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Article

Numerical Computation of Critical Binding Parameters of Screened Coulomb Potentials

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Abstract

For nearly a century screened Coulomb potentials have been of recognized importance in diverse areas of physics and chemistry. A key feature of interest in these potentials is the phenomenon of critical screening. This paper has three main purposes: To present an extensive, open-access, high accuracy (60 digit) benchmark reference data set of critical screening parameters, with validation; to confirm excellent past work in the field (to 30 digits), and to correct an historical oversight in its literature; and to present the essentials of our new approach, the "Phase Method" (PM), for computing them. Using the PM we calculate critical screening parameters, accurate to 60 decimal digits, for the Yukawa/Debye, Hulthén, Pseudo-Hulthén, and Exponential Cosine Screened Coulomb (ECSC) potentials. The practical feasibility of such calculations on inexpensive hardware opens up new possibilities in research and education. We highlight an apparently overlooked 1989 paper of Demiralp on critical screening parameters of the Yukawa potential, which accurately calculated them to 30 decimal digits. Our main results are computations of the critical screening parameters $\mu_c = 1/\mathcal{D}_c$ for screening lengths $\mathcal{D} \leq 1000$ au and angular momenta $l = 0 \dots 20$. The claimed accuracy of our results is supported by several independent lines of evidence: comparison with the most accurate (30 digit) values available in the print literature for the Yukawa, Hulthén, and ECSC potentials; comparison to 60 decimal digits accuracy with exactly known eigenvalues and critical binding parameters of the Pseudo-Hulthén potential; consistency tests between computed critical parameters, for various l -values for the Pseudo-Hulthén Potential, and known exact relations between eigenvalues; and application of a novel consistency test between results with different potential parameters, that exploits an exact scaling symmetry of this entire class of potentials. Similar calculations were done for ECSC and Yukawa potentials for screening lengths up to $\mathcal{D} \leq 10^5$ and $l \leq 12$, to 30 digit accuracy, which show interesting (and to our knowledge not previously reported) periodic structure in $\mathcal{D}_c(n, l)$ for the ECSC potential that is not observed for the Yukawa potential. The asymptotic scaling behavior for the Yukawa and Hulthén potentials is explained quantitatively by simple semiclassical calculations.

Keywords: screened coulomb; critical screening; critical binding; Yukawa potential; Debye potential; Hulthén potential; ECSC potential; plasma physics; Schrödinger equation; phase method

1. Introduction

1.1. Background

The Coulomb interaction between two point charges tends to become exponentially screened when the charges are embedded in a medium with mobile or virtual charges. In plasmas, semiconductors, or electrolyte solutions the characteristic length over which this occurs is called the Debye (or screening) length. The screening phenomenon appears in many areas of science and engineering, including nuclear and particle physics; dark matter; cosmology; astrophysics; plasma physics; inertial confinement fusion; condensed matter physics; biophysics; chemistry; semiconductor device physics; nanotechnology; and nanophotonics.

This behavior can be modeled by Screened Coulomb Potentials, such as Yukawa/Debye: $Ze^{-\mu r}/r$; Exponential Cosine Screened Coulomb (ECSC): $Ze^{-\mu r} \cos(\mu r)/r$; and Hulthén $Ze^{-\mu r}/\rho$ with $\rho \equiv$

$(1 - e^{-\mu r})/\mu$; for the moment, for simplicity, we suppress writing the centrifugal potential. μ is the screening parameter, and Z is a strength parameter (coupling constant, or atomic number, depending on context). As is well-known, and we will show below, Z can be set to 1 without loss of generality - doing so effectively rescales the screening length.

In this paper we have a simple goal: to directly determine the number of bound quantum states there are for these potentials as a function of screening length $\mathcal{D} = 1/\mu$ and angular momentum l . Our initial motivation for addressing this subjects was as a physically important and nontrivial application of our "Phase Method" [1] (PM). The PM accurately, robustly, and automatically calculates quantum energy eigenvalues and wave functions for a broad range of potentials, including discontinuous and singular ones, as described in [1]. It also turns out to be quite effective (with slight variations) and accurate for our present task of directly calculating critical binding parameters. The exact solutions for the Pseudo-Hulthén Potential provide important benchmarks against which to test our methods to high accuracy, as do several other analytical results (e.g. eigenvalues for pure Coulomb potential), and also other internal cross-checks and exact scaling relations, which are described below.

1.2. Preliminaries

It is well-known e.g. [8] that the three-dimensional time independent Schrödinger equation can be separated into an effective radial one-dimensional equation, and another, angular, equation, yielding spherical harmonics as solutions, by adding a centrifugal potential term $l(l+1)/(2r^2)$ into the radial equation: $-1/2 \Phi''(r) + (U(r) + l(l+1)/(2r^2))\Phi(r) = \mathcal{E}\Phi(r)$, with $\Phi(r)$ being the solution to the radial differential equation and l being the angular momentum quantum number. A proper wave function solution vanishes at $r = 0$ and $r \rightarrow \infty$, but we make use of more general divergent solutions in the Phase Method. With no loss of generality we also will use atomic-like units in which $\hbar = m = 1$, where m is the reduced mass of the two-particle (e.g. electron-ion) system. In the limit of infinite ion mass this is equivalent to atomic units and henceforth we will refer to our units as au.

Following Lam and Varshni [4] we define a variable ρ that serves as an approximation to r that is accurate at short distances, which allows the problem to be exactly solved analytically [6] for $l = 0$ (i.e. no centrifugal potential), but which deviates significantly from r at large distances. This Hulthén Potential is not exactly soluble for $l > 0$, but following Greene and Aldrich [7] it can be extended to what we call the "Pseudo-Hulthén" potential (Greene and Aldrich call it the "Hulthén effective potential") which is identical to the Hulthén potential except that it replaces the $1/r^2$ factor in the centrifugal potential with $1/\rho^2$ while also multiplying it by a $e^{-\mu\rho}$ damping factor, which has the effect of partially compensating for the fact that $1/\rho^2 \rightarrow 1/\mu^2$ rather than $1/r^2$ at large distances. Evidently for $l = 0$ the "Pseudo-Hulthén" and Hulthén' potentials are identical. Although this modifies the asymptotic potential, it renders the problem *exactly* soluble [6] in terms of Hypergeometric Functions for all l , for the energy eigenvalues and critical binding parameters, making it invaluable for our benchmark comparisons.

We first observe that for the screened Coulomb Potentials considered here, the change of variable $r \rightarrow s/Z$ transforms our original Schrödinger equation

$$-\frac{1}{2} \frac{d^2\Phi(r)}{dr^2} + \left(Z \frac{e^{-\mu r}}{r} + \frac{l(l+1)}{2r^2} \right) \Phi(r) = \mathcal{E}\Phi(r),$$

to

$$-\frac{1}{2} \frac{d^2\Phi(s)}{ds^2} + \left(\frac{e^{-\bar{\mu}s}}{s} + \frac{l(l+1)}{2s^2} \right) \Phi(s) = \bar{\mathcal{E}}\Phi(s),$$

with $\bar{\mu} \equiv \mu/Z$ and $\bar{\mathcal{E}} \equiv \mathcal{E}/Z^2$. The coefficient Z of the potential is absorbed in the rescaling; if the (reduced) mass m is not set to 1, the product of Zm sets the scale. It is easily shown in a similar manner that the Hulthén potential, ECSC, and pseudo-Hulthén potentials have this same scaling property, indeed any potential of the form $f(\mu r)/r$ or $f(\mu r)/\rho$ + centrifugal potential. Without loss of generality

we therefore take $Z = 1$, unless otherwise noted. Below we will use a variant of this scaling symmetry as an internal check on the accuracy of our higher precision calculations.

The attractive Coulomb potential $U(r) = -1/r$ possesses an infinite number of bound states. General arguments [8] show that potentials that decrease faster than $1/r^2$ at large distances only have a finite number of bound states. The Yukawa potential at short distances is Coulomb-like, as are the other potentials considered here, with the same infinite attractive $-1/r$ singularity at $r = 0$. At large distances the exponential decay factor $U(r) = -e^{-\mu r}/r$ reduces the attraction relative to a Coulomb potential, so that it admits only a finite number of bound states, the number depending critically on the value of μ (or equivalently its inverse $\mathcal{D} = 1/\mu$), and the angular momentum l . For example, for the Yukawa/Debye potential and $l = 0$, the critical screening parameter at which a single s-state just becomes bound is $\mu_c \approx 1.1906124 \dots$; a second state becomes bound for $l = 0$ at $\mu_c \approx .310209 \dots$

A key point regarding the critical binding parameters is that new physical processes/channels turn on or off at these critical values. This strongly affects photoionization processes in plasma physics and astrophysics [9,10], affecting spectra and opacity of stellar atmospheres. The related potentials Hulthén and Exponential Cosine Screened Coulomb (ECSC) have analogous behavior, but with their own critical binding parameters. These have been studied for many decades. Past work has been reviewed by Roy [11], and more recently by Jiao et al [3].

Here we present direct calculations of these parameters to substantially greater accuracy than in the past, i.e. 60 decimal digits, and demonstrate agreement with independent calculations [2,3] at their upper limit of 30 digits. This consensus, using quite different methods, is a strong validation of all of these results, over the range of digits that they happen to agree.

Although our computational approach calculates critical binding parameters directly, without the need to compute sequences of eigenvalues as a function of μ and extrapolating them to zero energy, it is still edifying to show eigenvalues can be accurately calculated by our methods for these potentials. We compare our eigenvalues with those of Stubbins [12] and Vrscay [13] for their selected values of μ and l for the Yukawa and Hulthén potentials, and also compare with exact values for Pseudo-Hulthén potential, with excellent agreement.

It should be noted that our methods and code readily can be used to calculate a greater number of accurate digits (if there is a need), higher l values, longer screening lengths, and other potentials – it is simply a matter of computation time. Our main data set presented here includes critical binding parameters to 60 accurate digits for Yukawa/Debye, Hulthén, Pseudo-Hulthén, and Exponential Cosine Screened Coulomb (ECSC) for $l = 0-20$, and critical screening lengths up to $\mathcal{D} = 10^3$. We also compute to the lower level of accuracy of 30 digits all critical binding parameters for Yukawa/Debye and ECSC for $l = 0-12$ and up to $\mathcal{D} = 10^5$ au and find intriguing structure in \mathcal{D}_c vs n in the latter.

There has been some disagreement in the literature about these values in recent years. Edwards [14] claimed highest accuracy to 10 digits, which disagreed with our own early (unpublished) calculations at the time, which were accurate to 30 digits, as well as Demiralp's [2], Rogers [15], and others. To help clarify this confusion our main original purpose in these investigations was to help clarify the problem by independently computing the critical parameters, and secondarily to demonstrate the method that we have developed to obtain them.

We initially found (in preliminary unpublished work) that our results were almost entirely in agreement with the work of Demiralp [2], which claimed 30 digits of accuracy. This very high degree of accuracy at the time (1989) was attained using minimal computational resources by modern standards, on a DEC VAX 11-780 using quadruple-precision arithmetic, by developing sophisticated variational methods with basis sets that were specifically constructed and optimized for each state so as to obtain rapid convergence. Our results, which are relatively simple and direct, but have the benefit of modern computational tools that we use in unconventional ways, also agree (to within their stated accuracies) with those of Rogers [15] (5 digits, 1970), Diaz [20] (15 digits, 1991), Singh [16] (15 digits, 1993), Napsuciale [17] (10 digits, 2021), Del Valle [18] (17 digits, 2018); yet disagree in the least few significant digits with those of Gomes et al [19] (1994), and Edwards et al [14] (10 digits, 2017).

In the literature the discrepancies in these claimed values appeared to have caused some investigators to doubt some results that actually were correct at higher accuracy, such as those of Diaz [20]. Yet the singular paper of Demiralp [2] appears to have been overlooked by all, and is rarely if ever cited. The high degree of agreement with Demiralp's tables of results independently calculated (to 30+) digits by two other groups, using quite different methods, indicates that his work continued to be the most accurate for decades until matched by the recent work of Jiao et al [3], and our own unpublished work, which were of the same level of accuracy as [2]. Secondly, the observed agreement between our results and Demiralp's also lends support to our new methods, which were independently developed before we became aware of either Demiralp's or Jiao's prior work.

The Phase Method [1] allows calculations to much higher accuracy than previously have been published on this topic to date. Here we present critical exponents to 60 decimal digits for screened coulomb potentials up to $l = 20$ and $\mathcal{D} = 1000$. They are validated by comparison with exact results (Coulomb, Hulthén $l = 0$, and Pseudo-Hulthén for arbitrary l); with prior publications up to their maximum of 30 digits; and by independent calculations using our ζ -scaling test. We also find that we obtain the same level of accuracy for other potentials including Hulthén, extended Hulthén, and Exponential Cosine Screened Coulomb (ECSC). More generally, we observe that the Phase Method can be conveniently applied to many other potentials when more normal levels of accuracy (say, 15-30 digits) are required. Our Mathematica [23] code is freely available for download for independent verification and use at <http://gbxafs.iit.edu/phase-method/>.

2. Materials and Methods

2.1. Preliminaries

The Yukawa/Debye/Static Screened Coulomb potential is a central (isotropic) potential. It is well known [8] that for non-zero angular momentum l , by addition of a centrifugal potential term, the three-dimensional problem becomes reducible to solving an equivalent one-dimensional potential

$$U(r) = -\frac{e^{-\mu r}}{r} + \frac{l(l+1)}{2r^2}$$

in the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2\Psi(r)}{dr^2} + U(r)\Psi(r) = \mathcal{E}\Psi(r).$$

Here $\Psi(r) = rR(r)$ and $R(r)$ is the radial wave function. The appropriate boundary conditions are $\Psi(0) = 0$ and $\Psi(r \rightarrow \infty) = 0$. We solve this here through direct calculation by a variant of the Phase Method [1]. As shown in [1] this approach is equally applicable to many potentials including other Yukawa/Debye-related potentials such as Hulthén and Exponential Cosine Screened Coulomb (ECSC).

2.2. Computational Approach

To calculate \mathcal{D}_c values we use essentially the same logic (and only slightly modified code) as the Phase Method [1] for finding the eigenvalues. Solving for all the energy eigenvalues \mathcal{E} for a fixed potential parameter μ (or equivalently, \mathcal{D}) within a given range of energies is isomorphic to the problem of solving for all the \mathcal{D}_c values within a specified range of \mathcal{D} , for fixed $\mathcal{E} = 0$. As described below, there are differences, however, when it comes to the choice of data range.

In this work we directly calculate the critical parameters μ_c (or \mathcal{D}_c). In contrast to most other approaches, we do not determine them by calculating a sequence of eigenvalues as a function of μ and extrapolating it to zero energy. Our approach is similar in spirit to Schey and Schwartz [22], and Edwards [14], but our methods are distinct from theirs. As recognized by these authors, calculation of $\mathcal{D}_c = 1/\mu_c$ is less computationally demanding than that of general eigenvalues, because the integration occurs very close to zero energy, where the asymptotic behavior as $r \rightarrow \infty$ is very smooth ($\approx r^{l+1}$) – it is close to a straight line for $l = 0$. Because of this, large steps can be taken in that region by the adaptive ODE solver, and only modest amounts of computation time is spent in doing so. In contrast,

integrating the ODE for arbitrary energies is much more time consuming, because of the oscillatory nature of the solutions. In that case choosing the range more carefully can substantially reduce the time required. In the Phase Method, as applied to general energy levels (rather than just those near the continuum limit), a useful heuristic in such cases is to calculate the classical turning points at the highest considered energy, and multiply them by “turning point scale factors” to estimate a sufficient range at that energy. In this work, for energies near zero, such care is not needed to obtain practical computation times over extremely long ranges.

2.3. Number of Accurate Digits

A peculiar but very useful characteristic of the PM, as presently implemented, is that if the range is sufficient, the ultimate number of accurate digits is reliably half the working precision (as specified in numbers of decimal digits). This is related to the ODE solver being unable to continue (i.e. not registering a legitimate transition/zero appearing within the data range) when the square of the deviation of the test energy from the exact eigenvalue becomes too small to represent at the selected working precision. With arbitrary precision arithmetic we are not much concerned about wasting digits of precision, as long as the ultimate accuracy and run time are reasonable. A very useful consequence is that the number of accurate digits can be “dialed-in” before running the job simply by choosing the working precision, i.e. just by changing a single number at runtime, provided the x -range is sufficient. This applies to calculating eigenvalues in the PM, but also to calculating critical screening lengths D_c .

The merits of calculating such quantities to high precision of 60 digits are several-fold. Taking in quantities that are nearly equal can stretch the limits of accuracy of computed numerical results. Very accurate numerical results may suggest patterns to investigate, or to provide critical tests of analytical results. More generally, the methods demonstrated here may be of use in much broader contexts in physics, and mathematics.

The probability of getting 9 digits correct randomly is one in a billion. The probability of randomly getting 60 digits correct is one in a quadrillion quadrillion quadrillion quadrillion. If this happens, you’re probably onto something.

2.4. Working Precision, Data Range, and Accuracy

Computationally, to avoid divide by zero errors, we must approximate the singular $1/r$ potential. We do it here simply by starting the ODE solution at a very short distance above $r = 0$, r_{\min} at which we set the initial value of the ODE solution to be zero. The value of the initial slope is relatively unimportant as long as it is nonzero. Setting the initial function value to zero effectively places an infinitely high potential barrier at this short distance r_{\min} . To attain the desired high level of accuracy of D_c (or eigenvalues) for these potentials with the $1/r$ singularity it is necessary to choose a range that extends to extremely short distances, related to the desired accuracy, and also rather long distances.

The short distance criteria are different for $l = 0$ and $l > 0$ owing to the presence of the repulsive centrifugal potential. The ODE solution for $l = 0$ samples the attractive singular potential very close to $r = 0$. In contrast, for $l > 0$, the repulsive $1/r^2$ centrifugal potential shifts the weight of the solution away from $r = 0$, making it less sensitive to the potential at very small r . For $l = 0$ the minimum distance required for these $\approx 1/r$ potentials for n_{digits} precision is close to $< 10^{-n_{\text{digits}}}$; for 60 digits this is 10^{-60} , but there is little reason not to go further as a safety factor. For the calculations presented here we used $r_{\min} = 10^{-110}$ which is empirically found to be vastly more than sufficient. The short distance cutoff that is needed for $l > 0$ is much less stringent than this. To set the upper limit r_{\max} , rather than adjusting it incrementally upward in a series of calculations until sufficient convergence is obtained, we find it practical to choose a very large cutoff, and rely on the adaptive sampling of the ODE solver algorithm to obtain the desired accuracy within a reasonable time frame. This could be optimized but it is not clear there is a practical need to do so for this application.

For this reason we choose very (arguably, unnecessarily) large upper r -limits, yet still obtain reasonable computation times. The run time likely might be reduced somewhat by optimization, but we agree with the programming aphorism (usually attributed to Knuth) that “premature optimization

is the root of all evil', meaning, don't optimize code unless there is a practical reason to do so. For this reason we have not followed the conventional procedure of selecting sensible ranges and then incrementally extending them until the computed values converge to the desired accuracy. This can be time-consuming and as we have seen, unnecessary for this work. Rather, we have started with very large ranges and decreased them if needed. Convergence still can be assessed by reducing the range moderately and comparing results, but this need be done only rarely.

The ranges that ultimately were used in this work were indexed to the desired accuracy via the working precision "WP" as $[10^{(10-WP)}, 10^{(WP-10)}]$ for $l = 0$, and $[10^{(-WP/4)}, 10^{(WP/2)}]$ for $l = 1 \dots 20$. These ranges are purely heuristic and not especially optimized, and so might warrant some systematic study. Actual values for $WP = 120$ were $[10^{-110}, 10^{110}]$ for $l = 0$ and $[10^{-30}, 10^{60}]$ for $l > 0$. For $l < 3$ the lower cutoff that was initially used was 10^{-40} , but the calculation spontaneously aborted for $l \geq 4$ using that cutoff, presumably because of numerics associated with the $l(l+1)/(2r^2)$ centrifugal potential; relaxing the lower cutoff by increasing it to 10^{-30} for $l \geq 4$ was found to be both stable and sufficient for the desired accuracy up to $l = 20$, at least. These ranges were checked by comparing with exact results for Pseudo-Hulthén potential, then consistently used for the other potentials considered here.

