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Slip Certificates for the Riemann ξ–Function via Poisson Forcing and Carleson Tents: A Stagewise One-Dimensional Criterion for Zero-Free Rectangles

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Abstract

Let $\xi(s)$ be the completed Riemann zeta function and $\Xi(z) = \xi(\frac{1}{2} + \mathrm{i}z)$. We study the logarithmic derivative field

$$m(z) := -\frac{\Xi'(z)}{\Xi(z)}$$

and introduce a one-dimensional functional along horizontal scan lines $z = t + \mathrm{i}\eta$: for a bounded interval $I \subset \mathbb{R}$ and $\eta > 0$,

$$\mathsf{Slip}^+_\eta(I) := \int_I \big(-\operatorname{Im} m(t + \mathrm{i}\eta)\big)_+ \,\mathrm{d}t,$$

with $\mathsf{Slip}^+_\eta(I) = +\infty$ if $\Xi(t + \mathrm{i}\eta) = 0$ for some $t \in I$. Writing $s = \frac{1}{2} + \mathrm{i}z = (\frac{1}{2} - \eta) + \mathrm{i}t$, one has

$$-\operatorname{Im} m(t + \mathrm{i}\eta) = \operatorname{Re}\left(\frac{\xi'}{\xi}(s)\right) = \frac{\partial}{\partial\sigma} \log|\xi(\sigma + \mathrm{i}t)| \qquad (\sigma = \tfrac{1}{2} - \eta),$$

so $-\operatorname{Im} m$ is a vertical derivative of $\log|\xi|$. Our first main result is a local coercivity principle with a Poisson/harmonic-measure interpretation: if Ξ has a zero $z_0 = t_0 + \mathrm{i}\eta_0$ of multiplicity k, then on every scan line just below z_0 the positive-part argument-variation on the symmetric window $[t_0 - d, t_0 + d]$ is bounded below by $k\pi/4$. Geometrically, a zero forces a quantized defect on the Carleson tent (cone) directly beneath it. We then prove a stagewise transducer from one-dimensional slip control to two-dimensional zero-free rectangles: if $\mathsf{Slip}^+_\eta(I) < \pi/4$ holds for all scan heights $\eta \in [\eta_\star, \frac{1}{2}]$ and for all I in a fixed two-shift unit-interval cover of $[T, 2T]$, then Ξ is zero-free in the corresponding upper-half-plane window, and hence every ξ-zero $\beta + \mathrm{i}\gamma$ with $T < \gamma < 2T$ satisfies $|\beta - \frac{1}{2}| < \eta_\star$. A finite-mesh reduction replaces the continuum of scan heights by finitely many scan lines using height-Lipschitz control; we also record perturbation stability of slip. To support formally checkable certificates, we record an explicit validated-numerics layer: disk enclosure arithmetic, Euler–Maclaurin remainder bounds for ζ, ζ', ζ'', and Stirling-type remainder bounds for ψ and ψ'. Appendices include certificate-format completeness and optional adaptive scan-height propagation.

Keywords: Riemann zeta function; Riemann xi-function; logarithmic derivative; Poisson kernel; harmonic measure; Carleson tents; argument principle; validated numerics; Euler–Maclaurin summation; Stirling remainder bounds; interval arithmetic

MSC: 11M26; 30D15; 31A05; 42B30; 65G20

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1. Introduction and Main Results

1.1. The Completed Zeta Function and Ξ

The completed zeta function is

$$\xi(s) := \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2} \Gamma\!\left(\frac{s}{2}\right) \zeta(s), \tag{1}$$

and we use the standard even entire normalization

$$\Xi(z) := \xi\Big(\tfrac{1}{2} + \mathrm{i}z\Big). \tag{2}$$

Then ξ and Ξ are entire and satisfy the symmetries

$$\xi(1-s) = \xi(s), \quad \xi(\bar{s}) = \overline{\xi(s)}, \qquad \Xi(-z) = \Xi(z), \quad \Xi(\bar{z}) = \overline{\Xi(z)}.$$

We write points in the Ξ–plane as

$$z = t + \mathrm{i}\eta, \qquad t \in \mathbb{R},\ \eta \in \mathbb{R},$$

and set

$$s = \tfrac{1}{2} + \mathrm{i}z = \Big(\tfrac{1}{2} - \eta\Big) + \mathrm{i}t. \tag{3}$$

Thus $\eta > 0$ corresponds to $\mathrm{Re}(s) < \frac{1}{2}$. A nontrivial zero $\rho = \beta + \mathrm{i}\gamma$ of ζ corresponds to a zero of Ξ at

$$z = \gamma + \mathrm{i}\Big(\tfrac{1}{2} - \beta\Big).$$

Consequently, proving that Ξ is zero-free for $\eta \geq \eta_0 > 0$ in a height window $t \in [T, 2T]$ implies that all ξ-zeros with $\Im\rho \in (T, 2T)$ satisfy $|\,\mathrm{Re}\,\rho - \frac{1}{2}| < \eta_0$.

1.2. Windows

For $T > 0$ and $\eta_0 \in (0, \frac{1}{2})$ define the closed and open Ξ–windows

$$R_{T,\eta_0} := \{t + \mathrm{i}\eta :\ T \leq t \leq 2T,\ \eta_0 \leq \eta \leq \tfrac{1}{2}\}, \qquad \Omega_{T,\eta_0} := \{t + \mathrm{i}\eta :\ T < t < 2T,\ \eta_0 < \eta < \tfrac{1}{2}\}. \tag{4}$$

Definition 1 (Zero sets). *For a function F and a set $U \subset \mathbb{C}$, define*

$$Z(F; U) := \{z \in U :\ F(z) = 0\}.$$

1.3. The Slip Observable (One-Sided Argument Variation)

Define the logarithmic derivative field

$$m(z) := -\frac{\Xi'(z)}{\Xi(z)} \qquad (z :\ \Xi(z) \neq 0). \tag{5}$$

The observable driving this paper is the nonnegative quantity $(-\,\mathrm{Im}\, m(t + \mathrm{i}\eta))_+$.

Definition 2 (Positive-part slip / positive argument-variation). *For $\eta > 0$ and a bounded interval $I \subset \mathbb{R}$, define*

$$\mathsf{Slip}^+_\eta(I) := \int_I (-\,\mathrm{Im}\, m(t + \mathrm{i}\eta))_+ \,\mathrm{d}t \in [0, +\infty],$$

with the convention $\mathsf{Slip}^+_\eta(I) = +\infty$ if $\Xi(t + \mathrm{i}\eta) = 0$ for some $t \in I$.

Remark 1 (Endpoint padding for interval banks). *For the two-shift unit-interval cover used below, some covering intervals may extend slightly beyond $[T, 2T]$ (by at most $1/2$). Thus a numerical certificate that verifies slip bounds on those intervals requires enclosures for m (and typically some modulus of continuity in η) on a slightly padded t–range. This is only a bookkeeping issue; the analytic transduction argument is unchanged.*

1.4. *Slip as Positive Variation of* $\arg \Xi$

Definition 3 (Positive variation)**.** *Let $\theta : I \to \mathbb{R}$ be absolutely continuous on a bounded interval $I \subset \mathbb{R}$. Define the* positive variation *of θ on I by*

$$\mathrm{Var}^{+}(\theta; I) := \int_I (\theta'(t))_+ \,\mathrm{d}t.$$

Proposition 1 (Slip equals the positive variation of $\arg \Xi$ along a scan segment)**.** *Fix $\eta > 0$ and a bounded interval $I \subset \mathbb{R}$. Assume $\Xi(t + \mathrm{i}\eta) \neq 0$ for all $t \in I$. Then there exists an absolutely continuous function $\theta : I \to \mathbb{R}$ such that*

$$\Xi(t + \mathrm{i}\eta) = |\Xi(t + \mathrm{i}\eta)|\, e^{\mathrm{i}\theta(t)} \qquad (t \in I),$$

and for almost every $t \in I$,

$$\theta'(t) = \mathrm{Im}\left(\frac{\Xi'}{\Xi}(t + \mathrm{i}\eta)\right) = -\,\mathrm{Im}\, m(t + \mathrm{i}\eta).$$

In particular,

$$\mathsf{Slip}^{+}_{\eta}(I) = \int_I (-\,\mathrm{Im}\, m(t + \mathrm{i}\eta))_+ \,\mathrm{d}t = \mathrm{Var}^{+}(\theta; I).$$

Proof. Define $f(t) := \Xi(t + \mathrm{i}\eta)$ for $t \in I$. Then f is C^1 and nonvanishing on I. Hence there exists an absolutely continuous lift $\theta : I \to \mathbb{R}$ such that $f(t) = |f(t)|e^{\mathrm{i}\theta(t)}$ on I.

Differentiate $f(t) = |f(t)|e^{\mathrm{i}\theta(t)}$ at points where θ' exists (a.e.):

$$\frac{f'(t)}{f(t)} = \frac{\mathrm{d}}{\mathrm{d}t} \log |f(t)| + \mathrm{i}\,\theta'(t).$$

Taking imaginary parts gives

$$\theta'(t) = \mathrm{Im}\left(\frac{f'(t)}{f(t)}\right) = \mathrm{Im}\left(\frac{\Xi'}{\Xi}(t + \mathrm{i}\eta)\right) = -\,\mathrm{Im}\, m(t + \mathrm{i}\eta).$$

Integrating the positive part yields the claim. □

1.5. *Carleson Tents (the Cone Geometry Beneath an Interval)*

A standard geometric architecture in harmonic analysis is the *Carleson tent* (or Carleson box) above an interval. Our local forcing theorem naturally produces a cone condition of this type beneath a zero.

Definition 4 (Carleson tent above an interval (upper half-plane))**.** *For a bounded interval $I \subset \mathbb{R}$ with center c_I and length $|I|$, define its (open) Carleson tent by*

$$\mathcal{T}(I) := \Big\{ t + \mathrm{i}\eta \in \mathbb{C} :\ 0 < \eta < |I|/2,\ |t - c_I| < |I|/2 - \eta \Big\}.$$

Equivalently, $\mathcal{T}(I)$ is the interior of the isosceles cone of slope 1 with apex at c_I on the real axis and top edge at height $|I|/2$. For $\eta_\star \in (0, 1/2)$ we also write the truncated tent

$$\mathcal{T}_{\geq \eta_\star}(I) := \mathcal{T}(I) \cap \{\eta \geq \eta_\star\}.$$

Remark 2. *The local window in the forcing theorem is exactly the base interval of a tent with top height equal to the vertical separation d from the zero: if $z_0 = t_0 + \mathrm{i}\eta_0$ and $\eta = \eta_0 - d$, then $[t_0 - d, t_0 + d]$ is the interval whose tent contains z_0 and whose top boundary passes through the scan line $\Im z = \eta$.*

1.6. *Slip vs. Argument Principle: Explicit Non-Equivalence*

The argument principle measures *net* winding on a closed contour. Slip measures *one-sided* variation on open scan segments, discarding cancellation. This distinction yields a local coercivity mechanism (zeros force positive variation) and a stagewise transducer that does not require contour nonvanishing data.

Proposition 2 (Slip bounds are not necessary for zero-freeness (non-equivalence))**.** *There exist entire nonvanishing functions F such that for every $\eta > 0$ and every unit interval $I \subset \mathbb{R}$,*

$$\int_I \left(-\operatorname{Im}\left(-\frac{F'(t+\mathrm{i}\eta)}{F(t+\mathrm{i}\eta)} \right) \right)_+ \mathrm{d}t \;>\; \frac{\pi}{4}.$$

In particular, unit-bank slip inequalities are not a reformulation of the argument principle for general holomorphic functions.

Proof. Let $F(z) = e^{\mathrm{i}Az}$ with $A > \pi/4$. Then F is entire and never zero, and

$$-\frac{F'(z)}{F(z)} = -\mathrm{i}A \quad \Longrightarrow \quad \left(-\operatorname{Im}(-F'/F)\right)_+ = (A)_+ = A.$$

Hence the integral over a unit interval equals $A > \pi/4$. □

Proposition 3 (Slip transduction needs no boundary nonvanishing checks)**.** *The hypotheses of the stagewise slip transducer (Theorem 2) involve only the scan-line values $\{-\operatorname{Im} m(t+\mathrm{i}\eta) : t \in I\}$ (and, for finite-mesh reduction, a height modulus of continuity). No evaluation of Ξ on $\partial R_{T,\eta_\star}$ is required, unlike an argument-principle certificate which must first verify $\Xi \neq 0$ on the full boundary.*

Proof. Immediate from the statements of Definition 2 and Theorems 2 and 3. □

1.7. A Finite-Sampling Barrier (Why Staged Certificates Are Natural)

Remark 3 (Finite samples do not decide winding / zero counts)**.** *Let $\Gamma : S^1 \to \mathbb{C}^\times$ be a continuous loop. Its winding number $\mathrm{wind}_0(\Gamma) \in \mathbb{Z}$ is a global invariant. In particular, for any fixed finite set of sample points $\theta_0, \dots, \theta_m \in S^1$, the sampled values $\Gamma(\theta_j)$ do not determine $\mathrm{wind}_0(\Gamma)$: one may modify Γ between sample points by inserting arbitrarily many additional turns around 0 while keeping all sample values fixed.*

Consequently, any checkable *certificate family that decides a winding/zero-count statement must either (i) impose additional regularity and a separation margin from 0 (enabling a mesh refinement argument), or (ii) use a structured inequality family that can be propagated (as in slip certificates). This explains why staged/mesh formats occur intrinsically and are not merely organizational.*

1.8. Main Results

Theorem 1 (Quantized Poisson/tent forcing near a zero)**.** *Assume Ξ has a zero at $z_0 = t_0 + \mathrm{i}\eta_0$ of multiplicity $k \geq 1$ with $\eta_0 > 0$. Then there exists $d_0 \in (0, \eta_0)$ such that for every $d \in (0, d_0)$,*

$$\mathsf{Slip}^+_{\eta_0 - d}\big([t_0 - d, t_0 + d]\big) \;\geq\; k\pi/4.$$

Equivalently: for all sufficiently small $d > 0$, the zero z_0 forces a slip defect on the base interval of the Carleson tent whose apex is (t_0, η_0) and whose top boundary is the scan line $\Im z = \eta_0 - d$.

The next theorem is the transduction mechanism from one-dimensional slip control to two-dimensional zero-free rectangles.

Definition 5 (Two-shift unit interval cover)**.** *Fix a real interval $J \subset \mathbb{R}$ and set $\mathcal{U} := \{0, \frac{1}{2}\}$. Define $\mathscr{I}_J$ to be the collection of unit intervals*

$$I = [m + u, m + u + 1], \qquad m \in \mathbb{Z},\ u \in \mathcal{U},$$

such that $I \cap J \neq \varnothing$.

Theorem 2 (Stagewise slip transducer: continuum-height form)**.** *Fix $T > 0$, $\eta_\star \in (0, \frac{1}{2})$, and let $J = [T, 2T]$. Assume that for every $\eta \in [\eta_\star, \frac{1}{2}]$ and every $I \in \mathscr{I}_J$,*

$$\mathsf{Slip}^+_\eta(I) \;<\; \frac{\pi}{4}.$$

Then

$$Z(\Xi; \Omega_{T,\eta_\star}) = \varnothing.$$

Corollary 1 (Strip control for ξ-zeros)**.** *Under the hypotheses of Theorem 2, every zero $\rho = \beta + \mathrm{i}\gamma$ of ξ with $T < \gamma < 2T$ satisfies*

$$\left|\beta - \tfrac{1}{2}\right| \;<\; \eta_\star.$$

For computational certification one typically reduces the continuum of scan heights to a finite mesh.

Theorem 3 (Finite-mesh slip transducer (Lipschitz-height form))**.** *Fix $T > 0$, $\eta_\star \in (0, \frac{1}{2})$, and $J = [T, 2T]$. Assume there exist:*

(i) *a margin $\delta \in (0, \pi/4)$ and a finite mesh*

$$\eta_\star = \tau_0 < \tau_1 < \cdots < \tau_M = \tfrac{1}{2};$$

(ii) *for each $I \in \mathscr{I}_J$, a constant $L_I < \infty$ such that the map $\eta \mapsto \mathsf{Slip}^+_\eta(I)$ is Lipschitz on $[\eta_\star, \frac{1}{2}]$ with Lipschitz constant L_I, i.e.*

$$\left|\mathsf{Slip}^+_\eta(I) - \mathsf{Slip}^+_\tau(I)\right| \;\le\; L_I\,|\eta - \tau| \qquad (\eta, \tau \in [\eta_\star, \tfrac{1}{2}]);$$

(iii) *mesh spacing satisfying, for each $I \in \mathscr{I}_J$,*

$$L_I \cdot \max_{0 \le j < M} (\tau_{j+1} - \tau_j) \;\le\; \delta;$$

(iv) *slip bounds on all mesh heights:*

$$\mathsf{Slip}^+_{\tau_j}(I) \;<\; \frac{\pi}{4} - \delta \qquad \text{for all } I \in \mathscr{I}_J,\ 0 \le j \le M.$$

Then $Z(\Xi; \Omega_{T,\eta_\star}) = \varnothing$, and hence Corollary 1 holds.

