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Article

A Closed-Form Inverse Laplace Transform of Shifted Quasi-Rational Spectral Functions via Generalized Hypergeometric and Kampé de Fériet Functions

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Abstract

Closed-form analytic inverses allow explicit tracking of parameter effects, facilitate interpretation of experimental signals, and support solving inverse problems. Here, we derive a rigorous closed-form expression for the inverse Laplace transform of a class of shifted quasi-rational spectral functions with a square-root radical and a power-law decaying factor. These functions appear in coupled diffusion processes in physics and in the analysis of electromagnetic signal propagation through electrically cascaded networks, signal processing, and related areas. The transform is expressed as a finite sum of three generalized hypergeometric functions—two Kummer functions and one five-parameter Kampé de Fériet function—each multiplied by a monomial depending on the decay parameter. The validity of the result is confirmed by direct Laplace transformation, which recovers the original spectral function. Several known inverse transforms appear as limiting cases, illustrating the generality of the solution. Additionally, reduction formulas for a subclass of Kampé de Fériet functions demonstrate how the general solution encompasses previously known results and highlight the generality of the method.

Keywords: inverse laplace transform; generalized hypergeometric functions; kummer function; Kampé de Fériet function; integral transforms; reduction formulas; quasi-rational spectral functions; coupled diffusion

1. Introduction

Laplace transforms represent a powerful analytical tool widely used to solve a broad class of problems in physics and engineering, from transient circuit behavior and control systems [1,2] to heat transfer, mass and particle transport, vibration analysis, and other physical phenomena [3–12]. The inverse Laplace transform is equally important, as it enables the return to the time domain, providing direct insight into the evolution of physical systems [6–12].

However, applying Laplace transform methods presents numerous mathematical and practical challenges. Inversion typically requires contour integration in the complex plane, which becomes complicated when the function exhibits multiple singularities or branch points [13,14]. The choice of an appropriate integration contour and the definition of branch cuts are especially delicate when radicals or other multivalued expressions are present [14,15]. Moreover, no general criterion guarantees the convergence of the Laplace transform or its inverse in all cases.

Problems governed by fractional differential equations, systems of coupled equations of motion, or (fractional) partial differential equations—such as diffusion, anomalous diffusion, vibrational systems, fluid dynamics, and electromagnetic wave propagation—often lead to spectral functions of irrational, quasi-rational, or transcendental form when solved using Laplace transforms [6,7,12]. These spectral functions are generally not amenable to classical inversion techniques (circular, hyperbolic, or parabolic contours), and their inverse transforms typically require special analytical [15–17] or numerical methods [18–23]. Some analytically obtained inverses of such functions have been compiled

into reference tables [24–26], while many are expressed in terms of special functions of mathematical physics [27–31], enabling deeper insights into the associated physical processes.

One important application where Laplace transforms yield complex spectral functions that are difficult to invert analytically by standard techniques is the interpretation of experimentally measured time-domain signals in photothermal (PT) and photoacoustic (PA) techniques [32–35]. These signals arise from the interplay of coupled physical processes, where one time-dependent phenomenon serves as a stimulus for another [36–40]. In PT measurements, the signal evolution depends on the temporal profile of optical excitation, heat transport properties, acoustic wave propagation in the surrounding medium, and the transfer function of the acoustic detector. When semiconducting materials are involved, the dynamics of photoexcited charge carriers further contribute to the overall response [36–39].

In such cases, finding an inverse Laplace transform in closed form is crucial for interpreting experimental data and understanding the underlying dynamics, yet for many of these functions no general analytical solution exists. Closed-form analytical inverses enable explicit tracking of parameter dependencies, facilitate physical interpretation of measured signals, and support the formulation and solution of inverse problems. This motivates the need for closed-form analytic inverses, as they allow explicit assessment of how individual system parameters affect observed signals.

The present work addresses a specific class of such spectral functions and derives, for the first time, a closed-form analytical expression for their inverse Laplace transform, expressed explicitly as a sum of two Kummer functions and one five-parameter Kampé de Fériet (KdF) function. Recent advances in the theory of generalized hypergeometric functions—particularly in reduction formulas and structural properties of the KdF function—have renewed interest in explicit analytical representations arising in integral transforms (see, e.g., [41–45]). This solution represents the main contribution of the paper, as no closed-form inverse exists for this class of spectral functions in the literature.

To illustrate the generality of the solution, we show that in certain limiting cases the general expression reduces to previously known forms expressed via Bessel functions, error functions (erf, erfc), and even a case involving the Mittag-Leffler (ML) function. Establishing these connections required the derivation of two reduction formulas for the five-parameter Kampé de Fériet function. These reduction formulas are secondary, serving solely to demonstrate that the general solution encompasses known results, and are not the main contribution.

Overall, the derived solution provides a unified analytical framework linking inverse Laplace transforms and generalized hypergeometric functions, while indicating broader connections to coupled diffusion-type phenomena. The structure of the paper is as follows. Section 2 summarizes the preliminary mathematical results and definitions used throughout the paper. Section 3 presents the derivation of the inverse Laplace transform by reformulating the complex contour integral as a convolution and expressing it in terms of generalized hypergeometric functions. Section 4 verifies the result by reapplying the Laplace transform. Section 5 demonstrates several known results as special cases, and Section 6 concludes the work and outlines directions for future research.

2. Preliminaries

In this section, we summarize the basic properties of Laplace transform, definitions of special functions and series and other mathematical remainder that will be used throughout the paper. For convenience, all theorems of Laplace transform are stated in the form of equivalences, as they will be employed in both directions, particularly when calculating inverse Laplace transforms.

