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Article

Inversion of Characteristic Functions Without Imaginary Unit

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Abstract

We start from a pure vector update of the concepts of complex number and characteristic function of a probability distribution, free from the imaginary unit, and reprove some basic properties of the latter. In particular, we derive vector inverse formulas for probability distributions and density functions, provide a new geometric proof of the convergence of suitably centered and normalized vector-powers of characteristic functions, and thus provide an update to the proof of the central limit theorem.

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1. Introduction

In the context of quantum physics, it is debated whether imaginary numbers are necessary or whether physics can do without them.

In mathematics, if one says that a classically defined characteristic function was merely formally developed in a series, what does it mean when one later says that this characteristic function is uniquely determined in this way even though it contains a completely unknown, merely symbolic or imaginary quantity?

Through practice and habitation, a mathematical ability can become a mathematical skill or even mastery. How should this process be evaluated if the capability is controversial from the outset? What does it mean when the skeptical voices fall silent? Anyone who recalls the history of solving cubic equations and compares it to the current state of our knowledge in this area can give themselves a personal answer to these questions.

The range of values of a classically defined characteristic function is generally completely unknown. That is because the characteristic functions of probability laws usually make essential use of the so-called imaginary unit. However, there is no standard mathematical definition for this symbolic or even mystic quantity where the definability of a mathematical concept means that it can be uniquely characterized using elementary language [1]. Nevertheless, the application of classically defined characteristic functions has led to fundamental theoretical and demonstrably practical results; regarding a literature review, reference is made to [2]. However, a reliable concrete numerical treatment of such a characteristic function typically only becomes possible when it satisfies a representation without the use of imaginary quantities.

When classical complex numbers are introduced, it is usually assumed that the imaginary unit is not a real number and there is a method of squaring it which yields the real number minus one. Neither the first nor the second assumption characterizes how the set of these complex numbers to be constructed contains the known set of real numbers as a subset, or how the method of squaring can be extended to the unknown set of these complex numbers other than simply doing so symbolically. These statements appear to be true if one does not accept that, in a classical definition of complex numbers, the real number one is equated with a particular vector of R^2. Such "lenient" or "generous" approach, however, has become widespread in the classical literature. Actually, considering the set of real numbers as a subset of the set of classical complex numbers cannot be accepted from the point of view of usual mathematical rigor. Rather, the set of ordered pairs or vectors, one component of which is any real number and the second component of which is zero, represents a one-dimensional subspace of the two-dimensional vector space of revised complex numbers. Regarding, for example, some consequences when solving quadratic equations, we refer to work [3] which should even be of interest to early semesters students of mathematics and teaching.

Furthermore, several examples in [2] show that revised characteristic functions can be evaluated numerically when one switches from the classical representation to the vectorial representation, i.e. representation freed from the imaginary unit. Another innovation arising for characteristic functions from the application of their vector representations, as opposed to classical representations, is that in Section 2 the effect of a distribution's symmetry on the characteristic function is updated. Whereas previously a distribution was considered symmetric if and only if its characteristic function was real, a result in Section 3 states that a distribution is symmetric if and only if the second component of its characteristic function is equal to zero. In a sense, this might be considered being practically the same, but this would not meet the nature of the matter. Instead, it speaks of the far-reaching intuition of mathematicians like Gauss and Cauchy in the establishment of the concept of complex numbers, which was not yet considered complete even in their time.

In a series of papers, the author has demonstrated that by constructively addressing the aforementioned shortcomings, additionally a vast new mathematical world of non-classically generalized complex numbers can be opened up. Some new insights will be applied here to initial fundamental properties of revised characteristic functions of probability distributions but still without leaving the realm of revised complex numbers.

To be more specific, some fundamental properties of the complex number plane and characteristic functions, both freed from the imaginary unit, are summarized in Section 2, and new integral and local inversion formulas for characteristic functions are derived in Section 3. In this process, classic techniques, such as those to be found, e.g., in [5–7] as well as in numerous subsequent books are vectorially reinterpreted and reformulated. As a result, all of our formulas are completely free of any mystical, symbolic or imaginary quantity. Section 4 follows with an updated vectorial, in part geometrical, proof of the central limit theorem, and the discussion in Section 5 closes this work.