Remark 4 (Obtaining L_I from bounds on m' (a sufficient condition))**.** *A convenient sufficient condition for hypothesis* (ii) *is the existence of a bound $B_I < \infty$ such that m is holomorphic on an open neighborhood of*

$$U(I) := \{t + \mathrm{i}\eta :\ t \in I,\ \eta \in [\eta_\star, \tfrac{1}{2}]\}$$

and $\sup_{z \in U(I)} |m'(z)| \le B_I$. Then Lemma 6 gives (ii) *with*

$$L_I = |I|\, B_I.$$

Remark 5 (Novelty in analytic form)**.** *The forcing mechanism Theorem 1 is a local Poisson/harmonic-measure coercivity principle: a zero forces a quantized positive-variation defect on scan lines in the tent below it. The transducer Theorem 2 then converts robust one-dimensional inequalities into a two-dimensional zero-free window without boundary nonvanishing checks.*

2. The Logarithmic-Derivative Dictionary

Proposition 4 (Dictionary between Ξ and ξ)**.** *Let* $z = t + \mathrm{i}\eta$ *and set* $s = \frac{1}{2} + \mathrm{i}z = (\frac{1}{2} - \eta) + \mathrm{i}t$*. Then, on* $\{\Xi(z) \neq 0\}$,

$$m(z) = -\mathrm{i}\,\frac{\xi'(s)}{\xi(s)}. \tag{6}$$

Consequently,

$$-\operatorname{Im} m(t + \mathrm{i}\eta) = \operatorname{Re}\bigg(\frac{\xi'}{\xi}\big((\tfrac{1}{2} - \eta) + \mathrm{i}t\big)\bigg). \tag{7}$$

Moreover, wherever $\xi(\sigma + \mathrm{i}t) \neq 0$,

$$\operatorname{Re}\bigg(\frac{\xi'}{\xi}(\sigma + \mathrm{i}t)\bigg) = \frac{\partial}{\partial\sigma}\log|\xi(\sigma + \mathrm{i}t)|.$$

Proof. Define $s(z) = \frac{1}{2} + \mathrm{i}z$. Then $\Xi(z) = \xi(s(z))$ and $s'(z) = \mathrm{i}$. Hence $\Xi'(z) = \mathrm{i}\,\xi'(s(z))$, so

$$m(z) = -\frac{\Xi'(z)}{\Xi(z)} = -\mathrm{i}\,\frac{\xi'(s(z))}{\xi(s(z))}.$$

If $w = \xi'(s)/\xi(s) = a + \mathrm{i}b$, then $-\mathrm{i}w = b - \mathrm{i}a$, so $-\operatorname{Im}(-\mathrm{i}w) = a = \operatorname{Re} w$. This gives (7). The last identity follows from Cauchy–Riemann applied to $\log|\xi|$. □

Lemma 1 (Meromorphic structure of m)**.** *The function* $m = -\Xi'/\Xi$ *is meromorphic on* $\mathbb{C}$ *and its poles are precisely the zeros of* Ξ*. If* z_0 *is a zero of* Ξ *of multiplicity* $k \geq 1$*, then* m *has a simple pole at* z_0 *with residue* $-k$.

Proof. Since Ξ is entire, Ξ'/Ξ is meromorphic with possible poles only at zeros of Ξ. If $\Xi(z) = (z - z_0)^k G(z)$ with $G(z_0) \neq 0$, then

$$\frac{\Xi'(z)}{\Xi(z)} = \frac{k}{z - z_0} + \frac{G'(z)}{G(z)},$$

hence $m(z) = -\frac{k}{z - z_0} - \frac{G'(z)}{G(z)}$ and the residue is $-k$. □

2.1. A Poisson Superposition Identity from the Hadamard Product

A key conceptual point is that $\Re(\xi'/\xi)$ is (up to explicit archimedean terms) a *Poisson superposition* of contributions from the zeros. This makes $-\Im m$ into a Poisson kernel field on scan lines.

Proposition 5 (Hadamard–Poisson superposition for $\Re(\xi'/\xi)$)**.** *There exists a real constant* $B \in \mathbb{R}$ *such that for all* $s = \sigma + \mathrm{i}t \in \mathbb{C}$ *with* $\xi(s) \neq 0$,

$$\frac{\xi'}{\xi}(s) = B + \sum_{\rho}\bigg(\frac{1}{s - \rho} + \frac{1}{\rho}\bigg), \tag{8}$$

where the sum ranges over all zeros ρ *of* ξ *(equivalently, the nontrivial zeros of* ζ*), and is understood in the standard symmetric sense (pairing* ρ *with* $1 - \rho$ *and with* $\overline{\rho}$*). Taking real parts yields*

$$\operatorname{Re}\bigg(\frac{\xi'}{\xi}(\sigma + \mathrm{i}t)\bigg) = B + \sum_{\rho = \beta + \mathrm{i}\gamma}\bigg(\frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} + \operatorname{Re}\frac{1}{\rho}\bigg). \tag{9}$$

In the Ξ*-coordinates* $z = t + \mathrm{i}\eta$ *with* $s = \frac{1}{2} + \mathrm{i}z = (\frac{1}{2} - \eta) + \mathrm{i}t$*, one has*

$$-\operatorname{Im} m(t + \mathrm{i}\eta) = B + \sum_{\rho = \beta + \mathrm{i}\gamma}\bigg(\frac{(\frac{1}{2} - \eta) - \beta}{((\frac{1}{2} - \eta) - \beta)^2 + (t - \gamma)^2} + \operatorname{Re}\frac{1}{\rho}\bigg). \tag{10}$$

Remark 6 (Interpretation)**.** *Each zero $\rho = \beta + \mathrm{i}\gamma$ contributes a Poisson kernel*

$$(t, \sigma) \mapsto \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2}$$

to $\Re(\xi'/\xi)(\sigma + \mathrm{i}t)$. The forcing theorem below may be read as: a single *Poisson source beneath a scan line forces a definite amount of positive-part mass on a symmetric interval; the transducer then propagates this local obstruction stagewise via an interval bank (a discretized Carleson-tent architecture).*

Proof. This is standard from the Hadamard product representation of ξ as an entire function of order 1. One form (see, e.g., [8, Ch. 2] or [17, Ch. II]) is

$$\xi(s) = \xi(0)\, e^{Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

with $B \in \mathbb{R}$. Taking logarithmic derivatives gives (8) wherever $\xi(s) \neq 0$, with the sum interpreted symmetrically. Taking real parts yields (9). Finally, (10) is (7) rewritten with $\sigma = \frac{1}{2} - \eta$. □

3. Quantized Slip Forcing (Poisson/Tent Coercivity)

Lemma 2 (Positive-part comparison)**.** *For every real u one has $(u)_+ \geq u$.*

Proof. If $u \geq 0$ then $(u)_+ = u$, while if $u < 0$ then $(u)_+ = 0 > u$. □

Lemma 3 (Poisson kernel mass and harmonic measure)**.** *For $d > 0$ the function $u \mapsto \dfrac{d}{u^2 + d^2}$ is (up to the factor π) the Poisson kernel of the upper half-plane evaluated at height d. In particular,*

$$\int_{-d}^{d} \frac{d}{u^2 + d^2}\, \mathrm{d}u = \frac{\pi}{2},$$

which equals π times the harmonic measure of $[-d, d]$ as seen from $\mathrm{i}d$ *in the upper half-plane.*

Proof. Compute directly:

$$\int_{-d}^{d} \frac{d}{u^2 + d^2}\, \mathrm{d}u = \Big[\arctan(u/d)\Big]_{-d}^{d} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

The harmonic-measure statement is the standard identification of the Poisson kernel with boundary harmonic measure. □

Proof of Theorem 1. Let $z_0 = t_0 + \mathrm{i}\eta_0$ be a zero of multiplicity $k \geq 1$. Factor

$$\Xi(z) = (z - z_0)^k G(z),$$

where G is holomorphic near z_0 and satisfies $G(z_0) \neq 0$. By Lemma 1,

$$m(z) = -\frac{k}{z - z_0} - \frac{G'(z)}{G(z)}.$$

Choose $r > 0$ such that G is nonvanishing on the closed disk $\overline{B}(z_0, r)$. Then G'/G is holomorphic on a neighborhood of this disk and hence bounded there:

$$M := \sup_{z \in \overline{B}(z_0, r)} \left| \frac{G'(z)}{G(z)} \right| < \infty.$$

If $M = 0$, set
$$d_0 := \min\left\{\frac{r}{2}, \frac{\eta_0}{2}\right\}.$$
If $M > 0$, set
$$d_0 := \min\left\{\frac{r}{2}, \frac{\eta_0}{2}, \frac{k\pi}{8M}\right\}.$$
In either case $d_0 \in (0, \eta_0)$.

Fix $d \in (0, d_0)$ and set $\eta := \eta_0 - d$ and $I_d = [t_0 - d, t_0 + d]$. Along the scan line $z = t + \mathrm{i}\eta$, write $u = t - t_0 \in [-d, d]$. Then
$$z - z_0 = (t - t_0) + \mathrm{i}(\eta - \eta_0) = u - \mathrm{i}d.$$
Compute the pole contribution:
$$-\operatorname{Im}\left(-\frac{k}{u - \mathrm{i}d}\right) = \operatorname{Im}\left(\frac{k}{u - \mathrm{i}d}\right) = \operatorname{Im}\left(\frac{k(u + \mathrm{i}d)}{u^2 + d^2}\right) = \frac{k\,d}{u^2 + d^2}.$$
Integrating and using Lemma 3 gives
$$\int_{I_d} -\operatorname{Im}\left(-\frac{k}{z - z_0}\right)\,\mathrm{d}t = k\int_{-d}^{d} \frac{d}{u^2 + d^2}\,\mathrm{d}u = k \cdot \frac{\pi}{2}. \tag{11}$$
For the holomorphic remainder term, note that $|z - z_0| \le \sqrt{2}\,d < r$ on I_d, so $z \in \overline{B}(z_0, r)$ and therefore
$$\left|\operatorname{Im}\left(\frac{G'(z)}{G(z)}\right)\right| \le \left|\frac{G'(z)}{G(z)}\right| \le M.$$
Thus
$$\left|\int_{I_d} \operatorname{Im}\left(\frac{G'(t + \mathrm{i}\eta)}{G(t + \mathrm{i}\eta)}\right)\mathrm{d}t\right| \le \int_{I_d} M\,\mathrm{d}t = 2dM. \tag{12}$$
Combining (11) and (12),
$$\int_{I_d} -\operatorname{Im} m(t + \mathrm{i}\eta)\,\mathrm{d}t \ge k\frac{\pi}{2} - 2dM.$$
If $M = 0$ this is $k\pi/2$. If $M > 0$, then $d < d_0 \le k\pi/(8M)$ gives $2dM \le k\pi/4$, hence
$$\int_{I_d} -\operatorname{Im} m(t + \mathrm{i}\eta)\,\mathrm{d}t \;\ge\; k\pi/4.$$
Finally, Lemma 2 yields
$$\mathsf{Slip}^{+}_{\eta}(I_d) = \int_{I_d} (-\operatorname{Im} m(t + \mathrm{i}\eta))_{+}\,\mathrm{d}t \ge \int_{I_d} -\operatorname{Im} m(t + \mathrm{i}\eta)\,\mathrm{d}t \ge k\pi/4.$$
□

Proposition 6 (Sharpness on the pure pole model)**.** *Let* $F(z) = (z - z_0)^k$ *with* $z_0 = t_0 + \mathrm{i}\eta_0$ *and* $k \ge 1$. *Define* $m_F := -F'/F = -k/(z - z_0)$. *Then for every* $d > 0$ *and* $\eta = \eta_0 - d$,
$$\mathsf{Slip}^{+}_{F,\eta}([t_0 - d, t_0 + d]) = k\frac{\pi}{2}.$$
In particular, the pole contribution supplies exactly $k\pi/2$ *on the symmetric window; the constant* $\pi/4$ *in Theorem 1 arises by reserving half of this mass to dominate the holomorphic remainder.*

Proof. Along $z = t + \mathrm{i}(\eta_0 - d)$ with $u = t - t_0$ one has
$$-\operatorname{Im} m_F(t + \mathrm{i}(\eta_0 - d)) = \frac{k\,d}{u^2 + d^2} > 0,$$

so the positive part is redundant and

$$\mathsf{Slip}^{+}_{F,\eta}([t_0-d,t_0+d]) = k\int_{-d}^{d}\frac{d}{u^2+d^2}\,\mathrm{d}u = k\frac{\pi}{2}.$$

☐

Remark 7 (Tent geometry (Carleson cone beneath a zero))**.** *Fix $z_0 = t_0 + \mathrm{i}\eta_0$ and $d > 0$. The symmetric base interval $[t_0 - d, t_0 + d]$ is precisely the interval whose Carleson tent contains z_0 and whose top boundary meets the scan line $\Im z = \eta_0 - d$. Thus Theorem 1 may be read as:* a zero forces a quantized positive-variation defect on every tent slice directly beneath it, *with harmonic-measure normalization.*

3.1. Perturbation Stability of Slip

Lemma 4 (Slip stability under perturbations of $\operatorname{Im} m$)**.** *Fix $\eta > 0$ and a bounded interval $I \subset \mathbb{R}$. Let m_1, m_2 be functions on $\{t + \mathrm{i}\eta : t \in I\}$ such that*

$$\sup_{t\in I}\big|\operatorname{Im} m_1(t+\mathrm{i}\eta) - \operatorname{Im} m_2(t+\mathrm{i}\eta)\big| \le \varepsilon.$$

Then the corresponding slips satisfy

$$\big|\mathsf{Slip}^{+}_{\eta,1}(I) - \mathsf{Slip}^{+}_{\eta,2}(I)\big| \;\le\; \varepsilon\,|I|.$$

Proof. For each $t \in I$, the map $x \mapsto (x)_+$ is 1-Lipschitz on $\mathbb{R}$, hence

$$\big|(-\operatorname{Im} m_1(t+\mathrm{i}\eta))_+ - (-\operatorname{Im} m_2(t+\mathrm{i}\eta))_+\big| \le |-\operatorname{Im} m_1(t+\mathrm{i}\eta) + \operatorname{Im} m_2(t+\mathrm{i}\eta)| \le \varepsilon.$$

Integrating over I yields the claim. ☐

Corollary 2 (Stability under multiplicative holomorphic twists)**.** *Let H be holomorphic on a neighborhood of $\{t + \mathrm{i}\eta : t \in I\}$ and set $\widetilde{\Xi}(z) := \Xi(z)e^{H(z)}$. Where Ξ is nonzero on the scan segment, the corresponding logarithmic derivatives satisfy*

$$m_{\widetilde{\Xi}}(z) = m_{\Xi}(z) - H'(z).$$

Consequently, if $\sup_{t\in I} |\operatorname{Im} H'(t + \mathrm{i}\eta)| \le \varepsilon$, then

$$\big|\mathsf{Slip}^{+}_{\eta,\widetilde{\Xi}}(I) - \mathsf{Slip}^{+}_{\eta,\Xi}(I)\big| \le \varepsilon|I|.$$

Proof. Differentiate $\widetilde{\Xi} = \Xi e^{H}$ to get $\widetilde{\Xi}'/\widetilde{\Xi} = \Xi'/\Xi + H'$. Negating gives $m_{\widetilde{\Xi}} = m_{\Xi} - H'$. Apply Lemma 4. ☐

4. Tent Energies and Local Counting Consequences

4.1. Shifted Carleson Tents Above a Scan Line

The forcing theorem is naturally expressed in the geometry of Carleson tents. For counting and energy statements it is convenient to use tents based at an arbitrary scan height η_0 (rather than the real axis).

Definition 6 (Shifted Carleson tent above an interval)**.** *Let $I \subset \mathbb{R}$ be a bounded interval with center c_I and length $|I|$, and let $\eta_0 \ge 0$. Define the shifted (open) tent above I with base height η_0 by*

$$\mathcal{T}_{\eta_0}(I) := \Big\{\, t + \mathrm{i}\eta \in \mathbb{C} :\ \eta_0 < \eta < \eta_0 + \frac{|I|}{2},\ |t - c_I| < \frac{|I|}{2} - (\eta - \eta_0)\Big\}.$$

Equivalently, after the vertical translation $z \mapsto z - \mathrm{i}\eta_0$, $\mathcal{T}_{\eta_0}(I)$ is the standard Carleson tent $\mathcal{T}(I)$ in the upper half-plane.

Remark 8. *If* $z_0 = t_0 + \mathrm{i}\eta_1$ *lies in* $\mathcal{T}_{\eta_0}(I)$*, then setting* $d := \eta_1 - \eta_0$ *gives*

$$t_0 \in I, \qquad [t_0 - d, t_0 + d] \subset I.$$

This is the geometric reason the forcing interval in Theorem 1 fits inside the bank interval.

4.2. A Tent Energy (Carleson-Measure Architecture)

The slip functional is one-dimensional (scan-line based). A standard two-dimensional object associated to the upper half-plane and tents is a Carleson-type energy with the measure $\mathrm{d}\eta/\eta$.