2.1. Laplace Transform and Inverse Relations

For a sufficiently regular function $f(t)$, the Laplace transform and its inverse are connected by the pairs of equivalent relations Eqs.(1–2), where Eq.(2) denotes the so-called Mellin formula, Bromwich integral, or Mellin inversion formula [1–3]:

$$\bar{F}(s) = \mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} e^{-st} f(t) dt; \quad \operatorname{Re}\{s\} > 0 \quad (1)$$

$$f(t) = \mathcal{L}^{-1}\{\bar{F}(s)\}(s) = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \bar{F}(s) ds; \quad t > 0 \quad (2)$$

The following equivalences will be frequently used in the sequel (often in the inverse direction to compute explicit time-domain expressions):

Linearity

$$\mathcal{L}\left\{\sum_{k=1}^n c_k f_k(t)\right\}(s) = \sum_{k=1}^n c_k \bar{F}_k(s) \Leftrightarrow \mathcal{L}^{-1}\left\{\sum_{k=1}^n c_k \bar{F}_k(s)\right\}(t) = \sum_{k=1}^n c_k f_k(t), \quad c_k \in \mathbb{R} \quad (3)$$

Shift in the s -domain (complex/frequency shift):

$$\mathcal{L}\{e^{At} f(t)\}(s) = \bar{F}(s - A) \Leftrightarrow \mathcal{L}^{-1}\{\bar{F}_k(s - A)\}(t) = e^{At} f(t) \quad (4)$$

Convolution theorem:

$$\mathcal{L}\{(f * g)\}(s) = \bar{F}(s)\bar{G}(s) \Leftrightarrow \mathcal{L}^{-1}\{\bar{F}(s)\bar{G}(s)\} = \int_0^t f(\tau)g(\tau - s) d\tau \quad (5)$$

In this paper, we also use selected table values for causal systems ($\operatorname{Re}\{s\} > 0$) [24,26]: for $q > 0, q \in \mathbb{R}$

$$\mathcal{L}\left\{\frac{t^{q-1}}{\Gamma(q)} h(t)\right\} = \frac{1}{s^q} \Leftrightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^q}\right\} = \frac{t^{q-1}}{\Gamma(q)} h(t) \quad (6)$$

where $\Gamma(\dots)$ denotes the Gamma function [27].

For $B > 0$:

$$\mathcal{L}\{e^{B^2 t} \operatorname{erfc}(b\sqrt{t})\} = \frac{1}{\sqrt{s+B}} \Leftrightarrow \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+B}}\right\} = e^{B^2 t} \operatorname{erfc}(b\sqrt{t}) \quad (7)$$

where $\operatorname{erfc}(\dots)$ denotes the complementary error function [27,31] that is defined by

$$\operatorname{erfc}(y) = 1 - \operatorname{erf}(y) \quad (8)$$

2.2. Kummer's Confluent Hypergeometric Function and KdF Function: Definitions

The confluent hypergeometric function of the first kind (Kummer's function) is defined by the infinite series [41]:

$$M(r_1, r_2, y) = \sum_{k=0}^{\infty} \frac{r_1^{(k)} y^k}{r_2^{(k)} k!} \quad (9)$$

valid for $y \in \mathbb{R}$. It also admits the integral representation:

$$M(r_1, r_2, y) = \frac{\Gamma(r_2)}{\Gamma(r_1)\Gamma(r_2 - r_1)} \int_0^1 e^{yz} y^{r_1-1} (1-y)^{r_2-r_1-1} dy \quad (10)$$

The Kummer transformation relates the Kummer function of positive and negative arguments:

$$M(r_1, r_2, y) = e^y M(r_2 - r_1, r_2, -y) \quad (11)$$

The KdF function is defined as a double hypergeometric series [42–46]:

$$F_{1;1;1}^{1;1;1}\left(\begin{matrix} a & b & c \\ d & e & f \end{matrix}; y_1, y_2\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{y_1^n}{n!} \frac{a^{(n+k)}}{d^{(n+k)}} \frac{b^{(n)}}{e^{(n)}} \frac{c^{(k)}}{f^{(k)}} \frac{y_2^k}{k!} \quad (12)$$

This function generalizes many known bivariate hypergeometric functions.

In Eqs.(5), (6) the Pochhammer symbol $\alpha^{(n)}$ represents the rising factorial [47]:

$$\alpha^{(0)} = 1, \quad \alpha^{(n)} = \alpha(\alpha + 1) \cdots (\alpha + n - 1) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \quad (13)$$

Similarly, the falling factorial is defined as [47]:

$$\alpha_{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1), \quad \text{with } \alpha_{(0)} = 1 \quad (14)$$

and appears in the definition of binomial coefficients [48]:

$$\binom{\alpha}{n} = \frac{\alpha_{(n)}}{n!} \quad (15)$$

The relationship between rising and falling factorials is given in [49]:

$$\alpha^{(n)} = (-1)^n (-\alpha)_{(n)} \quad (16)$$

Additionally, the following identity holds:

$$\alpha^{(n+k)} = (\alpha + n)^{(k)} \quad (17)$$

Useful contiguous relations for Kummer function that are used in the paper are [41,49]

$$r_2 M(r_1, r_2, y) - r_2 M(r_1 - 1, r_2, y) - y M(r_1, r_2 + 1, y) = 0 \quad (18)$$

$$(r_1 - r_2 + 1) M(r_1, r_2, y) - r_1 M(r_1 + 1, r_2, y) + (r_2 - 1) M(r_1, r_2 - 1, y) = 0 \quad (19)$$

Other identities used from the literature are given by the following equations [49]:

$$M(r_1, r_2, 0) = 1 \quad (20)$$

$$M(1/2, 1, y) = e^{y/2} I_0(y/2) \quad (21)$$

$$M(3/2, 3, y) = e^{y/2} \frac{4}{y} I_1(y/2) \quad (22)$$

$$M(1/2, 3/2, -y) = \frac{\sqrt{\pi}}{2\sqrt{y}} \operatorname{erf}(\sqrt{y}) \quad (23)$$

$$M(1, 3/2, y) = e^y \frac{\pi}{2y} \operatorname{erf}(y) \quad (24)$$

$$M(1, 2, y) = \frac{e^y - 1}{y} \quad (25)$$

where I_0 and I_1 are modified Bessel functions of zero and first order [27], respectively. For background on the KdF function, its reduction formulas, and related summation identities, we refer to classical sources and recent contributions such as [42–46]. Some reduction identities used in the paper that reduce specific KdF functions to Kummer's functions are derived in Appendix 2 and they are given by following equations:

$$F_{1;1}^{1;1} \left(\begin{matrix} 3/2 \\ 2 \end{matrix}; \begin{matrix} 1/2 \\ 3/2 \end{matrix}; -y, y \right) = M(1, 2, y) \quad (26)$$

$$F_{1;1}^{1;1} \left(\begin{matrix} 3/2 \\ 5/2 \end{matrix}; \begin{matrix} 1/2 \\ 3/2 \end{matrix}; -y, y \right) = \frac{3}{2y} (M(1, 3/2, y) - 1) \quad (27)$$