2. Revised Complex Number Plane

The revised complex number plane can be defined as an algebraic structure $\mathbf{C} = (R^2, \oplus, \circledast, \cdot, \mathbf{0}, \mathbf{1}, \mathbf{I})$ where R^2 represents the two-dimensional vector space of ordered pairs of real numbers, $\oplus$ denotes the corresponding well-known component-wise vector space addition and $\mathbf{0} \in R^2$ the additive neutral element of this space. The commutative vector multiplication is adopted from usual complex multiplication,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \circledast \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1y_1 - x_2y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix},$$

$\mathbf{1} \in R^2$ is the neutral element with respect to this vector operation, meaning that $\mathbf{1} \circledast x = x \circledast \mathbf{1} = x$, $\cdot : R \otimes R^2 \to R^2$ represents componentwise multiplication of a vector with a scalar, and it is set $\lambda \cdot z = \lambda z$, for short. The vector space elements $\mathbf{1}$ and $\mathbf{I}$ are assumed to be linear independent and to satisfy the equation $\mathbf{I} \circledast \mathbf{I} = -\mathbf{1}$. This multiplication consists of a mathematically positive oriented rotation and a scaling. To finally realize the desired outcomes for the vectors $\mathbf{1}$ and $\mathbf{I}$, let

$$\mathbf{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{I} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Furthermore, we observe the validity of the distributive law

$$z_1 \circledast (z_2 - z_3) = z_1 \circledast z_2 - z_1 \circledast z_3$$

and define the k-th $\circledast$-power of $z \in R^2$ as

$$z^{\circledast k} = z^{\circledast(k-1)} \circledast z, k = 1, 2, 3...; z^{\circledast 0} = \mathbf{1}.$$

Based upon this, we introduce the vector-valued $\circledast$-exponential function as

$$\exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^{\circledast k}, z \in R^2.$$

This function has the following $\circledast$-based multiplication property. For $x \in R, y \in R$,

$$\exp((x+y)\mathbf{I}) = \begin{pmatrix} \cos(x+y) \\ \sin(x+y) \end{pmatrix} = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} \circledast \begin{pmatrix} \cos y \\ \sin y \end{pmatrix} = \exp(x\mathbf{I}) \exp(y\mathbf{I}).$$

In particular, we have the Euler type formula

$$\exp(x\mathbf{I}) = (\cos x)\mathbf{1} + (\sin x)\mathbf{I},$$

and the frequently used asymptotic relation

$$\exp(th\mathbf{I}) - \mathbf{1} = \begin{pmatrix} \cos(th) - 1 \\ \sin(th) \end{pmatrix} \to \mathbf{0} \text{ as } h \to 0.$$

Remark 1. *A generalization of the above used multiplication property is given by*

$$\begin{pmatrix} \cos_p(t) \\ \sin_p(t) \end{pmatrix}^{\odot_p k} = \begin{pmatrix} \cos_p(kt) \\ \sin_p(kt) \end{pmatrix}^{\odot_p k}$$

where

$$\cos_p(t) = \frac{\cos t}{N_p(t)}, \ \sin_p(t) = \frac{\sin t}{N_p(t)} \text{ with } N_p(t) = (|\cos t|^p + |\sin t|^p)^{1/p}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \odot_p \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left[\frac{(|x_1|^p + |x_2|^p)(|y_1|^p + |y_2|^p)}{(|x_1 y_1 - x_2 y_2|^p + |x_1 y_2 + y_1 x_2|^p)} \right]^{\frac{1}{p}} \begin{pmatrix} x_1 y_1 - x_2 y_2 \\ x_1 y_2 + y_1 x_2 \end{pmatrix},$$

which is used at a crucial point when one wants to leave the realm of revised complex numbers, going beyond the scope of this work.

Let (Ω, A, P) now be a probability space and $X : \Omega \to R$ a real random variable.

Definition 1. *The characteristic function of X is defined as the mapping of R into the complex plane according to*

$$\varphi_X(t) = \mathbb{E} \exp(tX\mathbf{I}), t \in R$$

where $\mathbb{E}$ means mathematical expectation.

For some basic properties of this uniquely defined function, we refer to [2].