2.5. Phase Method

2.5.1. PM in a nutshell

Here we give a brief outline of key aspects of the Phase Method for solving the Time Independent Schrödinger Equation for 1D potentials, and central/separable potentials (and for good measure, the Helmholtz Equation). First, eigenvalues are accurately found, then wave functions are generated from them, as needed. The PM is described in more detail below (with the adaptations used for the critical binding parameter problem), and in expanded form with many applications in [1].

- Phase Method is related to Shooting Method, but upside down and backwards.
- Sturm's separation theorem implies the number of zeros in the solution doesn't change even if the initial conditions do: their locations just move around.
- You don't have to find a proper convergent wave function to find eigenvalues.
- "Shoot First" Method - uses nominal initial slope/value conditions for specified energy parameter (no tweaking initial conditions/boundary conditions is needed).
- Choose x -range so as to *promote divergent* solutions by placing initial point far enough into the classically forbidden region. It actually is beneficial for reasons described below and in [1]
- Automatic x -range setting is/can be done by scaling computed classical turning points at highest energy (easily overridden if desired). The small x (aka r) criterion is different for potentials singular at $x = 0$, see below.
- "Phase Plot" digitizes divergent ODE solution for the purpose of counting transitions/ # of bound states, and visualizing to check appropriate x -range.
- Use Adaptive ODE solver (no fixed grids!) with Stiffness Switching for robustness.
- Parallel interval-based evolutionary solver finds all eigenvalues accurately and efficiently, *no initial guessing is needed, as it is in Shooting Method*.
- Automatic maximum and minimum \mathcal{E} -search range setting is/can be calculated based on potential (easily overridden if desired).
- Use Arbitrary Precision Arithmetic - really big numbers are fine. Take care not to pollute high precision numbers with low precision.
- The number of accurate digits of the computed eigenvalues is reliably half the number of digits of working precision, iff range is adequate. The reason is explained below. *User chooses the accuracy, you just have to wait for the answer.*
- "Shoot First" method can be used to compute wave functions starting with the very accurate eigenvalues from above, with automatic divergence detection/truncation at large x .

- “Self-healing” of differences in the logarithmic derivatives owing to different initial conditions occurs well into classically forbidden region (see below); solution is robust even if transient overflow occurs.
- Automatic validation of wave functions: Virial Theorem tests, expectation value vs eigenvalue; wave function convergence tests.

2.5.2. Expanded Description

In this paper we borrow some ideas and methods from the Phase Method (PM) [1] an approach that we have developed for solving the Schrödinger equation in 1D, and central and separable potentials. It automatically maps out all of the (finite number of) eigenvalues within a designated energy range, without discretizing the potential as in matrix methods, or needing to guess locations of eigenvalues or tweak boundary conditions as in the Shooting Method, to which the PM is most closely related. It uses standard highly performant ODE solvers with stiffness switching algorithms (originally LSODA, currently, StiffnessSwitching) that automatically switch between stiff/nonstiff algorithms, according to the numerical conditions encountered during the solution process. This is basically an insurance policy so as to obtain a robust solution – many ODEs can become stiff under conditions that are not obvious to the user. Algorithms that are appropriate under stiff conditions (e.g. Backward Differentiation Formula, implicit methods) are computationally expensive, and so are used only when needed, on a step by step basis. Otherwise non-stiff algorithms are used where they are appropriate.

The PM is closely related to Shooting Methods, but in a sense it is a conceptual inversion of them. Importantly, it does *not* seek to find convergent wave functions from the start. We call this the “Shoot First Method” (as in “shoot first and ask questions later”) because we just don’t worry about the initial conditions: we can just use nominal initial conditions appropriate to the general class of potentials (e.g. singular at the origin, or not). Indeed the PM actively *promotes* divergent solutions, from which it extracts useful information, allowing the algorithm to accurately locate the eigenvalues. In a most basic sense it is node-counting, but not of a proper wave function: to paraphrase, the key point from Sturm’s separation theorem is that the number of zeroes in the integration range (if it is set wide enough) is invariant if the initial conditions are changed. Convergent wave functions are not needed or even sought in the determination of energy eigenvalues. We count zeroes, or equivalently, transitions in the Phase Plot described below. Once the eigenvalues are accurately determined (through a sort of parallel evolutionary binary search), proper wave functions then can be readily computed if they are needed, but again, without the need to adjust initial conditions.

2.5.3. “Self-healing” Logarithmic Derivative

It is not difficult to show that as long as the initial starting point for the ODE solver is placed well into the classically forbidden region (well before the lowest (in x) classical turning point), the difference between logarithmic derivatives of two solutions starting with different initial conditions rapidly vanishes in the ODE integration process.

Consider two solutions $\Phi_1(x)$ and $\Phi_2(x)$ of the time independent Schrödinger equation with the same energy \mathcal{E} and potential $U(x)$ but with different boundary conditions. We have then $\Phi_1''(x) = -2(\mathcal{E} - U)\Phi_1(x)$, and $\Phi_2''(x) = -2(\mathcal{E} - U)\Phi_2(x)$. Multiplying the first equation by $\Phi_2(x)$ and the second by $\Phi_1(x)$, and subtracting the two equations gives

$$\Phi_2(x)\Phi_1''(x) - \Phi_1(x)\Phi_2''(x) = 0.$$

Recognizing the left side of the equation as the derivative of $\Phi_2(x)\Phi_1'(x) - \Phi_1(x)\Phi_2'(x)$ and integrating, we get $\Phi_2(x)\Phi_1'(x) - \Phi_1(x)\Phi_2'(x) = \text{constant}$; dividing by $\Phi_1(x)\Phi_2(x)$ we get $\Phi_1'(x)/\Phi_1(x) - \Phi_2'(x)/\Phi_2(x) = \text{constant}/(\Phi_1(x)\Phi_2(x))$.

In the classically forbidden region the solutions $\Phi_1(x)$ and $\Phi_2(x)$ diverge exponentially and the product of their inverses rapidly approaches zero. We also recognize the left side of the equation as the difference of the logarithmic derivatives of the two solutions, which therefore rapidly approach

zero. Integrating, we find $\ln(\Phi_1(x)/\Phi_2(x)) \rightarrow \text{another constant}$ which implies the ratio of the two solutions is a proportionality constant, which is normalized away for a wave function. Another way to describe this is the Wronskian of the two functions rapidly approaches zero when the ODE solution is started well into the classical forbidden region, so the two solutions rapidly become linearly dependent.

This “self-healing” of differences between two solutions with different initial conditions when started in the classically forbidden region is the basic reason we don’t have to fuss with precise initial conditions in the PM. It is a great operational simplification. It also provides more robustness to the solution (e.g. in the event of transient numeric overflow) because any such glitch is just a difference of initial conditions for subsequent points at larger x , so the solution tends to “heal” itself.

2.5.4. Phase Plot

The PM makes use of what we call the *Phase Plot*, which is simply the tanh (hyperbolic tangent) of the diverging solutions of the Schrödinger equation. Alternatively $\frac{2}{\pi} \arctan$ could be used instead of tanh, whence the “phase”. We use fixed nominal initial conditions, for reasons described above. Reining in the very large magnitude exponentially diverging solutions by using the tanh (or arctan) effectively digitizes the solution, with useful results, as described below. Other compression functions could be used but it is helpful for visualization purposes to have ones that are differentiable.

In the Phase Method, for most potentials, the ODE integration, treated as an initial value problem, is started and ended well past the extremal classical turning points, i.e. far into the classically forbidden region where $\mathcal{E} < U(x)$ (in general for the PM we use the independent variable x because we are not only concerned with central potentials, for which r is conventional). However, for “one-sided potentials”, i.e. those in which the independent variable is restricted to $x \geq 0$, as in the potentials considered here, the ODE solution is started at a very small positive value x_{min} , at which the solution is set to zero, and the initial slope is set to some larger but nominal value. In the case of Coulomb-like potentials when high accuracy is sought, the value of x_{min} must be quite small. However, it is not necessary to adjust or tweak the initial conditions for the particular solution.

If there are n bound states below the specified energy the Phase Plot will show n step-like transitions, jumping between $\approx \pm 1$. These are easily counted by a human by eye, or by a computer – they are most directly counted within the ODE solver itself (in Mathematica [23], by using WhenEvents). The Phase Plot essentially digitizes the analog ODE solution, much like digital circuits, which in reality are just analog circuits that are driven into saturation, yet in doing so, afford new kinds of uses. The morphology (e.g. sharp vs rounded transitions) of the Phase Plot tells the user at a glance if the range $[x_{min}, x_{max}]$ is insufficient and needs to be extended. Another way to say this is the transition counting process rapidly evaluates the integrated density of states at the specified energy, typically in milliseconds. This is the sharp computational tool that provides the accuracy, along with arbitrary precision arithmetic.

2.5.5. Arbitrary Precision Arithmetic

If we were limited to machine precision (≈ 16 decimal digits, 64 bit real numbers, divided up between exponent and mantissa) other methods may be more accurate than the PM. For smooth potentials, spectral methods are especially rapidly convergent. Example code to implement these can be found in [1]. On the other hand, these other rapidly convergent methods may break for non-smooth potentials. The PM handles everything with about equal facility.

There are unique advantages that come with using arbitrary precision arithmetic in the PM. It allows us to better handle diverging solutions (which are beneficial) without encountering floating point overflow. If overflow is encountered, the range just needs to be reduced slightly. Also, we can effectively dial-in the accuracy of the results we desire.

Modern implementations of arbitrary precision are surprisingly efficient. Of course greater accuracy involves somewhat longer run times, but we find that these are quite acceptable on modern computers. Effectively we reduce the human user time in exchange for increased computer time. To use arbitrary precision in Mathematica we simply specify WP, the working precision, given in decimal

digits. However attention still needs to be given to not polluting high precision numbers with lower precision ones.

A simple python port has been written and posted, giving quite acceptable performance running in the Pyto environment on a Gen8 iPad, using machine precision. This demonstrates a good degree of portability. The Julia language/environment seems especially promising as an open source alternative to Mathematica for the PM, because of its integrated support for arbitrary precision arithmetic, sophisticated ODE solvers, and parallel evaluation.

2.5.6. Evolutionary Search: Sifting and Refining

The ability to quickly count the number of bound states that exist below a given energy allows us to rapidly determine if an energy interval contains one or more eigenvalues. This is then used to systematically map out all of the energy eigenvalues within a specified initial search interval, using a simple kind of parallel evolutionary binary search process. We define a simple function that takes a given energy interval as input; divides it in half; and tests if each subinterval contains one or more eigenvalues, by applying the state counting function to each end of each subinterval. If there are eigenvalues within a subinterval, it is retained and passed along to a common pool (list) of intervals; if not, the subinterval is discarded. This function then is repeatedly mapped in parallel (using Mathematica's ParallelMap) across the list of intervals until a fixed point is attained, or otherwise it is determined that sufficient accuracy is achieved. It is very helpful to cache ("memoize") previously computed values to avoid the need to recompute them, which speeds up the process about threefold. This is easy to do in Mathematica using the idiom e.g. $f[x_] := f[x] = \dots$

The list of intervals initially roughly doubles in length each iteration, but then rapidly converges to only those intervals containing a single eigenvalue, or nearly so (a process we call "sifting", which normally is done serially, but could be parallelized). Those intervals are then "refined" in parallel using the same basic process, bringing it to completion. A great virtue of this approach is that it automatically and efficiently separates even nearly-degenerate eigenvalues, and no guessing is needed whatsoever (as is normally the case for the shooting method). Basically the user defines the potential to solve (analytically or numerically), puts an upper limit on the energy below which to look for eigenvalues (or defines a specific energy range to look within), specifies the accuracy desired, and the algorithm does the rest automatically. In this paper, it is this Phase Method algorithm that we have adapted to calculate critical binding parameters to high accuracy, instead of the eigenvalues.

Once all of the energy eigenvalues are determined to sufficient accuracy, the wave functions are easy to calculate, again without any need to match boundary conditions. With accurate eigenvalues in hand, the ODE solver is started well into the classically forbidden region, using only nominal initial conditions. The "self-healing" of the solution described above comes into play, giving an accurate solution where it is substantial. Then, in the final classically forbidden region at large x , at some point, the solution will tend to exponentially diverge again, if only because of the errors associated with representation of real numbers on a computer by a finite number of bits. At that point the program simply detects the location of first divergence using a simple algorithm (find the local minimum closest to x_{max} of $\log |\Phi(x)|$), and truncates the solution there, setting the solution to zero above that point. The resulting approximate wave function is then normalized by dividing by the integral of its square. These approximate wave functions are satisfactory for many purposes, and refinements to them for improving accuracy are feasible, if needed.

As described above, it is a peculiar but very useful property that the ultimate accuracy of the eigenvalues in the PM is reliably equal to, or better than, the working precision (number of decimal digits) *divided by two*. This behavior is related to the following observation: in the ODE solution, the transitions will stop registering (i.e. zeros are not counted) in the ODE solution process when $(\mathcal{E} - \mathcal{E}_*)^2 < \text{numres}$, where \mathcal{E} is the test energy, \mathcal{E}_* is the eigenvalue, and numres is the numerical epsilon, i.e. the minimum difference representable by numbers at the designated working precision.

For example if one wants 30 digits of accuracy, just use `WorkingPrecision=60`. This is straightforward to do using arbitrary-precision arithmetic, where adding more digits of precision simply requires longer execution time (and a little more memory), which are readily available on modern computers.

The computations in this paper were mostly done on M4 Mac Mini computers with 16GB RAM which is more than enough: each process running on a single cores takes about 200MB RAM on Mac OS, so 10 cores requires only about 2 GB RAM. With arbitrary precision arithmetic in Mathematica, the penalty for using WP of twice the desired accuracy is very modest, at least for the range accuracy we are most interested in. Our approach to this issue has the benefit of allowing the user to simply specify up-front the minimum accuracy that is needed, by specifying the WP. The computer then chips away until the eigenvalues are revealed, which usually takes from minutes to hours on a desktop or laptop computer. It takes longer to process a greater number of eigenvalues, more or less in proportion to their number, as does computation of very high n ODE solutions, with their highly oscillatory structure.

But unlike most matrix methods, it is not necessary to calculate all the states, if only a few are needed, because the algorithm searches within an energy range that is specified by the user. The user only has to ensure the x -ranges are large enough, and the Phase Plot is a useful guide to that. But for most eigenvalue problems, a better heuristic is to define a range that is scaled from the the extremal classical turning points, which are automatically calculated, as described in [1]. In our PM software this is the default option, but easily can be overridden if desired. The whole process naturally parallelizes, and runs very well on modern multicore desktop computers. The computations readily could be farmed across a compute cluster, or even supercomputers if hundreds of thousands, or millions of accurate eigenvalues were needed. The code should be readily portable to the Julia language; a very basic Python port is currently downloadable for eigenvalues.

2.6. Eigenvalues and Critical Binding Parameters

The energy levels for the Yukawa and similar potentials as a function of μ (or \mathcal{D} are quite straightforward to determine by the Phase Method, as described above. Determining the critical screening parameters by the appearance and disappearance of transitions in the Phase Plot as a function of screening parameter (with a fixed upper energy limit of zero), is essentially the same process as finding the eigenvalues. Rough intervals in μ that contain transitions are automatically calculated from the Phase Plot, and then refined further. The evolutionary iterative bisection mechanic is virtually the same as in the standard Phase Method, and is also done in parallel.

Only a few user-inputs are needed for the critical binding parameter calculation: the potential; the angular momentum (needed to calculate the centrifugal potential); the desired accuracy (via working precision parameter); and the range of \mathcal{D} values within which to look for bound states. Applying the ζ -scaling test is another option. The rest of the process is automatic, typically taking a few hours, depending on the accuracy desired, on an inexpensive desktop computer to obtain critical exponents for *all* of the bound states of the specified angular momentum l .

2.7. Adapting the Phase Method for Determining Critical Screening Lengths $\mathcal{D}_c = 1/\mu_c$

It is helpful to use $\mathcal{D} = 1/\mu$ as our screening parameter, as in Rogers [15]. In applying the Phase Method (PM) to calculating eigenvalues of the Yukawa potential we vary the energy for a fixed \mathcal{D} and l and look for changes in the number of transitions in the Phase Plot that reflect changes in the number of bound states. This is done without any need to guess locations of eigenvalues or fiddle with boundary conditions. The task we are now faced with is the flip side of this process for finding eigenvalues: we fix $\mathcal{E} = 0$ and vary \mathcal{D} to see when new states become bound/unbound. The computational apparatus is essentially the same as the usual PM. For the purpose we simply adapted our a simple example code for computing eigenvalues which is posted for download at <https://gbxafs.iit.edu/phase-method/>. Here we adapted it to very accurately compute, for a given l , all the \mathcal{D}_c values that are less than a specified \mathcal{D}_{\max} . This problem is isomorphic to that of computing all the \mathcal{E} eigenvalues that are less than a specified \mathcal{E}_{\max} . Note that this variation on the PM allows us to find critical screening lengths

without the necessity to calculate eigenvalues for $\mathcal{E} \neq 0$, although those also readily can be calculated using the PM, if needed.

2.8. ζ -Scaling Test

As mentioned above, multiplying the prefactor of the exponential term in the potential (i.e. coupling constant or atomic number, which is here by default taken as 1, with no loss of generality) by a scaling factor ζ , while dividing the screening length D by the same factor, has the sole effect of multiplying the eigenvalues by ζ^2 . This is physically plausible because increasing the coupling constant (or atomic charge Ze) and decreasing screening length D have opposite effects on the propensity to bind states. But for these potentials, indeed for any that scale as $1/r$ using a single length scale parameter (here $D = 1/\mu$) describing the potential, they are exactly equivalent.

Here we exploit this symmetry as one check on the accuracy (or at least, self-consistency) of our calculations. We multiply the exponential term by a factor ζ while dividing the screening length by the same factor ζ . In doing so, the energy eigenvalues get rescaled by ζ^2 , but in the case of critical binding, these are identically zero. This implies that this double-rescaling transformation (e.g. coupling constant increased, screening length decreased by the same amount) should leave the critical screening parameters invariant, exactly. By doing two independent calculations, with unscaled ($\zeta = 1$) and scaled ($\zeta = 10$), which result in quite different spatial dependencies and computational numerics, we have an independent test as to whether our computed critical screening lengths are identical at the claimed accuracy. This zeta-scaling test is built into our code as a clickable option. Our tests have passed every time they have been applied.

The explicit scaled forms of the potential used in the calculations, including the centrifugal potential term (which is modified for the Pseudo-Hulthén potential), and including ζ -rescaling form of $\rho \rightarrow D/\zeta \left(1 - e^{-\frac{\zeta x}{D}}\right)$ are:

- Yukawa: $-\frac{\zeta e^{-\frac{\zeta x}{D}}}{x} + \frac{l(l+1)}{2x^2}$
- Hulthén: $-\frac{\zeta^2 e^{-\frac{\zeta x}{D}}}{D \left(1 - e^{-\frac{\zeta x}{D}}\right)} + \frac{l(l+1)}{2x^2}$
- Pseudo-Hulthén: $-\frac{\zeta^2 e^{-\frac{\zeta x}{D}}}{D \left(1 - e^{-\frac{\zeta x}{D}}\right)} + \frac{\zeta^2 l(l+1) e^{-\frac{\zeta x}{D}}}{2D^2 \left(1 - e^{-\frac{\zeta x}{D}}\right)^2}$
- ECSC: $-\frac{\zeta e^{-\frac{\zeta x}{D}} \cos\left(\frac{\zeta x}{D}\right)}{x} + \frac{l(l+1)}{2x^2}$.