2.3. Other Special Functions Used in the Paper, Defined by Infinite Series

The one-parameter ML function is defined by the series [50–53]:

$$E_{1/2}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k/2 + 1)k!} \quad (28)$$

There exists a relationship between the ML function and the complementary error function [50]:

$$E_{1/2}(\sqrt{y}) = e^y \operatorname{erfc}(-\sqrt{y}) \quad (29)$$

2.4. Series Reminders

We recall the definitions of binomial and infinite geometric series [54,56]:

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n = (1+z)^\alpha, \quad \alpha \in \mathbb{R}, z \in \mathbb{C}, |z| < 1, n \in \mathbb{N} \quad (30)$$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad z \in \mathbb{C}, |z| < 1, n \in \mathbb{N} \quad (31)$$

3. Main Results

In his section provides a detailed derivation of the inverse Laplace transform for a combined shifted quasi-rational functions with square-root radical and power-law decay term.

Theorem 1. Let complex function $\bar{G}(s)$ is given by

$$G(s) = \frac{1}{s^q(\sqrt{s+A+B})} \quad (32)$$

where s is a complex variable, A and B are shift parameters (internal and external, respectively), and $q > 0$. Then, the inverse Laplace transform of $\bar{G}(s)$ is given by

$$\begin{aligned} g(t) = \mathcal{L}^{-1}\{\bar{G}(s)\} &= \frac{t^{q-1/2}}{\Gamma(q+1/2)} M\left(1/2, q+1/2, -At\right) + \\ &+ \frac{B^2 t^{q+1/2}}{\Gamma(q+3/2)} F_{1;1}^{1;1}\left(\begin{matrix} 3/2 \\ q+3/2 \end{matrix}; \begin{matrix} 1/2 \\ 3/2 \end{matrix}; -B^2 t, (B^2 - A)t\right) - \\ &- \frac{B t^q}{\Gamma(q+1)} M(1, q+1, (B^2 - A)t) \end{aligned} \quad (33)$$

where $M(r_1, r_2, y)$ denotes the confluent hypergeometric function (Kummer's function), the generalized two-argument five-parameter KdF function is

$$F_{1;1}^{1;1}\left(\begin{matrix} a \\ d \end{matrix}; \begin{matrix} b \\ e \end{matrix}; -B^2 t, (B^2 - A)t\right) = F_{1;1;1}^{1;1;1}\left(\begin{matrix} a \\ d \end{matrix}; \begin{matrix} b \\ e \end{matrix}; \begin{matrix} c \\ c \end{matrix}; -B^2 t, (B^2 - A)t\right).$$

proof of the Theorem 1. (\Rightarrow). The proof proceeds as follows: firstly, it is shown that the inverse Laplace transform of the given function can be reduced to solving a specific convolution integral. The resulting convolution integral is then solved analytically.

The given complex function Equation (32) can be written as a product of two complex functions:

$$\bar{G}(s) = \frac{1}{s^q} \frac{1}{\sqrt{s+A+B}} = \bar{G}_1(s) \bar{G}_2(s) \quad (34)$$

where:

$$\bar{G}_1(s) = \frac{1}{s^q}, \quad (35)$$

$$\bar{G}_2(s) = \frac{1}{\sqrt{s+A+B}}. \quad (36)$$

Using the Convolution Theorem (Eq.(5)), the inverse Laplace transform of $\bar{G}(s)$ can be found as:

$$g(t) = \mathcal{L}^{-1}\{\bar{G}(s)\} = \int_0^t g_1(t-u)g_2(u)du \quad (37)$$

where

$$g_1(t) = \mathcal{L}^{-1}\{\bar{G}_1(s)\}, \quad (38)$$

$$g_2(t) = \mathcal{L}^{-1}\{\bar{G}_2(s)\}. \quad (39)$$

The inverse Laplace transform for $\bar{G}_1(s)$ (Eq.(35)) can be found in many reference tables [24,26] and is given in Eq.(6) of the Preliminaries section.

The inverse Laplace transform of $\bar{G}_2(s)$ (Eq.(36)) can be obtained using the complex shifting property (Eq.(4)) and solutions in tables of inverse Laplace transforms of irrational functions [25]:

$$g_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+A+B}}\right\} = \frac{1}{\pi t} e^{-At} - Be^{(B^2-A)t} \operatorname{erfc}(B\sqrt{t}). \quad (40)$$

By substituting Eqs.(6) and (40) into Eq.(37), we obtain:

$$g(t) = \mathcal{L}^{-1}\{\bar{G}(s)\} = \int_0^t \frac{(t-u)^{q-1}}{\Gamma(q)} \left(\frac{1}{\sqrt{\pi u}} e^{-Au} - Be^{(B^2-A)u} \operatorname{erfc}(B\sqrt{u}) \right) du. \quad (41)$$

The convolution integral in Eq.(41) can be solved by replacing Eq.(8) and decomposing the integral into three definite integrals:

$$g(t) = i_1(t) + i_2(t) - i_3(t) \quad (42)$$

breaking the problem into smaller parts:

$$i_1(t) = \frac{1}{\Gamma(q)\sqrt{\pi}} \int_0^t \frac{(t-u)^{q-1}}{\sqrt{u}} e^{-Au} du, \quad (43)$$

$$i_2(t) = \frac{B}{\Gamma(q)} \int_0^t (t-u)^{q-1} e^{(B^2-A)u} \operatorname{erf}(B\sqrt{u}) du, \quad (44)$$

$$i_3(t) = \frac{B}{\Gamma(q)} \int_0^t (t-u)^{q-1} e^{(B^2-A)u} du. \quad (45)$$

Each integral is solved by introducing the substitution:

$$x = \frac{u}{t}, \quad dx = \frac{du}{t} \quad (46)$$

Then the solution for the integral $i_1(t)$ becomes:

$$i_1(t) = \frac{1}{\sqrt{\pi}} \frac{t^{q-1/2}}{\Gamma(q)} \int_0^1 e^{-Atx} x^{-1/2} (1-x)^{q-1} dx = \frac{t^{q-1/2}}{\Gamma(q+1/2)} M(1/2; q+1/2; -At) \quad (47)$$

where a confluent hypergeometric function (Kummer's function, Eq.(10)) is identified in the definite integral over the interval $[0, 1]$.