Example 1. **A** *Let X be standard Gaussian distributed. The literature describes several different methods for calculating the characteristic function of X as*

$$\varphi_X(t) = e^{-t^2/2}, \ t \in R.$$

Many authors who choose the classical complex number approach treat thereby the imaginary unit as a constant, although this contradicts the fundamental assumption that this formal quantity can be any thing but a real number. This deficit in mathematical rigor can be considered compensated for by the fact that, in one way or another, contrary to all rules of mathematical rigor, the real number one and the vector $\begin{pmatrix}1\\0\end{pmatrix}$ *are equated when defining the classical complex numbers. Other authors refer to Cauchy's integral theorem. In their derivation, the authors in [5] state that*

$$\varphi_X(t) = e^{-t^2/2} J(t) \text{ with } J(t) = \int_{x=-\infty-it}^{\infty-it} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx,$$

where i denotes the imaginary unit from classical complex numbers, and it is assumed that it is known that the symbolic integral $J(t)$ *satisfies the equation*

$$J(t) = 1.$$

Without especially mentioning or even emphasizing this fact, the authors thereby indicate that a two-dimensional auxiliary consideration in the complex plane is used to treat the characteristic function of a random variable, which takes on values in the one-dimensional space R. A somewhat different, temporary, and auxiliary transition to a two-dimensional consideration can also be found in [8] where again, nevertheless, the classical approach to complex numbers is generally followed.

B *We now follow the consistently two-dimensional approach of the present work,*

$$\varphi_X(t) = E\begin{pmatrix}\cos tX\\ \sin tX\end{pmatrix} = \frac{1}{\sqrt{2\pi}}\begin{pmatrix}\int_{-\infty}^{\infty} \cos txe^{-x^2/2}dx\\ \int_{-\infty}^{\infty} \sin txe^{-x^2/2}dx\end{pmatrix} = \begin{pmatrix}\varphi(t)\\ 0\end{pmatrix}.$$

Differentiation and subsequent integration by parts lead to

$$\varphi'(t) = -t\varphi(t)$$

and with $\varphi_X(0) = \begin{pmatrix}1\\0\end{pmatrix}$*, it finally follows that* $\varphi(t) = e^{-t^2/2}$*, so that we have obtained an update of the classical claim,*

$$\varphi_X(t) = e^{-t^2/2}\begin{pmatrix}1\\0\end{pmatrix}, t \in R.$$

3. Inversion Formulas

The ideas of proof for our statements in this section are substantially inspired by those in [5], [6], [7]. Instead of the formulations used there, however, which all use the imaginary unit, we will here switch to pure vector representations free of any symbolic quantities. We denote the probability distribution of X by P^X.

Theorem 1 (Integral inversion formula)**.** *Let* $a < b$*, then*

$$\begin{pmatrix}P^X((a,b)) + \frac{1}{2}P^X(\{a,b\})\\ 0\end{pmatrix} = \lim_{R\to\infty} I(R)$$

where

$$I(R) = \frac{1}{2\pi}\int_{-R}^{R} (-\mathbf{I}) \circledast (\exp(-ta\mathbf{I}) - \exp(-tb\mathbf{I})) \circledast \frac{\varphi_X(t)}{t} dt.$$

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Proof. Change of order of integration results in

$$
\begin{aligned}
I(R) =& \frac{1}{2\pi} \int_{x=-\infty}^{\infty} \int_{t=-R}^{R} (-\mathbf{I}) \circledast (\exp(-ta\mathbf{I}) - \exp(-tb\mathbf{I})) \circledast \exp(tx\mathbf{I}) \frac{dt}{t} P^X(dx) \\
=& -\frac{1}{2\pi} \int_{x=-\infty}^{\infty} \int_{t=-R}^{R} \begin{pmatrix} \sin(-tb+tx) - \sin(-ta+tx) \\ \cos(-ta+tx) - \cos(-tb+tx) \end{pmatrix} \frac{dt}{t} P^X(dx) \\
=& -\frac{1}{\pi} \begin{pmatrix} \int_{x=-\infty}^{\infty} (\int_{t=0}^{R} \frac{\sin t(x-b)}{t} dt - \int_{t=0}^{R} \frac{\sin t(x-a)}{t} dt) P^X(dx) \\ 0 \end{pmatrix} \\
=& \frac{1}{\pi} \begin{pmatrix} \int_{x=-\infty}^{\infty} I^*(R) P^X(dx) \\ 0 \end{pmatrix} \text{ where } I^*(R) = \int_{\mu=R(x-b)}^{R(x-a)} \frac{\sin\mu}{\mu} d\mu.
\end{aligned}
$$

Note that

$$
\lim_{R\to\infty} I^*(R) = \begin{cases} 0 & \text{if } x > b \\ \int_0^{\infty} \frac{\sin\mu}{\mu} d\mu & \text{if } x = b \\ \int_{-\infty}^{\infty} \frac{\sin\mu}{\mu} d\mu & \text{if } a < x < b \\ \int_{-\infty}^{0} \frac{\sin\mu}{\mu} d\mu & \text{if } x = a \\ 0 & \text{if } x < a \end{cases}
$$

Changing order of integration and taking the limit finishes the proof □

Let F_X denote the cumulative distribution function of the random vriable X.

