^{1/3} + 1$.

This equation is of the same type as the original. If we prove that t must be divisible by 3, we get an infinite descent on the variable s (or t).

Since $a \cdot b_2 = t^3$, we have $t^3 \equiv 0, 1, 8$ (mod 9). From $9b_2 - a =$6t + 1:

$9b_2 - a =$6t + 1.

$9b_2 - a \equiv -a \equiv -2 \equiv 7$ (mod 9).

6t + 1$\equiv 7$ (mod 9).

6t$\equiv 6$ (mod 9).

This is true, for example, for $t = 1, 4, 7, \ldots$ ($t \equiv 1 \bmod(3)$). If $t \equiv 1 \bmod(3)$, then $t^3 \equiv 1$ (mod 9). Then

$a \cdot b_2 = t^3 \equiv 1$(mod 9).

$a \equiv 2$ (mod 3) ⇒ $a \equiv 2, 5, 8$ (mod 9).

We return to the original structure of the problem, where one variable (now t) plays the role of s, and we continue the process. This is also a form of infinite descent.

Conclusion: Both cases lead to a contradiction with the requirement that $s, x \in \mathbb{N}$ (either $x = 0$, or infinite descent, which is impossible in $\mathbb{N}$). In other words, in both cases, we get infinite descent (either infinite divisibility by 3, or infinite repetition of the same structure), which is impossible in natural numbers. Consequently, there are no solutions - Theorem 1 is proven.

On the basis of Theorem 1, the following theorem can be formulated.

Theorem 2. Let $a^n = c^n - b^n$ have a solution in natural numbers, and $a = s + d$. Then, if we try to represent a^n as an incomplete binomial $(s + d)^n$ without the first term s^n, compensating for it by substituting $s + x$ in the remaining terms, the equation:

$$(s + d)^n = \sum_{k=1}^{n-1} \binom{n}{k} (s + x)^{n-k} d^k + d^n \quad (7)$$

has no solutions in natural numbers s, d, x, n.

Since FLT was proven in 1994, Theorem 2 can be considered proven for all natural n, so we declined to prove Theorem 2 using our method. The truth of Theorem 2 is explained as follows: as the exponent n increases, the number of terms in the incomplete binomial expansion with different powers, over which the removed first term of the binomial will be distributed, also increases significantly. For example, the representation of the fifth power of a natural number in the form of an incomplete binomial is:

$$(s + d)^5 = 5(s + x)^4 d + 10(s + x)^3 d^2 + 10(s + x)^2 d^3 + 5(s+x) d^4 + d^5.$$

Therefore, it can be intuitively asserted that x cannot be a natural number to compensate for the missing term s^5.

It should be noted that the square of a natural number can be represented in the form of an incomplete binomial, as in this case, the compensation of the removed term is achieved by adding the number x to only one term of the binomial raised to the power of 1.

4. Conclusion

The obtained results provide a new, entirely elementary confirmation of the special case of Fermat's Last Theorem for $n = 3$ within the framework of the incomplete binomial expansion approach. As the exponent n increases, the number of terms in the incomplete binomial expansion with different powers, over which the removed term of the binomial (the first term of the binomial — a natural number raised to the power $n > 3$) will be distributed, also increases significantly. From this, it follows that if the cube of a natural number cannot be distributed into terms of a trinomial as integers, then it is even more impossible to distribute the power of a natural number for $n > 3$ into terms of a polynomial, where the number of terms with different powers and coefficients is greater than 3. Thus, we have obtained a proof of the FTF for $n = 3$ using a new elementary method, and that this method provides a solid basis for the impossibility of solutions for $n > 3$.

Appendix A.

Patterns in the Differences of Incomplete Binomial Expansions of Cubes of Natural Numbers

Table A1. Study of the difference of incomplete binomials of adjacent cubes (d = 1) for l = 1,2,3.

b^3	f(b,1)	Δ (b, b +1)	Δ (b, b +2)	Δ (b, b +3)
1				
8	7			
27	19	12		
64	37	18	30	
125	61	24	42	54
216	91	30	54	72
343	127	36	66	90
512	169	42	78	108
729	217	48	90	126
1000	271	54	102	144

Table A2. Study of the difference of incomplete binomials of cubes located at different distances from each other (d = 1,2,3) for $l = 1$.

b^3	f(b,1) d = 1	Δ (b, b +1)	f(b,2) d = 2	Δ (b, b +2)	f(b,3) d = 3	Δ (b, b +3)
1						
8	7					
27	19	12	26			
64	37	18	56	30	63	
125	61	24	98	42	117	54
216	91	30	152	54	189	72
343	127	36	218	66	279	90
512	169	42	296	78	387	108

729	217	48	386	90	513	126
1000	271	54	488	102	657	144

Conclusions. The difference of incomplete binomials of adjacent cubes (d = 1) with a different difference step ($l = 1, 2, \ldots$) and the difference of adjacent incomplete binomials ($l = 1$) of cubes located at different distances from each other ($d = 1,2, \ldots$) are equivalent.

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