Definition 7 (Tent energy of the positive-part field)**.** *Let* $I \subset \mathbb{R}$ *be a bounded interval and* $\eta_0 \geq 0$*. Define the tent energy of the field* $(-\operatorname{Im} m)_+$ *above the base height* η_0 *by*

$$\mathcal{E}_{\eta_0}(I) := \iint_{\mathcal{T}_{\eta_0}(I)} (-\operatorname{Im} m(t + \mathrm{i}\eta))_+ \, \frac{\mathrm{d}t\, \mathrm{d}\eta}{\eta}.$$

If Ξ *vanishes anywhere in* $\mathcal{T}_{\eta_0}(I)$*, interpret* $\mathcal{E}_{\eta_0}(I) = +\infty$*.*

Proposition 7 (Tent-energy logarithmic blow-up near a zero)**.** *Assume* Ξ *has a zero at* $z_0 = t_0 + \mathrm{i}\eta_0$ *of multiplicity* $k \geq 1$ *with* $\eta_0 > 0$*. Let* I *be any interval such that* $z_0 \in \mathcal{T}_0(I)$ *(equivalently,* $z_0 \in \mathcal{T}(I)$*). Then there exist constants* $c, C > 0$ *and* $r_0 \in (0,1)$ *(depending on* z_0 *and* I *but not on* ε*) such that for all* $\varepsilon \in (0, r_0)$*,*

$$\iint_{\mathcal{T}_0(I)\setminus\overline{B}(z_0,\varepsilon)} (-\operatorname{Im} m(z))_+ \, \frac{\mathrm{d}A(z)}{\operatorname{Im} z} \;\geq\; c\, k\, \log\!\left(\frac{1}{\varepsilon}\right) - C,$$

where $\mathrm{d}A$ *denotes planar Lebesgue measure. In particular,* $\mathcal{E}_0(I) = +\infty$ *whenever* $\mathcal{T}(I)$ *contains a zero.*

Proof sketch. Write $\Xi(z) = (z - z_0)^k G(z)$ near z_0 with $G(z_0) \neq 0$, so $m(z) = -k/(z - z_0) - G'(z)/G(z)$. In a sufficiently small disk around z_0, G'/G is bounded, hence contributes at most $O(1)$ to the integral.

For the principal part $m_{\mathrm{pp}}(z) = -k/(z - z_0)$, one checks

$$-\operatorname{Im} m_{\mathrm{pp}}(z) = k\, \frac{\eta_0 - \eta}{(t - t_0)^2 + (\eta - \eta_0)^2},$$

so $(-\operatorname{Im} m_{\mathrm{pp}})_+$ dominates a positive multiple of $k/|z - z_0|$ on a fixed angular sector below z_0. Restrict to a half-annulus $\{\, \varepsilon < |z - z_0| < r \,\}$ inside the tent and with η bounded below by a positive constant, so that $1/\eta$ is comparable to a constant. In polar coordinates about z_0 this yields a lower bound $\gg k \int_\varepsilon^r \frac{1}{\rho}\, \mathrm{d}\rho = k \log(r/\varepsilon)$, giving the claimed logarithmic blow-up. ☐

Remark 9 (Interpretation)**.** *Proposition 7 places the slip framework inside a standard harmonic-analysis architecture: zeros create non-Carleson mass in the tent-energy built from the Poisson-type field* $-\Im m$*. This is qualitatively different from the contour-winding viewpoint of the argument principle.*

4.3. A Local Counting Inequality from Scan-Line Slip

The forcing theorem shows that a single zero produces a quantized slip defect on scan lines below it. The next proposition turns this into a local *counting inequality* under a mild holomorphic-remainder control. (One can regard this as a "soft" local zero-density statement in tent geometry.)

Proposition 8 (Local tent counting from one scan-line slip, with a holomorphic remainder bound)**.** *Fix* $\eta > 0$ *and a bounded interval* $I \subset \mathbb{R}$*. Let* $\mathcal{T}_\eta(I)$ *be the shifted tent above* I*. Assume that* Ξ *has only finitely many zeros in* $\mathcal{T}_\eta(I)$*, say*

$$z_j = t_j + \mathrm{i}\eta_j \in \mathcal{T}_\eta(I), \qquad \operatorname{mult}(z_j) = k_j, \qquad 1 \leq j \leq N,$$

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and that Ξ has no zeros on the closed scan segment $\{t+\mathrm{i}\eta : t \in I\}$.

Define the meromorphic principal-part subtraction

$$H(z) := m(z) + \sum_{j=1}^{N} \frac{k_j}{z - z_j}.$$

Assume H is holomorphic on a neighborhood of $\{t+\mathrm{i}\eta : t \in I\}$ and satisfies the bound

$$\sup_{t\in I} \left|\operatorname{Im} H(t+\mathrm{i}\eta)\right| \ \le\ M.$$

Then one has the scan-line lower bound

$$\mathsf{Slip}^{+}_{\eta}(I) \ \ge\ \frac{\pi}{2}\sum_{j=1}^{N} k_j \ -\ M|I|.$$

Consequently, if $\mathsf{Slip}^{+}_{\eta}(I) \le S$, then

$$\sum_{j=1}^{N} k_j \ \le\ \left\lfloor \frac{2(S+M|I|)}{\pi} \right\rfloor.$$

Proof sketch. Using $(u)_+ \ge u$ and integrating along the scan line $\Im z = \eta$ gives

$$\mathsf{Slip}^{+}_{\eta}(I) \ge \int_I \big(-\operatorname{Im} m(t+\mathrm{i}\eta)\big)\ \mathrm{d}t = \sum_{j=1}^{N} \int_I -\operatorname{Im}\!\left(-\frac{k_j}{(t+\mathrm{i}\eta) - z_j}\right) \mathrm{d}t \ -\ \int_I \operatorname{Im} H(t+\mathrm{i}\eta)\ \mathrm{d}t.$$

The remainder term is bounded below by $-M|I|$.

For each pole term, set $d_j := \eta_j - \eta > 0$ and $u = t - t_j$. Then

$$-\operatorname{Im}\!\left(-\frac{k_j}{u - \mathrm{i}d_j}\right) = k_j \frac{d_j}{u^2 + d_j^2}.$$

Since $z_j \in \mathcal{T}_\eta(I)$, one has $[t_j - d_j, t_j + d_j] \subset I$, hence

$$\int_I k_j \frac{d_j}{(t-t_j)^2 + d_j^2}\ \mathrm{d}t \ \ge\ \int_{t_j-d_j}^{t_j+d_j} k_j \frac{d_j}{(t-t_j)^2 + d_j^2}\ \mathrm{d}t = k_j \frac{\pi}{2}.$$

Combining yields the stated $\pi/2$ coefficient. ☐

Remark 10. *The hypothesis on $\operatorname{Im} H$ is automatically satisfied (with some $M < \infty$) if the scan segment $\{t+\mathrm{i}\eta : t \in I\}$ lies in a compact set disjoint from the zero set of Ξ and the subtraction removes all poles in a neighborhood. In applications, M can be obtained by validated enclosures for m on a small scan neighborhood.*

5. The Stagewise Slip Transducer and Finite-Mesh Reduction

Lemma 5 (Two-shift margin lemma). *Let $J \subset \mathbb{R}$ be an interval and $\mathscr{I}_J$ be as in Definition 5. For every $t \in J$ there exists $I \in \mathscr{I}_J$ such that $t \in I$ and*

$$\operatorname{dist}(t, \partial I) \ge \frac{1}{4}.$$

Proof. Write $t = n + \alpha$ with $n \in \mathbb{Z}$ and $\alpha \in [0,1)$.

If $\alpha \in [1/4, 3/4]$, choose $I = [n, n+1]$. If $\alpha \in [0, 1/4)$, choose $I = [n - \frac{1}{2}, n + \frac{1}{2}]$. If $\alpha \in (3/4, 1)$, choose $I = [n + \frac{1}{2}, n + \frac{3}{2}]$.

Then $t \in I$ and $\operatorname{dist}(t, \partial I) \ge 1/4$ in each case. Moreover $I \cap J \ne \varnothing$ since $t \in I \cap J$. ☐

Proof of Theorem 2. Assume for contradiction that Ξ has a zero $z_0 = t_0 + \mathrm{i}\eta_0$ with

$$t_0 \in (T, 2T), \qquad \eta_0 \in (\eta_\star, \tfrac{1}{2}).$$

By Lemma 5, choose $I \in \mathscr{I}_J$ such that $t_0 \in I$ and $\operatorname{dist}(t_0, \partial I) \geq 1/4$.

Apply Theorem 1 to z_0 and obtain $d_0 > 0$. Choose $d \in (0, d_0)$ such that $d \leq 1/8$ and $d \leq (\eta_0 - \eta_\star)/2$. Then $I_d = [t_0 - d, t_0 + d] \subset I$ and $\eta := \eta_0 - d \geq \eta_\star$.

Since the slip integrand is nonnegative and $I_d \subset I$,

$$\mathsf{Slip}^+_\eta(I) \;\geq\; \mathsf{Slip}^+_\eta(I_d).$$

By Theorem 1, $\mathsf{Slip}^+_\eta(I_d) \geq \pi/4$, contradicting the hypothesis $\mathsf{Slip}^+_\eta(I) < \pi/4$ for all $\eta \in [\eta_\star, 1/2]$ and all $I \in \mathscr{I}_J$.

Hence $Z(\Xi; \Omega_{T,\eta_\star}) = \varnothing$. □

Proof of Corollary 1. Assume $\xi(\rho) = 0$ with $\rho = \beta + \mathrm{i}\gamma$ and $T < \gamma < 2T$. Set $z := \gamma + \mathrm{i}(\frac{1}{2} - \beta)$. Then $\Xi(z) = \xi(\rho) = 0$.

If $\frac{1}{2} - \beta > \eta_\star$, then $z \in \Omega_{T,\eta_\star}$, contradicting Theorem 2. If $\frac{1}{2} - \beta = \eta_\star$, then $\Xi(\gamma + \mathrm{i}\eta_\star) = 0$, hence for any interval $I \ni \gamma$ one has $\mathsf{Slip}^+_{\eta_\star}(I) = +\infty$ by Definition 2, again contradicting the slip hypothesis $\mathsf{Slip}^+_{\eta_\star}(I) < \pi/4$. Therefore $\frac{1}{2} - \beta < \eta_\star$, i.e. $\beta > \frac{1}{2} - \eta_\star$.

For the other inequality, apply $\xi(1 - s) = \xi(s)$: if $\xi(\beta + \mathrm{i}\gamma) = 0$ then $\xi(1 - \beta + \mathrm{i}\gamma) = 0$, and the previous argument gives $1 - \beta > \frac{1}{2} - \eta_\star$, i.e. $\beta < \frac{1}{2} + \eta_\star$. Hence $|\beta - \frac{1}{2}| < \eta_\star$. □

5.1. A Lipschitz Bound in Scan Height

Lemma 6 (Vertical Lipschitz bound for slip)**.** *Fix a bounded interval $I \subset \mathbb{R}$ and heights $\eta_1 \leq \eta_2$ in $(0, \frac{1}{2}]$. Assume m is holomorphic on a neighborhood of*

$$U(I; \eta_1, \eta_2) := \{t + \mathrm{i}\eta : \; t \in I, \; \eta \in [\eta_1, \eta_2]\},$$

and set

$$B := \sup_{z \in U(I;\eta_1,\eta_2)} |m'(z)| < \infty.$$

Then for all $\eta, \tau \in [\eta_1, \eta_2]$,

$$\left|\mathsf{Slip}^+_\eta(I) - \mathsf{Slip}^+_\tau(I)\right| \;\leq\; |I|\, B\, |\eta - \tau|.$$

Proof. Fix $t \in I$ and $\eta, \tau \in [\eta_1, \eta_2]$. Since m is holomorphic, along $u \mapsto t + \mathrm{i}u$ we have

$$\frac{\mathrm{d}}{\mathrm{d}u} m(t + \mathrm{i}u) = \mathrm{i}\, m'(t + \mathrm{i}u).$$

Hence

$$m(t + \mathrm{i}\eta) - m(t + \mathrm{i}\tau) = \int_\tau^\eta \mathrm{i}\, m'(t + \mathrm{i}u)\, \mathrm{d}u,$$

so $|m(t + \mathrm{i}\eta) - m(t + \mathrm{i}\tau)| \leq B|\eta - \tau|$. Taking imaginary parts yields

$$|-\operatorname{Im} m(t + \mathrm{i}\eta) + \operatorname{Im} m(t + \mathrm{i}\tau)| \leq B|\eta - \tau|.$$

Since $x \mapsto (x)_+$ is 1-Lipschitz on $\mathbb{R}$,

$$\left|(-\operatorname{Im} m(t + \mathrm{i}\eta))_+ - (-\operatorname{Im} m(t + \mathrm{i}\tau))_+\right| \leq B|\eta - \tau|.$$

Integrate over $t \in I$ to obtain the stated inequality. □

Proof of Theorem 3. Fix $I \in \mathscr{I}_J$. By hypothesis, $\eta \mapsto \mathsf{Slip}^+_\eta(I)$ is Lipschitz on $[\eta_\star, 1/2]$ with Lipschitz constant L_I.

Fix any $\eta \in [\eta_\star, 1/2]$. Choose j with

$$|\eta - \tau_j| \le \max_{0 \le r < M} (\tau_{r+1} - \tau_r).$$

Then

$$\begin{aligned} \mathsf{Slip}^+_\eta(I) &\le \mathsf{Slip}^+_{\tau_j}(I) + L_I\, |\eta - \tau_j| \\ &< \left(\frac{\pi}{4} - \delta\right) + L_I \max_{0 \le r < M} (\tau_{r+1} - \tau_r) \\ &\le \left(\frac{\pi}{4} - \delta\right) + \delta = \frac{\pi}{4}, \end{aligned}$$

using the mesh-spacing hypothesis $L_I \cdot \max_{0 \le r < M}(\tau_{r+1} - \tau_r) \le \delta$.

Since I and η were arbitrary, the hypotheses of Theorem 2 hold, and hence $Z(\Xi; \Omega_{T,\eta_\star}) = \varnothing$. □

Remark 11 (Tent/Carleson viewpoint on the transducer). *If a zero lies in $\Omega_{T,\eta_\star}$, it lies in the truncated Carleson tent above some unit interval in the two-shift bank (by Lemma 5). The forcing theorem then creates a slip defect on a scan line below the zero on that interval, contradicting the strict $< \pi/4$ slip hypothesis. This is a geometric "tent detection" mechanism rather than a contour winding computation.*

6. Slip-Flux Littlewood Lemmas and Classical Strip Consequences

6.1. A Slip-Flux Formulation of Littlewood's Lemma

We fix a symmetric truncation convention for sums over zeros (as in Definition A8).

Definition 8 (Slip-flux functional between two vertical lines). *Let $\sigma_1 < \sigma_2$ be real numbers, and let $a < b$ be real. Assume $\xi(\sigma + \mathrm{i}t) \neq 0$ for all $(\sigma, t) \in [\sigma_1, \sigma_2] \times [a, b]$. Define the* (signed) slip-flux *across the strip slab by*

$$\mathsf{Flux}_\xi(\sigma_1, \sigma_2; [a,b]) := \int_a^b \Big(\log|\xi(\sigma_2 + \mathrm{i}t)| - \log|\xi(\sigma_1 + \mathrm{i}t)| \Big)\, \mathrm{d}t.$$

Equivalently (since $\partial_\sigma \log|\xi| = \Re(\xi'/\xi)$ wherever $\xi \neq 0$),

$$\mathsf{Flux}_\xi(\sigma_1, \sigma_2; [a,b]) = \int_{\sigma_1}^{\sigma_2} \int_a^b \mathrm{Re}\left(\frac{\xi'}{\xi}(\sigma + \mathrm{i}t) \right) \mathrm{d}t\, \mathrm{d}\sigma.$$

Theorem 4 (Slip-flux Littlewood lemma (Hadamard form)). *Let $\sigma_1 < \sigma_2$ and $a < b$, and assume $\xi(\sigma + \mathrm{i}t) \neq 0$ for all $(\sigma, t) \in [\sigma_1, \sigma_2] \times [a, b]$. Let $B \in \mathbb{R}$ be the Hadamard constant for ξ so that*

$$\frac{\xi'}{\xi}(s) = B + \sum_\rho \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right) \quad \textit{(symmetric summation).}$$

Then one has the identity

$$\mathsf{Flux}_\xi(\sigma_1, \sigma_2; [a,b]) = B(\sigma_2 - \sigma_1)(b - a) + \lim_{R \to \infty} \sum_\rho^{(R)} \int_a^b \left(\log \frac{|(\sigma_2 + \mathrm{i}t) - \rho|}{|(\sigma_1 + \mathrm{i}t) - \rho|} + (\sigma_2 - \sigma_1)\, \mathrm{Re}\, \frac{1}{\rho} \right) \mathrm{d}t, \quad (13)$$

where $\rho = \beta + \mathrm{i}\gamma$ and the sum is truncated symmetrically in $|\gamma| \le R$.

Moreover, for each fixed R one has the exact truncated identity

$$\mathsf{Flux}_\xi(\sigma_1, \sigma_2; [a,b]) = B(\sigma_2 - \sigma_1)(b - a) + \sum_\rho^{(R)} \int_a^b \left(\log \frac{|(\sigma_2 + \mathrm{i}t) - \rho|}{|(\sigma_1 + \mathrm{i}t) - \rho|} + (\sigma_2 - \sigma_1)\, \mathrm{Re}\, \frac{1}{\rho} \right) \mathrm{d}t + \varepsilon_R, \quad (14)$$

where $\varepsilon_R \to 0$ as $R \to \infty$.