Corollary 1. *If F_X is continuous and $a < b$ then*

$$
\begin{pmatrix} F_X(b) - F_X(a) \\ 0 \end{pmatrix} = \lim_{R\to\infty} I(R).
$$

Let us now assume that X has density f_X and let $||.||$ denote the Euclidean norm in R^2.

Corollary 2 (Lokal inversion formula)**.** *If F_X is absolutely continuous and $||\varphi_X||^2$ integrable then*

$$
\begin{pmatrix} f_X(x) \\ 0 \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_X(t) \circledast \exp(-tx\mathbf{I}) dt.
$$

Moreover, f_X is bounded and uniformly continuous.

Proof. The above results indicate that

$$
\begin{pmatrix} \frac{1}{2h}(F_X(x+h) - F_X(x-h)) \\ 0 \end{pmatrix} = \lim_{R\to\infty} I^+(R)
$$

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where

$$I^+(R) = \frac{1}{2\pi}\int_{-R}^{R} (-\mathbf{I}) \circledast (-\exp(-t(x+h)\mathbf{I}) + \exp(-t(x-h)\mathbf{I})) \circledast \varphi_X(t)\frac{dt}{2ht}$$
$$= \frac{1}{2\pi}\int_{-R}^{R} (-\mathbf{I}) \circledast (\exp(-tx\mathbf{I}) \circledast (2\sin(th)\cdot \mathbf{I}) \circledast \exp(-tx\mathbf{I}))\varphi_X(t)\frac{dt}{2ht}$$
$$= \frac{1}{2\pi}\int_{-R}^{R} (-\mathbf{I}^{\circledast 2}) \circledast \exp(-tx\mathbf{I}) \circledast \varphi_X(t)\frac{2\sin(th)}{2th}dt$$

Changing the order of taking limits leads to

$$\lim_{h\to 0}\begin{pmatrix}\frac{1}{2h}(F_X(x+h) - F_X(x-h))\\ 0\end{pmatrix} = \lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\exp(-tx\mathbf{I})\circledast\varphi_X(t)dt.$$

Moreover,

$$\begin{pmatrix} f_X(x+h) - f_X(x) \\ 0\end{pmatrix}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_X(t)\circledast(\exp(-tx\mathbf{I})\circledast\exp(-th\mathbf{I}) - \exp(-tx\mathbf{I})\circledast \mathtt{1})dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-tx\mathbf{I})\circledast(\exp(-th\mathbf{I}) - \mathtt{1})\circledast\varphi_X(t)dt \to \mathtt{0} \text{ as } h\to 0$$

□

Example 2. *An application of Corollary 2 to the standard Gaussian characteristic function*

$$\varphi_X(t) = e^{-\frac{t^2}{2}}\begin{pmatrix}1\\0\end{pmatrix}, t\in R$$

indeed yields

$$\begin{pmatrix} f_X(x)\\ 0\end{pmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\begin{pmatrix}1\\0\end{pmatrix}\circledast\exp(-tx\mathbf{I})dt$$
$$= \begin{pmatrix}\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(-tx)e^{-\frac{t^2}{2}}dt\\ 0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\\ 0\end{pmatrix}.$$

Corollary 3. *The distribution of a random variable is symmetric if and only if the second component of its characteristic function is constantly equal to zero.*

Proof. Assume that X has a symmetric probability distribution, that is $F_X(x) = 1 - F_X(x+0)$ for all $x\in R$ where $F_X(x+0)$ means right side limit of F_X at point x, then

$$\varphi_X(t) = \mathtt{E}\begin{pmatrix}\cos tX\\ \sin tX\end{pmatrix} = \begin{pmatrix}\int_{-\infty}^{\infty}\cos(tx)dF_X(x) + F_X(+0) - F_X(0)\\ 0\end{pmatrix}.$$