Note: for $l = 0$ the Hulthén and Pseudo-Hulthén potentials are identical.

In summary, critical screening lengths are invariant under changes in ζ . We use this to verify the accuracy of our calculations to the indicated number of digits by comparing independent runs with different instances of the potential $\zeta = 1$ and $\zeta = 10$. In all of the tests, our results were confirmed to the claimed accuracy.

2.9. Pseudo-Hulthén Internal Self-Consistency Tests

The μ_c values for the PH potential are exactly known and highly regular: $\mu_c = 2/(n + l)^2$. This symmetry is evident from inspection of the PH data tables. This structure implies that incrementing the index l while decrementing n in the table by the same amount should give the same μ_c to within the tolerance of 10^{-60} , determined by setting the working precision to 120. We have programmatically checked this relation for 192 distinct pairs of l, n values for the $l = 0 \dots 20$ tables, and find all discrepancies in μ_c to be below the specified tolerance. As these correspond to independent computational runs for different l , with different centrifugal potentials, this is a stringent test of our computational methods.

The agreement in all cases with the exact known values is even more so. All 714 (for $l = 0 \dots 20$) μ_c values for the Pseudo-Hulthén Potential that are presented in the tables in the Appendix were compared programmatically with exact values, and agree to the claimed 60 decimal digits (i.e. the

absolute value of the error $\leq 10^{-60}$). The code and parameters used to generate these results were then applied to calculation of the Yukawa, Hulthén, and ECSC tables.

2.10. Alternative Method for Calculating Critical Screening Lengths D_c

We also implemented an alternative approach based on considerations that are also shared by our own methods used here, but have been exploited by few investigators tackling this problem. The central point is that the asymptotic dependence of solutions near zero energy are very smooth and therefore relatively fast to calculate with an adaptive ODE solver, even over extremely long ranges. At zero energy, in spherical coordinates, the wave function at large distances has the asymptotic form $a/r^l + br^{l+1}$, which at large r is dominated by r^{l+1} . Dividing the ODE solution by r^l produces a straight line. A zero slope line demarcates bound from unbound states. These considerations are invoked by Schey and Schwartz [22], and also Edwards [14]. Using this observation, as an alternative to the Phase method, we sought the solution with zero slope, which is straightforward to calculate numerically in our usual way out to very large distances. However, extrapolating the slope solely from smaller values of r isn't sufficient or reliable: a long lever arm (range) is needed to achieve high accuracy of the slope estimation. Further, we must exclude the region at low r to ensure that we are in an asymptotically appropriate region. Here we chose to use the range $r_{\max}/2$ to r_{\max} for slope determination.

We sought the value of μ that gives zero slope when the energy is set to zero. This was done in an evolutionary search process like that used in the Phase Method. As before, we defined an interval in μ within which to search for critical values. Our metric was simple: we varied μ and (programmatically) looked for changes in the sign of the slope corresponding to μ values at the ends of the μ interval. As in the PM, we then bisect the interval, apply the sign change criterion to see in which half the desired μ_c resides, retain the viable subinterval if it exists, and discard the other, and iterate (using ParallelMap to map across the common list of intervals). As in the PM we "memoize" (cache) previously computed results to avoid unnecessarily recomputing them.

The initial bracketing intervals were determined by applying the state-locating function to roughly map out the transitions, evaluating (in parallel) on a simple grid search. These brackets compared well to those of Bylicki's numerical tables [21] that are posted online; the bound-unbound transitions were judged by the values at which the energies first acquired an imaginary part. This alternative approach worked well (to 30 digit accuracy) when paying attention to the "points on accuracy" listed above.

However, our adapted Phase Method that was used to generate all of the tables in this work is simpler and more efficient, mostly because the evolutionary binary search automates the entire process, so no bracketing of the μ transition locations by other means is needed.

2.11. Recommendations for Obtaining Accurate Calculations

It may be helpful to outline certain details that have needed attention to achieve the claimed accuracy in our present computing environment.

- Use arbitrary precision arithmetic sufficient to get the desired accuracy. It is not necessary to limit oneself to machine arithmetic or to use GPUs, which normally support at most double precision floating point. Our code for this class of problems (and more generally, Phase Method) primarily uses CPU cores, not GPU.
- Take care to avoid polluting high precision numerics by introducing lower precision numbers into the calculation, intentionally or inadvertently. For example "1./2." \neq "1/2": the real number 1./2. is machine precision, while the rational fraction 1/2 has infinite precision. Mathematica/Wolfram Language automatically tracks precision and accuracy of real numbers in its computations, a powerful feature. SetPrecision is your friend.
- For robustness use adaptive ODE solvers with stiffness detection and switching. Alternatively change variables so the function being integrated is rather smooth. Or do both.
- Use very short and reasonably long distance cutoffs sufficient to obtain the desired accuracy. For singular potentials, and for $l = 0$, the short distance cutoff needs to be much shorter than it does

for $l > 0$ because in the latter the centrifugal potential pushes the wave function to larger r . For near-zero energies the time cost is very modest to use an absurdly long range with an adaptive ODE solver. For $l = 0$, the solution at large r where $U \approx 0$, is close to a straight line. But to determine a near-zero slope of a computed solution a long lever-arm (i.e. long range) is still needed.

- Compare with known exact values as benchmarks (and of course literature values) to validate the computational apparatus. Use ζ -scaling tests and internal cross checks as applicable.

3. Results

In this section we first present evidence supporting our accuracy claims for the PM-calculated results. We then present numerical results in 84 tables in the Appendix, with some limited graphics and limited analysis.

- Compare PM eigenvalues to 60 digit accuracy with exact values for Coulomb and Pseudo-Hulthén potentials.
- Compare PM eigenvalues to 30 digit accuracy with Stubbins's variational calculations for the Yukawa, Hulthén, and Pseudo-Hulthén potentials for $\mu = 10^{-8}$, confirming the correct ordering of levels at small μ , and Stubbins's values.
- Compare our μ_c results to 30 digits with Demiralp's for the Yukawa and Hulthén potentials.
- Compare our μ_c results to 30 digits to those of Jiao's values printed in the paper, for several potentials.
- Compare PM μ_c results to 60 digit accuracy with exact values for Pseudo-Hulthén potentials.
- Present our tables at 60 digits accuracy of μ_c values for PseudoHulthén, Hulthén, Yukawa, and ECSC potentials for all states up to $\mathcal{D} = 1000$ and $l = 0 \dots 20$.
- These are followed by our computed \mathcal{D}_c values at 30 digits accuracy up to $\mathcal{D} = 10^5$ for $l = 0 \dots 12$ for the Yukawa potential and ECSC potentials. Over this range the asymptotic dependence is clear. The ECSC values show interesting structure that is not present in the former. Asymptotic convergence is clear.
- Confirm the correct ordering of $\mu_c(n, l)$ for the various potentials in all cases: $\mu_c(\text{hulthén}) \geq \mu_c(\text{pseudo-hulthén}) > \mu_c(\text{yukawa})$
- Using semiclassical approximation, we analytically derive the asymptotic forms of \mathcal{D}_c vs n_l for Yukawa and Hulthén potentials, which agree with fits to the numerically calculated behavior.

3.1. Phase Method Eigenvalues - Validation Tests

Here we give some examples of the accuracy of eigenvalue calculation using the PM. PM-calculated Energy eigenvalues to 60 digits. Comparing to exact eigenvalues of the Coulomb and Pseudo-Hulthén potential, we confirm PM calculations are accurate to the 60 digit accuracy chosen by setting $WP = 120$. Specifically what is meant by this is the worst case absolute value of the difference between computed and exact value is of order 10^{-60} . The median of the errors of the eigenvalues typically is one or two orders of magnitude lower than the maximum.

3.1.1. Comparison with Exact Pseudo-Hulthén Potential Eigenvalues

The computed results for μ_c for $l = 0 \dots 20$ are tabulated for $D \leq 1000$, at working precision $WP = 120$. Our claimed accuracy for these tables is 60 digits, specifically meaning the absolute value of the difference between the calculated μ_c values and the exact values are all $\leq 10^{-60}$. As support for this, we compare to exact solutions for the eigenvalues of the Pseudo-Hulthén potential. These have the simple form $\frac{(2 - \mu(n+l)^2)^2}{8(n+l)^2}$ for $n = 1 \dots (\lfloor \sqrt{\frac{2}{\mu}} \rfloor - l)$. Setting $\mathcal{E} = 0$ and solving for μ gives $\mu_c = 2/(n+l)^2$.

3.1.2. Comparison with Exact Coulomb Eigenvalues, Confirmation of SO(4) Degeneracy

Test calculations were done to verify accuracy of eigenvalues for coulomb potentials with various angular momenta. We conclude that the full 60 digit accuracy is readily attained, provided the working precision is set to 120 and the r -ranges are set appropriately. We simply give a summary of the results here.

A test calculation was made for $l = 0$ for the Coulomb potential $(-1/x + l(l+1)/(2x^2))$, including centrifugal potential) over the r -range $10^{-60} \dots 10^4$ at $WP = 120$, which gave maximum error of order 10^{-62} (median error 10^{-64}) for all the states that were calculated, which were the 12 s-states below the designated upper energy limit of $-.003$ au. Another calculation was done for $l = 1$ over the range $10^{-40} \dots 10^4$ at $WP = 120$ which gave maximum error of order 10^{-64} for all the states that were calculated, i.e. the lowest 11 p-states below $-.003$ au. Choices of these ranges for $l = 0$ and $l > 0$ depend on desired accuracy and working precision as described above.

We note that, to the same level of accuracy, the energy eigenvalues corresponding to the different l values are degenerate, a consequence of the well-known SO(4)-symmetry of the Coulomb potential. This is a broken symmetry for the screened Coulomb potentials, Yukawa et al, which do not have energy-degenerate levels for different l .

It is important to note that each of our calculations for different l involves a numerically different centrifugal potential, and an independent computation, so the high degree of consistency among the degenerate eigenvalues for different l also is a significant confirmation of numerical consistency and robustness of the Phase Method.

To verify the accuracy (here at WP60, giving accuracy of 10^{-30} , to compare with earlier work) for high angular momentum states, a calculation for $l = 20$, and also a repeat run of $l = 0$, were done with the lower energy search limit set to -1 , so as to include all the bound states. For $l = 20$, the 50 energy eigenvalues for states $n = 21 \dots 70$ were automatically located and calculated to an accuracy of order 10^{-35} . For $l = 0$, all energy eigenvalues for $n = 1 \dots 70$ were located and calculated with an accuracy better than 10^{-33} . These calculations took about 3.6 hours on a Mac M1 Studio Ultra, and would be perhaps twice that on a M4 Mac Mini computer; fewer eigenvalues would take less time. These results show that accurate eigenvalues can be obtained close to the continuum limit for the Coulomb case, and for higher angular momentum states (up to $l = 20$).

3.1.3. Comparison with Exact Coulomb Eigenvalues Near $\mathcal{E} = 0$

The critical binding parameter problem addressed in this work focusses on states at the continuum limit. Unlike the screened-Coulomb potentials that are the focus of this paper, which have a finite number of bound states for $\mu > 0$, the Coulomb Potential has an infinite number of bound states, accumulating near $\mathcal{E} = 0^-$. For this reason in these calculations we set an upper energy limit below $\mathcal{E} = 0$, to avoid requiring the PM to compute an infinite number of eigenvalues, which could take a while.

Ordinarily, to compute eigenvalues we often employ the heuristic of choosing the r -range by scaling the classical turning points, evaluated at the highest computed energy. For attractive potentials like those considered here, which are singular at $r = 0$, r_{\min} is chosen as small as the numerics allow at the specified WP. For Coulomb-like potentials this is $r_{\min} \leq 10^{-WP/2}$. For the upper range limit, we use the turning point scaling recipe. The upper classical turning point for energy limit 10^{-4} au is $r = 10^4$ au, and we scale that by a factor of 10, to set $r_{\max} = 10^5$ (the exact value is not critical, as long as it is large enough). For this test, because we are particularly interested in states near $\mathcal{E} = 0$, we set the *lower* end of the energy search interval to 10^{-3} au, i.e. reasonably close to the continuum.

For this test we calculated energy eigenvalues for $l = 0, 1, 2$ at WP60 (expected accuracy of 30 digits, iff the r -range is sufficient). The r -range was set to $10^{-32} \leq r \leq 10^5$ au with an energy search range from $-10^{-3} \leq \mathcal{E} \leq -10^{-5}$ au. For $l=0$ the PM, using its sifting and refining process, computed eigenvalues for all 48 states from $n = 23 \dots 70$ with an energy error 10^{-33} au or better, which, as is typical, is slightly more accurate than the nominal expected 30 digits. This illustrates the importance

of a using a sufficiently small r_{\min} . The calculations took about half an hour to calculate on a Mac M1 Studio Ultra.

3.1.4. Comparison of PM-calculated Energy eigenvalues to Stubbins, and Vrscay

Using the Phase Method also we readily confirm the full 30 digits of accuracy of the tabulation by Stubbins [12] of the Yukawa and Hulthén potentials, for the smallest n s-levels, as well as the claimed 10^{-7} accuracy of his upper levels. We also readily confirm Vrscay's [13] 20-digit values for all levels.

3.1.5. Comparison of PM eigenvalues with Stubbins for $\mu = 10^{-8}$; Perturbation Theory

Stubbins's paper [12] reported accurate variational calculations of eigenvalues for selected states, for Yukawa and Hulthén potentials over an assortment of screening parameters μ , the values being specific to each state. In contrast, our Phase Method approach automatically computes all the energy eigenvalues up to the specified upper energy for a selected $\mu = 1/D$ and l (or if desired, only those within a chosen energy window).

The $D = 10^8$ ($\mu = 10^{-8}$) case is an extreme test self-imposed by Stubbins. We applied the same test to the PM. Our computed eigenvalues for $\mu = 10^{-8}$ for 1s and 2p for Yukawa and Hulthén potentials agree with Stubbins's to all of his listed digits (with the slight discrepancy of his 1s value, in the least significant digit). Other values up to $n = 8$ were also computed for $l = 0$ and $l = 1$.

Stubbins [12] also showed that his computed eigenvalues conformed to the correct ordering of eigenvalues: $\mathcal{E}_{\text{coulomb}} < \mathcal{E}_{\text{hulthén}} < \mathcal{E}_{\text{yukawa}}$ even at the small values of $\mu = 10^{-8}$, supporting the accuracy of his calculations in this extreme limit, where the variational basis set might become inadequate. We confirm his results with our (non-variational) PM calculations finding $\mathcal{E}_{\text{coulomb}} < \mathcal{E}_{\text{hulthén}} < \mathcal{E}_{\text{pseudo-hulthén}} < \mathcal{E}_{\text{yukawa}}$, and also find it is quantitatively consistent with first order perturbation theory, as we show next.

Using the exact Coulomb eigenvalues as references, in first order perturbation theory, the shift in eigenvalues for the other potentials is just the expectation value of the perturbation Hamiltonian ΔH , evaluated in the relevant eigenstates of the unperturbed (Coulomb) system. ΔH is the difference in the potential from (Yukawa, Hulthén) from the Coulomb potential. The perturbation Hamiltonian for $l > 0$ must include the centrifugal potential, which is slightly different in form for Pseudo-Hulthén than the others, so that term doesn't cancel out. Expanding those perturbations to first order in $\mu = 10^{-8}$ simply gives the constant μ for the Yukawa potential, and $\mu/2$ for Hulthén/Pseudo-Hulthén. This makes evaluation of the expectation value trivial. We conclude the corresponding eigenvalues (when they exist) are shifted from the Coulomb eigenvalues upward by μ (for Yukawa/Debye) or $\mu/2$ (Hulthén and Pseudo-Hulthén) potentials. Our computed results agree with this fully, with residual differences of second order in μ , or 10^{-16} .

Other eigenvalues for various choices of μ in Stubbins's article were also computed and compared with excellent agreement to Stubbins's listed digits, which are variable in number, up to 30 digits. This mutually confirms the accuracy of Stubbins's variational eigenvalues, and our own PM-calculated ones.

3.2. Critical Binding Parameters

3.2.1. Comparison with Exact Pseudo-Hulthén Potential Eigenvalues

The computed results for μ_c for $l = 0 \dots 20$ are tabulated for $D \leq 1000$, at working precision $WP = 120$. Our claimed accuracy for these tables is 60 digits, specifically meaning the absolute value of the difference between the calculated μ_c values and the exact values are all $\leq 10^{-60}$. As support for this, we compare to exact solutions for the eigenvalues of the Pseudo-Hulthén potential. These have the simple form $\frac{(2-\mu(n+l)^2)^2}{8(n+l)^2}$ for $n = 1 \dots (\lfloor \sqrt{\frac{2}{\mu}} \rfloor - l)$. Setting $\mathcal{E} = 0$ and solving for μ gives $\mu_c = 2/(n+l)^2$.

3.2.2. Comparison with Demiralp's 1989 μ_c calculations for the Yukawa Potential

A remarkable paper [2] apparently has been overlooked in subsequent literature. In 1989 Demiralp [2] developed a sophisticated variational approach by constructing basis sets specially "tuned" to provide optimal convergence for each state. His Table 1 presents critical binding parameters μ_c to 30 digit accuracy for the first 44 bound states of the Yukawa potential, with μ_c ranging from ≈ 1.1906 to ≈ 0.0115 (roughly $\mathcal{D} \approx 0.84\text{--}87.0$). Over this range the following states become bound: (1s–10s); (2p–10p); (3d–10d); (4f–9f); (5g–9g); (6h–8h); (7i–8i); (8k).

Comparing with our own earliest results on this problem calculated to 30 digits (which were calculated, but not published, before learning of Demiralp's or Jiao's results) we found full agreement with Demiralp's claimed 30 digits for the entire $l = 0, 1, 2$ manifolds (27 s,p,d states), with the only discrepancies being in the least significant digits (28–30) for $l > 2$ manifolds (17 f,g,h,i,k states), and a single clerical error described below. The number of digits of agreement between our results and Demiralp's can be summarized approximately as $30 - \lfloor l - 2 \rfloor$ digits. Since our initial (unpublished) 30 digit calculations were done, we have subsequently performed 45 digit and 60 digit calculations (those presented here) with the same conclusions.

We note that in Demiralp's table 1 there appears to be a single clerical error (transposition in the eighth– ninth decimal places) for the 9s state: 0.015673723828... (compared to our value 0.015673732828...). As our own numbers and typeset tables are manipulated solely by machine (but checked by the author, who reportedly is human) we think such clerical errors in our table are highly unlikely.

As is common in this subfield, Demiralp used state labeling that is conventional for atomic physics and plasma physics, and chemistry (except the $l = 7$ state is mistakenly labeled "j" rather than the spectroscopic convention "k"). As this numbering scheme is not appropriate for nuclear systems, and as this work is relevant to them, we simply label the states with the angular momentum quantum number $l = 0, 1, \dots$, and within a given l manifold, number them sequentially from their lowest energy level as $n = 1, 2, \dots$. This also is the natural way in which they are computed, one l -manifold at a time. The first state of angular momentum l appears at $l = \bar{n} - 1$, where \bar{n} is the principal quantum number, so our sequence number $n = \bar{n} - l$.

The complete agreement of our results with the (totally different, computationally) variational calculations of Demiralp for $0 \leq l \leq 2$ and the near-complete agreement for $l \geq 3$ is a strong confirmation of both results to the consensus accuracy. In 1989, and for three decades afterward, no other known published computations came close to the accuracy of Demiralp's results. It is unfortunate that they had been largely overlooked over that time frame. We expect the full set of Jiao's recent results which are expected to be accurate to 30 digits will agree with our own that are presented here. However, to ensure independence, we have kept a "firewall" (and a paywall) between them, for others to independently verify.