Proof. Step 1: Start from the Hadamard log-derivative. Fix $R \geq 1$ and define the truncated meromorphic approximation

$$S_R(s) := B + \sum_{\rho}^{(R)} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

Standard theory of the Hadamard product for ξ (entire of order 1) implies that on any compact set K avoiding zeros,

$$\sup_{s \in K} \left| \frac{\xi'}{\xi}(s) - S_R(s) \right| \xrightarrow[R\to\infty]{} 0. \tag{15}$$

Here we take

$$K := \{\sigma + \mathrm{i}t : \ \sigma \in [\sigma_1, \sigma_2], \ t \in [a,b]\},$$

which is compact and disjoint from the zero set by hypothesis.

Step 2: Integrate in σ (termwise for fixed R). For each fixed $\rho = \beta + \mathrm{i}\gamma$, for fixed t, we have

$$\frac{\partial}{\partial \sigma} \log |(\sigma + \mathrm{i}t) - \rho| = \mathrm{Re}\left(\frac{1}{(\sigma + \mathrm{i}t) - \rho} \right),$$

since $\log |z|$ has gradient equal to the real part of $1/z$ in one complex variable. Therefore,

$$\int_{\sigma_1}^{\sigma_2} \mathrm{Re}\left(\frac{1}{(\sigma + \mathrm{i}t) - \rho} \right) \mathrm{d}\sigma = \log \frac{|(\sigma_2 + \mathrm{i}t) - \rho|}{|(\sigma_1 + \mathrm{i}t) - \rho|}.$$

Also,

$$\int_{\sigma_1}^{\sigma_2} \mathrm{Re}\left(\frac{1}{\rho} \right) \mathrm{d}\sigma = (\sigma_2 - \sigma_1)\, \mathrm{Re}\, \frac{1}{\rho}.$$

Hence, for fixed R,

$$\int_{\sigma_1}^{\sigma_2} \mathrm{Re}(S_R(\sigma + \mathrm{i}t))\, \mathrm{d}\sigma = B(\sigma_2 - \sigma_1) + \sum_{\rho}^{(R)} \left(\log \frac{|(\sigma_2 + \mathrm{i}t) - \rho|}{|(\sigma_1 + \mathrm{i}t) - \rho|} + (\sigma_2 - \sigma_1)\, \mathrm{Re}\, \frac{1}{\rho} \right).$$

Integrate in $t \in [a,b]$ to obtain (14) with

$$\varepsilon_R := \int_{\sigma_1}^{\sigma_2} \int_a^b \left(\mathrm{Re}\left(\frac{\xi'}{\xi}(\sigma + \mathrm{i}t) \right) - \mathrm{Re}(S_R(\sigma + \mathrm{i}t)) \right) \mathrm{d}t\, \mathrm{d}\sigma.$$

Step 3: Control the truncation error. By (15) on K,

$$\sup_{(\sigma,t)\in[\sigma_1,\sigma_2]\times[a,b]} \left| \mathrm{Re}\left(\frac{\xi'}{\xi}(\sigma + \mathrm{i}t) \right) - \mathrm{Re}(S_R(\sigma + \mathrm{i}t)) \right| \leq \sup_{s\in K} \left| \frac{\xi'}{\xi}(s) - S_R(s) \right| \xrightarrow[R\to\infty]{} 0.$$

Therefore

$$|\varepsilon_R| \leq (\sigma_2 - \sigma_1)(b-a) \cdot \sup_{s\in K} \left| \frac{\xi'}{\xi}(s) - S_R(s) \right| \xrightarrow[R\to\infty]{} 0.$$

Taking $R \to \infty$ in (14) yields (13). ☐

Remark 12 (Classical Littlewood lemma viewpoint). *Formula* (13) *is a Hadamard-product (sum-over-zeros) representation for the* difference of vertical mean logs *of ξ between σ_1 and σ_2. It is equivalent in spirit to Littlewood-type lemmas (mean log identities) but expressed directly as a σ-flux of the scan derivative field $\partial_\sigma \log |\xi| = \Re(\xi'/\xi)$, which is precisely the field $-\Im m$ in the Ξ-coordinates.*

6.2. Harmonic-Measure Corollary (Arctan Increments)

Differentiating the slip-flux identity with respect to the horizontal endpoint b recovers the harmonic-measure (arctan) increments that appear in the Poisson kernel analysis.

Corollary 3 (Endpoint derivative gives harmonic-measure increments)**.** *Under the hypotheses of Theorem 4, fix $a < b_0$. Then for almost every $b \in (a, b_0)$ one has*

$$\frac{\partial}{\partial b}\,\mathsf{Flux}_\xi(\sigma_1,\sigma_2;[a,b]) = \log|\xi(\sigma_2+ib)| - \log|\xi(\sigma_1+ib)|.$$

Moreover, for each fixed R and each b such that $\xi(\sigma_j+\mathrm{i}b)\neq 0$,

$$\log|\xi(\sigma_2+\mathrm{i}b)|-\log|\xi(\sigma_1+\mathrm{i}b)| = B(\sigma_2-\sigma_1)+\sum_{\rho}^{(R)}\left(\log\frac{|(\sigma_2+\mathrm{i}b)-\rho|}{|(\sigma_1+\mathrm{i}b)-\rho|}+(\sigma_2-\sigma_1)\,\mathrm{Re}\,\frac{1}{\rho}\right)+o_R(1),$$

and differentiating in σ yields, for almost every b,

$$\mathrm{Re}\left(\frac{\xi'}{\xi}(\sigma+\mathrm{i}b)\right) = B+\sum_{\rho}\left(\frac{\sigma-\beta}{(\sigma-\beta)^2+(b-\gamma)^2}+\mathrm{Re}\,\frac{1}{\rho}\right)\quad \textit{(symmetric summation)},$$

so integrating in t recovers the arctan/harmonic-measure increments of Theorem A2.

Proof. The first identity is the fundamental theorem of calculus for absolutely continuous functions: $b\mapsto\int_a^b(\log|\xi(\sigma_2+it)|-\log|\xi(\sigma_1+it)|)\,\mathrm{d}t$ has a.e. derivative equal to the integrand.

For the second identity, apply Theorem 4 to the interval $[a,b]$ and then differentiate the right-hand side in b at points where all functions are differentiable; the differentiation passes through the finite sum, and the truncation error $o_R(1)$ arises from $\varepsilon_R\to 0$ uniformly on compact b-ranges. The final displayed identity is the real-part form of the Hadamard log-derivative, and the arctan increment identity is Theorem A2. □

6.3. A Classical Strip Consequence with Explicit Constants (Certificate Template)

The next proposition is a concrete, explicit-constant instantiation of the finite-mesh transducer. It is designed to read as a classical statement about zeros in $(T,2T)$ with explicit strip width, conditional on verifying explicit inequalities for m on finitely many scan lines.

Proposition 9 (Explicit-constant strip control from a finite record)**.** *Fix $T>0$ and set $J=[T,2T]$. Fix an explicit strip width parameter*

$$\eta_\star := \frac{1}{10}.$$

Let $\mathscr{I}_J$ be the two-shift unit bank on J. Assume the following three items have been verified (e.g. by validated numerics):

(i) (Holomorphy and derivative bound on the scan slab.) *For every $I\in\mathscr{I}_J$, the function m is holomorphic on a neighborhood of*

$$U(I) := \{t+\mathrm{i}\eta:\ t\in I,\ \eta\in[\eta_\star,\tfrac{1}{2}]\},$$

and satisfies the uniform bound

$$\sup_{z\in U(I)}|m'(z)| \le B \quad \textit{with} \quad B := 10.$$

(ii) (A margin.) *Fix*

$$\delta := \frac{\pi}{100}.$$

(iii) (Mesh and mesh slip inequalities.) *Let the mesh spacing be*

$$h := \frac{\delta}{|I|B} = \frac{\pi/100}{1\cdot 10} = \frac{\pi}{1000},$$

and take any mesh $\eta_\star = \tau_0 < \tau_1 < \cdots < \tau_M = \frac{1}{2}$ *with* $\max_j(\tau_{j+1} - \tau_j) \le h$*. Assume that for every* $I \in \mathscr{I}_J$ *and every mesh height* τ_j *one has*

$$\mathsf{Slip}^+_{\tau_j}(I) \; < \; \frac{\pi}{4} - \delta.$$

Then Ξ *is zero-free in* $\Omega_{T,\eta_\star}$*, hence every zero* $\rho = \beta + \mathrm{i}\gamma$ *of* ξ *with* $T < \gamma < 2T$ *satisfies the explicit strip bound*

$$|\beta - \tfrac{1}{2}| \; < \; \frac{1}{10}.$$

Proof. By (i) and Lemma 6, for each $I \in \mathscr{I}_J$ the map $\eta \mapsto \mathsf{Slip}^+_\eta(I)$ is Lipschitz on $[\eta_\star, \frac{1}{2}]$ with Lipschitz constant $L_I \le |I|B = 10$.

Since $\max_j(\tau_{j+1} - \tau_j) \le h = \delta/(|I|B)$, for any $\eta \in [\eta_\star, \frac{1}{2}]$ one can choose a mesh point τ_j with $|\eta - \tau_j| \le h$, and then

$$\mathsf{Slip}^+_\eta(I) \le \mathsf{Slip}^+_{\tau_j}(I) + L_I|\eta - \tau_j| < \left(\frac{\pi}{4} - \delta\right) + (|I|B) \cdot \frac{\delta}{|I|B} = \frac{\pi}{4}.$$

Thus the hypotheses of the continuum-height transducer hold, so $Z(\Xi; \Omega_{T,\eta_\star}) = \varnothing$. The strip statement for zeros of ξ follows exactly as in Corollary 1. ☐

Remark 13 (How this functions as a "small certified example")**.** *Proposition 9 packages the certificate data into* explicit constants*:* $\eta_\star = 0.1$*, margin* $\delta = \pi/100$*, derivative bound* $B = 10$*, mesh step* $h = \pi/1000$*. What remains (and is well-scoped for validated numerics) is to verify the concrete inequalities in (i) and (iii) for a chosen height window* $[T, 2T]$*. Including at least one actual numerical window (even modest* T*) for which these inequalities are checked would turn this template into a full certified example theorem.*

6.4. A Classical-Number-Theory Corollary (Off-Critical Zero Counting Bound)

We now state a consequence in classical number-theoretic language: a bound on total bankwise slip at one scan height controls how many zeros can lie to the left of a given vertical line within a height window.

Theorem 5 (Off-critical zero counting bound from bankwise slip mass)**.** *Fix* $T > 0$ *and set* $J = [T, 2T]$ *and* $\eta \in (1/4, \frac{1}{2})$*. Let* $\sigma := \frac{1}{2} - \eta$*. Assume* $\Xi(t + \mathrm{i}\eta) \ne 0$ *for all* $t \in [T-1, 2T+1]$*. Let* $\mathscr{I}_J$ *be the two-shift unit bank on* J*.*

Define the left-of-line zero multiset

$$\mathcal{R}(T, \eta) := \{\rho = \beta + \mathrm{i}\gamma : \; \xi(\rho) = 0, \; T < \gamma < 2T, \; \beta \le \sigma\},$$

counting multiplicity. Assume that for each $I \in \mathscr{I}_J$ *the hypothesis (iii) of Theorem A4 holds on the scan line* $\Im z = \eta$ *with the same bound* M *(i.e. the pole-subtracted remainder has* $\sup_{t \in I} |\Im H(t + \mathrm{i}\eta)| \le M$*). Then*

$$\sum_{\rho \in \mathcal{R}(T,\eta)} \mathrm{mult}(\rho) \; \le \; \left\lfloor \frac{2}{\pi} \left(\sum_{I \in \mathscr{I}_J} \mathsf{Slip}^+_\eta(I) + M \sum_{I \in \mathscr{I}_J} |I| \right) \right\rfloor.$$

In particular, if $M = 0$ *(pure pole model at the scan level) then*

$$\sum_{\rho \in \mathcal{R}(T,\eta)} \mathrm{mult}(\rho) \; \le \; \left\lfloor \frac{2}{\pi} \sum_{I \in \mathscr{I}_J} \mathsf{Slip}^+_\eta(I) \right\rfloor.$$

Proof. Fix $I \in \mathscr{I}_J$. Apply Theorem A4 to the shifted tent $\mathcal{T}_\eta(I)$ to obtain

$$\sum_{z_j \in \mathcal{T}_\eta(I)} k_j \leq \left\lfloor \frac{2(\mathsf{Slip}_\eta^+(I) + M|I|)}{\pi} \right\rfloor.$$

Now sum this bound over $I \in \mathscr{I}_J$. Every zero $\rho = \beta + i\gamma$ with $\beta \leq \sigma = \frac{1}{2} - \eta$ and $T < \gamma < 2T$ corresponds to a Ξ-zero at $z = \gamma + i(\frac{1}{2} - \beta)$ with $\Im z \geq \eta$ and $\Re z \in (T, 2T)$. Write $d := \Im z - \eta = (\frac{1}{2} - \beta) - \eta \geq 0$. Since $0 \leq \beta \leq 1$ for zeros of ζ, we have $d \leq \frac{1}{2} - \eta < \frac{1}{4}$ (using $\eta > 1/4$). By the two-shift margin lemma applied at $t = \gamma$, choose $I \in \mathscr{I}_J$ such that $\gamma \in I$ and $\mathrm{dist}(\gamma, \partial I) \geq 1/4 > d$. Then $z \in \mathcal{T}_\eta(I)$. Thus the total multiplicity of $\mathcal{R}(T, \eta)$ is bounded by the sum of multiplicities counted in the tents. Dropping the floor and collecting constants yields the stated inequality. □

Remark 14. *The restriction $\eta > 1/4$ ensures that any point at height $\Im z \in [\eta, 1/2]$ lies within the shifted tent above some* unit *bank interval with two-shift margin. For smaller η, one should use a scaled bank (Definition A7 and Theorem A1) or a multiscale (dyadic) Carleson decomposition.*

Remark 15 (Status)**.** *The hypothesis that a uniform remainder bound M is available is natural in the validated-numerics layer: it amounts to bounding the holomorphic part of m on a scan neighborhood once all poles in the tent are accounted for. The inequality itself is a classical-looking local zero-count statement in $(T, 2T)$ with a left-of-line condition $\beta \leq \frac{1}{2} - \eta$, expressed in terms of the scan observable.*

7. Baseline Zero-Free Certificates via the Argument Principle (Contrast)

This section records the classical argument-principle certificate as a point of contrast. As emphasized in Section 1.6 and Propositions 2 and 3, slip certificates are not a reformulation of the argument principle: they are based on one-sided argument variation and local Poisson/tent coercivity, and they do not require verifying boundary nonvanishing on $\partial R_{T,\eta_\star}$.

Definition 9 (Boundary contour)**.** *Let $\partial R_{T,\eta_\star}$ denote the positively oriented boundary contour of the closed rectangle $R_{T,\eta_\star}$.*

Theorem 6 (Argument principle on $R_{T,\eta_\star}$)**.** *Assume $\Xi(z) \neq 0$ for all $z \in \partial R_{T,\eta_\star}$. Define*

$$N(R_{T,\eta_\star}) := \frac{1}{2\pi i} \int_{\partial R_{T,\eta_\star}} \frac{\Xi'(z)}{\Xi(z)}\, dz = -\frac{1}{2\pi i} \int_{\partial R_{T,\eta_\star}} m(z)\, dz.$$

Then $N(R_{T,\eta_\star}) \in \mathbb{Z}_{\geq 0}$ and

$$N(R_{T,\eta_\star}) = \sum_{z_0 \in Z(\Xi; \Omega_{T,\eta_\star})} \mathrm{mult}(z_0).$$

In particular,

$$Z(\Xi; \Omega_{T,\eta_\star}) = \varnothing \iff N(R_{T,\eta_\star}) = 0.$$

Proof. Since Ξ is entire, Ξ'/Ξ is meromorphic with simple poles precisely at zeros of Ξ, and the residue at a zero of multiplicity k equals k. Under the hypothesis that Ξ has no zeros on the contour, the residue theorem yields the stated identity. □

Remark 16 (Why slip certificates behave differently)**.** *Argument-principle certificates are global and require boundary nonvanishing information; they measure net winding and can be numerically unstable when zeros approach the boundary. Slip certificates instead rely on local Poisson/tent forcing: a zero creates a one-sided variation defect on nearby scan lines which cannot be canceled away. This yields a different (and composable) certificate format for establishing zero-free windows.*

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8. A Boundary-to-Bulk Obstruction Viewpoint: K-Theory and Poincaré–Lelong

This section records a conceptual reformulation of the classical zero-counting invariant for a holomorphic function on a rectangle: it is the boundary-to-bulk connecting class in topological K-theory, and its curvature refinement is the divisor current (Poincaré–Lelong). We then note how slip certificates force vanishing of this obstruction.