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If, vice versa, the second component of φ_X is zero, then $\varphi_X(t) = \varphi_{\bar{X}}(t)$ where

$$t \to \varphi_{\bar{X}}(t) = \mathtt{E}\begin{pmatrix} \int\limits_{-\infty}^{\infty} \cos(tx)dF_X(x) \\ -\int\limits_{-\infty}^{\infty} \sin(tx)dF_X(x) \end{pmatrix}$$

denotes the conjugate function of $t \to \varphi_X(t), t \in R$. It follows from

$$\varphi_{\bar{X}}(t) = \mathtt{E}\begin{pmatrix} \cos tX \\ -\sin tX \end{pmatrix} = \begin{pmatrix} \mathtt{E}\cos(-tX) \\ \mathtt{E}\sin(-tX) \end{pmatrix} = \varphi_{-X}(t)$$

that

$$\varphi_X(t) = \varphi_{-X}(t), t \in R.$$

According to Theorem 1, the characteristic function of a random variable uniquely determines its distribution function, thus,

$$F_X(x) = F_{-X}(x), x \in R$$

□

Remark 2. *This proof shows that, to overcome the dependence on the imaginary unit, one could basically simply follow a known proof from the literature and replace the real and imaginary parts of the classical characteristic function that appear there with the two components of the real vector $\varphi_X(t)$ considered here.*

4. Central Limit Theorem

Let $X_1, ..., X_n$ be independent and identically distributed random variables defined on a common probability space, $S_n = X_1 + ... + X_n, \sigma > 0$ and $\sigma^2 = \mathtt{V}(X_1) < \infty$ the finite variance of X_1. Furthermore, $\mathtt{E}Y$ means the mathematical expectation of a random variable Y.

Theorem 2. *The characteristic functions $t \to \varphi_n(t)$ where*

$$\varphi_n(t) = E\exp(t\frac{S_n - ES_n}{\sigma\sqrt{n}}I), t \in R$$

converge to the standard Gaussian characteristic function for every $t \in R$ as n approaches infinity.

Proof. Since we do not represent characteristic functions using imaginary numbers in this work, but prefer vector representations, the possibility arises to provide a new, geometric, proof for the well-known statement of the theorem. To this end, let

$$\varphi(t) = \mathtt{E}\exp(t(X_1 - \mathtt{E}X_1)\mathbf{I}), t \in R$$

denote the characteristic function of the centered random variable X_1. According to [2],

$$\varphi(t) = \begin{pmatrix} 1 - \frac{t^2}{2}\sigma^2 + \theta_1(t) \\ \theta_2(t) \end{pmatrix}$$

where here and below, in the sense of Landau, and for $k = 1, 2, ...,$

$$\theta_k(t) = o(t^2) \text{ as } t \to 0.$$

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The polar representation of $\varphi(t)$ is given by

$$\varphi(t) = r(t)\begin{pmatrix}\cos\phi(t)\\ \sin\phi(t)\end{pmatrix}$$

where

$$\begin{aligned} r(t) &= \sqrt{(1-\frac{t^2}{2}\sigma^2+\theta_1(t))^2+(\theta_2(t))^2} \\ &= \sqrt{1-t^2\sigma^2+\theta_3(t)} = 1-\frac{t^2}{2}\sigma^2+\theta_4(t) \end{aligned}$$

and

$$\cos\phi(t) = \frac{1-\frac{t^2}{2}\sigma^2+\theta_1(t)}{r(t)} = 1+\theta_5(t);\ \sin\phi(t) = \frac{\theta_2(t)}{r(t)} = \theta_6(t).$$

Therefore,

$$\tan\phi(t) = \frac{\theta_6(t)}{1+\theta_5(t)}$$

and

$$\phi(t) = \arctan\frac{\theta_6(t)}{1+\theta_5(t)} = \arctan\theta_7(t).$$

Finally, it follows from [4] that

$$\varphi_n(t) = \varphi^{\circledast n}(\frac{t}{\sigma\sqrt{n}}),$$

thus

$$\varphi_n(t) = (r(\frac{t}{\sigma\sqrt{n}}))^n\begin{pmatrix}\cos(n\phi(\frac{t}{\sigma\sqrt{n}}))\\ \sin(n\phi(\frac{t}{\sigma\sqrt{n}}))\end{pmatrix}.$$

Note that, for all $t \in R$,

$$\begin{aligned} (r(\frac{t}{\sigma\sqrt{n}}))^n &= (1-(\frac{t}{\sigma\sqrt{n}})^2\frac{\sigma^2}{2}+\theta_4(\frac{t}{\sigma\sqrt{n}}))^n \\ &= (1-\frac{t^2/2}{n}+o(\frac{1}{n}))^n \to e^{-\frac{t^2}{2}} \text{ as } n\to\infty. \end{aligned}$$