3.3. Comparison with Singh and Varshni's [16] Calculations of μ_c for the ECSC Potential

Singh and Varshni in 1983 published their calculations to an accuracy of 8 digits (10 for the 1s state) of critical screening parameters for the ECSC potential. Their tabulations covered bound states ranging from $l = 0 \dots 8$ and principal quantum number $n = (l + 1) \dots 8$, explicitly states 1s...8k. Our computed values fully agree with theirs to their accuracy of 8...10 digits.

3.4. Comparison with μ_c of Jiao et al [3] for Yukawa, ECSC, and Hulthén Potentials

In addition to their extensive benchmark calculations of μ_c to an accuracy of 30 digits, and interesting observations, Jiao et al published a rather thorough review of the chronology of the apparently steady advances in accuracy of past calculations of critical parameters. Unfortunately, the community and review articles seem to have missed Demiralp's 1989 paper [2], which gave results accurate in most cases to 30 digits, which was a significant advance at the time. The historical record will need to be corrected.

The computational method Jiao et al have used (Generalized Pseudo-Spectral Method), like previous investigators such as Bylicki et al [21] and Roy [11], calculates the critical parameters, energies of the bound states, as well as resonances and widths at positive energies. This is quite different in method from the results presented here, which directly determines the critical parameters.

We have compared our critical parameters μ_c to those explicitly listed in the paper, for $1s$ and $2p$ (our $l = 0, n = 1$, and $l = 1, n = 1$) states of the Yukawa, ECSC, and Hulthén potentials. To maintain independence (and because their supplementary data is behind a paywall) we have not yet compared our full tabulations. However there is perfect agreement between our results and their printed values to their full listed 30 digits that are presented in their Tables 1–3. Our values to 60 digits below are quoted directly from our tables in the appendix.

Yukawa $1s$:

1.190 612 421 060 617 705 342 777 106 361 046 347 275 901 572 981 749 063 530 790

Yukawa $2p$:

0.220 216 806 606 573 040 405 041 463 289 577 110 508 548 104 023 305 084 754 657

ECSC $1s$:

0.720 524 085 881 953 095 871 917 136 918 578 087 183 481 757 107 097 035 500 102

ECSC $2p$:

0.148 205 032 642 758 419 285 886 459 123 248 041 030 459 181 523 505 707 250 28

Hulthén $1s$:

2.000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000

Hulthén $2p$:

0.376 935 996 093 545 491 107 886 597 464 844 838 761 579 551 432 087 617 184 727

3.5. ζ -Scaling Tests

As described above, the screened Coulomb potentials considered here (Yukawa, ECSC, Pseudo-Hulthén, Hulthén) all have the same exact scaling symmetry: multiplying the coupling constant/atomic number (i.e. the coefficient of the $e^{-\mu x}$ term) by a factor ζ is exactly equivalent to multiplying the screening length \mathcal{D} by the same factor, with the only consequence being a rescaling of the energy eigenvalues by a factor of ζ^2 . If $\zeta > 1$ the rescalings both have the effect of increasing the propensity to bind states.

Our search for critical binding parameters refers specifically to the case of $\mathcal{E} = 0$ energy eigenvalues, and we can conclude that multiplying the exponential factor(s) by ζ while simultaneously dividing the screening length \mathcal{D} by ζ must leave the critical parameters strictly invariant. We use this symmetry as an exacting test of our numerical values. The rescaling also applies to the exponential terms in $\rho = (1 - e^{-\mu r})/\mu$ and to the exponential decay factor of the centrifugal term in the Pseudo-Hulthén potential.

The scaled and unscaled potentials are quite different numerically in these two cases if the scale factor is substantial. In our case we use $\zeta = 10$, which gives dramatically different spatial dependencies in the two cases, and the detailed numerics during the calculations in the two cases are inevitably quite different. We apply this critical ζ -scaling test to support our claim of accuracy of critical exponents to the full 60 digits.

3.6. Our Tables

Here we present our tables of μ_c for Pseudo-Hulthén, Hulthén, Yukawa, and ECSC potentials, from $l = 0 \dots 20$ at 60 digit accuracy, for all states up to screening lengths of $\mathcal{D} = 1000$ au. Elsewhere we will present similar tables accurate to 30 digits up to $\mathcal{D} = 10^5$ au for all four of these potentials. We have also computed a few up to $\mathcal{D} = 10^6$ au which produces $1124\mu_c$ values. These calculations are straightforward, but take about 40 hours to compute all the μ_c values to 30 digits for a single l , on a M4 Mac Mini, because their large number. The computation is easily job-farmed, or it could be sped up by clustering if there a need.

We start with the Pseudo-Hulthén potential [7] as a benchmark, as it is similar to the other screened Coulomb potentials, but was contrived to be exactly soluble for both critical binding parameters and energy eigenvalues, for all l for which there is a bound state.

In these PH tables, it will be noted that many of the entries for angular momentum $l = 0 \dots 20$ are equal (to within rounding to 10^{-60} in the last digit) but are shifted upward for each increase in l by 1 by dropping the top row. The exact expression is [7] $\mu_c = 2/(n+l)^2$, $n = 1 \dots n_{\max}$, where $n_{\max} = (\lfloor (2/\mu)^{1/2} \rfloor - l)$. All of the presented results agree with the theoretical expressions to the claimed accuracy. Since each of the calculations for the 21 different l values is computationally independent of the others (they are executed in different jobs, and often, on different computers), and as each run employs different l and different centrifugal potential terms, this demonstrates a remarkable consistency of the tabulated values. This supports our claims of accuracy to the indicated number of digits. Each of the following tables in the Appendix were done in the same way.

Our results for μ_c at $l = 0 \dots 20$ are tabulated for $D \leq 1000$, using working precision $WP = 120$. We claim accuracy to 60 digits, specifically meaning the difference between the calculated μ_c values and the exact values are claimed to be $\leq 10^{-60}$. For this potential, the maximum error of 10^{-60} was found for $l = 0$ and $l = 1$; the median of the errors of all the 21 values of l was 0.2×10^{-61} .

3.7. Comparison with Rogers et al

Rogers et al [15], in their figure 5, showed useful relationships between the total number of bound states (n^*), and with the square of the number of bound s-states (g^*), both expressed as a linear functions of the critical screening parameter \mathcal{D} . Our tables reproduce their figure quite accurately, as seen in Figure 1, but also extend it to a much larger range. In this we have limited the \mathcal{D} range in the plot and fits to that for which angular momenta $l > 12$ do not contribute (as they were not included in this dataset for $\mathcal{D}_{\max} = 10^3$). The inset box in the figure represents the approximate range covered in figure 5 of Rogers et al.

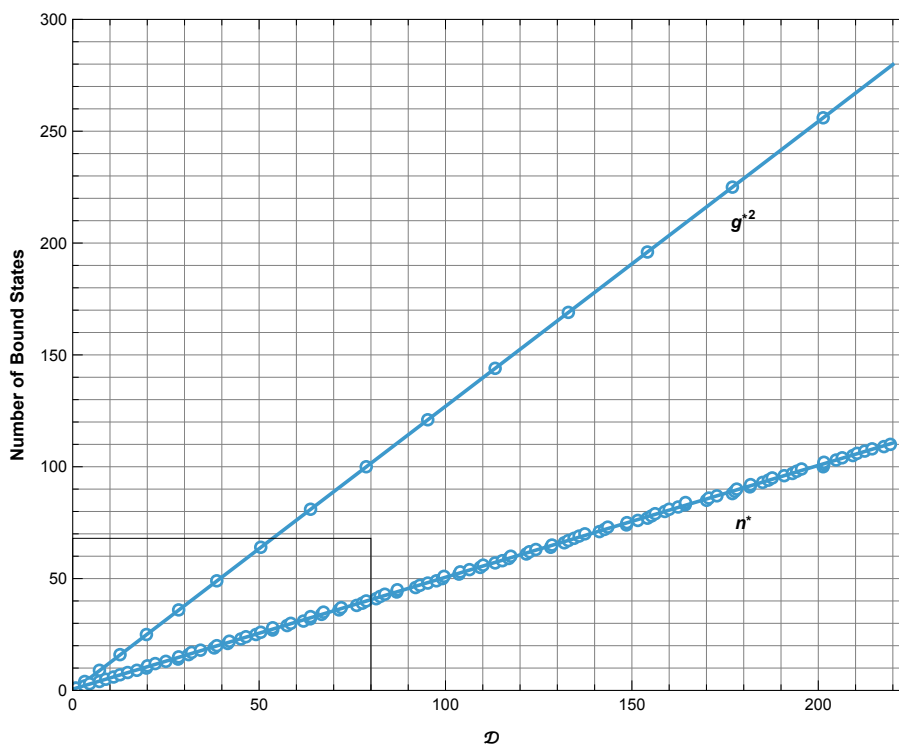


Figure 1. Number of Bound States vs \mathcal{D} for Yukawa potential. The inset rectangle shows the range covered by Rogers et al [15].

We also have performed least squares fits (shown as the straight lines in the figure) using their same parameterizations: $n^* = a_1 + a_2\mathcal{D}$; $(g^*)^2 = b_1\mathcal{D}$; and $(g^*)^2 = b_2 + b_3\mathcal{D}$. The best fit values

are: $b_1 = 1.27297(1.27294, 1.2300)$; $b_2 = -0.289(-0.314, -0.264)$; $b_3 = 1.27309(1.27308, 1.27312)$; $a_1 = 0.573(0.412, 0.733)$; $a_2 = .50012(.49885, .50138)$, where the quantities in the parentheses are 95% confidence intervals. These values are quite close to those of both Rogers [15] et al and Schey and Schwartz [22].

We show below by semiclassical quantization of the phase space integral that the number of s -states scales as $\mathcal{D} \approx \frac{\pi}{4}n^2$ as previously observed, and confirmed here. Further, the coefficient $\frac{4}{\pi} \approx 1.2732$ agrees very well with our numerical fits above. These fits may be useful for roughly predicting the total number of bound states for other values of \mathcal{D} .

3.8. Yukawa $\mathcal{D}_c(l, n)$

It is clear from the data shown in Figure 3 that the Yukawa, Hulthén, and of course, the exactly known Pseudo-Hulthén critical screening lengths $\mathcal{D}_c = 1/\mu_c$ are rather smooth functions of l and n . The ECSC potential is also, but with some added rough structure we will show over an extended range below. This is made especially clear by plotting $\sqrt{\mathcal{D}_c}$ vs $l + 1$ and n in Figure 4, for the Yukawa Potential. The maximum n for the higher l states is limited by D_{\max} and l so we have selected a square region for $(l + 1|n) = 1 \dots 17$. The Yukawa critical values evidently are not functions only of $(n + l)$, as is the case for Pseudo-Hulthén potential: for example, exchanging values of n and l gives a different value for \mathcal{D}_c . This is seen in the contour plot of Figure 2. The contours are labeled with $\sqrt{\mathcal{D}_c}$, so, for example, the $l + 1 = 10, n = 10$ value is about 19, and $\mathcal{D}_c(l = 9, n = 10) \approx 19^2 \approx 360$ au.

For Pseudo-Hulthén, $\sqrt{2\mathcal{D}_c} = (n + l)$, exactly. In this case the coefficients of the linear relation for between $\sqrt{\mathcal{D}_c}$ and n and l are precisely equal and constant, which is not the case for the Yukawa Potential.

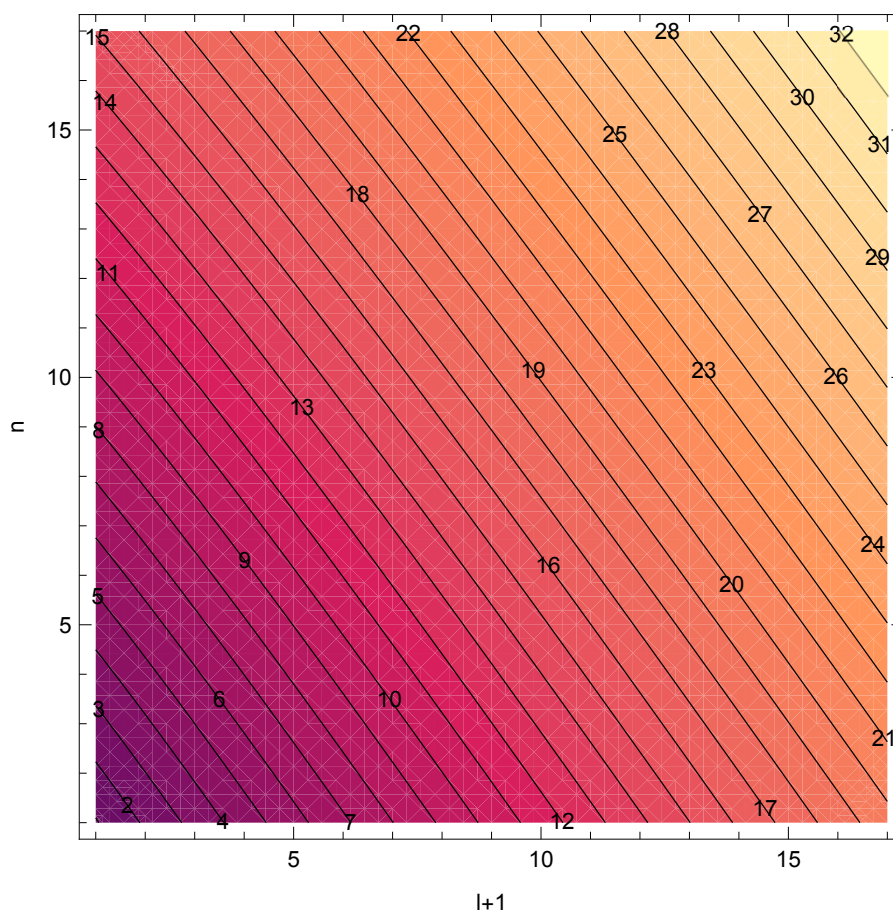


Figure 2. Contour map of interpolated values of $\sqrt{\mathcal{D}_c}$ for the Yukawa Potential as a function of $l + 1$ and n , for ranges $(n|l + 1) = 1 \dots 17$. The nearly equally-spaced parallel contours show that $\sqrt{\mathcal{D}_c}$ is approximately a linear function of both l and n , but with unequal coefficients. Each contour is labeled with the value of $\sqrt{\mathcal{D}_c}$.

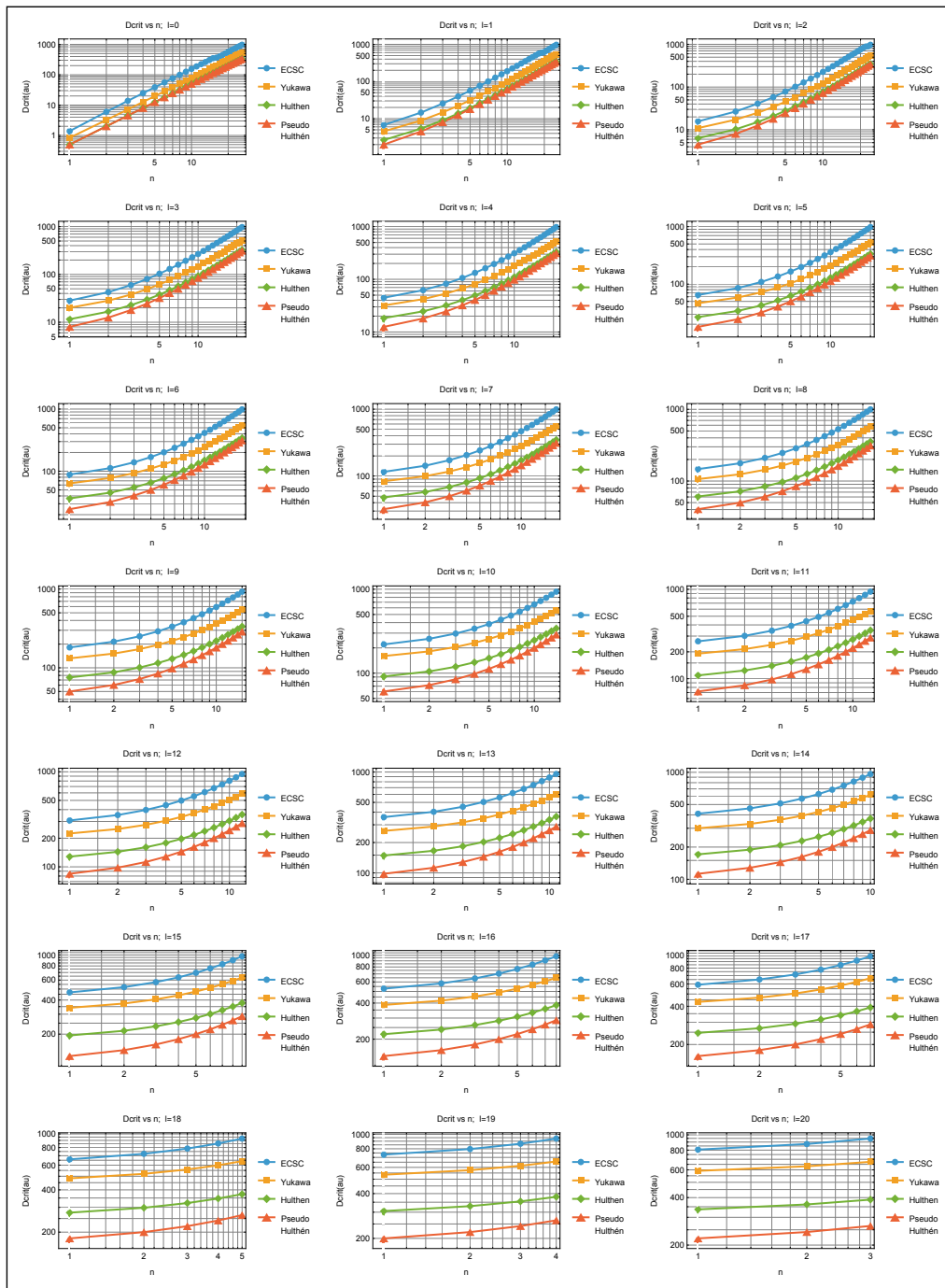


Figure 3. Log-log Plots of $D_c = 1/\mu_c$ vs n for ECSC, Yukawa, Hulthen, and Pseudo-Hulthen potentials vs n for all 21 values of l . The correct ordering of all n and l values is evident.

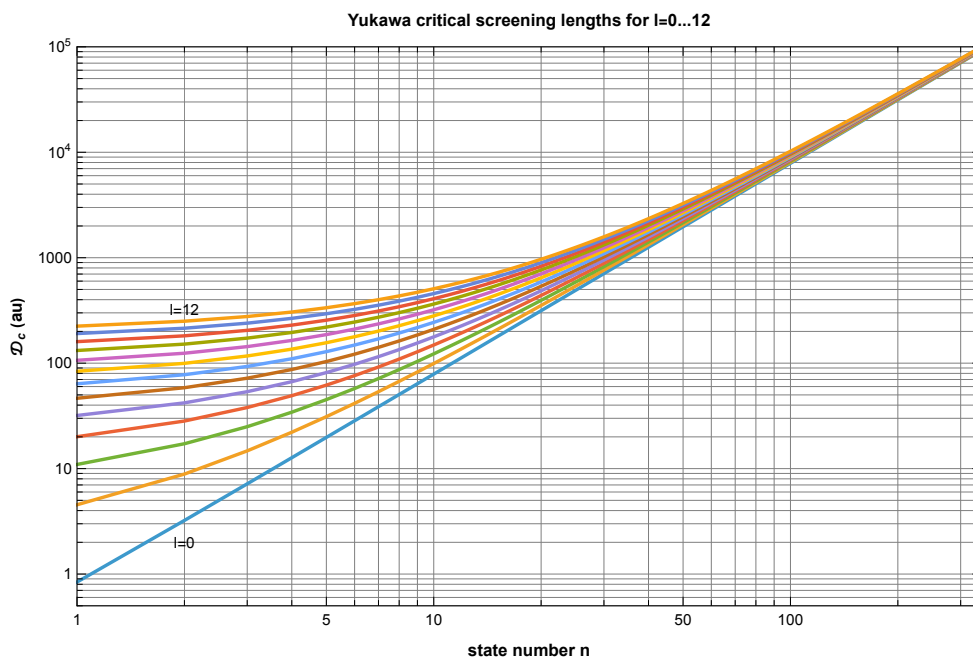


Figure 4. \mathcal{D}_c vs n_l for Yukawa potential up to $\mathcal{D} = 10^5$ au calculated to 30 digits.