8.1. The Boundary K^1 Class of a Nonvanishing Holomorphic Function

Definition 10 (Boundary unitary and boundary K^1 class)**.** *Let $R \subset \mathbb{C}$ be a compact rectangle with positively oriented boundary $\Gamma = \partial R$. Let F be holomorphic on a neighborhood of R and assume $F \neq 0$ on Γ. Define the boundary unitary*

$$u_F : \Gamma \to U(1), \qquad u_F(z) := \frac{F(z)}{|F(z)|}.$$

Its homotopy class defines a K^1 class

$$[F]_\partial \;:=\; [u_F] \;\in\; K^1(\Gamma) \;\cong\; K^1(S^1) \;\cong\; \mathbb{Z}.$$

Theorem 7 (K-theoretic argument principle (connecting homomorphism equals zero count))**.** *Let R, Γ, F be as in Definition 10. Let*

$$\delta :\; K^1(\Gamma) \longrightarrow K^0(R, \Gamma)$$

be the connecting homomorphism in the long exact sequence of the pair (R, Γ). Under the canonical identifications $K^1(\Gamma) \cong \mathbb{Z}$ and $K^0(R, \Gamma) \cong \widetilde{K}^0(S^2) \cong \mathbb{Z}$, one has

$$\delta([F]_\partial) \;=\; \sum_{z_0 \in Z(F; \operatorname{int} R)} \operatorname{mult}(z_0).$$

Equivalently,

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{F'(z)}{F(z)}\,\mathrm{d}z \;=\; \delta([F]_\partial) \;\in\; \mathbb{Z}.$$

Proof sketch. Since $R/\Gamma \simeq S^2$ and $\Gamma \simeq S^1$, the groups $K^1(\Gamma)$ and $K^0(R, \Gamma)$ are both canonically isomorphic to $\mathbb{Z}$, and δ identifies the generator in $K^1(S^1)$ with the Bott class in $\widetilde{K}^0(S^2)$. The integer represented by $[F]_\partial$ is the winding number of u_F, which equals $\frac{1}{2\pi \mathrm{i}} \int_\Gamma F'/F$. By the residue theorem (argument principle), this winding equals the number of zeros of F in $\operatorname{int} R$ counted with multiplicity. □

8.2. Poincaré–Lelong: Curvature of the Obstruction Is the Divisor Current

Theorem 8 (Poincaré–Lelong formula (divisor current))**.** *Let $U \subset \mathbb{C}$ be open and let F be holomorphic on U, not identically zero. Then, in the sense of distributions (currents) on U,*

$$\frac{1}{2\pi}\, dd^c \log|F| \;=\; \sum_{z_0 \in Z(F; U)} \operatorname{mult}(z_0)\, \delta_{z_0},$$

where $d^c := \frac{1}{2\mathrm{i}}(\bar\partial - \partial)$ and δ_{z_0} denotes the Dirac point mass at z_0. In particular, if $R \Subset U$ is a rectangle whose boundary avoids the zeros of F, then

$$\sum_{z_0 \in Z(F; \operatorname{int} R)} \operatorname{mult}(z_0) = \frac{1}{2\pi} \int_R dd^c \log|F| = \frac{1}{2\pi \mathrm{i}} \int_{\partial R} \frac{F'}{F}\,\mathrm{d}z.$$

Remark 17. *The last identity is Stokes' theorem applied to $dd^c \log|F|$, together with the local model $\log|F(z)| = k \log|z - z_0| + O(1)$ near a zero of multiplicity k. This can be viewed as a differential-form/current refinement of Theorem 7.*

8.3. Slip Certificates as Vanishing of a Boundary-to-Bulk Obstruction

Corollary 4 (Slip-certified zero-free rectangles force vanishing of the boundary K^1 class)**.** *Let* $R = R_{T,\eta_\star}$ *be the closed rectangle in the* Ξ*-plane as in Equation* (4)*. Assume a slip certificate (e.g. via Theorem 2 or Theorem 3) establishes that* Ξ *has no zeros in* $\operatorname{int} R$*, and assume moreover* $\Xi \neq 0$ *on* ∂R*. Then the boundary class* $[\Xi]_\partial \in K^1(\partial R)$ *is null:*

$$[\Xi]_\partial = 0 \in K^1(\partial R),$$

equivalently,

$$\delta([\Xi]_\partial) = 0 \in K^0(R, \partial R).$$

Proof. By Theorem 7 applied to $F = \Xi$, one has $\delta([\Xi]_\partial) = N_R(\Xi)$, the number of zeros in $\operatorname{int} R$ counted with multiplicity. If R is zero-free in its interior, then $N_R(\Xi) = 0$, hence $\delta([\Xi]_\partial) = 0$. Since $\delta : K^1(\partial R) \to K^0(R, \partial R)$ is an isomorphism for a rectangle pair, $[\Xi]_\partial = 0$. □

Remark 18 (Conceptual summary)**.** *The argument principle shows that zero counting is a boundary-to-bulk obstruction class in K-theory (and its Poincaré–Lelong curvature refinement is the divisor current). Slip certificates supply a* different *mechanism for forcing the vanishing of this obstruction: they certify a two-dimensional zero-free window by one-dimensional tent/scan inequalities.*

9. Extensions to Completed L-Functions and Automorphic Forms

The slip forcing and transducer results are fundamentally function-theoretic: they use only local factorization near zeros (for forcing) and one-dimensional-to-two-dimensional propagation (for transduction). Consequently, the same framework applies to many other entire functions, including the Ξ-type normalizations of completed L-functions.

9.1. A General Ξ-Normalization for Completed L-Functions

We recall a standard framework (see, e.g., [10,22,23]). Let $L(s)$ be an L-function with analytic conductor $Q > 0$ and archimedean factors specified by parameters $\mu_1, \ldots, \mu_d \in \mathbb{C}$ (degree d). Define

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big), \qquad \Lambda(s) := Q^{s/2}\prod_{j=1}^{d}\Gamma_{\mathbb{R}}(s+\mu_j)\, L(s).$$

In many automorphic settings (e.g. for unitary representations, after standard normalization), $\Lambda(s)$ admits analytic continuation and satisfies a functional equation of the form

$$\Lambda(s) = \varepsilon\,\overline{\Lambda(1-\bar{s})}, \qquad |\varepsilon| = 1. \tag{16}$$

In the sequel we assume Λ is entire, since the forcing and transducer arguments are local and require holomorphy away from zeros.

Definition 11 (Ξ-normalization of a completed L-function)**.** *Assume* $\Lambda(s)$ *is entire and satisfies* (16)*. Define the associated* Ξ*-function by*

$$\Xi_\Lambda(z) := \Lambda\Big(\tfrac{1}{2} + \mathrm{i}z\Big).$$

Then Ξ_Λ *is entire and satisfies the* twisted conjugation symmetry

$$\Xi_\Lambda(\bar{z}) = \varepsilon\,\overline{\Xi_\Lambda(z)} \qquad (z \in \mathbb{C}). \tag{17}$$

In particular, its zero set is invariant under complex conjugation.

Define also

$$m_\Lambda(z) := -\frac{\Xi'_\Lambda(z)}{\Xi_\Lambda(z)} \qquad (\Xi_\Lambda(z) \neq 0).$$

Proof. Let $z \in \mathbb{C}$ and set $s = \frac{1}{2} + \mathrm{i}\bar{z}$. Then $1 - \bar{s} = \frac{1}{2} + \mathrm{i}z$, and (16) gives

$$\Lambda\Big(\tfrac{1}{2} + \mathrm{i}\bar{z}\Big) = \Lambda(s) = \varepsilon\,\overline{\Lambda(1-\bar{s})} = \varepsilon\,\overline{\Lambda(\tfrac{1}{2} + \mathrm{i}z)}.$$

That is $\Xi_\Lambda(\bar{z}) = \varepsilon\,\overline{\Xi_\Lambda(z)}$. □

Remark 19 (Removing the twist)**.** *If one chooses a square root $\varepsilon^{1/2}$ and defines $\widetilde{\Xi}_\Lambda(z) := \varepsilon^{-1/2}\Xi_\Lambda(z)$, then $\widetilde{\Xi}_\Lambda(\bar{z}) = \overline{\widetilde{\Xi}_\Lambda(z)}$. This renormalization does not affect zeros or logarithmic derivatives, and we do not use it below.*

Proposition 10 (Log-derivative dictionary for Ξ_Λ)**.** *Let $z = t + \mathrm{i}\eta$ and set $s = \frac{1}{2} + \mathrm{i}z = (\frac{1}{2} - \eta) + \mathrm{i}t$. Wherever $\Xi_\Lambda(z) \neq 0$,*

$$m_\Lambda(z) = -\mathrm{i}\,\frac{\Lambda'}{\Lambda}(s), \qquad -\operatorname{Im} m_\Lambda(t + \mathrm{i}\eta) = \operatorname{Re}\bigg(\frac{\Lambda'}{\Lambda}(s)\bigg) = \frac{\partial}{\partial\sigma}\log|\Lambda(\sigma + \mathrm{i}t)|.$$

9.2. Slip Transduction for Ξ_Λ

Define the positive-part slip for Ξ_Λ by, for $\eta > 0$ and bounded $I \subset \mathbb{R}$,

$$\mathsf{Slip}^{+}_{\Lambda,\eta}(I) := \int_I \big(-\operatorname{Im} m_\Lambda(t + \mathrm{i}\eta)\big)_+ \,\mathrm{d}t,$$

with $\mathsf{Slip}^{+}_{\Lambda,\eta}(I) = +\infty$ if $\Xi_\Lambda(t + \mathrm{i}\eta) = 0$ for some $t \in I$.

Theorem 9 (Slip forcing and transduction for general Ξ_Λ)**.** *Assume Λ is entire and define Ξ_Λ as in Definition 11. Then:*

(i) (Quantized forcing.) *If Ξ_Λ has a zero at $z_0 = t_0 + \mathrm{i}\eta_0$ of multiplicity $k \geq 1$ with $\eta_0 > 0$, then for all sufficiently small $d > 0$,*

$$\mathsf{Slip}^{+}_{\Lambda,\eta_0 - d}([t_0 - d, t_0 + d]) \geq k\pi/4.$$

(ii) (Stagewise transducer.) *Fix $T > 0$, $\eta_\star \in (0, \frac{1}{2})$, and let $J = [T, 2T]$. If for every $\eta \in [\eta_\star, \frac{1}{2}]$ and every $I \in \mathscr{I}_J$ one has*

$$\mathsf{Slip}^{+}_{\Lambda,\eta}(I) < \frac{\pi}{4},$$

then $Z(\Xi_\Lambda; \Omega_{T,\eta_\star}) = \varnothing$.

Proof. Both parts are proved exactly as Theorems 1 and 2. The proofs use only that Ξ_Λ is entire and that $m_\Lambda = -\Xi'_\Lambda/\Xi_\Lambda$ has a simple pole with residue $-k$ at a zero of multiplicity k. □

9.3. Consequences for Zeros of $\Lambda(s)$ and $L(s)$

A zero $\rho = \beta + \mathrm{i}\gamma$ of $\Lambda(s)$ corresponds to a zero of Ξ_Λ at

$$z = \gamma + \mathrm{i}(\tfrac{1}{2} - \beta).$$

Thus a zero-free region for Ξ_Λ with $\eta \geq \eta_\star$ translates into a one-sided strip constraint for ρ.

Corollary 5 (One-sided strip control for zeros of $\Lambda(s)$ from slip bounds)**.** *In the setting of Theorem 9(ii), every zero $\rho = \beta + \mathrm{i}\gamma$ of $\Lambda(s)$ with $T < \gamma < 2T$ satisfies*

$$\beta > \tfrac{1}{2} - \eta_\star.$$

Remark 20 (Two-sided strip control and self-duality). *For the Riemann ξ-function, the functional equation $\xi(1-s) = \xi(s)$ yields symmetric control $|\beta - \frac{1}{2}| < \eta_\star$. For a general automorphic L-function $L(s,\pi)$, the functional equation typically relates $\Lambda(s,\pi)$ to $\Lambda(1-s,\tilde{\pi})$.*

- *If π is self-dual (so $\tilde{\pi} \simeq \pi$) and the root number is $\varepsilon = \pm 1$, then $\Lambda(s,\pi) = \varepsilon\,\Lambda(1-s,\pi)$ and*

  $$\Xi_\Lambda(-z) = \varepsilon\,\Xi_\Lambda(z),$$

  so zeros are symmetric about $\mathrm{Re}(s) = 1/2$ for the same *function and one obtains two-sided strip control.*
- *If π is not self-dual, one obtains the symmetric strip statement by applying the slip certificate to both $\Lambda(s,\pi)$ and $\Lambda(s,\tilde{\pi})$ (or, in the Dirichlet case, to χ and $\overline{\chi}$).*

10. Complex Disks and Enclosure Arithmetic

We use closed disks in $\mathbb{C}$ as explicit enclosures for validated numerics.

Definition 12 (Closed disks). *A* (closed) disk *is a set*

$$\overline{B}(c,r) := \{w \in \mathbb{C} : \ |w-c| \le r\},$$

with center $c \in \mathbb{C}$ and radius $r \in [0,\infty)$.

Definition 13 (Primitive disk operations). *Let $D_1 = \overline{B}(c_1,r_1)$ and $D_2 = \overline{B}(c_2,r_2)$ be disks.*

(i) Addition and subtraction:

$$D_1 + D_2 := \overline{B}(c_1+c_2,\ r_1+r_2), \qquad -D_1 := \overline{B}(-c_1,r_1), \qquad D_1 - D_2 := D_1 + (-D_2).$$

(ii) Multiplication:

$$D_1 \cdot D_2 := \overline{B}\big(c_1c_2,\ |c_1|r_2 + |c_2|r_1 + r_1r_2\big).$$

(iii) Reciprocal and division: *if $0 \notin D_2$ (equivalently $|c_2| > r_2$), define*

$$\frac{1}{D_2} := \overline{B}\left(\frac{1}{c_2},\ \frac{r_2}{|c_2|\,(|c_2| - r_2)}\right), \qquad D_1/D_2 := D_1 \cdot (1/D_2).$$

(iv) Exponential:

$$\exp(D_1) := \overline{B}\Big(\exp(c_1),\ |\exp(c_1)|\,(e^{r_1} - 1)\Big).$$

Lemma 7 (Soundness of primitive disk operations). *Let D_1, D_2 be disks as in Definition 13.*

(a) *If $w_j \in D_j$, then $w_1 + w_2 \in D_1 + D_2$ and $w_1w_2 \in D_1 \cdot D_2$.*
(b) *If $0 \notin D_2$ and $w_1 \in D_1$, $w_2 \in D_2$, then $w_1/w_2 \in D_1/D_2$.*
(c) *If $w \in D_1$, then $e^w \in \exp(D_1)$.*

Proof. Write $w_j = c_j + \delta_j$ with $|\delta_j| \le r_j$ and estimate directly. □

Remark 21 (Branch control for Log). *When Stirling-type formulas are used via $\mathrm{Log}\,z$, one must ensure the enclosure domain avoids the branch cut. In our applications, $z = s/2$ has large imaginary part and lies well inside a half-plane, so the principal branch is stable.*

11. Euler–Maclaurin Enclosures for ζ, ζ', and ζ''

11.1. Bernoulli Numbers and Periodic Bernoulli Functions

Definition 14 (Bernoulli numbers and Bernoulli polynomials). *The Bernoulli numbers* $(B_n)_{n\ge 0}$ *are defined by*

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty} B_n\frac{x^n}{n!}.$$

The Bernoulli polynomials $B_n(x)$ *are defined by*

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},\qquad \text{so that}\quad B_n(0)=B_n.$$

Define the periodic Bernoulli function by $\widetilde{B}_n(x):=B_n(\{x\})$.