In the computation of the integral Eq. (43), the series representation of the erf(y) function through its Taylor expansion [54–56] is used:

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{n!(2n+1)} \quad (48)$$

Remark. Appendix A shows that the series in Eq.(48) has an infinite radius of convergence, implying that the function is entire and Eq.(48) is valid over the entire domain.

By substituting Eq.(48) into Eq.(44), changing the order of summation and integration, and recognizing confluent hypergeometric functions under the infinite sum, the function $i_2(t)$ becomes:

$$\begin{aligned} i_2(t) &= \frac{2B^2}{\Gamma(q)\sqrt{\pi}} \sum_{n=0}^{\infty} \left(\frac{(-1)^n B^{2n}}{n!(2n+1)} t^{q+n+1/2} \int_0^1 e^{(B^2-A)tx} x^{n+1/2} (1-x)^{q-1} dx \right) = \\ &= \frac{2B^2 t^{q+1/2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-B^2 t)^n}{n!(2n+1)} \frac{\Gamma(n+3/2)}{\Gamma(n+q+3/2)} M(n+3/2; n+q+3/2; (B^2-A)t) \end{aligned} \quad (49)$$

Using the infinite series representation of confluent hypergeometric functions (Eq.(9)) and substituting into Eq.(49), the solution is expressed as a double infinite series:

$$i_2(t) = \frac{B^2 t^{q+1/2}}{\Gamma(q+3/2)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-B^2 t)^n}{n!} \frac{(1/2)^{(n)} (3/2)^{(n+k)}}{(3/2)^{(n)} (q+3/2)^{(n+k)}} \frac{((B^2-A)t)^k}{k!} \quad (50)$$

In this series, the specific five-parameter KdF hypergeometric function is identified (Eq.(12)). The final analytical solution for the integral $i_2(t)$ is:

$$i_2(t) = \frac{B^2 t^{q+1/2}}{\Gamma(q+3/2)} F_{1;1}^{1;1} \left(\begin{matrix} 3/2 \\ q+3/2 \end{matrix}; \begin{matrix} 1/2 \\ 3/2 \end{matrix}; -B^2 t, (B^2-A)t \right) \quad (51)$$

where the solution does not depend on the parameter c .

By substituting Eq.(46) into Eq.(45) and using Eq.(10), the solution for $i_3(t)$ is obtained:

$$\begin{aligned} i_3(t) &= \frac{Bt^q}{\Gamma(q)} \int_0^1 (1-x)^{q-1} e^{(B^2-A)tx} dx = \\ &= \frac{Bt^q}{\Gamma(q)} \int_0^1 (1-x)^{q-1} x^{1-1} e^{(B^2-A)tx} dx = \\ &= \frac{Bt^q}{\Gamma(q+1)} M(1, q+1, (B^2-A)t) \end{aligned} \quad (52)$$

Based on Eqs. (47), (51), (52), and (42), the analytical solution of the convolution integral, i.e., the function $g(t)$ is:

$$\begin{aligned} g(t) &= \frac{t^{q-1/2}}{\Gamma(q+1/2)} M(1/2, q+1/2, -At) + \\ &+ \frac{B^2 t^{q+1/2}}{\Gamma(q+3/2)} F_{1;1}^{1;1} \left(\begin{matrix} 3/2 \\ q+3/2 \end{matrix}; \begin{matrix} 1/2 \\ 3/2 \end{matrix}; -B^2 t, (B^2-A)t \right) - \\ &- \frac{Bt^q}{\Gamma(q+1)} M(1, q+1, (B^2-A)t) \end{aligned} \quad (53)$$

Eq.(53) completes the proof of **Theorem 1** (Eq.(33)) in the forward direction (\Rightarrow). \square

4. Verification of the Solution

Let us now prove Theorem 1 on the inverse Laplace transform for a combined shifted quasi-rational functions with square-root radical and power-law decay term in the opposite direction, by finding the Laplace transform of Eq.(53).

Proof of the Theorem 1. (\Leftarrow). Since the function $g(t)$ given by Eq.(53) is represented as the sum of three functions, by using the linearity of the Laplace transform (Eq.(3)), its Laplace transform can be written as:

$$\bar{G}(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{i_1(t)\} + \mathcal{L}\{i_2(t)\} - \mathcal{L}\{i_3(t)\} \quad (54)$$

where $i_1(t)$, $i_2(t)$, and $i_3(t)$ are given by Eqs.(47), (51), and (52), respectively.

To find the Laplace transform of each of the three functions (Eqs.(47), (51), and (52)), we use the series representations of Kummer's and KdF functions (Eqs. (9), (12)), the known Laplace transform of a monomial (Eq.(6)), the relations between Pochhammer symbols and binomial coefficients (Eqs.(14)-(17)), as well as the definitions of binomial and infinite geometric series (Eqs.(30), (31)).