3.9. Inequality Plots

As is well-known, there is a rigorous ordering of the energy levels in these potentials, owing to the strict inequality of the potential energies at all values of r , for the same μ and l . As a consequence there is the reverse ordering of the μ_c values, and the same ordering of \mathcal{D}_c values. Figure 3 shows this clearly for all computed μ_c . These represent 84 independent runs and 2167 different levels – all conform to the known strict ordering of the levels.

It will be noticed that the $l = 0$ curves for all potentials are close to a straight line on a log-log plot, which indicates a power law relation, in this case, n^2 . This is explained below. However, note the “kinks” in the ECSC curves for $l = 0 \dots 2$: they are not artifacts – they are real. A larger range of \mathcal{D}_{\max} is needed to see this structure clearly, which is done below.

4. Discussion

4.1. Asymptotic Behavior of $\mathcal{D}(n_l, l)$ vs n_l

For the Yukawa Potential, the log-log plot of $\mathcal{D}(n_l, l)$ vs n_l in Figures 4 and 5 show convergence to the $l = 0$ curve at large n_l for all l values. Fits to our computed values are very close to $\propto n^2$ behavior, with a coefficient that is close to $\pi/4$. Similar fits in the Hulthén (Figure 3) and Pseudo-Hulthén cases give the same n^2 dependence, with the coefficient close to $1/2$. The ECSC potential shows a similar trend as the Yukawa case, but with additional structure superimposed on it, which does not smoothly converge.

(Note: in the tables and figures below, for simplicity we suppress the l subscript in n_l . n simply numbers the states for each l manifold; it is not generally equal the conventional principal quantum number from atomic physics, which ties state numbering to angular momentum.)

The convergence of the $l > 0$ curves in Figures 4–7 to the $l = 0$ curve at sufficiently large n can be rationalized by a simple argument. Large n corresponds to large \mathcal{D}_c , and therefore small μ_c , which is the $\mu \rightarrow 0$ Coulomb limit of both the Yukawa and ECSC potentials (similarly for Hulthén and Pseudo-Hulthén. Coulomb potential eigenvalues depend on l and n_l (state number for a given l) through their sum, as $\mathcal{E}_{n_l, l} = -1/(2(n_l + l)^2)$, $n_l = 1, 2, \dots$. For a given l , if $n_l \gg l$ the eigenvalues approach the $l = 0$ limit as n_l increases, regardless of l .

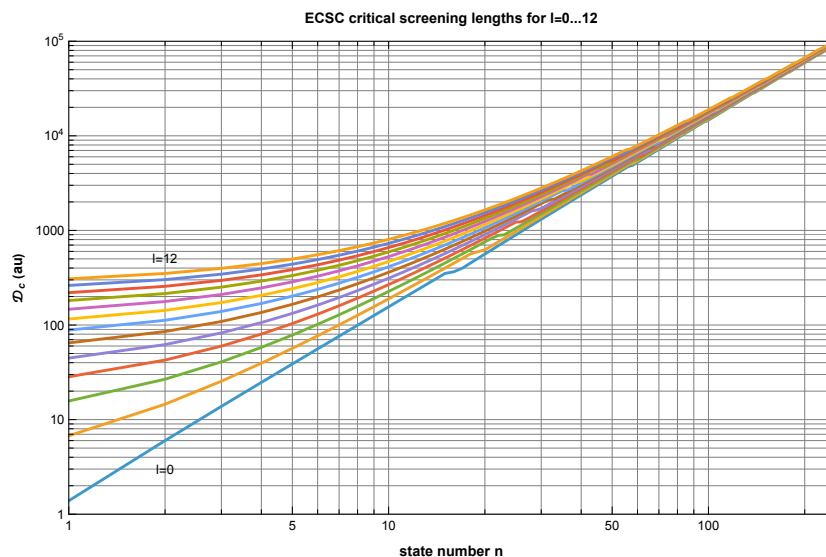


Figure 5. D_c vs n_l for ECSC potential up to $D = 10^5$ au calculated to 30 digits.

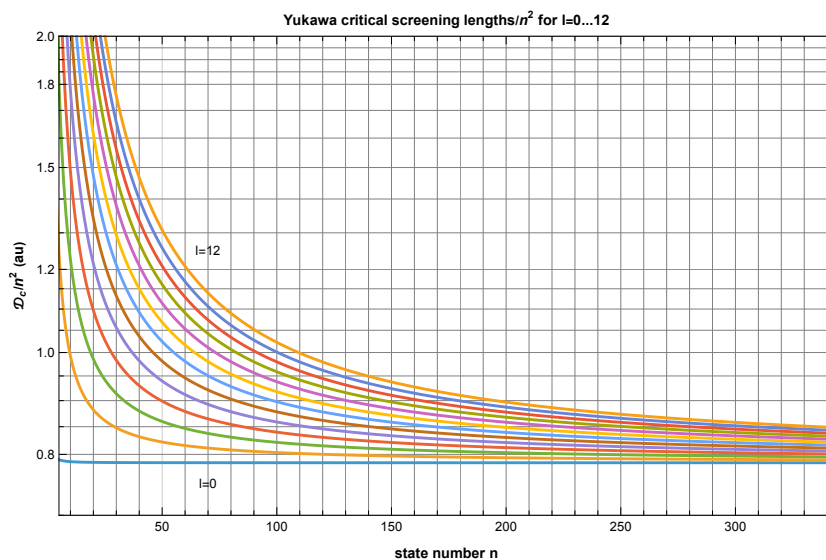


Figure 6. D_c/n^2 vs n_l for Yukawa potential up to $D = 10^5$ au calculated to 30 digits.

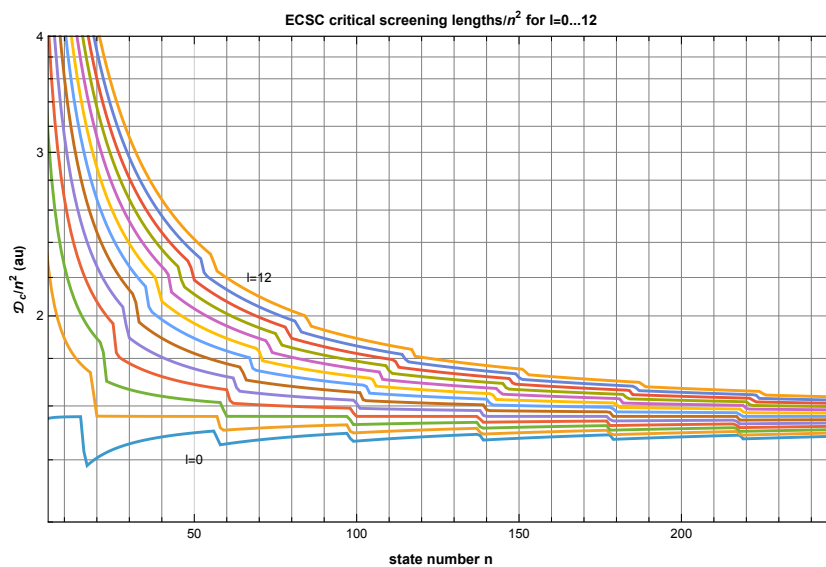


Figure 7. D_c/n^2 vs n_l for ECSC potential up to $D = 10^5$ au calculated to 30 digits.

We can explain the Yukawa and (Pseudo)-Hulthén $l = 0$ fitting results quantitatively using semiclassical quantization of the phase space integral in units of Planck's constant h , which becomes increasingly accurate as n increases. As usual, we employ units in which $\hbar = m = 1$ where m is the reduced mass of the two body system. The phase space integral is

$$\mathcal{J}(\mathcal{E}) = 2 \int_{r_1(\mathcal{E})}^{r_2(\mathcal{E})} \sqrt{2(\mathcal{E} - U(r))} dr,$$

where $r_1(\mathcal{E})$ and $r_2(\mathcal{E})$ are the classical turning points for energy \mathcal{E} , i.e. the extremal points at which $U(r) = \mathcal{E}$. For the Yukawa Potential (with $l = 0$, i.e. the centrifugal potential is zero), $r_1 = 0$, and $\mu r_2(\mathcal{E}) = W(-\mu/\mathcal{E})$ where W is the (principal branch of the) Lambert W function (ProductLog in Mathematica). For the $l = 0$ Hulthén/Pseudo-Hulthén Potential we have $\mu r_2(\mathcal{E}) = \log(1 - \mu/\mathcal{E})$.

Quantizing $\mathcal{J}(\mathcal{E})$ in multiples of Planck's constant $h = 2\pi\hbar$, and taking $\hbar = 1$ as usual, we have $2\pi(n - \nu) = \mathcal{J}(\mathcal{E})$. Here ν is the negative of a Maslov Index (we number states $n = 1, 2, \dots$), a refinement we can neglect in the limit of large n . This condition amounts to requiring an integer number of half-oscillations of the solution to reside between the classical turning points. Its energy parameter provides an upper bound to the true energy eigenvalue; the true eigenvalues is reduced below that limit by tunneling [1]. This quantization condition gives an approximate relationship between \mathcal{E} and n . Taking $\mathcal{E} \rightarrow 0$, $r_1 \rightarrow 0$, $r_2 \rightarrow \infty$ we get

$$2\pi(n - \nu) = 2 \int_0^\infty \sqrt{-2U(r)} dr.$$

For these potentials, the $\mathcal{J}(\mathcal{E} = 0)$ integral can be factored into a dimensionless definite integral (pure number) times $1/\sqrt{\mu} = \sqrt{\mathcal{D}}$. Equating it to $2\pi n$ implies that $\mathcal{D} \propto n^2$, and the question is, what are the proportionality constants? The integrals for the Yukawa and Hulthén/Pseudo-Hulthén potentials are straightforward to work out, and they evaluate respectively to $\mathcal{J} = 4\sqrt{\frac{\pi}{\mu}}$ and $\mathcal{J} = 2\pi\sqrt{\frac{2}{\mu}}$, which for large n gives the observed $\mathcal{D} \propto \frac{\pi}{4}n^2$ and $\mathcal{D} \propto \frac{1}{2}n^2$ behaviors.

The phase space integral for the ECSC potential is more involved, as is the physics it represents. The $l = 0$ ECSC potential crosses $\mathcal{E} = 0$ at $r = \frac{\pi}{2}\mathcal{D}$, unlike the others, which smoothly approach $\mathcal{E} = 0$ as $r \rightarrow \infty$. A detailed analysis of this interesting behavior of the bound states and resonant states of the ECSC potential is out of the scope of the present paper, which is focussed on critical binding parameters at $\mathcal{E} = 0$.

5. Conclusions

This work has had three main purposes: to validate and confirm excellent historical work, some done to as much as 30 digit accuracy with very limited computational resources; to build on all past work to extend the database to 60 digits over extended parameter ranges; and to introduce and explain the "Phase Method" [1] which allows such computations to be easily done in hours on ordinary desktop computers.

Specifically we have calculated tables of critical exponents μ_c for four potentials: Yukawa/Debye, Exponential Cosine Screened Coulomb (ECSC); Hulthén; and Pseudo-Hulthén (PH) for $l = 0 \dots 20$ for screening lengths up to $\mathcal{D}_c = 1/\mu_c \leq 1000$. The PH potential has the advantage of being exactly soluble and is used as a benchmark, and one of several checks on the accuracy of our calculations. We have highlighted the apparently-overlooked 1989 paper of Demiralp, which accurately calculated μ_c for s and p states for the Yukawa potential up to $n = 10$ with an accuracy to 30 digits, and other angular momenta from $l = 3 \dots 7$ to 27–29 digits. We further test our values by comparison with the most accurate values currently available in print by Jiao et al [3], and by applying cross checks between many independent calculations with different potential parameters by employing a well-known scaling relation in a novel way. Similar calculations (tables to be presented elsewhere) for the same potentials and l values at 30 digits accuracy also were performed for 100-fold larger screening lengths $\mathcal{D}_c \leq 10^5$ and for a couple, up to $\mathcal{D}_c \leq 10^6$ to demonstrate feasibility, with initial results presented here.

We find interesting not-yet-unexplained roughly periodic structure in plots of \mathcal{D}_c/n^2 vs n for the ECSC potential, while corresponding plots of \mathcal{D}_c/n^2 vs n plots for Yukawa are smooth. These imply that there are certain ranges of state number n in which the rate of state-binding suddenly but temporarily increases, perhaps owing to shape resonances. This should prove interesting to explore.

The Phase Method [1] and variations described here have a vast array of other potential applications, among them the study of related potentials (e.g. generalized ECSC potential), and unrelated ones; semiclassical limiting behavior and approximations; and dimensionalities $D \neq 3$, including fractal dimensions. The demonstrated ability to easily compute eigenvalues and critical binding parameters to exceptional accuracy, as well as wave functions, on inexpensive hardware, offers new possibilities for research and education in diverse fields. Mathematica code and data files are freely available for download. It is our hope that the data and methods presented here will prove to be of lasting value.

Appendix A. Data Tables

Appendix A.1. Yukawa Potential μ_c vs l for $\mathcal{D} = 1/\mu \leq 1000$ au

Table A1. Critical binding parameters μ_c (au⁻¹) for Yukawa potential, $l=0$

1	1.190	612	421	060	617	705	342	777	106	361	046	347	275	901	572	981	749	063	530	790
2	0.310	209	282	713	936	939	110	112	212	953	178	705	349	599	031	944	443	568	807	467
3	0.139	450	294	064	178	013	882	357	954	889	722	517	358	682	070	623	404	646	786	613
4	0.078	828	110	273	171	565	170	282	204	980	085	098	973	351	019	260	888	746	422	916
5	0.050	583	170	374	558	799	782	284	408	667	887	460	545	239	317	224	034	890	845	443
6	0.035	183	477	367	820	588	148	390	875	853	262	885	268	142	234	028	596	118	621	973
7	0.025	876	416	481	121	578	169	930	945	718	830	202	789	078	031	434	488	317	564	745
8	0.019	826	307	429	825	264	337	059	425	553	555	077	160	575	657	815	051	433	639	569
9	0.015	673	732	828	474	905	372	994	735	928	228	583	279	580	081	823	547	690	160	729
10	0.012	700	950	763	810	386	771	388	389	902	420	733	627	024	086	520	424	984	123	071
11	0.010	500	024	455	619	102	409	170	099	944	982	569	310	719	743	670	339	995	597	072
12	0.008	825	198	121	484	659	811	270	303	976	175	766	844	867	057	703	402	578	304	334
13	0.007	521	262	452	007	728	805	322	863	142	424	514	822	184	317	460	547	814	979	975
14	0.006	486	286	938	808	751	846	027	057	159	306	710	502	220	964	478	728	953	456	964
15	0.005	651	091	789	872	167	090	289	697	309	407	460	341	038	027	574	898	977	790	528
16	0.004	967	387	416	043	937	005	279	974	157	702	092	628	637	459	319	849	508	630	310
17	0.004	400	638	186	614	534	200	164	624	753	101	341	961	775	493	049	429	592	111	617
18	0.003	925	616	252	642	987	361	919	801	781	687	855	256	624	734	089	581	709	077	023
19	0.003	523	546	047	303	059	302	930	807	006	543	848	657	268	654	756	338	445	013	676
20	0.003	180	220	857	157	501	549	875	970	900	787	366	240	278	974	498	709	193	057	221
21	0.002	884	730	898	748	748	370	510	575	417	920	197	208	411	065	838	540	918	481	283
22	0.002	628	586	099	141	690	992	345	132	094	391	618	422	797	663	117	277	445	964	482
23	0.002	405	099	565	976	777	462	613	742	086	301	436	629	914	052	722	366	960	447	330
24	0.002	208	946	829	749	002	585	609	890	805	134	123	005	098	755	544	739	839	203	438
25	0.002	035	845	837	319	507	460	598	299	223	590	573	273	328	352	843	900	479	099	061
26	0.001	882	321	319	013	588	185	399	202	323	956	567	238	298	970	220	511	031	022	951
27	0.001	745	529	031	596	967	394	013	752	883	138	912	204	659	566	564	432	926	568	473
28	0.001	623	123	100	112	845	045	101	426	154	562	803	419	344	175	396	382	738	941	081
29	0.001	513	154	790	434	057	475	846	749	997	753	711	787	895	410	525	806	583	407	642
30	0.001	413	994	481	430	354	847	091	995	213	563	143	574	858	903	425	925	220	597	261
31	0.001	324	270	953	535	356	533	721	572	017	564	643	842	847	503	263	856	721	647	933
32	0.001	242	823	737	179	600	438	576	044	256	794	487	115	870	099	004	541	118	628	533
33	0.001	168	665	406	436	293	443	089	992	126	289	954	735	444	623	102	366	797	661	583
34	0.001	100	951	514	625	698	631	606	189	558	994	126	701	492	245	400	524	937	647	872
35	0.001	038	956	451	786	215	690	706	155	028	365	487	321	455	601	213	643	108	615	177

Table A2. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=1$

1	0.220	216	806	606	573	040	405	041	463	289	577	110	508	548	104	023	305	084	754	657
2	0.112	710	498	359	524	944	973	972	952	155	224	913	892	868	301	875	890	014	593	875
3	0.067	885	376	100	579	552	788	417	968	577	926	855	528	188	107	427	403	091	549	403
4	0.045	186	248	071	624	990	093	706	122	691	149	700	082	081	699	976	692	806	777	344
5	0.032	174	932	293	205	003	538	581	132	479	497	170	537	750	623	981	126	930	117	511
6	0.024	047	639	235	996	140	851	390	943	514	030	035	675	238	783	333	920	290	131	907
7	0.018	640	705	347	623	634	654	280	763	091	951	419	532	127	297	367	838	085	448	127
8	0.014	865	869	356	224	286	239	220	545	742	341	923	021	754	151	982	978	073	475	837
9	0.012	128	229	513	755	452	397	915	667	879	939	264	611	177	011	460	670	670	828	612
10	0.010	080	687	145	923	836	466	765	910	015	546	439	569	913	105	433	368	902	613	307
11	0.008	509	830	499	710	479	069	873	094	008	358	997	383	848	480	474	272	922	583	324
12	0.007	278	668	379	888	337	373	490	037	851	784	478	448	770	322	700	178	950	709	026
13	0.006	296	036	702	746	696	030	864	002	654	933	252	283	498	242	423	890	647	828	468
14	0.005	499	381	839	660	386	381	910	026	478	217	577	768	672	121	690	011	380	117	557
15	0.004	844	636	441	262	068	006	313	029	223	715	294	851	389	343	933	457	321	314	760
16	0.004	300	037	517	324	852	911	233	319	241	109	599	247	077	705	594	076	090	544	189
17	0.003	842	226	505	895	156	685	504	061	852	778	439	097	527	492	424	141	010	873	837
18	0.003	453	717	701	331	520	535	157	714	511	573	752	357	702	570	686	090	275	549	267
19	0.003	121	212	990	695	427	514	038	142	600	662	801	589	490	316	456	054	263	576	486
20	0.002	834	454	552	306	933	607	219	254	650	364	909	995	696	257	810	636	277	317	361
21	0.002	585	427	964	400	090	816	911	891	970	182	323	725	271	901	385	550	216	674	772
22	0.002	367	798	612	298	766	592	699	509	977	451	237	650	315	498	092	054	846	347	200
23	0.002	176	506	522	122	167	038	928	807	483	767	618	074	874	559	727	831	805	560	934
24	0.002	007	470	722	526	291	520	410	939	702	436	475	387	447	922	043	526	849	229	839
25	0.001	857	370	574	821	133	625	569	481	579	762	605	274	570	459	191	740	685	633	026
26	0.001	723	482	004	804	275	535	865	533	469	996	700	596	091	855	669	492	421	919	457
27	0.001	603	553	437	033	706	920	056	660	004	002	837	964	937	629	839	845	815	161	581
28	0.001	495	710	805	433	181	432	382	166	478	091	914	633	492	675	901	649	701	715	847
29	0.001	398	384	108	558	467	382	672	389	889	861	747	550	847	715	778	020	223	051	410
30	0.001	310	250	102	833	715	024	834	389	134	071	732	450	976	710	281	049	033	349	026
31	0.001	230	187	206	409	594	279	451	158	767	280	728	961	852	262	389	126	055	200	474
32	0.001	157	239	729	339	536	930	506	942	974	729	467	957	609	782	525	684	694	791	028
33	0.001	090	589	289	956	033	634	456	609	981	034	981	037	707	106	886	395	073	854	117
34	0.001	029	531	814	193	654	775	135	822	036	625	433	213	210	801	172	832	352	701	833