Lemma 8 (Uniform bound for periodic Bernoulli functions). *For each integer* $n\ge 1$ *and each* $x\in\mathbb{R}$ *one has the Fourier series*

$$\widetilde{B}_{2n}(x)=(-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^{2n}},$$

and consequently

$$\sup_{x\in\mathbb{R}}|\widetilde{B}_{2n}(x)|\le\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n)\le\frac{4(2n)!}{(2\pi)^{2n}}.$$

In particular,

$$|B_{2n}|=|\widetilde{B}_{2n}(0)|\le\frac{4(2n)!}{(2\pi)^{2n}}.$$

11.2. Euler–Maclaurin for ζ

Lemma 9 (Derivatives of x^{-s}). *Let* $s\in\mathbb{C}$. *For integers* $r\ge 0$ *and* $x>0$,

$$\frac{d^r}{dx^r}x^{-s}=(-1)^r\,(s)_r\,x^{-s-r}.$$

Theorem 10 (Euler–Maclaurin representation for $\zeta(s)$ with explicit remainder). *Fix integers* $N\ge 1$ *and* $m\ge 1$ *and let* $s\in\mathbb{C}$ *with* $s\ne 1$ *and* $\mathrm{Re}(s)>-2m$. *Then*

$$\zeta(s)=\sum_{n=1}^{N-1}n^{-s}+\frac{N^{1-s}}{s-1}+\frac{1}{2}N^{-s}+\sum_{k=1}^{m}\frac{B_{2k}}{(2k)!}\,(s)_{2k-1}\,N^{-s-(2k-1)}+R_{m,N}(s),\tag{18}$$

where

$$R_{m,N}(s)=-\frac{1}{(2m+2)!}\int_N^{\infty}\widetilde{B}_{2m+2}(x)\,\frac{d^{2m+2}}{dx^{2m+2}}(x^{-s})\,\mathrm{d}x,\tag{19}$$

and

$$|R_{m,N}(s)|\le\frac{4}{(2\pi)^{2m+2}}\cdot\frac{|(s)_{2m+2}|}{\mathrm{Re}(s)+2m+1}\,N^{-\mathrm{Re}(s)-2m-1}.\tag{20}$$

11.3. Differentiated Euler–Maclaurin Bounds

Lemma 10 (Tail integrals with log factors). *Let* $\alpha>1$, $N\ge 2$, *and* $q\in\{0,1,2\}$. *Then*

$$I_q(\alpha,N):=\int_N^{\infty}x^{-\alpha}(\log x)^q\,\mathrm{d}x$$

admits the closed forms

$$I_0(\alpha,N)=\frac{N^{1-\alpha}}{\alpha-1},$$

$$I_1(\alpha, N) = \frac{N^{1-\alpha}}{\alpha - 1}\left(\log N + \frac{1}{\alpha - 1}\right),$$

$$I_2(\alpha, N) = \frac{N^{1-\alpha}}{\alpha - 1}\left((\log N)^2 + \frac{2\log N}{\alpha - 1} + \frac{2}{(\alpha - 1)^2}\right).$$

Lemma 11 (Pochhammer derivative bounds)**.** *Let* $r \geq 1$ *and write* $(s)_r = \prod_{j=0}^{r-1}(s + j)$*. Then*

$$\left|\frac{d}{ds}(s)_r\right| \leq |(s)_r| \sum_{j=0}^{r-1} \frac{1}{|s + j|},$$

$$\left|\frac{d^2}{ds^2}(s)_r\right| \leq |(s)_r| \left[\left(\sum_{j=0}^{r-1} \frac{1}{|s + j|}\right)^2 + \sum_{j=0}^{r-1} \frac{1}{|s + j|^2}\right].$$

Theorem 11 (Euler–Maclaurin remainder bounds for $\zeta'(s)$ and $\zeta''(s)$)**.** *Fix integers* $N \geq 2$ *and* $m \geq 1$ *and let* $s \in \mathbb{C}$ *with* $s \neq 1$ *and* $\mathrm{Re}(s) > -2m$*. Define* $\zeta_{N,m}(s)$ *to be the finite part of* (18) *(omit* $R_{m,N}$*), and define* $\zeta^{(q)}_{N,m}(s)$ *for* $q = 1, 2$ *by termwise differentiation of that finite expression. Then*

$$\zeta^{(q)}(s) = \zeta^{(q)}_{N,m}(s) + R^{(q)}_{m,N}(s) \qquad (q = 1, 2),$$

where

$$|R^{(1)}_{m,N}(s)| \leq \mathcal{R}^{(1)}_{m,N}(s), \qquad |R^{(2)}_{m,N}(s)| \leq \mathcal{R}^{(2)}_{m,N}(s),$$

with explicit majorants

$$\begin{aligned}
\mathcal{R}^{(0)}_{m,N}(s) &:= \frac{4}{(2\pi)^{2m+2}}\, |(s)_{2m+2}|\, I_0(\alpha, N),\\
\mathcal{R}^{(1)}_{m,N}(s) &:= \frac{4}{(2\pi)^{2m+2}}\Big(\big|\tfrac{d}{ds}(s)_{2m+2}\big|\, I_0(\alpha, N) + |(s)_{2m+2}|\, I_1(\alpha, N)\Big),\\
\mathcal{R}^{(2)}_{m,N}(s) &:= \frac{4}{(2\pi)^{2m+2}}\Big(\big|\tfrac{d^2}{ds^2}(s)_{2m+2}\big|\, I_0(\alpha, N) + 2\big|\tfrac{d}{ds}(s)_{2m+2}\big|\, I_1(\alpha, N) + |(s)_{2m+2}|\, I_2(\alpha, N)\Big),
\end{aligned}$$

where $\alpha := \mathrm{Re}(s) + 2m + 2 > 1$ *and* I_q *are as in Lemma* 10*.*

Definition 15 (Point-disks for ζ, ζ', and ζ'')**.** *For* $s \in \mathbb{C}$ *with* $s \neq 1$ *and* $\mathrm{Re}(s) > -2m$ *define disk enclosures*

$$D_{\zeta^{(q)}}(s; N, m) := \overline{B}\big(\zeta^{(q)}_{N,m}(s),\ \mathcal{R}^{(q)}_{m,N}(s)\big), \qquad q \in \{0, 1, 2\}.$$

12. Stirling-Type Enclosures for ψ and ψ'

Definition 16 (Digamma and trigamma)**.** *Define* $\psi(z) := \Gamma'(z)/\Gamma(z)$ *and* $\psi'(z) = \frac{d}{dz}\psi(z)$*.*

Lemma 12 (Half-angle lower bound)**.** *Let* $z \in \mathbb{C} \setminus \{0\}$ *have argument* $\theta \in (-\pi, \pi)$ *and modulus* $r = |z|$*. Then for all* $t \geq 0$*,*

$$|t + z| \ \geq\ (t + r)\cos\left(\frac{|\theta|}{2}\right).$$

Theorem 12 (Stirling expansions for ψ and ψ' with explicit remainder bounds)**.** *Let* $m \geq 1$ *and let* $z \in \mathbb{C} \setminus \{0\}$ *with principal argument* $\theta = \mathrm{Arg}(z) \in (-\pi, \pi)$*. Then*

$$\psi(z) = \mathrm{Log}\, z - \frac{1}{2z} - \sum_{k=1}^{m-1} \frac{B_{2k}}{2k\, z^{2k}} + R^{\psi}_m(z),$$

where

$$R_m^{\psi}(z) = \int_0^{\infty} \frac{\widetilde{B}_{2m}(t)}{(t+z)^{2m+1}}\,\mathrm{d}t, \qquad |R_m^{\psi}(z)| \le \frac{4(2m)!}{2m(2\pi)^{2m}}\,|z|^{-2m}\,\sec^{2m+1}\!\Big(\frac{|\theta|}{2}\Big).$$

Moreover,

$$\psi'(z) = \frac{1}{z} + \frac{1}{2z^2} + \sum_{k=1}^{m-1} \frac{B_{2k}}{z^{2k+1}} + R_m^{\psi'}(z),$$

where

$$R_m^{\psi'}(z) = -(2m+1)\int_0^{\infty} \frac{\widetilde{B}_{2m}(t)}{(t+z)^{2m+2}}\,\mathrm{d}t, \qquad |R_m^{\psi'}(z)| \le \frac{4(2m)!}{(2\pi)^{2m}}\,|z|^{-2m-1}\,\sec^{2m+2}\!\Big(\frac{|\theta|}{2}\Big).$$

13. Enclosures for m and m' via ξ'/ξ

The slip observable depends on m and, for finite-mesh reduction, on moduli of continuity in height. A convenient sufficient condition is a uniform bound on m' on scan slabs; we record identities expressing m and m' in terms of ζ, ζ', ζ'' and ψ, ψ'.

13.1. Logarithmic Derivatives

Wherever $\xi(s) \neq 0$,

$$\frac{\xi'}{\xi}(s) = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\,\psi\Big(\frac{s}{2}\Big) + \frac{\zeta'}{\zeta}(s). \tag{21}$$

Differentiating gives

$$\Big(\frac{\xi'}{\xi}\Big)'(s) = -\frac{1}{s^2} - \frac{1}{(s-1)^2} + \frac{1}{4}\,\psi'\Big(\frac{s}{2}\Big) + \Big(\frac{\zeta'}{\zeta}\Big)'(s), \tag{22}$$

and

$$\Big(\frac{\zeta'}{\zeta}\Big)'(s) = \frac{\zeta''(s)}{\zeta(s)} - \Big(\frac{\zeta'(s)}{\zeta(s)}\Big)^2, \tag{23}$$

wherever $\zeta(s) \neq 0$.

13.2. From s to z

Recall $s = \frac{1}{2} + \mathrm{i}z$ and $m(z) = -\Xi'(z)/\Xi(z)$. By Proposition 4,

$$m(z) = -\mathrm{i}\,\frac{\xi'}{\xi}\Big(\tfrac{1}{2} + \mathrm{i}z\Big). \tag{24}$$

Moreover, since $\frac{d}{dz}(\frac{1}{2} + \mathrm{i}z) = \mathrm{i}$,

$$m'(z) = \Big(\frac{\xi'}{\xi}\Big)'\Big(\tfrac{1}{2} + \mathrm{i}z\Big). \tag{25}$$

Remark 22 (Validated numerics workflow). *Equations (21)–(25) reduce pointwise enclosures for m and m' to enclosures for ζ, ζ', ζ'' at s and for ψ, ψ' at $s/2$. These in turn are enclosed using the Euler–Maclaurin and Stirling remainder bounds of Sections 11 and 12, together with disk arithmetic (Section 10).*

Appendix A Certificate Format Completeness on Zero-Free Closures

Appendix A.1 Strict Bounds Imply Slack (Compactness)

Lemma A1 (Uniform slack from strict slip inequalities). *Fix a finite family of bounded intervals $\mathscr{I}$ and a compact scan-height interval $\mathcal{H} = [\eta_\star, \frac{1}{2}] \subset (0, \frac{1}{2}]$. Assume m is holomorphic on a neighborhood of*

$$U := \{t + \mathrm{i}\eta : \ t \in \bigcup_{I \in \mathscr{I}} I,\ \eta \in \mathcal{H}\}.$$

Assume further that

$$\mathsf{Slip}^{+}_{\eta}(I) < \frac{\pi}{4} \qquad \textit{for all } \eta \in \mathcal{H} \textit{ and all } I \in \mathscr{I}.$$

Then there exists $\varepsilon > 0$ *such that*

$$\mathsf{Slip}^{+}_{\eta}(I) \le \frac{\pi}{4} - \varepsilon \qquad \textit{for all } \eta \in \mathcal{H} \textit{ and all } I \in \mathscr{I}.$$

Proof. Since m is holomorphic on a neighborhood of the compact set U, the derivative m' is continuous on U and hence bounded:

$$B := \sup_{z \in U} |m'(z)| < \infty.$$

By Lemma 6, for each fixed $I \in \mathscr{I}$ the map $\eta \mapsto \mathsf{Slip}^{+}_{\eta}(I)$ is Lipschitz (hence continuous) on $\mathcal{H}$. Since $\mathcal{H}$ is compact, each $\mathsf{Slip}^{+}_{\eta}(I)$ attains a maximum on $\mathcal{H}$:

$$M_I := \max_{\eta \in \mathcal{H}} \mathsf{Slip}^{+}_{\eta}(I).$$

The strict inequality hypothesis implies $M_I < \pi/4$. Since $\mathscr{I}$ is finite, the finite maximum

$$M := \max_{I \in \mathscr{I}} M_I$$

satisfies $M < \pi/4$. Take $\varepsilon := \pi/4 - M > 0$. ☐

Appendix A.2 Slack Implies Existence of a Finite Mesh

Proposition A1 (Finite mesh exists under uniform slack and height-Lipschitz constants). *Fix a finite family of bounded intervals* $\mathscr{I}$ *and* $\mathcal{H} = [\eta_\star, \frac{1}{2}]$. *Assume that for each* $I \in \mathscr{I}$ *there exists* $L_I < \infty$ *such that* $\eta \mapsto \mathsf{Slip}^{+}_{\eta}(I)$ *is Lipschitz on* $\mathcal{H}$ *with constant* L_I. *Assume there exists* $\varepsilon > 0$ *such that*

$$\mathsf{Slip}^{+}_{\eta}(I) \le \frac{\pi}{4} - \varepsilon \qquad \textit{for all } \eta \in \mathcal{H} \textit{ and all } I \in \mathscr{I}.$$

Then there exist $\delta \in (0, \varepsilon)$ *and a finite mesh*

$$\eta_\star = \tau_0 < \tau_1 < \cdots < \tau_M = \tfrac{1}{2}$$

such that for every $I \in \mathscr{I}$:

(i) $L_I \cdot \max_j(\tau_{j+1} - \tau_j) \le \delta$;
(ii) $\mathsf{Slip}^{+}_{\tau_j}(I) \le \frac{\pi}{4} - \varepsilon < \frac{\pi}{4} - \delta$ *for all mesh heights.*

Consequently, the finite-mesh hypotheses of Theorem 3 can be met (on $\mathscr{I}$ *and* $\mathcal{H}$*) with this choice of* δ *and mesh.*

Proof. Choose $\delta := \varepsilon/2$. Since $\mathscr{I}$ is finite, $L_{\max} := \max_{I \in \mathscr{I}} L_I < \infty$. If $L_{\max} = 0$, take the trivial mesh $\tau_0 = \eta_\star$, $\tau_1 = \frac{1}{2}$. Otherwise choose a uniform mesh on $\mathcal{H}$ with maximal spacing

$$\max_j (\tau_{j+1} - \tau_j) \le \delta / L_{\max}.$$

Then $L_I \cdot \max_j(\tau_{j+1} - \tau_j) \le \delta$ for all I and the slack gives the mesh inequalities. ☐

Remark A1 (A sufficient condition for height-Lipschitz constants). *A convenient sufficient condition for the Lipschitz property in Proposition A1 is the existence of bounds* $B_I < \infty$ *such that* m *is holomorphic on a neighborhood of* $U_I := \{t + \mathrm{i}\eta : t \in I, \eta \in \mathcal{H}\}$ *and* $\sup_{z \in U_I} |m'(z)| \le B_I$. *Then Lemma 6 yields Lipschitz constants* $L_I = |I| B_I$.

Appendix B Uniform Disk Enclosures on Neighborhoods (Module for $\sup|m'|$)

Definition A1 (Real-part bounds on a disk)**.** *For $D = \overline{B}(c, r)$ define*

$$\sigma_-(D) := \inf_{w\in D} \mathrm{Re}(w) = \mathrm{Re}(c) - r, \qquad \sigma_+(D) := \sup_{w\in D} \mathrm{Re}(w) = \mathrm{Re}(c) + r.$$

Definition A2 (Uniform remainder radii on disks for $\zeta^{(q)}$)**.** *Let $D_s = \overline{B}(c_s, r_s)$ and assume $1 \notin D_s$ and $\sigma_-(D_s) > -2m$. Set $\sigma_- := \sigma_-(D_s)$, $\alpha_- := \sigma_- + 2m + 2 > 1$, and $r_0 := 2m + 2$.*

Define the Pochhammer disk (computed using disk arithmetic) by

$$D_{(s)_{r_0}} := \prod_{j=0}^{r_0-1} (D_s + j).$$

Writing $D_{(s)_{r_0}} = \overline{B}(c_P, r_P)$, set $P_{\max} := |c_P| + r_P$.

For $j = 0, 1, \ldots, r_0 - 1$, set $d_j := |c_s + j| - r_s$ and assume $d_j > 0$ for all j. Define

$$S_1 := \sum_{j=0}^{r_0-1} \frac{1}{d_j}, \qquad S_2 := \sum_{j=0}^{r_0-1} \frac{1}{d_j^2},$$

and $P'_{\max} := P_{\max} S_1$, $P''_{\max} := P_{\max}(S_1^2 + S_2)$.

Define uniform remainder radii:

$$\begin{aligned}
\overline{\mathcal{R}}^{(0)}(D_s; N, m) &:= \frac{4}{(2\pi)^{2m+2}} P_{\max}\, I_0(\alpha_-, N), \\
\overline{\mathcal{R}}^{(1)}(D_s; N, m) &:= \frac{4}{(2\pi)^{2m+2}} \Big(P'_{\max}\, I_0(\alpha_-, N) + P_{\max}\, I_1(\alpha_-, N) \Big), \\
\overline{\mathcal{R}}^{(2)}(D_s; N, m) &:= \frac{4}{(2\pi)^{2m+2}} \Big(P''_{\max}\, I_0(\alpha_-, N) + 2P'_{\max}\, I_1(\alpha_-, N) + P_{\max}\, I_2(\alpha_-, N) \Big).
\end{aligned}$$

Appendix C Relational Propagation in Scan Height (Optional)

Fix $\eta_\star \in (0, \frac{1}{2})$ and set $\mathcal{H} := [\eta_\star, \frac{1}{2}]$.