The Laplace transform of $i_1(t)$ is:

$$\begin{aligned} \mathcal{L}\{i_1(t)\} &= \mathcal{L}\left\{\frac{t^{q-1/2}}{\Gamma(q+1/2)} M\left(1/2, q+1/2, -At\right)\right\} = \\ &= \frac{1}{\Gamma(q+1/2)} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)} (-A)^n}{n!(q+1/2)^{(n)}} \mathcal{L}\{t^{q-1/2+n}\} = \\ &= \frac{1}{\Gamma(q+1/2)} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)} (-A)^n \Gamma(q+1/2+n)}{n!(q+1/2)^{(n)} s^{q+1/2+n}} = \\ &= \frac{1}{s^{q+1/2}} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!} \left(-\frac{A}{s}\right)^n = \frac{1}{s^q \sqrt{s+A}} \end{aligned} \quad (55)$$

The Laplace transform of $i_2(t)$ is:

$$\begin{aligned}
\mathcal{L}\{i_2(t)\} &= \mathcal{L}\left\{\frac{2B^2 t^{q+1/2}}{\sqrt{\pi}} F_{1;1}^{1;1}\left(\frac{3}{2}; \frac{1}{2}; -B^2 t, (B^2 - A)t\right)\right\} = \\
&= \frac{2B^2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(3/2)}{\Gamma(q+3/2)} \frac{(3/2)^{(n+k)}}{(q+3/2)^{(n+k)}} \frac{(1/2)^{(n)}}{(3/2)^{(n)}} \frac{(-B^2)^n (B^2 - A)^k}{n! k!} \mathcal{L}\{t^{n+k+q+1/2}\} = \\
&= \frac{2B^2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(3/2)}{1} \frac{(3/2)^{(n+k)}}{1} \frac{(1/2)^{(n)}}{(3/2)^{(n)}} \frac{(-B^2)^n (B^2 - A)^k}{n! k!} \frac{1}{s^{n+k+q+1/2}} = \\
&= \frac{B^2}{s^{q+3/2}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(3/2)^{(n)}(3/2+n)^{(k)}}{n!k!} \frac{(1/2)^{(n)}}{(3/2)^{(n)}} \left(\frac{-B^2}{s}\right)^n \left(\frac{B^2 - A}{s}\right)^k = \\
&= \frac{B^2}{s^{q+3/2}} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!} \left(\frac{-B^2}{s}\right)^n \sum_{k=0}^{\infty} \frac{(3/2+n)^{(k)}}{k!} \left(\frac{B^2 - A}{s}\right)^k = \\
&= \frac{B^2}{s^{q+3/2}} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!} \left(\frac{-B^2}{s}\right)^n \left(1 - \frac{B^2 - A}{s}\right)^{-3/2-n} = \\
&= \frac{B^2}{s^{q+3/2}} \left(1 - \frac{B^2 - A}{s}\right)^{-3/2} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!} \left(\frac{-B^2}{s - B^2 + A}\right)^n = \\
&= \frac{B^2}{s^q} \frac{1}{(s - B^2 + A)^{3/2}} \left(1 - \frac{-B^2}{s - B^2 + A}\right) = \\
&= \frac{B^2}{s^q (s - B^2 + A) \sqrt{s + A}}
\end{aligned} \tag{56}$$

The Laplace transform of $i_3(t)$ is:

$$\begin{aligned}
\mathcal{L}\{i_3(t)\} &= \mathcal{L}\left\{\frac{Bt^q}{\Gamma(q+1)} M(1; q+1; (B^2 - A)t)\right\} = \\
&= \frac{B}{\Gamma(q+1)} \sum_{n=0}^{\infty} \frac{n!(B^2 - A)^n}{n!(q+1)^{(n)}} \mathcal{L}\{t^{q+n}\} = \\
&= \frac{B}{\Gamma(q+1)} \sum_{n=0}^{\infty} \frac{(B^2 - A)^n \Gamma(q+n+1)}{(q+1)^{(n)} s^{q+n+1}} = \\
&= \frac{B}{\Gamma(q+1)} \sum_{n=0}^{\infty} \frac{(B^2 - A)^n \Gamma(q+1)(q+1)^{(n)}}{(q+1)^{(n)} s^{q+n+1}} = \\
&= \frac{B}{s^{q+1}} \sum_{n=0}^{\infty} \left(\frac{B^2 - A}{s}\right)^n = \frac{B}{s^{q+1}} \frac{1}{1 - \frac{B^2 - A}{s}} = \\
&= \frac{B}{s^q (s - B^2 + A)}
\end{aligned} \tag{57}$$

Finally, combining Eqs.(54)–(57), we recover the original complex function $\bar{G}(s)$:

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \frac{1}{s^q \sqrt{s+A}} - \frac{B}{s^q (s - B^2 + A)} + \frac{B^2}{s^q (s - B^2 + A) \sqrt{s+A}} = \\
&= \frac{1}{s^q} \frac{s - B^2 + A - B\sqrt{s+A} + B^2}{(s - B^2 + A) \sqrt{s+A}} = \\
&= \frac{1}{s^q} \frac{\sqrt{s+A}(\sqrt{s+A} - B)}{(\sqrt{s+A} - B)(\sqrt{s+A} + B) \sqrt{s+A}} = \\
&= \frac{1}{s^q} \frac{1}{\sqrt{s+A} + B} = \bar{G}(s)
\end{aligned} \tag{58}$$

Eq.(58) completes the proof of Theorem 1 in the Laplace transform direction (\Leftarrow). \square

This proof confirms that the derived inverse Laplace transform in Eq.(53) is consistent and exact.

5. Analyzis and Discussion

In this section, we demonstrate that many well-known analytical solutions reported in the literature are contained in the general solution derived in this work as special cases Table 1 summarizes several special cases that are embedded in the complex function considered in Eq.(1). For each case, the corresponding inverse Laplace transform is provided together with its known analytical form from the literature [25].

Starting from the general solution Eq.(33), it is shown that in all these specific parameter regimes, the obtained expression naturally reduces to the classical analytical results.

Table 1. Known solutions of the inverse Laplace transform for special cases of combined shifted quasi-rational functions with square-root radicals and power-law decay terms [25].