Table A3. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=2$

1	0.091	345	120	771	732	184	927	710	066	860	260	994	943	058	993	864	195	867	962	724
2	0.058	105	052	754	469	264	181	224	714	848	364	071	796	955	340	541	865	049	333	535
3	0.040	024	353	938	324	274	958	258	960	950	658	181	673	118	690	753	654	208	576	940
4	0.029	166	650	229	397	650	381	902	551	150	091	217	129	157	580	019	769	556	909	594
5	0.022	161	826	355	339	786	360	688	903	637	457	024	877	612	036	345	657	214	230	915
6	0.017	390	648	079	030	682	359	871	838	426	184	182	502	979	485	340	826	520	343	673
7	0.013	999	880	572	892	515	518	358	979	691	514	812	098	789	437	273	737	551	341	386
8	0.011	506	513	742	042	353	053	318	933	727	190	168	758	587	209	157	943	458	105	434
9	0.009	620	998	940	890	081	042	424	057	801	069	027	814	090	471	439	898	512	911	102
10	0.008	161	438	126	668	077	044	523	912	919	975	242	516	373	935	542	734	899	348	321
11	0.007	009	014	659	248	687	450	049	068	390	492	017	656	833	274	589	352	446	825	790
12	0.006	083	513	307	343	769	558	596	719	084	217	149	872	553	287	208	443	598	072	485
13	0.005	329	226	409	206	182	617	348	572	775	155	650	190	945	604	476	957	984	410	940
14	0.004	706	506	949	866	792	512	495	422	435	716	029	963	101	377	318	013	909	277	734
15	0.004	186	527	200	410	042	742	554	435	951	858	045	443	435	632	667	950	375	315	840
16	0.003	747	926	383	550	933	875	136	224	697	423	279	581	813	662	973	078	313	615	282
17	0.003	374	608	580	805	902	649	837	659	210	475	717	015	240	132	062	620	663	303	514
18	0.003	054	261	568	450	447	774	006	055	618	720	207	075	788	231	222	821	048	916	356
19	0.002	777	339	272	194	660	833	012	768	207	848	573	859	863	430	645	052	250	329	466
20	0.002	536	349	311	124	928	089	182	086	459	601	707	039	529	840	356	104	537	778	129
21	0.002	325	345	514	945	833	738	861	095	969	664	614	107	779	838	637	455	001	225	593
22	0.002	139	560	761	545	996	571	936	013	070	195	328	858	393	703	719	224	502	165	369
23	0.001	975	137	529	514	510	354	406	032	056	886	739	893	061	243	972	697	606	823	909
24	0.001	828	927	566	628	524	923	101	211	142	942	527	640	333	770	967	660	694	246	099
25	0.001	698	341	150	409	933	953	733	046	490	190	128	000	806	523	805	651	869	848	548
26	0.001	581	232	403	959	194	139	432	095	443	641	835	760	096	746	988	833	129	612	474
27	0.001	475	811	146	311	305	909	447	665	928	269	328	257	730	663	861	902	163	531	699
28	0.001	380	574	492	052	148	496	398	574	889	651	133	405	696	540	251	054	686	840	630
29	0.001	294	253	304	826	113	370	665	428	389	193	914	411	563	199	890	931	647	151	226
30	0.001	215	769	932	371	477	893	693	222	805	826	863	312	587	198	270	663	636	942	743
31	0.001	144	204	588	312	792	283	726	073	175	272	439	118	444	679	484	764	438	558	006
32	0.001	078	768	418	048	793	116	474	516	325	958	752	052	503	431	043	757	851	736	456
33	0.001	018	781	773	061	749	791	129	320	870	422	123	075	885	617	400	474	730	729	266

Table A4. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=3$

1	0.049	831	132	318	645	225	242	887	831	002	756	419	440	415	254	341	194	982	502	589
2	0.035	389	389	799	948	414	321	063	141	286	063	362	448	208	083	941	289	892	157	971
3	0.026	350	671	639	829	827	843	633	383	335	035	383	631	935	032	612	947	309	330	267
4	0.020	342	170	661	307	635	762	223	228	450	065	484	246	066	940	406	974	771	477	223
5	0.016	156	534	483	956	706	186	410	014	699	551	483	726	817	563	207	573	518	227	065
6	0.013	129	670	383	752	190	264	262	157	917	649	320	668	913	997	462	501	300	189	673
7	0.010	872	967	863	956	656	342	020	988	496	407	073	181	307	093	545	132	692	686	993
8	0.009	147	265	755	199	268	525	577	924	628	974	040	607	565	331	808	752	619	513	417
9	0.007	799	091	791	686	795	710	131	452	595	299	498	840	659	022	373	640	501	679	706
10	0.006	726	427	312	139	240	861	214	910	759	578	845	633	802	104	201	004	082	855	607
11	0.005	859	383	443	402	889	698	289	236	251	622	438	930	953	446	493	153	408	211	647
12	0.005	148	820	047	526	225	185	045	857	181	995	018	473	506	632	864	251	408	032	433
13	0.004	559	392	437	530	779	396	185	247	922	013	097	837	524	042	575	992	551	325	080
14	0.004	065	165	471	558	655	012	064	088	436	545	336	755	705	403	583	745	763	678	618
15	0.003	646	768	376	314	233	606	573	161	941	208	620	684	968	227	283	273	284	885	643
16	0.003	289	502	633	216	048	192	574	672	661	599	916	090	238	618	463	782	825	826	848
17	0.002	982	055	486	788	724	398	334	477	829	435	111	025	837	466	831	923	805	881	408
18	0.002	715	607	664	343	344	053	292	853	654	935	292	143	165	268	196	142	338	690	688
19	0.002	483	203	307	516	971	681	859	424	549	993	353	399	544	774	111	150	280	516	139
20	0.002	279	297	761	250	375	290	750	408	181	634	900	677	086	689	129	248	825	673	886
21	0.002	099	428	166	628	839	463	777	361	230	566	958	396	211	396	595	939	409	513	926
22	0.001	939	970	232	034	424	082	634	325	271	630	512	685	964	158	271	561	600	936	121
23	0.001	797	956	386	187	015	984	241	369	615	094	918	116	640	509	463	088	195	930	211
24	0.001	670	938	253	277	169	233	056	284	762	099	518	961	964	715	630	545	691	685	442
25	0.001	556	881	538	131	045	939	860	272	845	354	331	649	503	836	819	868	632	852	472
26	0.001	454	084	889	238	817	876	719	857	784	205	570	323	572	494	521	815	472	080	959
27	0.001	361	116	694	648	767	942	391	343	942	070	473	192	981	666	859	256	063	304	039
28	0.001	276	765	425	737	156	972	514	713	243	123	304	066	935	453	062	826	160	937	355
29	0.001	200	000	312	916	096	874	039	942	749	637	128	322	028	179	146	716	691	838	248
30	0.001	129	939	970	418	023	721	190	916	915	081	194	137	505	946	742	094	008	155	363
31	0.001	065	827	187	541	120	970	799	660	693	252	073	378	937	543	174	659	633	147	968
32	0.001	007	008	540	723	110	453	312	126	542	731	245	616	310	673	817	018	596	127	931

Table A5. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=4$

1	0.031	343	552	436	538	045	217	407	715	645	768	660	311	933	402	418	847	475	743	263
2	0.023	799	103	968	969	544	816	131	058	135	674	890	103	275	339	606	635	424	001	953
3	0.018	646	215	359	623	339	252	011	442	573	715	064	507	502	876	468	540	810	216	313
4	0.014	980	862	636	386	665	583	951	297	163	710	250	509	181	187	055	193	530	056	221
5	0.012	286	145	678	137	967	681	387	938	523	522	620	191	693	243	195	219	177	937	507
6	0.010	250	170	296	275	649	468	436	766	130	425	113	885	709	328	055	826	184	193	602
7	0.008	676	175	408	226	172	756	104	823	814	983	799	375	234	985	868	143	563	265	324
8	0.007	435	300	540	333	087	455	670	244	870	848	665	496	368	206	209	200	831	141	662
9	0.006	440	420	274	157	059	035	813	828	138	976	926	729	839	730	138	662	290	521	431
10	0.005	630	986	001	798	835	787	482	560	241	030	160	589	384	983	606	159	460	085	160
11	0.004	963	899	986	689	285	023	671	739	067	813	660	301	093	770	576	999	050	383	628
12	0.004	407	834	292	379	346	775	981	460	150	484	206	993	251	408	889	639	783	510	227
13	0.003	939	588	917	617	898	199	511	666	827	816	230	941	987	338	954	401	749	467	705
14	0.003	541	695	809	262	418	358	258	147	367	359	623	653	826	794	552	270	646	362	825
15	0.003	200	805	693	900	593	119	738	431	699	814	780	398	592	430	672	227	477	178	786
16	0.002	906	579	239	013	156	530	468	604	345	146	439	426	563	268	240	182	853	225	791
17	0.002	650	910	514	033	720	453	595	914	372	563	973	142	472	175	288	326	162	078	179
18	0.002	427	373	893	045	390	161	197	983	762	647	703	935	875	061	769	968	045	301	990
19	0.002	230	823	996	545	080	929	538	238	810	685	217	329	795	098	841	005	140	742	654
20	0.002	057	102	228	651	278	914	860	670	805	894	614	101	325	945	070	929	080	116	926
21	0.001	902	818	711	700	931	943	500	899	464	615	193	280	029	922	177	292	786	280	541
22	0.001	765	188	310	404	463	977	532	706	312	135	700	843	445	432	440	677	821	678	107
23	0.001	641	905	968	565	860	687	998	562	610	571	267	191	073	159	591	782	109	715	999
24	0.001	531	050	964	936	382	621	915	017	201	729	241	419	582	532	335	106	883	030	812
25	0.001	431	012	681	834	144	623	893	901	004	263	830	213	268	506	751	292	894	441	657
26	0.001	340	432	544	322	203	056	858	066	029	621	562	426	282	241	850	269	848	435	058
27	0.001	258	158	232	834	184	595	386	951	198	066	015	565	160	644	568	413	565	290	276
28	0.001	183	207	296	185	967	337	294	495	763	263	849	295	289	186	151	270	185	354	239
29	0.001	114	738	025	867	771	817	695	593	650	189	441	877	894	889	358	012	798	757	952
30	0.001	052	025	984	167	231	875	696	220	132	270	494	924	454	652	523	992	900	254	242

Table A6. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=5$

1	0.021 524 548 401 894 416 262 524 364 177 194 466 883 459 401 283 133 469 882 964
2	0.017 095 135 615 471 857 578 885 040 840 919 499 801 798 948 713 700 365 836 399
3	0.013 883 519 722 165 163 911 660 034 621 283 169 354 094 048 610 278 678 681 690
4	0.011 485 753 776 518 393 844 790 229 238 468 285 898 336 202 595 467 314 010 449
5	0.009 651 169 953 126 909 789 885 247 571 489 616 892 028 780 410 282 245 333 635
6	0.008 217 950 625 200 664 573 659 593 141 928 815 118 771 500 759 643 948 700 606
7	0.007 078 067 894 672 385 758 249 398 468 330 578 213 541 539 077 331 769 492 458
8	0.006 157 289 717 717 264 083 983 436 752 308 686 701 235 871 906 210 149 887 915
9	0.005 403 310 239 195 924 377 948 458 162 320 486 561 340 754 020 756 555 114 501
10	0.004 778 452 977 369 578 427 843 543 209 776 905 086 275 950 283 958 298 715 241
11	0.004 255 045 931 632 468 597 930 204 878 918 933 581 370 096 914 792 533 797 334
12	0.003 812 410 019 081 560 541 376 616 973 747 959 789 581 467 145 671 212 298 198
13	0.003 434 850 400 625 823 596 884 068 924 773 710 870 715 305 834 701 631 222 466
14	0.003 110 287 613 363 367 249 668 363 724 596 225 397 863 864 119 980 324 260 224
15	0.002 829 306 488 249 314 222 267 699 555 230 084 518 224 153 198 314 635 148 236
16	0.002 584 483 667 168 114 591 408 090 672 988 871 186 707 634 914 066 011 347 108
17	0.002 369 904 476 527 537 655 587 306 026 469 561 773 643 203 596 593 552 724 741
18	0.002 180 810 754 362 922 030 335 467 276 039 768 274 768 400 860 354 828 146 860
19	0.002 013 340 690 275 879 305 713 698 173 754 484 542 071 162 966 539 039 961 279
20	0.001 864 334 266 333 522 836 843 629 739 326 951 840 936 945 440 294 940 645 613
21	0.001 731 186 100 864 983 916 478 409 186 550 342 529 460 006 215 731 338 455 608
22	0.001 611 732 973 217 795 354 972 561 069 801 488 155 896 597 061 731 706 907 160
23	0.001 504 167 015 542 677 685 663 134 156 682 101 083 471 501 251 179 337 945 783
24	0.001 406 968 104 807 650 291 938 986 799 094 882 623 110 873 166 541 491 209 999
25	0.001 318 850 761 471 931 366 935 521 687 288 262 886 399 263 684 656 422 046 673
26	0.001 238 722 111 175 958 741 106 528 387 527 442 548 124 584 930 089 359 351 818
27	0.001 165 648 357 161 920 355 078 830 465 363 854 866 268 059 076 312 224 660 822
28	0.001 098 827 853 746 689 400 098 131 476 345 162 889 793 445 839 640 324 245 726
29	0.001 037 569 339 197 314 004 070 695 060 671 416 900 743 803 965 226 039 302 431

Table A7. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=6$

1	0.015 691 083 667 701 475 990 456 075 684 746 765 747 489 404 601 734 787 901 617
2	0.012 871 464 312 728 093 468 171 295 596 442 391 982 070 946 861 585 607 789 714
3	0.010 736 147 820 911 701 644 601 236 426 176 375 751 136 815 181 112 002 957 379
4	0.009 082 952 537 639 986 592 173 519 469 425 260 061 210 063 874 681 677 853 359
5	0.007 778 558 385 748 099 860 568 617 229 592 920 963 732 773 442 984 439 165 276
6	0.006 732 353 063 648 933 088 218 291 811 417 569 808 795 221 292 410 943 256 321
7	0.005 881 113 049 997 633 587 222 379 272 302 724 611 627 362 644 607 029 731 117
8	0.005 179 699 336 067 246 152 066 822 873 198 613 720 324 719 998 403 885 533 717
9	0.004 595 224 082 090 593 670 655 069 129 500 111 771 590 454 663 000 802 115 566
10	0.004 103 289 551 746 336 045 236 479 761 089 145 821 626 326 391 572 275 271 006
11	0.003 685 502 078 183 432 613 848 026 724 480 319 397 226 799 715 082 004 232 711
12	0.003 327 791 850 616 851 030 946 771 392 307 570 073 403 204 865 250 588 835 334
13	0.003 019 254 370 704 045 405 100 107 894 847 934 365 216 826 600 425 717 922 485
14	0.002 751 337 025 318 293 645 611 605 483 383 159 272 492 518 694 612 383 642 882
15	0.002 517 258 503 219 957 472 361 646 990 312 717 964 172 347 734 977 582 491 226
16	0.002 311 588 141 227 367 805 687 371 801 852 145 599 005 234 191 687 550 509 396
17	0.002 129 936 928 410 114 323 618 093 925 413 704 598 706 333 879 689 458 989 013
18	0.001 968 727 644 463 920 474 961 017 078 205 818 210 063 787 578 277 629 413 853
19	0.001 825 021 861 825 776 108 217 755 748 210 604 827 336 228 789 316 702 443 580
20	0.001 696 388 333 153 405 088 760 399 813 924 033 723 010 018 874 919 369 208 788
21	0.001 580 801 857 137 708 186 019 817 847 203 073 940 253 520 802 461 123 550 281
22	0.001 476 564 837 985 978 917 946 554 967 291 369 576 368 378 027 928 163 740 730
23	0.001 382 245 915 970 997 146 866 276 768 844 677 041 367 786 669 679 381 308 544
24	0.001 296 631 562 737 812 939 991 713 168 852 874 674 022 478 107 014 205 935 236
25	0.001 218 687 611 196 305 109 054 563 454 085 246 030 565 701 681 447 409 186 200
26	0.001 147 528 462 123 847 933 901 358 975 320 835 668 992 662 965 501 945 912 553
27	0.001 082 392 269 659 998 708 355 882 407 289 444 738 037 739 985 958 892 019 706
28	0.001 022 620 818 021 923 148 802 760 553 537 832 576 361 364 590 294 753 906 219

Table A8. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=7$

1	0.011 944 531 306 208 677 525 206 755 417 428 052 609 604 230 955 715 860 040 182
2	0.010 039 758 846 369 438 096 516 015 494 947 267 187 183 302 585 495 974 384 002
3	0.008 548 707 757 897 152 348 756 881 521 307 298 365 141 920 595 361 534 141 093
4	0.007 361 194 112 631 079 076 347 189 566 228 968 718 840 491 876 599 791 123 747
5	0.006 401 071 548 888 573 906 330 788 559 918 396 686 014 424 537 717 086 018 226
6	0.005 614 442 239 208 413 159 709 270 561 099 207 434 217 271 667 245 719 361 398
7	0.004 962 343 085 609 065 409 656 260 924 390 065 448 038 125 229 781 172 440 915
8	0.004 416 074 969 810 907 970 526 810 303 391 857 539 024 529 062 176 258 296 404
9	0.003 954 142 236 476 666 278 910 079 407 729 959 324 487 961 737 574 895 102 379
10	0.003 560 200 686 378 626 401 516 941 411 662 448 040 775 307 248 261 227 861 611
11	0.003 221 653 053 543 648 048 330 227 390 599 074 874 493 085 016 890 365 522 492
12	0.002 928 669 560 826 826 632 022 464 633 536 633 113 740 169 635 337 708 252 136
13	0.002 673 493 239 101 928 947 187 252 838 587 510 787 004 031 230 656 008 188 266
14	0.002 449 939 550 974 127 343 712 029 795 611 406 158 755 376 656 118 303 407 692
15	0.002 253 030 840 615 853 724 080 841 480 624 583 265 634 794 356 498 716 826 658
16	0.002 078 725 790 036 004 285 545 341 425 663 415 987 616 421 329 332 653 921 718
17	0.001 923 716 778 004 823 219 501 204 934 975 253 444 581 424 321 461 687 395 113
18	0.001 785 276 409 311 969 253 523 874 710 291 995 241 563 215 851 679 803 189 196
19	0.001 661 140 083 797 948 053 942 338 662 309 362 734 028 635 253 433 789 636 994
20	0.001 549 415 279 942 150 535 491 443 715 032 469 653 715 616 223 506 549 922 763
21	0.001 448 510 849 228 748 610 672 403 975 814 168 547 968 894 458 655 795 618 476
22	0.001 357 081 447 051 376 572 600 916 108 003 374 093 039 601 210 173 054 157 875
23	0.001 273 983 518 393 097 469 888 128 550 702 987 930 710 495 597 365 791 767 027
24	0.001 198 240 180 023 735 532 641 905 984 169 525 657 320 986 228 295 521 609 803
25	0.001 129 013 007 892 089 769 639 940 394 720 602 695 449 573 396 633 295 040 416
26	0.001 065 579 224 880 297 293 717 724 330 859 592 767 107 420 065 434 649 352 879
27	0.001 007 313 142 309 918 256 032 572 433 927 596 105 623 097 654 112 398 700 745