Definition A3 (Good-height sets with margin)**.** *Let $I \subset \mathbb{R}$ be bounded and let $\Delta \geq 0$. Define*

$$G_I(\Delta) := \Big\{ \eta \in \mathcal{H} : \ \mathsf{Slip}_\eta^+(I) < \frac{\pi}{4} - \Delta \Big\}.$$

Definition A4 (Height relations and transfer)**.** *A* height relation *on $\mathcal{H}$ is a subset $\mathcal{R} \subseteq \mathcal{H} \times \mathcal{H}$. For $A \subseteq \mathcal{H}$, define its* transfer *(relational image) under $\mathcal{R}$ by*

$$\mathsf{T}_{\mathcal{R}}(A) := \{ \eta \in \mathcal{H} : \ \exists \tau \in A \text{ such that } (\tau, \eta) \in \mathcal{R} \}.$$

Definition A5 (Composition of height relations)**.** *Given height relations $\mathcal{R}, \mathcal{R}' \subseteq \mathcal{H} \times \mathcal{H}$, define their composition $\mathcal{R} \circ \mathcal{R}'$ by*

$$(\tau, \eta) \in (\mathcal{R} \circ \mathcal{R}') \iff \exists \nu \in \mathcal{H} \text{ such that } (\tau, \nu) \in \mathcal{R} \text{ and } (\nu, \eta) \in \mathcal{R}'.$$

Lemma A2 (Transfer respects relation composition)**.** *For all $A \subseteq \mathcal{H}$ one has*

$$\mathsf{T}_{\mathcal{R}\circ\mathcal{R}'}(A) = \mathsf{T}_{\mathcal{R}'}(\mathsf{T}_{\mathcal{R}}(A)).$$

Proof. Unwind the definitions. □

Definition A6 (Local hop relation on a slab)**.** *Let* $H = [\alpha, \beta] \subseteq \mathcal{H}$ *and let* $h \geq 0$*. Define the hop relation*

$$\mathcal{R}(H; h) := \{(\tau, \eta) \in \mathcal{H} \times \mathcal{H} : \tau, \eta \in H, |\eta - \tau| \leq h\}.$$

Lemma A3 (One-hop propagation with controlled margin loss)**.** *Let* $I \subset \mathbb{R}$ *be bounded and let* $H = [\alpha, \beta] \subseteq \mathcal{H}$*. Assume* m *is holomorphic on a neighborhood of*

$$U(I; H) := \{t + i\eta : t \in I, \eta \in H\}, \qquad B := \sup_{z \in U(I;H)} |m'(z)| < \infty.$$

Fix $\varepsilon \geq 0$ *and set*

$$h := \begin{cases} \varepsilon/(|I|B), & B > 0, \\ +\infty, & B = 0. \end{cases}$$

Then for every $\Delta \geq 0$*,*

$$\mathsf{T}_{\mathcal{R}(H;h)}\big(G_I(\Delta + \varepsilon)\big) \subseteq G_I(\Delta).$$

Proof. Apply Lemma 6 and estimate margin loss. ☐

Appendix D Scaled Horizontal Banks and Cauchy Bounds for m' (Optional)

Appendix D.1 Scaled Two-Shift Interval Banks

Definition A7 (Two-shift ℓ-bank)**.** *Let* $J \subset \mathbb{R}$ *be bounded and let* $\ell > 0$*. Define the* two-shift ℓ-bank *by*

$$\mathscr{I}_J^{(\ell)} := \Big\{ I = [m\ell + u, m\ell + u + \ell] : m \in \mathbb{Z}, u \in \{0, \ell/2\}, I \cap J \neq \varnothing \Big\}.$$

Lemma A4 (Scaled two-shift margin lemma)**.** *Let* $J \subset \mathbb{R}$ *and* $\ell > 0$*. For every* $t \in J$ *there exists* $I \in \mathscr{I}_J^{(\ell)}$ *such that* $t \in I$ *and*

$$\operatorname{dist}(t, \partial I) \geq \frac{\ell}{4}.$$

Proof. Rescale the proof of Lemma 5. ☐

Theorem A1 (Scaled stagewise slip transducer)**.** *Fix* $T > 0$*,* $\eta_\star \in (0, \frac{1}{2})$*, and let* $J = [T, 2T]$*. Fix* $\ell > 0$*. Assume that for every* $\eta \in [\eta_\star, \frac{1}{2}]$ *and every* $I \in \mathscr{I}_J^{(\ell)}$*,*

$$\mathsf{Slip}_\eta^+(I) < \frac{\pi}{4}.$$

Then

$$Z(\Xi; \Omega_{T,\eta_\star}) = \varnothing.$$

Proof. Identical to the proof of Theorem 2, using Lemma A4. ☐

Appendix D.2 Cauchy Bounds for m'

Lemma A5 (Cauchy derivative estimate)**.** *Let* f *be holomorphic on a neighborhood of the closed disk* $\overline{B}(z_0, R)$*. Then for every* $0 < r < R$ *and every* z *with* $|z - z_0| \leq r$*,*

$$|f'(z)| \leq \frac{1}{R - r} \sup_{w \in \overline{B}(z_0, R)} |f(w)|.$$

Proof. Cauchy integral formula for f' on $\overline{B}(z, R - r)$. ☐

Proposition A2 (Bounding $\sup|m'|$ from bounds on $\sup|m|$)**.** *Let $U \subset \mathbb{C}$ be a compact set on which m is holomorphic on a neighborhood. Assume we have a finite family of disks $\overline{B}(z_j, r_j)$ covering U and radii $R_j > r_j$ such that $\overline{B}(z_j, R_j)$ lies in the holomorphy neighborhood of m. Assume further that for each j we have a bound*

$$M_j \;\geq \sup_{w \in \overline{B}(z_j, R_j)} |m(w)|.$$

Then for every $z \in U$,

$$|m'(z)| \;\leq\; \max_j \frac{M_j}{R_j - r_j}.$$

In particular,

$$\sup_{z \in U} |m'(z)| \;\leq\; \max_j \frac{M_j}{R_j - r_j}.$$

Proof. Apply Lemma A5 on a disk containing z and take the maximum. ☐

Appendix E Detailed Poisson/Tent Identities and Divisor-Current Proofs

Appendix E.1 Integrated Hadamard–Poisson Identity (Harmonic-Measure Form)

We use a symmetric truncation convention for sums over zeros.

Definition A8 (Symmetric truncation over zeros)**.** *Let $\{\rho\}$ denote the multiset of zeros of ξ (with multiplicity). For $R \geq 1$ define the truncated multiset*

$$\mathcal{Z}(R) := \{\rho = \beta + \mathrm{i}\gamma : \; |\gamma| \leq R\},$$

counting multiplicities. For an expression $A(\rho)$, write

$$\sum_{\rho}^{(R)} A(\rho) := \sum_{\rho \in \mathcal{Z}(R)} A(\rho).$$

A statement of the form $\sum_\rho A(\rho) = L$ is understood in the sense that $\lim_{R\to\infty} \sum_\rho^{(R)} A(\rho) = L$ and the limit exists.

Theorem A2 (Integrated Hadamard–Poisson identity)**.** *Let $\sigma \in \mathbb{R}$ and let $a < b$ be real numbers. Assume $\xi(\sigma + \mathrm{i}t) \neq 0$ for all $t \in [a, b]$. Let $B \in \mathbb{R}$ be the Hadamard constant for ξ so that*

$$\frac{\xi'}{\xi}(s) = B + \sum_\rho \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right) \quad \textit{(symmetric summation).} \tag{A1}$$

Then

$$\int_a^b \mathrm{Re}\!\left(\frac{\xi'}{\xi}(\sigma + \mathrm{i}t) \right) \mathrm{d}t = B(b-a) + \lim_{R\to\infty} \sum_{\rho}^{(R)} \left(\arctan\frac{b-\gamma}{\sigma-\beta} - \arctan\frac{a-\gamma}{\sigma-\beta} + (b-a)\,\mathrm{Re}\,\frac{1}{\rho} \right), \tag{A2}$$

where $\rho = \beta + \mathrm{i}\gamma$. Moreover, for each fixed R the truncated identity holds with equality:

$$\int_a^b \mathrm{Re}\!\left(\frac{\xi'}{\xi}(\sigma + \mathrm{i}t) \right) \mathrm{d}t = B(b-a) + \sum_{\rho}^{(R)} \int_a^b \mathrm{Re}\!\left(\frac{1}{(\sigma+\mathrm{i}t) - \rho} + \frac{1}{\rho} \right) \mathrm{d}t + \epsilon_R, \tag{A3}$$

where $\epsilon_R \to 0$ as $R \to \infty$.

Proof. Step 1: Termwise integration for a fixed truncation. Fix $R \geq 1$. Since ξ is entire of order 1, the canonical product representation implies that the truncated sum

$$S_R(s) := B + \sum_{\rho}^{(R)} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

is a meromorphic function with poles only at zeros $\rho \in \mathcal{Z}(R)$, and on compact subsets of $\mathbb{C}$ disjoint from those zeros it is holomorphic and bounded.

By the hypothesis $\xi(\sigma + \mathrm{i}t) \neq 0$ on $[a, b]$, the line segment

$$K := \{\sigma + \mathrm{i}t : \ t \in [a, b]\}$$

is a compact set disjoint from the zero set of ξ and hence disjoint from all but finitely many zeros. In particular, for all sufficiently large R the truncation includes all zeros with $|\Im\rho| \leq R$, but K still does not meet any zero. Thus, for each fixed R, the integrand

$$t \mapsto \mathrm{Re}\left(\frac{1}{(\sigma + \mathrm{i}t) - \rho} + \frac{1}{\rho} \right)$$

is continuous on $[a, b]$ and integrable.

Compute for a single $\rho = \beta + \mathrm{i}\gamma$:

$$\mathrm{Re}\left(\frac{1}{(\sigma + \mathrm{i}t) - \rho} \right) = \mathrm{Re}\left(\frac{1}{(\sigma - \beta) + \mathrm{i}(t - \gamma)} \right) = \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2}.$$

Hence

$$\int_a^b \mathrm{Re}\left(\frac{1}{(\sigma + \mathrm{i}t) - \rho} \right) \mathrm{d}t = \int_a^b \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} \, \mathrm{d}t = \arctan \frac{b - \gamma}{\sigma - \beta} - \arctan \frac{a - \gamma}{\sigma - \beta}.$$

Also

$$\int_a^b \mathrm{Re}\left(\frac{1}{\rho} \right) \mathrm{d}t = (b - a) \, \mathrm{Re} \, \frac{1}{\rho}.$$

Summing these equalities over $\rho \in \mathcal{Z}(R)$ yields the explicit expression for the truncated integral term.

Step 2: Passage to the limit. The identity (A1) (in symmetric summation) is standard from the Hadamard product for ξ (e.g. [17, Ch. II] or [8, Ch. 2]): for each compact set K avoiding zeros, one has

$$\lim_{R \to \infty} \sup_{s \in K} \left| \frac{\xi'}{\xi}(s) - S_R(s) \right| = 0.$$

Apply this with $K = \{\sigma + \mathrm{i}t : \ t \in [a, b]\}$ to obtain

$$\sup_{t \in [a,b]} \left| \frac{\xi'}{\xi}(\sigma + \mathrm{i}t) - S_R(\sigma + \mathrm{i}t) \right| \to 0.$$

Taking real parts and integrating gives an error $\epsilon_R \to 0$, yielding (A3), and then the limit formula (A2) follows by inserting the computed integrals. ☐

Remark A2 (Harmonic-measure interpretation). *For fixed $\sigma \neq \beta$, the increment*

$$\arctan \frac{b - \gamma}{\sigma - \beta} - \arctan \frac{a - \gamma}{\sigma - \beta}$$

is (up to normalization) the harmonic measure of the interval $[a, b]$ as seen from the point $\sigma + \mathrm{i}\gamma$ in the upper (or lower) half-plane depending on the sign of $\sigma - \beta$.

Appendix E.2 Tent Energies as a Carleson Obstruction (Rigorous Blow-Up)

Definition A9 (Shifted Carleson tent). *Let $I \subset \mathbb{R}$ be a bounded interval with center c_I and length $|I|$, and let $\eta_0 \geq 0$. Define*

$$\mathcal{T}_{\eta_0}(I) := \left\{ t + \mathrm{i}\eta : \ \eta_0 < \eta < \eta_0 + \frac{|I|}{2}, \ |t - c_I| < \frac{|I|}{2} - (\eta - \eta_0) \right\}.$$

Definition A10 (Tent energy). *Let $I \subset \mathbb{R}$ be bounded and $\eta_0 \geq 0$. Define*

$$\mathcal{E}_{\eta_0}(I) := \iint_{\mathcal{T}_{\eta_0}(I)} (-\operatorname{Im} m(t + \mathrm{i}\eta))_+ \, \frac{\mathrm{d}t \, \mathrm{d}\eta}{\eta},$$

with the convention $\mathcal{E}_{\eta_0}(I) = +\infty$ if Ξ vanishes anywhere in $\mathcal{T}_{\eta_0}(I)$.

Theorem A3 (Tent-energy blow-up near a zero). *Assume Ξ has a zero at $z_0 = t_0 + \mathrm{i}\eta_0$ of multiplicity $k \geq 1$ with $\eta_0 > 0$. Let I be a bounded interval such that $z_0 \in \mathcal{T}_0(I)$. Then $\mathcal{E}_0(I) = +\infty$. More quantitatively: there exist constants $C_0 > 0$ and $r_0 > 0$ such that for all $\varepsilon \in (0, r_0)$,*

$$\iint_{\mathcal{T}_0(I) \setminus \overline{B}(z_0, \varepsilon)} (-\operatorname{Im} m(z))_+ \, \frac{\mathrm{d}A(z)}{\operatorname{Im} z} \geq k\, C_0 \log\!\left(\frac{1}{\varepsilon}\right) - C_0.$$

Proof. Step 1: Local decomposition of m near the zero. Since z_0 is a zero of multiplicity k, there exists $r_1 > 0$ and a holomorphic nonvanishing function G on $\overline{B}(z_0, r_1)$ such that $\Xi(z) = (z - z_0)^k G(z)$ on this disk. Thus, on $\overline{B}(z_0, r_1) \setminus \{z_0\}$,

$$m(z) = -\frac{\Xi'(z)}{\Xi(z)} = -\frac{k}{z - z_0} - \frac{G'(z)}{G(z)}.$$

Set

$$h(z) := -\frac{G'(z)}{G(z)}.$$

Then h is holomorphic on $\overline{B}(z_0, r_1)$ and hence bounded there:

$$M_1 := \sup_{z \in \overline{B}(z_0, r_1)} |h(z)| < \infty.$$

Step 2: Geometric sector inside the tent. Because $z_0 \in \mathcal{T}_0(I)$ and $\eta_0 > 0$, there exists $r_2 \in (0, r_1)$ such that the truncated cone sector

$$S := \left\{ z = z_0 + \rho e^{\mathrm{i}\theta} : \ 0 < \rho < r_2, \ \theta \in \left[\tfrac{5\pi}{4}, \tfrac{7\pi}{4}\right] \right\}$$

lies inside $\mathcal{T}_0(I)$, and also satisfies $\Im z \geq \eta_0/2$ on S (choose $r_2 \leq \eta_0/2$). In particular, on S we have $\Im z \geq \eta_0/2$ (so $\Im z > 0$). Also, since $\theta \in [5\pi/4, 7\pi/4]$ implies $\sin\theta \leq 0$, we have $\Im z = \eta_0 + \rho \sin\theta \leq \eta_0$ on S. Hence

$$\frac{1}{\Im z} \geq \frac{1}{\eta_0}.$$

Step 3: Lower bound for the pole contribution on the sector. Write $z - z_0 = \rho e^{\mathrm{i}\theta}$ with $\theta \in [5\pi/4, 7\pi/4]$. Then

$$-\operatorname{Im}\!\left(-\frac{k}{z - z_0}\right) = \operatorname{Im}\!\left(\frac{k}{\rho e^{\mathrm{i}\theta}}\right) = \operatorname{Im}\!\left(\frac{k}{\rho} e^{-\mathrm{i}\theta}\right) = \frac{k}{\rho}\sin(-\theta) = \frac{k}{\rho}(-\sin\theta).$$

On $\theta \in [5\pi/4, 7\pi/4]$, one has $-\sin\theta \geq \frac{\sqrt{2}}{2}$, hence

$$-\operatorname{Im}\left(-\frac{k}{z-z_0}\right) \;\geq\; \frac{k}{\rho}\cdot\frac{\sqrt{2}}{2}. \tag{A4}$$

Step 4: Absorb the holomorphic remainder and integrate. On S we have $|\operatorname{Im} h(z)| \leq |h(z)| \leq M_1$. Therefore

$$-\operatorname{Im} m(z) = -\operatorname{Im}\left(-\frac{k}{z-z_0} + h(z)\right) = -\operatorname{Im}\left(-\frac{k}{z-z_0}\right) - \operatorname{Im} h(z) \geq \frac{k\sqrt{2}}{2\rho} - M_1.$$

Hence, for ρ small enough (say $\rho \leq r_3 := \min\{r_2, k\sqrt{2}/(4M_1)\}$ when $M_1 > 0$, and $r_3 = r_2$ if $M_1 = 0$), we have

$$-\operatorname{Im} m(z) \;\geq\; \frac{k\sqrt{2}}{4\rho} \;>\; 0,$$

so the positive part is redundant:

$$(-\operatorname{Im} m(z))_+ = -\operatorname{Im} m(z) \;\geq\; \frac{k\sqrt{2}}{4\rho} \qquad (z \in S,\ 0 < \rho \leq r_3).$$

Now integrate over $S \setminus \overline{B}(z_0, \varepsilon)$ in polar coordinates about z_0:

$$\iint_{S\setminus\overline{B}(z_0,\varepsilon)} (-\operatorname{Im} m(z))_+ \,\frac{\mathrm{d}A(z)}{\Im z} \;\geq\; \int_{\theta=5\pi/4}^{7\pi/4}\int_{\rho=\varepsilon}^{r_3} \left(\frac{k\sqrt{2}}{4\rho}\right)\cdot\left(\frac{1}{\eta_0}\right)\cdot\rho\,\mathrm{d}\rho\,\mathrm{d}\theta.$$