Special case number	Parameters	$\bar{G}(s)$	$g(t)$
1	$q \in (0, +\infty), A = 0, B = 0$	$\frac{1}{s^{q+1/2}}$	$\frac{t^{q-1/2}}{\Gamma(q+1/2)}$
2	$q = 1/2, A \neq 0, B = 0$	$\frac{1}{\sqrt{s}} \frac{1}{\sqrt{s+A}}$	$e^{-At/2} I_0\left(\frac{At}{2}\right)$
3	$q = 1, A \neq 0, B = 0$	$\frac{1}{s} \frac{1}{\sqrt{s+A}}$	$\frac{1}{\sqrt{A}} \operatorname{erf}(\sqrt{At})$
4	$q = 3/2, A \neq 0, B = 0$	$\frac{1}{s^{3/2}} \frac{1}{\sqrt{s+A}}$	$te^{-At/2} (I_0(At/2) + I_1(At/2))$
5	$q = 1/2, A = 0, B \neq 0$	$\frac{1}{\sqrt{s}} \frac{1}{\sqrt{s+B}}$	$e^{B^2 t} \operatorname{erfc}(B\sqrt{t})$
6	$q = 1, A = 0, B \neq 0$	$\frac{1}{s} \frac{1}{\sqrt{s+B}}$	$\frac{1}{B} \left(1 - e^{B^2 t} \operatorname{erfc}(B\sqrt{t})\right)$
7	$q = 1, A = 0, B \neq 0$	$\frac{1}{s} \frac{1}{\sqrt{s+B}}$	$\frac{1}{B} \left(1 - E_{1/2}(-B\sqrt{t})\right)$

Lemma 1.

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^{q+1/2}} \right\} = \frac{t^{q-1/2}}{\Gamma(q+1/2)} M(1/2, q+1/2, 0) = \frac{t^{q-1/2}}{\Gamma(q+1/2)}, \quad q \in (0, +\infty)$$

Proof. By substituting the identity given in Eq. (19) ($M(1/2, q+1/2, 0)$) into Eq. (33), the following relation is obtained:

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^{q+1/2}} \right\} = \frac{t^{q-1/2}}{\Gamma(q+1/2)} M(1/2, q+1/2, 0) = \frac{t^{q-1/2}}{\Gamma(q+1/2)}$$

Hence, the Lemma is proven. \square

Lemma 2.

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\frac{1}{\sqrt{s+A}}\right\} = M(1/2, 1, -At) = e^{-At/2}I_0(At/2)$$

Proof. The proof employs Kummer's transformation Eq. (11) together with the relation between Kummer's function and the modified Bessel function given in Eq. (21):

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\frac{1}{\sqrt{s+A}}\right\} &= M(1/2, 1, -At) = e^{-At}M(1/2, 1, At) = \\ &= e^{-At}e^{At/2}I_0(At/2) = e^{-At/2}I_0(At/2)\end{aligned}$$

Thus, Lemma 2 is thus proven. \square

Lemma 3.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{\sqrt{s+A}}\right\} = \frac{t^{1/2}}{\Gamma(3/2)}M(1/2, 3/2, -At) = \frac{1}{\sqrt{A}}\operatorname{erf}(\sqrt{At})$$

Proof. Starting from Eq. (33) and replacing the obtained Kummer function $M(1/2, 3/2, -At)$ using Eq. (23), the following identity follows:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{\sqrt{s+A}}\right\} &= \frac{t^{1/2}}{\Gamma(3/2)}M(1/2, 3/2, -At) = \frac{t^{1/2}}{(1/2)\Gamma(1/2)}\frac{\sqrt{\pi}}{2(\sqrt{At})}\operatorname{erf}(\sqrt{At}) = \\ &= \frac{1}{\sqrt{A}}\operatorname{erf}(\sqrt{At})\end{aligned}$$

Lemma 3 is thereby proven. \square

Lemma 4.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\frac{1}{\sqrt{s+A}}\right\} = tM(1/2, 2, -At) = te^{-At/2}(I_0(At/2) + I_1(At/2))$$

Proof. We use two contiguous relations for the confluent hypergeometric function (Eqs. (18), (19)), along with the connections between Kummer and modified Bessel functions (Eqs. (21), (22)).

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\frac{1}{\sqrt{s+A}}\right\} &= tM(1/2, 2, -At) = te^{-At}M(3/2, 2, At) = \\ &= te^{-At}\left[M(1/2, 1, At) + \frac{At}{4}M(3/2, 3, At)\right] = \\ &= te^{-At/2}(I_0(At/2) + I_1(At/2))\end{aligned}$$

Hence, Lemma 4 is proven. \square

Lemma 5.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}(\sqrt{s+B})}\right\} &= M(1/2, 1, 0) + B^2tF_{1,1}^{1,1}\left(\frac{3}{2}; \frac{1}{2}; -B^2t, B^2t\right) - \\ &= \frac{B\sqrt{t}}{\Gamma(3/2)}M(1, 3/2, B^2t) = e^{B^2t}\operatorname{erfc}(B\sqrt{t})\end{aligned}$$

Proof. We apply Eq. (20), Kummer's transformation (Eq. (11)), the relation between Kummer and error functions (Eq. (23)), and the reduction identity for the KdF function (Eq. (26), proven in Appendix B).

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}(\sqrt{s}+B)}\right\} &= M(1/2, 1, 0) + B^2 t F_{1;1}^{1;1}\left(\frac{3/2}{2}; \frac{1/2}{3/2}; -B^2 t, B^2 t\right) - \\
&\quad - \frac{B\sqrt{t}}{\Gamma(3/2)} M(1, 3/2, B^2 t) = \\
&= 1 + B^2 t \left(\frac{e^{B^2 t} - 1}{B^2 t}\right) - \frac{B\sqrt{t}}{\Gamma(3/2)} \Gamma(3/2) \frac{\operatorname{erf}(B\sqrt{t})}{B\sqrt{t}} e^{B^2 t} = \\
&= 1 + (e^{B^2 t} - 1) - e^{B^2 t} \operatorname{erf}(B\sqrt{t}) = e^{B^2 t} (1 - \operatorname{erf}(B\sqrt{t})) = \\
&= e^{B^2 t} \operatorname{erfc}(B\sqrt{t})
\end{aligned}$$

This completes the proof of Lemma 5. \square

Lemma 6.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s} \frac{1}{\sqrt{s}+B}\right\} &= \frac{\sqrt{t}}{\Gamma(3/2)} M(1/2, 3/2, 0) + \frac{B^2 t^{3/2}}{\Gamma(5/2)} F_{1;1}^{1;1}\left(\frac{3/2}{5/2}; \frac{1/2}{3/2}; -B^2 t, B^2 t\right) - \\
&\quad - \frac{Bt}{\Gamma(2)} M(1, 2, B^2 t) = \frac{1}{B} (1 - e^{B^2 t} \operatorname{erfc}(B\sqrt{t}))
\end{aligned}$$

Proof. We use Eq. (20), Eq. (25) (linking Kummer and exponential function), and the reduction formulas for KdF given by Eq. (27) (proven in Appendix B).