Table A9. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=8$

1	0.009 395 999 944 229 339 224 815 371 489 123 824 712 215 843 650 316 625 130 244
2	0.008 049 285 481 393 441 050 500 660 525 463 100 634 466 948 144 655 517 910 648
3	0.006 967 264 013 476 505 425 449 438 946 407 037 148 883 838 085 007 261 904 163
4	0.006 085 772 583 749 423 361 272 611 700 121 963 362 172 146 752 876 144 553 184
5	0.005 358 781 236 870 528 025 460 544 297 654 352 970 530 603 818 532 219 437 391
6	0.004 752 621 421 098 130 620 937 910 939 291 194 739 799 584 668 562 273 825 350
7	0.004 242 235 989 927 510 214 102 154 530 580 597 235 704 086 069 104 401 144 291
8	0.003 808 684 339 665 585 421 450 136 488 296 669 151 460 742 674 011 023 949 979
9	0.003 437 446 921 667 810 978 303 849 796 049 184 340 223 403 020 784 351 850 864
10	0.003 117 250 673 470 559 321 166 620 133 363 148 522 971 744 384 529 999 081 439
11	0.002 839 241 029 266 540 366 976 566 687 501 186 397 990 873 253 895 163 580 477
12	0.002 596 388 909 385 139 460 522 562 741 643 575 196 493 755 508 838 846 570 581
13	0.002 383 059 792 373 246 991 808 379 169 974 571 587 162 713 868 441 954 167 395
14	0.002 194 696 366 688 727 005 811 608 852 847 655 345 872 077 026 966 345 944 397
15	0.002 027 581 937 730 602 223 393 643 875 562 163 746 797 494 732 235 832 640 138
16	0.001 878 662 026 735 100 877 672 503 927 528 050 906 933 266 324 919 320 652 769
17	0.001 745 408 425 832 184 791 816 039 974 331 338 940 035 372 523 187 463 508 951
18	0.001 625 714 587 396 416 809 816 203 671 981 577 502 363 441 291 716 755 009 001
19	0.001 517 814 388 480 072 200 287 303 267 420 209 018 734 055 904 790 778 797 621
20	0.001 420 218 508 039 996 436 384 775 023 848 474 505 640 549 725 946 419 254 696
21	0.001 331 664 199 796 881 950 754 282 080 125 489 496 276 176 823 823 541 562 576
22	0.001 251 075 342 949 062 842 287 357 819 664 283 079 076 819 538 487 080 221 241
23	0.001 177 530 443 737 066 820 482 105 959 713 463 241 507 224 195 141 982 794 786
24	0.001 110 236 835 481 654 047 770 413 565 249 806 313 533 862 964 274 264 136 476
25	0.001 048 509 746 303 775 426 833 603 164 228 069 279 659 304 134 232 835 893 659

Table A10. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=9$

1	0.007 584 125 249 138 792 121 527 487 152 454 852 279 440 449 565 965 407 365 864
2	0.006 597 033 115 693 489 589 576 186 666 737 207 490 058 388 854 264 489 194 153
3	0.005 787 097 483 586 137 309 150 239 140 474 623 612 101 072 431 857 959 285 062
4	0.005 114 915 383 089 152 569 411 366 503 823 719 978 986 773 215 435 075 916 108
5	0.004 551 345 045 024 666 356 146 479 397 854 217 749 562 966 604 818 410 081 072
6	0.004 074 484 792 696 218 118 207 090 923 945 036 186 651 097 862 084 919 298 511
7	0.003 667 634 139 663 550 231 875 797 474 914 462 753 762 662 661 033 913 808 353
8	0.003 317 890 360 839 953 391 665 231 693 330 641 138 063 648 833 734 990 010 786
9	0.003 015 165 049 871 463 972 868 976 201 676 521 954 139 610 572 813 731 087 091
10	0.002 751 483 666 477 167 081 012 037 922 610 102 748 150 103 629 210 996 860 498
11	0.002 520 479 159 086 602 758 130 781 318 843 423 622 300 252 401 860 231 755 898
12	0.002 317 020 853 798 832 359 639 175 843 441 263 523 853 816 061 465 220 269 225
13	0.002 136 939 032 618 266 990 443 240 663 904 733 669 736 508 234 912 775 623 435
14	0.001 976 818 137 944 311 024 062 104 873 478 567 108 853 364 619 633 632 697 818
15	0.001 833 839 822 524 118 146 365 491 076 209 288 347 043 354 930 675 250 806 235
16	0.001 705 662 632 356 885 680 202 940 163 989 458 738 865 469 888 345 193 135 395
17	0.001 590 328 908 643 110 123 165 403 483 454 423 895 752 096 009 584 787 644 726
18	0.001 486 192 121 598 028 510 217 098 931 204 294 608 015 322 874 629 178 125 987
19	0.001 391 859 688 429 412 177 734 072 906 222 096 429 381 878 883 632 864 126 340
20	0.001 306 147 631 267 882 288 522 129 212 862 117 285 228 997 907 180 792 583 763
21	0.001 228 044 364 791 101 738 553 663 019 129 105 530 121 137 841 039 958 133 724
22	0.001 156 681 579 450 838 113 489 757 346 957 050 653 972 688 399 630 301 874 676
23	0.001 091 310 680 552 144 671 866 704 234 379 398 769 277 708 362 049 267 266 189
24	0.001 031 283 608 182 991 869 223 668 871 688 899 991 596 326 673 357 185 734 088

Table A11. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=10$

1	0.006 250 052 998 160 550 125 094 547 331 185 307 026 223 602 413 300 224 717 538
2	0.005 505 090 666 622 390 257 931 928 915 029 232 150 879 294 763 424 200 666 430
3	0.004 883 139 839 015 814 308 457 481 923 223 741 406 185 006 551 319 732 178 311
4	0.004 358 925 188 061 638 748 304 535 008 183 463 349 164 733 201 486 765 212 139
5	0.003 913 274 064 716 964 419 566 621 726 413 066 366 721 016 362 532 401 925 762
6	0.003 531 445 167 692 476 806 594 273 791 351 848 104 071 095 938 580 735 069 509
7	0.003 201 964 158 709 195 528 838 567 563 625 568 241 184 506 447 146 688 177 016
8	0.002 915 798 808 338 067 174 722 260 041 665 089 709 771 515 621 808 367 096 203
9	0.002 665 765 687 651 831 691 673 248 255 376 329 561 516 056 570 398 776 654 620
10	0.002 446 097 396 502 023 109 081 680 542 793 209 543 769 716 310 377 318 571 022
11	0.002 252 122 799 661 950 378 810 218 327 732 010 589 534 872 916 259 167 597 093
12	0.002 080 027 935 581 741 293 797 178 318 990 997 831 956 580 335 559 723 838 717
13	0.001 926 675 265 543 951 436 445 441 237 643 038 677 012 304 705 585 184 596 567
14	0.001 789 465 623 093 761 037 199 198 826 421 934 858 795 110 656 177 748 442 220
15	0.001 666 231 767 642 459 371 183 103 740 713 966 282 565 696 647 712 143 606 418
16	0.001 555 155 574 506 382 979 417 617 513 962 712 413 735 429 506 567 895 462 566
17	0.001 454 703 075 289 709 945 315 612 269 299 830 789 189 603 411 122 501 528 062
18	0.001 363 573 102 357 857 456 021 499 221 212 064 950 823 589 408 729 730 704 013
19	0.001 280 656 390 299 498 290 561 929 733 985 272 365 034 456 471 001 163 952 598
20	0.001 205 002 780 195 834 880 951 875 243 835 305 580 963 380 844 300 845 946 938
21	0.001 135 794 750 235 436 376 063 502 618 169 908 098 781 490 633 866 326 245 422
22	0.001 072 325 921 095 456 495 141 294 414 506 459 610 184 786 102 929 493 891 706
23	0.001 013 983 499 760 005 051 801 319 898 014 119 047 645 987 529 432 071 111 159

Table A12. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=11$

1	0.005 239 411 361 174 170 081 811 642 525 798 338 434 184 323 360 747 063 821 342
2	0.004 663 418 808 054 505 064 777 756 023 350 704 749 086 313 198 613 515 461 215
3	0.004 175 494 195 710 189 388 646 348 773 341 854 194 591 416 714 401 644 612 676
4	0.003 758 823 657 939 642 725 796 629 224 219 665 899 039 832 708 046 070 480 416
5	0.003 400 377 273 927 100 382 220 875 245 437 599 207 644 013 766 771 991 942 529
6	0.003 089 940 413 523 433 763 303 415 351 859 962 261 736 846 628 086 158 039 911
7	0.002 819 420 424 860 304 881 143 589 968 050 760 396 588 104 490 673 213 423 324
8	0.002 582 343 247 597 222 487 659 930 015 036 334 676 495 071 271 510 165 515 597
9	0.002 373 483 081 135 099 444 420 533 486 732 696 246 810 983 935 695 033 539 708
10	0.002 188 586 608 523 370 563 258 465 736 790 900 018 649 990 507 576 163 779 733
11	0.002 024 165 310 572 309 026 832 090 598 381 126 152 340 398 996 370 871 184 092
12	0.001 877 337 416 776 480 284 749 316 014 984 579 227 717 895 792 179 613 151 799
13	0.001 745 706 455 380 135 287 769 879 363 775 733 608 347 223 833 323 004 743 207
14	0.001 627 267 077 455 873 029 369 664 278 210 059 631 972 699 831 783 208 143 815
15	0.001 520 331 408 409 830 608 148 634 179 287 904 846 021 629 537 437 965 298 966
16	0.001 423 470 993 297 035 474 396 228 895 659 580 346 878 747 206 307 212 185 279
17	0.001 335 470 691 712 295 579 246 271 754 490 888 779 160 936 171 379 115 471 385
18	0.001 255 291 804 934 371 620 708 403 457 293 944 929 038 066 364 081 095 788 152
19	0.001 182 042 391 104 597 365 234 314 072 772 552 754 900 524 044 570 237 113 716
20	0.001 114 953 217 684 404 793 200 122 509 580 984 266 790 797 123 625 316 362 059
21	0.001 053 358 165 461 903 513 001 802 591 727 233 416 666 030 083 049 039 047 787

Table A13. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=12$

1	0.004 455 496 943 517 601 349 637 757 200 655 865 471 210 302 965 746 458 502 488
2	0.004 000 991 659 868 884 042 758 891 819 418 723 655 074 994 360 337 013 430 632
3	0.003 611 187 503 012 747 107 464 005 455 091 044 002 293 000 630 169 903 968 748
4	0.003 274 548 952 167 286 671 245 573 373 206 403 639 380 072 169 314 370 319 396
5	0.002 981 971 390 374 176 333 663 370 512 581 208 824 578 680 930 043 936 734 122
6	0.002 726 196 961 486 426 297 862 730 271 834 393 118 179 774 715 942 563 311 452
7	0.002 501 386 647 180 406 110 992 685 448 347 513 848 184 045 811 102 956 430 740
8	0.002 302 802 872 988 051 892 688 104 659 260 797 700 699 387 048 090 457 787 701
9	0.002 126 571 376 904 221 528 946 340 563 514 773 716 825 756 566 124 216 334 762
10	0.001 969 500 633 142 702 853 165 303 323 600 382 616 487 040 634 731 999 128 919
11	0.001 828 943 556 483 328 912 432 104 255 179 400 468 422 887 758 429 525 719 054
12	0.001 702 690 604 064 645 781 328 642 634 869 629 100 666 720 268 976 582 130 325
13	0.001 588 886 429 473 350 206 814 125 910 746 564 081 157 970 426 653 455 091 152
14	0.001 485 964 371 980 893 127 846 975 076 087 827 813 390 488 297 035 329 906 217
15	0.001 392 594 571 809 471 402 891 335 403 121 942 621 376 044 910 344 452 452 245
16	0.001 307 642 582 723 996 660 131 596 117 302 362 479 943 283 186 631 047 398 607
17	0.001 230 136 135 279 599 015 464 807 313 968 791 276 751 799 700 503 862 566 415
18	0.001 159 238 275 603 447 579 471 208 864 180 256 842 070 016 871 482 750 194 748
19	0.001 094 225 526 128 117 404 893 168 875 255 081 188 411 224 109 305 914 987 304
20	0.001 034 470 028 279 627 329 962 237 713 421 281 366 111 084 588 068 357 915 621

Table A14. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=13$

1	0.003 835 226 174 264 515 033 267 849 139 548 287 740 281 097 991 587 591 226 740
2	0.003 470 302 462 489 204 945 585 539 068 407 823 790 916 903 490 978 804 345 634
3	0.003 153 976 114 291 042 717 949 854 207 701 177 670 939 753 177 640 424 453 760
4	0.002 878 120 447 734 598 850 866 488 018 620 901 917 298 030 406 411 265 287 464
5	0.002 636 218 878 011 081 549 066 871 387 046 140 374 529 638 794 175 132 238 239
6	0.002 423 000 366 333 179 840 743 214 676 013 223 739 391 849 645 981 842 616 459
7	0.002 234 166 914 669 194 049 927 416 936 326 038 110 710 100 744 564 283 851 169
8	0.002 066 187 641 759 626 973 239 462 375 541 590 714 182 010 563 790 559 076 866
9	0.001 916 141 593 138 516 856 733 231 031 572 576 206 639 750 887 234 608 026 107
10	0.001 781 596 616 649 485 140 899 488 562 126 511 444 343 183 928 620 385 139 636
11	0.001 660 515 203 807 703 594 660 574 302 292 474 292 358 054 137 268 499 771 481
12	0.001 551 180 688 047 140 566 569 070 536 919 700 656 439 109 175 813 607 456 975
13	0.001 452 138 949 798 487 527 596 656 250 417 300 764 817 038 699 785 240 162 118
14	0.001 362 152 034 313 485 848 794 932 175 962 870 778 149 149 187 021 963 445 995
15	0.001 280 160 994 380 952 933 558 947 795 206 565 362 660 254 727 130 441 015 133
16	0.001 205 255 930 398 127 368 975 756 683 479 312 747 218 735 343 586 025 357 311
17	0.001 136 651 685 864 041 050 426 838 804 513 171 225 101 599 074 317 879 279 107
18	0.001 073 668 016 607 772 073 028 222 715 035 470 611 251 581 121 416 952 656 863
19	0.001 015 713 321 529 109 825 025 447 961 433 846 096 808 903 604 890 636 451 046

Table A15. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=14$

1	0.003 336 024 121 729 805 956 918 763 618 364 004 188 577 896 128 767 031 971 880
2	0.003 038 600 957 641 413 959 575 804 305 569 796 033 719 065 827 325 773 068 864
3	0.002 778 385 371 486 867 445 221 410 117 527 901 108 132 872 033 742 877 098 913
4	0.002 549 518 962 783 644 062 156 863 546 886 975 455 084 783 051 164 998 409 663
5	0.002 347 238 493 289 466 865 949 143 696 459 062 178 276 945 890 571 007 448 224
6	0.002 167 641 471 603 124 816 104 479 027 127 610 478 269 524 999 968 252 047 474
7	0.002 007 507 792 443 196 338 514 697 714 518 684 296 578 713 269 975 429 788 116
8	0.001 864 162 719 392 691 280 155 587 151 520 803 244 206 536 104 791 397 800 256
9	0.001 735 370 681 029 733 752 162 145 098 968 289 075 457 118 112 598 869 529 709
10	0.001 619 252 257 890 261 787 502 903 290 276 890 326 623 969 267 602 257 070 901
11	0.001 514 218 783 983 000 724 063 540 961 231 996 106 570 353 900 448 882 406 734
12	0.001 418 920 443 003 632 684 725 873 879 137 726 291 885 072 291 858 315 332 799
13	0.001 332 204 787 009 072 738 591 807 672 901 417 028 465 606 650 504 366 793 229
14	0.001 253 083 366 394 357 291 072 370 125 176 773 360 101 411 265 649 824 709 677
15	0.001 180 704 718 138 835 475 012 457 960 198 899 044 406 723 602 985 342 266 794
16	0.001 114 332 372 215 182 774 967 043 380 193 250 306 697 755 812 706 968 937 716
17	0.001 053 326 844 124 648 437 335 449 165 330 083 288 563 096 951 317 838 179 554

Table A16. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=15$

1	0.002 928 313 521 690 063 775 178 776 079 353 356 453 724 725 986 585 221 040 381
2	0.002 682 716 051 955 355 629 610 886 662 206 352 269 656 433 593 079 157 623 365
3	0.002 466 086 847 984 669 778 248 235 516 458 127 809 383 316 532 599 418 463 238
4	0.002 274 116 676 276 297 038 694 673 908 198 245 041 642 044 405 740 018 354 029
5	0.002 103 258 808 569 315 934 612 160 071 587 004 277 448 945 371 871 269 983 256
6	0.001 950 574 266 548 647 004 604 820 785 481 937 962 417 190 881 211 611 967 836
7	0.001 813 612 211 653 184 498 602 404 933 849 530 841 118 127 027 613 990 764 846
8	0.001 690 316 707 004 217 820 056 986 780 235 656 946 103 781 124 123 156 416 802
9	0.001 578 953 453 396 412 720 944 889 184 586 206 978 121 468 011 504 502 490 564
10	0.001 478 051 786 259 914 503 603 107 545 220 643 456 169 886 960 217 082 544 152
11	0.001 386 358 428 830 161 449 583 559 567 029 929 135 579 245 697 745 276 093 784
12	0.001 302 800 372 059 126 246 170 510 608 003 738 237 111 544 443 455 431 242 517
13	0.001 226 454 891 906 965 500 903 276 548 213 646 801 146 153 678 408 195 005 443
14	0.001 156 525 186 978 848 624 895 555 859 734 854 862 280 105 774 165 401 622 138
15	0.001 092 320 470 962 647 429 470 490 846 153 871 628 737 549 899 934 378 017 173
16	0.001 033 239 618 000 765 628 065 651 843 502 415 059 432 911 813 203 161 748 424

Table A17. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=16$

1	0.002 591 027 910 214 871 070 625 042 516 821 801 391 952 087 717 496 884 658 561
2	0.002 385 880 178 181 694 995 598 146 680 365 351 513 906 485 719 172 864 903 217
3	0.002 203 621 420 843 433 177 368 808 770 283 768 330 760 753 615 836 115 133 035
4	0.002 041 024 944 882 576 024 385 601 586 913 763 192 233 925 579 170 094 456 800
5	0.001 895 406 037 180 615 077 534 751 461 975 248 341 078 203 558 426 711 723 652
6	0.001 764 517 385 706 657 126 442 908 688 929 664 159 518 035 071 184 714 805 369
7	0.001 646 467 113 152 885 078 527 020 445 468 933 934 998 273 549 798 026 973 727
8	0.001 539 654 041 671 617 090 370 974 014 336 900 234 171 218 047 454 419 530 702
9	0.001 442 716 197 827 149 212 113 163 921 266 118 394 176 239 412 584 270 210 298
10	0.001 354 489 571 461 081 813 102 150 949 066 090 175 267 914 454 531 992 486 067
11	0.001 273 974 874 828 760 757 502 820 249 436 414 572 180 276 919 800 969 360 863
12	0.001 200 310 587 579 380 044 186 713 734 312 327 887 417 177 637 892 651 889 033
13	0.001 132 750 973 420 842 475 602 438 413 132 435 920 066 789 771 018 801 907 321
14	0.001 070 648 053 862 317 134 663 180 060 232 472 060 719 462 037 193 361 390 090
15	0.001 013 436 750 339 900 880 995 987 806 171 969 681 623 170 499 534 697 934 543