The ρ cancels, giving

$$\geq \frac{k\sqrt{2}}{4\eta_0}\cdot\left(\int_{5\pi/4}^{7\pi/4}\mathrm{d}\theta\right)\cdot\left(\int_{\varepsilon}^{r_3}\frac{\mathrm{d}\rho}{\rho}\right) = \frac{k\sqrt{2}}{4\eta_0}\cdot\frac{\pi}{2}\cdot\log\left(\frac{r_3}{\varepsilon}\right).$$

This is $kC_0\log(1/\varepsilon) - C_0$ with $C_0 := \frac{\pi\sqrt{2}}{8\eta_0}$ (absorbing $\log r_3$ into the additive constant). Since $S \subset \mathcal{T}_0(I)$, the same lower bound holds with S replaced by $\mathcal{T}_0(I)$, proving the claim. In particular the integral diverges as $\varepsilon \downarrow 0$, hence $\mathcal{E}_0(I) = +\infty$. ☐

Appendix E.3 Local Counting from Slip Mass (Rigorous, with Explicit Hypotheses)

Theorem A4 (Local counting from scan-line slip with pole subtraction control). *Fix $\eta > 0$ and a bounded interval $I \subset \mathbb{R}$. Assume:*

(i) $\Xi(t + \mathrm{i}\eta) \neq 0$ *for all* $t \in I$ *(so* $\mathsf{Slip}^+_\eta(I) < \infty$*);*

(ii) *there are finitely many zeros of* Ξ *in the shifted tent* $\mathcal{T}_\eta(I)$*, say*

$$z_j = t_j + \mathrm{i}\eta_j \in \mathcal{T}_\eta(I), \qquad \operatorname{mult}(z_j) = k_j, \qquad 1 \leq j \leq N;$$

(iii) *the function*

$$H(z) := m(z) + \sum_{j=1}^{N}\frac{k_j}{z - z_j}$$

extends holomorphically to a neighborhood of the closed scan segment $\{t + \mathrm{i}\eta :\ t \in I\}$ *and satisfies*

$$\sup_{t\in I}|\operatorname{Im} H(t + \mathrm{i}\eta)| \leq M.$$

Then the scan-line slip satisfies the lower bound

$$\mathsf{Slip}_\eta^+(I) \;\geq\; \frac{\pi}{2}\sum_{j=1}^{N} k_j \;-\; M|I|. \tag{A5}$$

Consequently, if $\mathsf{Slip}_\eta^+(I) \leq S$*, then*

$$\sum_{j=1}^{N} k_j \;\leq\; \left\lfloor \frac{2(S+M|I|)}{\pi} \right\rfloor.$$

Proof. Since $x \mapsto (x)_+$ satisfies $(x)_+ \geq x$ for all real x,

$$\mathsf{Slip}_\eta^+(I) = \int_I (-\operatorname{Im} m(t+\mathrm{i}\eta))_+ \,\mathrm{d}t \;\geq\; \int_I (-\operatorname{Im} m(t+\mathrm{i}\eta))\,\mathrm{d}t.$$

Using the pole subtraction definition of H on the scan segment (where all terms are holomorphic),

$$m(t+\mathrm{i}\eta) = -\sum_{j=1}^{N} \frac{k_j}{(t+\mathrm{i}\eta)-z_j} + H(t+\mathrm{i}\eta).$$

Therefore

$$\int_I (-\operatorname{Im} m(t+\mathrm{i}\eta))\,\mathrm{d}t = \sum_{j=1}^{N}\int_I -\operatorname{Im}\left(-\frac{k_j}{(t+\mathrm{i}\eta)-z_j}\right)\mathrm{d}t \;-\; \int_I \operatorname{Im} H(t+\mathrm{i}\eta)\;\mathrm{d}t.$$

The remainder term is bounded below by $-M|I|$.

Fix j and write $d_j := \eta_j - \eta > 0$ and $u := t - t_j$. Then

$$(t+\mathrm{i}\eta) - z_j = (t-t_j) + \mathrm{i}(\eta-\eta_j) = u - \mathrm{i}d_j,$$

and

$$-\operatorname{Im}\left(-\frac{k_j}{u-\mathrm{i}d_j}\right) = \frac{k_j d_j}{u^2+d_j^2}.$$

Since $z_j \in \mathcal{T}_\eta(I)$, by the tent geometry one has $\operatorname{dist}(t_j, \partial I) > d_j$, hence $[t_j-d_j, t_j+d_j] \subset I$. Therefore

$$\begin{aligned}\int_I \frac{k_j d_j}{(t-t_j)^2+d_j^2}\,\mathrm{d}t &\geq \int_{t_j-d_j}^{t_j+d_j} \frac{k_j d_j}{(t-t_j)^2+d_j^2}\,\mathrm{d}t \\ &= k_j \int_{-d_j}^{d_j} \frac{d_j}{u^2+d_j^2}\,\mathrm{d}u = k_j\cdot\frac{\pi}{2}.\end{aligned}$$

Summing over j yields

$$\int_I (-\operatorname{Im} m(t+\mathrm{i}\eta))\,\mathrm{d}t \;\geq\; \frac{\pi}{2}\sum_{j=1}^{N} k_j - M|I|.$$

Combining with $\mathsf{Slip}_\eta^+(I) \geq \int_I(-\operatorname{Im} m)$ gives (A5). Rearranging yields the multiplicity bound. □

Remark A3 (On the hypothesis that H is holomorphic near the scan segment)**.** *If the subtraction sum includes* all *poles of m in a neighborhood of the scan segment, then H is holomorphic there. In validated numerics one can verify such a condition by disk enclosures and a certified exclusion of zeros/poles in a covering.*

Appendix E.4 Poincaré–Lelong for Holomorphic Functions in One Complex Variable

We record the divisor-current identity in a form sufficient for our applications. Write $z = x + \mathrm{i}y$, $\partial = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$, $\bar{\partial} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$, and

$$d^c := \frac{1}{2\mathrm{i}}(\bar{\partial} - \partial), \qquad dd^c = \mathrm{i}\partial\bar{\partial}.$$

Lemma A6 (Distribution identity for $\log|z|$)**.** *In the sense of distributions on* $\mathbb{C}$*,*

$$dd^c \log|z| = 2\pi\,\delta_0,$$

where δ_0 *is the Dirac delta at* 0*.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{C})$. By Stokes' theorem on $\mathbb{C} \setminus \overline{B}(0,\varepsilon)$,

$$\int_{\mathbb{C}\setminus\overline{B}(0,\varepsilon)} \log|z|\; dd^c\varphi = \int_{\partial\overline{B}(0,\varepsilon)} \log|z|\; d^c\varphi - \int_{\partial\overline{B}(0,\varepsilon)} \varphi\; d^c \log|z|.$$

As $\varepsilon \downarrow 0$, the first boundary term tends to 0 since $\log|z| = \log\varepsilon$ and the boundary length is $O(\varepsilon)$. For the second term, on $|z| = \varepsilon$ one has $d^c \log|z| = d(\arg z)$, hence

$$\int_{\partial\overline{B}(0,\varepsilon)} d^c \log|z| = 2\pi.$$

More precisely,

$$\int_{\partial\overline{B}(0,\varepsilon)} \varphi\, d^c \log|z| \;\to\; \varphi(0) \int_{\partial\overline{B}(0,\varepsilon)} d^c \log|z| = \varphi(0)\cdot 2\pi,$$

consistent with the present normalization of d^c. Thus

$$\int_{\mathbb{C}} \log|z|\; dd^c\varphi = 2\pi\,\varphi(0),$$

which is exactly $dd^c \log|z| = 2\pi\delta_0$ in distributions. ☐

Theorem A5 (Poincaré–Lelong in one complex variable)**.** *Let* $U \subset \mathbb{C}$ *be open and let* F *be holomorphic on* U*, not identically zero. Then, as currents on* U*,*

$$\frac{1}{2\pi}\, dd^c \log|F| = \sum_{z_0 \in Z(F;U)} \mathrm{mult}(z_0)\,\delta_{z_0}.$$

In particular, if $R \Subset U$ *is a rectangle such that* $F \neq 0$ *on* ∂R*, then*

$$\sum_{z_0 \in Z(F;\mathrm{int}\,R)} \mathrm{mult}(z_0) = \frac{1}{2\pi}\int_R dd^c \log|F| = \frac{1}{2\pi\mathrm{i}}\int_{\partial R} \frac{F'(z)}{F(z)}\,\mathrm{d}z.$$

Proof. **Step 1: Local computation near a zero.** Fix $z_0 \in Z(F;U)$ of multiplicity k. Then in a neighborhood $V \Subset U$ of z_0 one can factor

$$F(z) = (z - z_0)^k G(z),$$

where G is holomorphic and nonvanishing on V. Hence

$$\log|F(z)| = k\log|z - z_0| + \log|G(z)|.$$

Apply dd^c:

$$dd^c \log|F| = k\, dd^c \log|z - z_0| + dd^c \log|G|.$$

By Lemma A6 translated to z_0, one has $dd^c \log|z-z_0| = 2\pi\,\delta_{z_0}$. Since G is holomorphic and nonvanishing, $\log|G|$ is harmonic, hence $dd^c \log|G| = 0$ as a current. Therefore on V,

$$\frac{1}{2\pi} dd^c \log|F| = k\,\delta_{z_0}.$$

Step 2: Globalization by partition of unity. The zeros of F in a compact set are finite. Cover $\operatorname{supp}(\varphi)$ for a test function $\varphi \in C_c^\infty(U)$ by finitely many neighborhoods V of the above form plus a neighborhood where F has no zeros (so $\log|F|$ is harmonic there). A partition of unity then yields

$$\left\langle \frac{1}{2\pi} dd^c \log|F|, \varphi \right\rangle = \sum_{z_0 \in Z(F;U)} \operatorname{mult}(z_0)\,\varphi(z_0),$$

which is exactly the current identity.

Step 3: The rectangle integral identity. If $F \neq 0$ on ∂R, then $\log F$ has a branch on a neighborhood of ∂R and

$$d\log F = \frac{F'}{F}\,\mathrm{d}z.$$

Using $dd^c = \mathrm{i}\partial\bar{\partial}$ and Stokes' theorem gives

$$\int_R dd^c \log|F| = \int_{\partial R} d^c \log|F|,$$

and $d^c \log|F|$ is (up to normalization) the real part of $d\log F$ rotated by $\pi/2$, yielding the standard boundary integral $\frac{1}{\mathrm{i}}\int_{\partial R} F'/F\,\mathrm{d}z$. Combining with the current identity gives the stated integer equality. ☐

Proposition A3 (A validated slip certificate on a nonvacuous height window (scaled bank))**.** *Let $T := 10$ and $\eta_\star := 0.49$, and consider the window $\Omega_{T,\eta_\star}$ from Equation (4). Let $\ell := 1/16$ and let $\mathscr{I}^{(\ell)}_{[T,2T]}$ be the two-shift ℓ-bank (Definition A7). Set the strict margin*

$$\delta := \frac{\pi}{2000}.$$

A validated computation using Arb ball arithmetic (python-flint, working precision 256 bits) verifies that for every $\eta \in [\eta_\star, \frac{1}{2}]$ and every $I \in \mathscr{I}^{(\ell)}_{[T,2T]}$,

$$\mathsf{Slip}^+_\eta(I) \;<\; \frac{\pi}{4} - \delta. \tag{A6}$$

Moreover, the maximal certified upper bound over all such η and I is

$$\max_{\substack{\eta \in [\eta_\star, 1/2] \\ I \in \mathscr{I}^{(\ell)}_{[T,2T]}}} \mathsf{Slip}^+_\eta(I) \;\le\; 0.6770630666\ldots \;<\; \frac{\pi}{4} - \delta.$$

Consequently,

$$Z(\Xi; \Omega_{10,0.49}) = \varnothing.$$

Equivalently, every zero $\rho = \beta + \mathrm{i}\gamma$ of ξ with $10 < \gamma < 20$ satisfies

$$|\beta - \tfrac{1}{2}| \;<\; 0.49.$$

Proof. The validated computation establishes the strict slip inequalities $\mathsf{Slip}^+_\eta(I) < \pi/4$ for every $\eta \in [\eta_\star, 1/2]$ and every $I \in \mathscr{I}^{(\ell)}_{[T,2T]}$. Therefore the hypotheses of the scaled stagewise slip transducer Theorem A1 are satisfied (with $T = 10$ and $\eta_\star = 0.49$), and it follows that $Z(\Xi; \Omega_{10,0.49}) = \varnothing$.

The strip statement for zeros of ξ follows from the Ξ–ξ dictionary (Equation (3) and the discussion in Section 1): a zero $\rho = \beta + \mathrm{i}\gamma$ corresponds to a Ξ-zero at $z = \gamma + \mathrm{i}(\frac{1}{2} - \beta)$, and the functional equation symmetry $\xi(1-s) = \xi(s)$ yields the two-sided bound exactly as in the proof of Corollary 1. □

Remark A4 (Certificate record (one-line summary))**.** *In the computation for Proposition A3 one has* $\ell = 1/16$ *and* $|\mathscr{I}^{(\ell)}_{[10,20]}| = 321$ *(covering a padded t-range* $[10 - \ell/2, 20 + \ell/2] = [9.96875, 20.03125]$*). The height interval* $[0.49, 0.5]$ *was partitioned into* 16 *subintervals of width* $1/1600$*, and the maximum certified upper bound* $0.6770\ldots$ *occurred on the band* $[0.49375, 0.494375]$ *and the bank interval* $[18.5, 18.5625]$*.*

Proposition A4 (Validated slip certificate on $(10, 20)$ (scaled bank))**.** *Let* $T := 10$ *and* $\eta_\star := 0.49$*. Let* $\ell := 1/16$ *and let* $\mathscr{I}^{(\ell)}_{[T,2T]}$ *denote the two-shift* ℓ*-bank (Definition A7) on* $[T, 2T] = [10, 20]$*. Set the strict margin*

$$\delta := \frac{\pi}{2000}.$$

A validated computation using Arb ball arithmetic (python-flint) at working precision 256 *bits verifies that for every scan height* $\eta \in [\eta_\star, \frac{1}{2}]$ *and every* $I \in \mathscr{I}^{(\ell)}_{[T,2T]}$*,*

$$\mathsf{Slip}^{+}_{\eta}(I) \;<\; \frac{\pi}{4} - \delta. \tag{A7}$$

Moreover, the maximal certified upper bound is

$$\max_{\substack{\eta \in [\eta_\star, 1/2] \\ I \in \mathscr{I}^{(\ell)}_{[10,20]}}} \mathsf{Slip}^{+}_{\eta}(I) \;\le\; 0.677063066556875 \;<\; \frac{\pi}{4} - \delta.$$

Consequently,

$$Z(\Xi; \Omega_{10,0.49}) = \varnothing.$$

Equivalently, every zero $\rho = \beta + \mathrm{i}\gamma$ *of* ξ *with* $10 < \gamma < 20$ *satisfies*

$$\left|\beta - \tfrac{1}{2}\right| \;<\; 0.49.$$

Proof. The validated computation establishes the strict slip inequalities $\mathsf{Slip}^{+}_{\eta}(I) < \pi/4$ for every $\eta \in [\eta_\star, 1/2]$ and every $I \in \mathscr{I}^{(\ell)}_{[10,20]}$. Therefore the hypotheses of the scaled stagewise slip transducer Theorem A1 are satisfied (with $T = 10$, $\eta_\star = 0.49$, and $\ell = 1/16$), and it follows that $Z(\Xi; \Omega_{10,0.49}) = \varnothing$.

The strip statement for zeros of ξ follows from the Ξ–ξ dictionary (Equation (3)) exactly as in the proof of Corollary 1. □

Remark A5 (Certificate record)**.** *In Proposition A4, the bank size is* $|\mathscr{I}^{(1/16)}_{[10,20]}| = 321$*, and the height interval* $[0.49, 0.5]$ *is partitioned into* $N_\eta = 16$ *subintervals. The maximal certified slip upper bound* $0.6770630665\ldots$ *occurs on the height band* $[0.49375, 0.494375]$ *and the bank interval* $[18.5, 18.5625]$*.*

Remark A6 (Negative example: a zero forces local failure of the slip inequality)**.** *Let* $z_0 = t_0 + \mathrm{i}\eta_0$ *with* $\eta_0 > 0$*, and let* $F(z) := (z - z_0)$*. Then* F *is entire and has a simple zero at* z_0*. By Proposition 6, for every* $d \in (0, \eta_0)$ *one has*

$$\mathsf{Slip}^{+}_{F,\eta_0 - d}([t_0 - d, t_0 + d]) = \frac{\pi}{2}.$$

In particular, no bank-based certificate of the form $\mathsf{Slip}^{+}_{\eta}(I) < \pi/4$ *can hold on any bank interval* I *that contains* $[t_0 - d, t_0 + d]$ *at height* $\eta = \eta_0 - d$*. This is the model-case manifestation of the quantized forcing theorem Theorem 1.*

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