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s} \frac{1}{\sqrt{s}+B}\right\} &= \frac{\sqrt{t}}{\Gamma(3/2)} M(1/2, 3/2, 0) + \frac{B^2 t^{3/2}}{\Gamma(5/2)} F_{1;1}^{1;1}\left(\frac{3/2}{5/2}; \frac{1/2}{3/2}; -B^2 t, B^2 t\right) - \\
&\quad - \frac{Bt}{\Gamma(2)} M(1, 2, B^2 t) = \\
&= \frac{B\sqrt{t}}{B\Gamma(3/2)} + \frac{(B\sqrt{t})^3}{B\Gamma(5/2)} \frac{3}{2B^2 t} (M(1, 3/2, B^2 t) - 1) - \frac{B^2 t e^{B^2 t} - 1}{B} = \\
&= \frac{B\sqrt{t}}{B\Gamma(3/2)} + \frac{B\sqrt{t}}{B\Gamma(3/2)} (e^{B^2 t} M(1/2, 3/2, -B^2 t) - 1) - \frac{1}{B} (e^{B^2 t} - 1) = \\
&= \frac{1}{B} e^{B^2 t} \operatorname{erf}(B\sqrt{t}) - \frac{1}{B} (e^{B^2 t} - 1) = \\
&= \frac{1}{B} (e^{B^2 t} (\operatorname{erf}(B\sqrt{t}) - 1) + 1) = \\
&= \frac{1}{B} (1 - e^{B^2 t} \operatorname{erfc}(B\sqrt{t}))
\end{aligned}$$

Lemma 6 is thus established. \square

Lemma 7.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s(\sqrt{s}+B)}\right\} &= \frac{\sqrt{t}}{\Gamma(3/2)} M(1/2, 3/2, 0) + \frac{B^2 t^{3/2}}{\Gamma(5/2)} F_{1;1}^{1;1}\left(\frac{3/2}{5/2}; \frac{1/2}{3/2}; -B^2 t, B^2 t\right) - \\
&\quad - \frac{Bt}{\Gamma(2)} M(1, 2, B^2 t) = \frac{1}{B} (1 - E_{1/2}(-B\sqrt{t}))
\end{aligned}$$

Proof. Using Lemma 6 and the relation between the ML and error function given by Eq. (29), we obtain:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(\sqrt{s+B})}\right\} &= \frac{\sqrt{t}}{\Gamma(3/2)} M(1/2, 3/2, 0) + \frac{B^2 t^{3/2}}{\Gamma(5/2)} F_{1;1}^{1;1}\left(\frac{3/2}{5/2}; \frac{1/2}{3/2}; -B^2 t, B^2 t\right) - \\ &- \frac{Bt}{\Gamma(2)} M(1, 2, B^2 t) = \frac{1}{B} \left(1 - e^{B^2 t} \operatorname{erfc}(B\sqrt{t})\right) = \\ &= \frac{1}{B} \left(1 - E_{1/2}(-B\sqrt{t})\right)\end{aligned}$$

Hence, Lemma 7 is proven. \square

6. Conclusion

We derived a rigorous closed-form analytical expression for the inverse Laplace transform of a class of shifted quasi-rational spectral functions with square-root radicals. This constitutes the main contribution of the paper, representing a solution not previously reported in the literature. The derived inverse transform is expressed in terms of generalized hypergeometric functions, including Kummer and five-parameter Kampé de Fériet functions, and exhibits a convolution structure that has not been represented analytically before. Reduction formulas for a subclass of the Kampé de Fériet function were obtained as a secondary result, demonstrating how the general solution reduces to known special cases. The closed-form solution is validated analytically by reapplying the Laplace transform, which fully recovers the original spectral function.

This work represents the first step of a modular research framework, which will be extended in future studies to include asymptotic analysis, numerical validation, and systematic applications in photothermal and photoacoustic signal analysis, as motivated by the broader context of coupled diffusion-type processes and inverse procedures highlighted in the Introduction, demonstrating the utility of analytic inverses in both theoretical and experimental investigations.

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Abbreviations

The following abbreviations are used in this manuscript:

PT	Photothermal
PA	Photoacoustic
KdF	Kampé de Fériet
ML	Mittag-Leffler

Appendix A. The Function $\operatorname{erf}(x)$ Is Entire Function-Proof

If $f(z)$ is equal to the sum of its Taylor series for all z in complex plane, it is called entire function. It means that the radius of convergence of Taylor series of entire function is infinite. Such functions are exponential function, trigonometric function sine and cosine, etc.

The radius of convergence of infinite series is defined by the following expression [56]. Let

$$(Tf)(z) = \sum_{n=0}^{\infty} a_n z^n. \quad (\text{A1})$$

The radius of convergence of this series is defined by

$$R_{Tf} = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|. \quad (\text{A2})$$

Using this definition and Taylor series expansion of error function Eq. (48), we obtain

$$R = \lim_{n \rightarrow \infty} \left| \frac{\frac{2(-1)^n}{\sqrt{\pi} n!(2n+1)}}{\frac{2(-1)^{n+1}}{\sqrt{\pi} (n+1)!(2n+3)}} \right|. \quad (\text{A3})$$

After simplification, this yields

$$R = \lim_{n \rightarrow \infty} \frac{(n+1)!(2n+3)}{n!(2n+1)} = \lim_{n \rightarrow \infty} \frac{(n+1)(2n+3)}{2n+1}. \quad (\text{A4})$$

Since

$$\lim_{n \rightarrow \infty} \frac{(n+1)(2n+3)}{2n+1} = +\infty, \quad (\text{A5})$$

we conclude that the radius of convergence is

$$R = +\infty. \quad (\text{A6})$$

Since the radius of convergence of the Taylor series expansion of the $\operatorname{erf}(\cdot)$ function is infinite, this power series can represent the $\operatorname{erf}(\cdot)$ function over its entire domain of definition.