Moreover, for all $t \in R$,

$$n\phi(\frac{t}{\sigma\sqrt{n}}) = n\arctan\theta_7(\frac{t}{\sqrt{n}}) = no(\frac{t^2}{n}) = o(1) \text{ as } n\to\infty.$$

Thus,

$$\cos(n\phi(\frac{t}{\sigma\sqrt{n}})) = 1+o(1),\ \sin(n\phi(\frac{t}{\sigma\sqrt{n}})) = o(1)$$

and, for all $t \in R$,

$$\varphi_n(t) \to e^{-\frac{t^2}{2}}\begin{pmatrix}1\\0\end{pmatrix} \text{ as } n\to\infty$$

□

Corollary 4 (Central Limit Theorem)**.** *The cumulative distribution function of* $\frac{S_n-ES_n}{\sigma\sqrt{n}}$ *converges weakly to that of the standard Gaussian probability law.*

Proof. To prove this Corollary without using imaginary numbers, one may basically follow the ideas in the literature using classical characteristic functions as, e.g., in [9]. According to Theorem 2, $\varphi_n(t)$ converges to the standard Gaussian characteristic vector function which was derived in Example 1 **B**.

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In those proofs in the literature which show that the corresponding sequence of distribution functions is then accordingly convergent to the standard normal distribution function, one can, without further technical difficulties, replace the real and imaginary parts of the classical characteristic functions by the first and second components of the vectorial characteristic functions considered in the present work, making the complete proof free of imaginary numbers □

5. Discussion

In classical theory of complex numbers, what is meant by the complex or Gaussian plane of numbers? For a graphic representation, many authors draw two arrows perpendicular to each other which are denoted by x or y in a group A and by x or iy in a group B. Here, i represents the symbolic or so-called imaginary unit, which is generally said to be not a real number and to satisfy the equation $i^2 = -1$. However, contrary to the precision and pedantry customary in mathematics, no constructive statement is made about the mathematical nature of i, nor is anything said about the method of squaring (which, incidently, must also include squaring of real numbers). This is evidently due to the unknown nature of the symbolic quantity i. Occasionally, confusion arises from the impression that the real plane is generated by the variables x and y, whereas this plane is actually spanned by the linear independent vectors **1** and **I**.

The magnitude or norm of the complex number $z = (x, y)$ or $z = x + iy$ is always defined as $|z| = \sqrt{x^2 + y^2}$. All authors interpret $|z|$ as the Euclidean length of the vector or arrow connecting the origin $\mathtt{0}$ and z.

For group A of authors, this means a correct geometric interpretation; for them, the set $\{z : |z| < \epsilon\}$ with $\epsilon > 0$ is equal to the set of all points in the plane with a Euclidean distance from the point $\mathtt{0}$ less than ϵ, that is a disc of radius ϵ. However, the imaginary unit does not appear at all in their representation of the number plane and their definition of the basic notion of norm although this would be natural or even necessary.

The authors in group B clearly indicate that the imaginary unit i is present; but for them, due to the desired geometric interpretation mentioned above, a different length determination of the vector connecting $\mathtt{0}$ and $x + iy$ would be more reasonable, namely $|z| = \sqrt{x^2 + (iy)^2}$, assuming a suitable squaring method exists. Then, the set $\{z : |z| < \epsilon\}$ with $0 < \epsilon < 1$ is equal to the set $\{(x, y)^T \in R^2 : x^2 - y^2 < \epsilon^2\}$. Unfortunately, one could fail to provide a reason for such definition of an environment.

The classical theory of complex numbers thus leads to problems and contradictions that should ideally be resolved. These shortcomings do not arise in the pure vector approach to the complex or Gaussian number plane presented in Section 1 in the form of the algebraic structure **C**. The resulting conclusions for several considerations from the literature are presented in this work, particularly in the case of inversion formula for characteristic functions and the convergence of suitably centered and normalized powers of such functions. Consequently, the two-dimensional considerations presented here replace classical considerations using mystical, symbolic or imaginary quantities, which, until proven otherwise, are subject to a latent uncertainty, from the point of view of mathematical rigor.

Finally, we would like to return to the debate that was mentioned at the beginning of this work. As long as there is no mathematical theory of imaginary numbers that meets the strictest standards of mathematical rigor, from a purely mathematical standpoint, it can be suggested that one considers using suitable vector representations instead of imaginary numbers.

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