Appendix B. The Proof of Reduction Identities for Specific Five Parameter KdF Function Given by Eq. (26) and Eq. (27)

In proving the identities given by Eq. (26) and Eq. (27) (reduction identity for specific five-parameter KdF functions that doesn't depend on fifth parameter and which arguments are of converse signs), the series representation of the confluent hypergeometric function (Eq. (9)) and the representation of the KdF function via double sum (Eq. (12)) are used. We also used Kummer's transformation given by Eq. (11) and identity given by Eq. (25), as well as definition of rising factorial, Eq. (13).

As part of argument, we apply diagonal summation, which allows us to rewrite the double sum as a single sum over total index $n = i + j$ [57]. The diagonal summation rule is given by:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(i, j, i+j) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} f(n-j, j, n)$$

where $n = i + j$, and each pair $(i, j) \in \mathbb{N}_0$ such that $n = i + j$ is included exactly once.

The proof of reduction identity for KdF function given by Eq. (26)

$$\begin{aligned}
F_{1;1}^{1;1} \left(\begin{matrix} 3/2, 1/2 \\ 2, 3/2 \end{matrix} ; -y, y \right) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(3/2)^{(n+k)} (1/2)^{(n)} (-y)^n (y)^k}{(2)^{(n+k)} (3/2)^{(n)} n! k!} = \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(3/2)^{(m)} (1/2)^{(m-k)} (-1)^{m-k} y^m}{(2)^{(m)} (3/2)^{(m-k)} k! (m-k)!} = \\
&= \sum_{m=0}^{\infty} y^m \frac{(3/2)^{(m)}}{(m+1)!} \sum_{k=0}^m \frac{(1/2)^{(m-k)} (-1)^{m-k}}{(3/2)^{(m-k)} k! (m-k)!} = \\
&= \frac{y^0}{1!} + \frac{y^1}{2!} \frac{3}{2} \left(-\frac{1/2}{3/2} + \frac{1/2}{1/2} \right) + \\
&+ \frac{y^2}{3!} \frac{3}{2} \frac{5}{2} \left(\frac{1/2}{5/2} \cdot \frac{1}{2!} - \frac{1/2}{3/2} \cdot \frac{1}{1!} + \frac{1/2}{1/2} \cdot \frac{1}{2!} \right) + \\
&+ \frac{y^3}{4!} \frac{3}{2} \frac{5}{2} \frac{7}{2} \left(-\frac{1/2}{7/2} \cdot \frac{1}{3!} + \frac{1/2}{5/2} \cdot \frac{1}{2!} - \frac{1/2}{3/2} \cdot \frac{1}{2!} + \frac{1/2}{1/2} \cdot \frac{1}{3!} \right) + \dots = \\
&= \frac{y^0}{1!} + \frac{y^1}{2!} + \frac{y^2}{3!} + \frac{y^3}{4!} + \dots = \\
&= \sum_{j=0}^{\infty} \frac{y^j}{(j+1)!} = \sum_{j=0}^{\infty} \frac{1^{(j)} y^j}{(2)^{(j)} j!} = \\
&= M(1, 2, y)
\end{aligned}$$

The proof of reduction identity for KdF function given by Eq. (27)

$$\begin{aligned}
F_{1;1}^{1;1} \left(\begin{matrix} 3/2, 1/2 \\ 5/2, 3/2 \end{matrix} ; -y, y \right) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(3/2)^{(n+k)} (1/2)^{(n)} (-y)^n (y)^k}{(5/2)^{(n+k)} (3/2)^{(n)} n! k!} = \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(3/2)^{(m)} (1/2)^{(m-k)} (-1)^{m-k} y^m}{(5/2)^{(m)} (3/2)^{(m-k)} k! (m-k)!} = \\
&= \sum_{m=0}^{\infty} y^m \frac{(3/2)^{(m)}}{(5/2)^{(m)}} \sum_{k=0}^m \frac{(1/2)^{(m-k)} (-1)^{m-k}}{(3/2)^{(m-k)} k! (m-k)!} = \\
&= \sum_{m=0}^{\infty} y^m \frac{3/2}{3/2 + m} \sum_{k=0}^m \frac{(1/2) (-1)^{m-k}}{(1/2 + m - k) k! (m-k)!} = \\
&= y^0 \frac{3/2 \cdot 1/2}{3/2 \cdot 1/2} + y^1 \frac{3/2}{5/2} \left(-\frac{1/2}{3/2} + \frac{1/2}{1/2} \right) + \\
&+ y^2 \frac{3/2}{7/2} \left(\frac{1/2}{5/2} \cdot \frac{1}{2!} - \frac{1/2}{3/2} \cdot \frac{1}{1!} + \frac{1/2}{1/2} \cdot \frac{1}{2!} \right) + \\
&+ y^3 \frac{3/2}{9/2} \left(-\frac{1/2}{7/2} \cdot \frac{1}{3!} + \frac{1/2}{5/2} \cdot \frac{1}{2!} - \frac{1/2}{3/2} \cdot \frac{1}{2!} + \frac{1/2}{1/2} \cdot \frac{1}{3!} \right) + \dots = \\
&= \frac{y^0}{1} + \frac{y^1}{\frac{5}{2}} + \frac{y^2}{\frac{5}{2} \cdot \frac{5}{2}} + \frac{y^3}{\frac{5}{2} \cdot \frac{5}{2} \cdot \frac{5}{2}} + \dots = \\
&= \sum_{j=0}^{\infty} \frac{y^j}{(5/2)^{(j)}} = \frac{3/2}{y} \sum_{j=0}^{\infty} \frac{y^{j+1}}{(3/2) (5/2)^{(j)}} = \\
&= \frac{3/2}{y} \sum_{j=1}^{\infty} \frac{y^j}{(3/2)^{(j)}} = \frac{3/2}{y} \left(\sum_{j=0}^{\infty} \frac{y^j}{(3/2)^{(j)}} - 1 \right) = \\
&= \frac{3/2}{y} \left(\sum_{j=0}^{\infty} \frac{1^{(j)} y^j}{(3/2)^{(j)} j!} - 1 \right) = \frac{3/2}{y} (M(1, 3/2, y) - 1)
\end{aligned}$$

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