Table A18. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=17$

1	0.002 308 833 222 819 251 305 393 823 458 478 520 412 738 118 434 683 615 356 495
2	0.002 135 717 343 983 354 159 268 680 237 108 460 381 556 393 113 449 530 811 179
3	0.001 980 925 766 084 668 600 031 353 933 963 489 446 763 952 374 477 196 269 075
4	0.001 842 003 769 572 564 369 779 671 735 944 871 780 562 890 724 271 672 246 597
5	0.001 716 889 033 898 471 921 650 512 446 077 803 400 060 195 097 028 852 852 498
6	0.001 603 839 476 804 630 299 879 620 071 795 149 748 435 675 871 056 817 837 230
7	0.001 501 375 983 520 056 025 591 051 605 214 880 848 652 060 635 872 157 718 699
8	0.001 408 236 639 971 366 472 134 146 230 261 841 958 290 253 105 088 624 838 458
9	0.001 323 339 921 134 082 814 911 830 860 634 712 843 620 326 811 858 111 759 176
10	0.001 245 754 900 097 509 247 517 112 174 144 328 487 953 376 926 426 731 630 042
11	0.001 174 676 998 415 608 161 074 980 197 588 318 007 373 487 207 612 780 987 687
12	0.001 109 408 138 028 501 672 998 169 009 518 614 105 548 240 218 211 658 298 502
13	0.001 049 340 410 662 235 302 905 866 437 581 598 356 920 764 543 549 490 843 477

Table A19. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=18$

1	0.002 070 352 844 679 897 614 495 441 669 136 892 931 398 468 609 960 641 853 274
2	0.001 922 931 852 759 094 572 122 737 689 935 592 122 263 198 421 706 279 433 545
3	0.001 790 353 932 832 950 673 795 976 717 716 089 407 901 662 026 777 056 789 307
4	0.001 670 724 929 796 019 945 359 780 980 425 894 368 443 162 203 757 990 583 833
5	0.001 562 439 526 454 969 032 963 454 611 679 019 750 898 250 485 659 529 910 467
6	0.001 464 130 510 016 686 332 072 337 568 012 760 170 969 023 460 851 522 707 787
7	0.001 374 628 049 506 160 772 569 763 701 379 100 893 663 008 242 081 398 232 787
8	0.001 292 926 804 729 776 999 401 559 486 542 685 595 281 072 835 719 699 846 180
9	0.001 218 159 203 740 478 951 038 795 448 596 084 445 808 664 738 397 142 238 330
10	0.001 149 573 610 383 056 133 553 135 113 784 158 917 837 424 136 818 840 162 853
11	0.001 086 516 392 272 896 290 349 807 564 942 162 531 889 116 605 486 236 316 965
12	0.001 028 417 118 005 185 830 611 777 856 294 072 034 096 468 291 969 115 216 280

Table A20. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=19$

1	0.001 867 002 301 807 143 147 367 477 477 175 839 127 059 813 897 511 474 826 454
2	0.001 740 432 085 714 494 766 618 749 910 344 181 368 717 341 095 412 673 593 907
3	0.001 626 012 652 566 334 066 945 568 353 921 336 292 824 883 904 764 058 563 939
4	0.001 522 263 571 120 434 435 122 982 839 227 962 511 169 813 271 724 573 691 566
5	0.001 427 920 214 894 725 198 896 519 639 304 917 090 752 264 736 128 772 946 204
6	0.001 341 897 484 241 425 923 224 047 443 920 693 635 227 789 760 609 768 153 593
7	0.001 263 260 387 071 954 520 950 575 705 920 612 356 955 695 952 653 482 257 935
8	0.001 191 200 046 163 457 354 029 723 387 689 329 409 204 757 633 429 524 534 075
9	0.001 125 014 026 707 656 558 506 083 741 900 672 030 646 948 638 932 626 656 671
10	0.001 064 090 123 644 600 827 231 234 340 636 881 975 122 160 199 411 677 219 969
11	0.001 007 892 935 272 570 139 329 408 980 714 769 959 701 898 931 372 777 388 281

Table A21. Critical binding parameters μ_c (au^{-1}) for Yukawa potential, $l=20$

1	0.001 692 205 505 855 946 304 800 054 144 464 956 352 161 730 464 131 127 903 368
2	0.001 582 731 867 386 316 112 478 910 396 305 038 907 858 874 002 536 245 066 974
3	0.001 483 300 477 159 519 142 548 717 572 457 230 921 615 285 826 781 361 004 702
4	0.001 392 740 768 833 757 926 733 056 359 754 201 725 588 191 350 359 843 775 041
5	0.001 310 045 611 120 833 511 083 142 693 031 038 823 413 917 694 890 419 001 051
6	0.001 234 344 964 670 920 319 656 163 401 600 728 820 574 128 224 096 630 292 019
7	0.001 164 884 319 100 886 140 579 087 794 814 772 305 293 486 867 485 166 674 060
8	0.001 101 006 951 302 174 403 967 410 491 462 692 480 063 832 713 055 964 672 680
9	0.001 042 139 255 881 033 275 848 251 861 680 954 126 427 918 280 134 416 603 927

Appendix A.2. Exponential Cosine Screened Coulomb (ECSC) Potential μ_c vs l for $\mathcal{D} = 1/\mu \leq 1000$ au**Table A22.** Critical binding parameters μ_c (au⁻¹) for ECSC potential, $l=0$

1	0.720 524 085 881 953 095 871 917 136 918 578 087 183 481 757 107 097 035 500 102
2	0.166 617 599 995 556 539 731 598 280 947 442 350 321 823 717 112 428 036 976 749
3	0.072 436 991 196 399 382 410 616 183 437 020 010 582 340 111 221 679 442 539 474
4	0.040 427 221 157 774 623 711 569 333 221 315 103 793 817 401 687 739 383 934 111
5	0.025 787 301 102 820 745 520 704 458 261 109 335 771 504 558 091 797 255 040 648
6	0.017 878 285 415 402 881 402 256 914 633 797 271 902 658 538 868 493 306 743 693
7	0.013 122 872 755 839 147 382 548 346 950 692 787 697 254 237 221 504 031 954 820
8	0.010 041 421 218 113 844 638 910 790 952 481 122 462 906 622 005 664 195 968 541
9	0.007 930 924 973 987 263 445 857 856 216 689 003 392 968 900 834 472 246 789 134
10	0.006 422 322 284 191 135 912 404 066 577 715 848 787 203 071 762 741 856 571 797
11	0.005 306 661 160 443 474 353 548 132 017 193 328 783 724 776 687 572 007 740 000
12	0.004 458 408 166 049 461 735 737 050 860 933 554 574 147 499 880 984 550 362 260
13	0.003 798 444 345 396 882 716 950 511 753 513 245 998 300 649 916 614 059 199 498
14	0.003 274 892 232 463 425 565 932 301 184 318 633 833 472 375 874 003 836 450 050
15	0.002 852 586 963 704 240 723 272 646 587 552 628 651 478 131 085 238 514 229 834
16	0.002 732 746 158 566 434 829 602 775 911 304 860 604 390 892 212 564 667 486 379
17	0.002 507 007 181 691 831 480 583 835 569 084 873 986 148 629 710 533 854 672 386
18	0.002 220 630 554 441 271 915 747 015 516 080 658 058 971 621 873 272 912 800 797
19	0.001 980 666 003 343 410 153 519 574 212 315 086 343 531 333 607 484 681 657 531
20	0.001 777 599 543 651 618 709 301 049 909 046 310 051 220 808 482 520 530 882 671
21	0.001 604 235 938 574 517 370 364 110 511 696 420 378 633 241 296 068 656 300 841
22	0.001 455 052 129 719 299 008 578 213 418 863 093 564 464 362 482 225 956 578 860
23	0.001 325 751 649 328 551 819 638 657 065 183 782 253 473 519 741 091 224 756 492
24	0.001 212 951 657 528 666 003 299 552 989 738 524 868 071 153 290 142 911 327 921
25	0.001 113 959 333 811 295 082 320 808 272 277 889 110 309 927 208 605 306 667 978
26	0.001 026 609 611 039 482 810 433 287 787 806 015 382 154 975 066 541 356 255 959

Table A23. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=1

1	0.148	205	032	642	758	419	285	886	459	123	248	041	030	459	181	523	505	707	250	286
2	0.068	712	143	689	454	437	828	240	788	113	432	765	979	858	246	762	388	460	980	918
3	0.039	263	401	179	219	453	864	149	118	304	166	829	024	724	592	932	343	315	105	173
4	0.025	315	625	317	701	391	098	514	960	031	651	610	563	594	714	223	194	493	595	469
5	0.017	652	070	207	413	558	721	254	671	787	887	029	810	186	348	432	592	643	242	289
6	0.013	001	063	990	474	074	365	275	868	082	517	618	307	972	808	851	539	823	804	174
7	0.009	970	087	244	432	186	697	472	955	683	956	352	071	545	854	209	255	719	599	895
8	0.007	886	405	586	786	030	126	440	977	459	839	315	291	719	821	005	917	171	286	125
9	0.006	393	114	816	456	147	513	817	777	567	377	854	253	854	876	997	136	904	017	402
10	0.005	286	711	315	486	548	551	953	185	978	343	312	406	080	888	649	281	772	603	305
11	0.004	444	321	287	407	293	710	003	874	183	722	928	778	055	633	141	971	560	937	257
12	0.003	788	216	176	793	633	137	648	687	186	220	637	200	149	682	029	129	640	949	128
13	0.003	267	287	414	827	172	235	861	365	493	012	856	744	007	400	025	629	246	069	751
14	0.002	846	815	793	550	234	128	074	943	903	196	462	595	060	575	902	199	231	132	784
15	0.002	502	548	890	296	272	760	943	467	729	367	024	110	386	311	218	834	033	360	778
16	0.002	217	132	123	848	092	917	031	749	982	157	147	421	172	437	883	353	221	832	813
17	0.001	977	882	470	444	294	750	928	243	167	632	479	483	843	630	285	588	126	142	715
18	0.001	775	357	278	046	269	682	741	200	694	524	841	569	009	615	393	563	744	040	130
19	0.001	679	417	892	412	599	929	152	047	424	655	148	848	853	134	731	694	848	761	737
20	0.001	602	409	543	901	829	364	115	713	307	738	670	886	419	423	668	371	848	237	331
21	0.001	453	549	510	557	440	109	116	263	808	362	929	007	553	410	611	237	133	811	728
22	0.001	324	504	135	177	119	067	759	812	640	848	046	265	874	044	867	246	037	100	112
23	0.001	211	907	337	084	608	546	000	383	666	111	407	950	195	845	117	431	426	729	476
24	0.001	113	078	471	423	532	574	095	024	410	859	055	509	913	135	595	083	509	313	953
25	0.001	025	861	441	470	766	528	539	329	580	482	966	401	944	182	744	148	121	407	307

Table A24. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=2

1	0.063	581	546	150	838	472	194	379	494	206	193	266	728	159	563	676	891	951	063	219
2	0.037	405	048	313	454	121	087	967	384	446	332	325	869	630	812	959	787	029	566	381
3	0.024	500	014	162	249	349	814	867	206	456	637	534	779	164	648	024	014	168	062	332
4	0.017	242	903	688	977	069	524	256	967	813	816	727	303	312	443	899	180	851	050	974
5	0.012	774	701	431	983	541	897	555	762	599	560	737	399	948	055	381	509	205	426	349
6	0.009	835	204	171	995	417	863	073	223	909	090	108	832	651	772	205	543	298	252	706
7	0.007	801	227	482	595	971	277	826	701	887	952	528	740	313	502	670	727	927	280	671
8	0.006	336	762	026	759	042	608	543	639	062	208	044	125	522	346	016	145	715	707	531
9	0.005	247	980	905	682	110	713	894	256	692	025	435	431	263	498	359	098	873	120	387
10	0.004	416	843	931	244	811	052	184	169	355	385	716	199	713	277	803	689	525	637	465
11	0.003	768	192	046	966	102	933	645	201	578	808	900	456	472	540	325	893	752	498	980
12	0.003	252	355	600	366	466	879	247	762	266	216	358	446	084	153	359	967	007	979	155
13	0.002	835	457	560	138	389	598	862	343	857	412	159	123	668	966	133	186	821	814	070
14	0.002	493	757	472	554	534	946	742	937	543	775	622	014	972	404	333	317	945	952	996
15	0.002	210	222	406	252	910	591	557	102	378	827	364	463	541	765	017	378	949	157	272
16	0.001	972	377	332	047	092	819	785	710	102	753	244	221	411	064	169	933	938	960	822
17	0.001	770	917	569	083	661	691	650	980	986	324	749	807	966	809	396	752	442	695	814
18	0.001	598	789	735	644	962	555	915	572	598	188	792	364	892	441	750	464	226	417	279
19	0.001	450	568	904	055	144	038	551	866	876	077	668	245	420	706	904	163	862	361	004
20	0.001	322	027	753	333	045	677	668	313	925	569	130	827	090	151	683	605	334	506	847
21	0.001	209	832	986	637	388	593	698	505	553	650	614	793	368	163	449	575	396	523	143
22	0.001	130	801	507	929	820	471	634	652	527	457	705	288	284	505	203	155	348	232	538
23	0.001	111	327	822	871	870	162	461	369	530	643	437	596	392	411	359	951	874	528	553
24	0.001	024	373	776	687	919	039	894	681	503	072	769	784	290	507	008	933	984	839	748

Table A25. Critical binding parameters μ_c (au^{-1}) for ECSC potential, $l=3$

1	0.035	241	242	180	742	251	639	977	833	793	880	265	961	039	430	645	317	095	033	654
2	0.023	482	156	409	613	228	006	999	976	191	869	900	316	143	339	517	169	753	362	402
3	0.016	708	150	087	855	378	948	364	481	307	490	624	513	542	311	298	158	739	811	321
4	0.012	469	382	412	504	385	986	631	331	976	288	202	151	206	855	107	784	421	818	533
5	0.009	649	192	213	468	040	846	731	725	995	694	154	201	584	716	608	806	979	937	820
6	0.007	681	858	994	576	573	231	448	514	773	371	551	579	742	784	246	021	765	815	270
7	0.006	256	839	407	887	471	963	956	587	592	522	627	972	816	286	671	023	246	871	993
8	0.005	192	548	540	600	130	789	259	456	920	460	452	448	079	727	326	129	421	032	010
9	0.004	377	237	553	391	555	417	209	413	880	934	652	814	580	269	483	148	999	599	860
10	0.003	739	166	314	111	031	766	228	431	693	843	363	047	899	639	860	358	725	237	422
11	0.003	230	613	293	401	157	772	989	544	149	319	638	151	086	206	089	343	744	580	327
12	0.002	818	857	623	875	297	936	481	933	164	970	799	363	090	277	525	289	369	742	717
13	0.002	480	869	673	272	555	322	626	059	315	644	972	125	223	374	836	023	433	574	525
14	0.002	200	067	255	259	246	161	906	335	636	826	900	145	700	647	222	482	911	651	250
15	0.001	964	269	073	671	345	830	080	211	575	966	068	027	454	164	885	221	854	862	277
16	0.001	764	366	561	120	792	626	792	451	274	812	678	462	729	152	572	357	819	995	631
17	0.001	593	440	143	914	171	565	152	193	171	763	884	438	509	961	687	319	889	791	853
18	0.001	446	157	990	118	769	278	152	582	633	798	942	686	498	693	400	587	070	710	313
19	0.001	318	358	701	872	416	919	767	133	870	794	596	737	587	507	311	708	814	305	189
20	0.001	206	756	417	855	208	420	797	555	245	185	050	991	694	045	918	395	592	998	639
21	0.001	108	728	992	099	847	850	130	300	380	941	350	429	678	459	354	018	227	900	180
22	0.001	022	163	568	905	894	038	762	537	116	805	475	543	268	522	293	648	495	141	694

Table A26. Critical binding parameters μ_c (au^{-1}) for ECSC potential, $l=4$

1	0.022	371	423	947	612	063	346	334	499	365	316	843	905	640	785	814	050	126	144	468
2	0.016	099	483	083	173	257	338	961	551	772	224	894	071	595	090	220	509	166	623	999
3	0.012	110	841	492	508	710	047	544	808	940	701	540	126	394	383	005	266	780	465	164
4	0.009	425	574	620	316	002	848	173	103	670	831	316	109	200	365	056	421	613	980	684
5	0.007	535	771	378	168	027	489	285	464	678	473	536	971	541	588	471	166	432	121	929
6	0.006	157	665	312	445	150	133	665	625	101	563	171	502	002	626	977	753	638	974	322
7	0.005	123	012	502	152	556	036	586	243	025	699	828	635	833	457	176	598	196	662	568
8	0.004	327	122	089	535	273	201	243	725	797	030	012	177	514	774	571	028	983	833	464
9	0.003	702	181	251	230	666	459	489	618	659	527	864	795	039	621	512	748	561	004	431
10	0.003	202	750	095	117	000	317	314	739	353	188	575	731	621	301	439	838	359	802	374
11	0.002	797	483	740	785	447	273	592	178	023	808	438	380	370	963	163	808	479	155	620
12	0.002	464	209	908	484	348	607	079	209	600	548	747	933	511	507	637	647	438	413	262
13	0.002	186	896	212	318	133	659	861	910	078	050	073	816	802	730	414	195	635	203	990
14	0.001	953	723	060	349	632	197	466	639	507	062	647	594	700	238	905	890	152	661	997
15	0.001	755	825	344	755	730	839	510	865	988	179	826	690	539	318	373	229	704	010	053
16	0.001	586	450	763	530	546	251	344	069	099	315	022	042	138	978	645	923	433	449	498
17	0.001	440	384	564	110	390	099	122	996	959	029	317	493	187	586	787	711	599	978	745
18	0.001	313	548	689	385	726	532	897	817	461	045	446	504	026	827	427	553	322	634	773
19	0.001	202	717	521	098	501	244	900	107	264	155	902	133	321	561	669	524	662	543	071
20	0.001	105	313	076	778	122	837	787	470	393	229	881	745	812	677	360	559	370	033	509
21	0.001	019	255	297	327	095	057	101	882	096	501	090	654	964	625	702	819	439	384	912

Table A27. Critical binding parameters μ_c (au^{-1}) for ECSC potential, $l=5$

1	0.015 455 476 970 671 009 960 309 373 172 367 170 228 417 216 386 096 488 422 508
2	0.011 720 488 823 891 314 751 277 604 695 796 127 246 961 264 173 711 822 782 017
3	0.009 176 572 196 297 732 699 725 173 634 343 668 712 418 750 757 551 074 121 927
4	0.007 370 163 443 404 617 186 906 242 323 907 129 233 784 517 476 295 983 242 328
5	0.006 043 615 630 932 783 746 352 895 328 830 631 883 368 358 941 415 505 260 925
6	0.005 042 113 368 849 732 762 709 113 596 969 543 536 182 919 356 615 760 040 050
7	0.004 268 262 232 267 991 151 728 793 887 547 163 904 762 306 043 649 832 443 918
8	0.003 658 402 503 356 846 797 672 098 426 392 347 638 590 808 357 774 453 529 303
9	0.003 169 554 099 284 664 144 114 713 855 068 637 055 170 361 464 830 365 526 319
10	0.002 771 880 176 036 274 253 639 455 792 941 146 543 767 467 911 919 661 452 508
11	0.002 444 161 398 272 089 091 211 541 576 700 769 205 375 698 840 410 978 402 998
12	0.002 170 983 911 650 292 963 994 406 496 885 474 328 434 224 656 037 472 813 664
13	0.001 940 939 310 232 447 018 430 410 437 367 072 559 709 094 313 040 736 160 339
14	0.001 745 441 763 489 914 590 621 348 942 183 194 255 624 426 408 469 893 247 826
15	0.001 577 932 365 462 606 196 855 634 559 229 177 883 568 281 259 067 503 575 915
16	0.001 433 332 662 833 002 802 775 354 060 159 609 740 744 634 099 515 269 402 433
17	0.001 307 662 208 726 982 429 445 474 493 595 163 941 055 112 825 844 115 420 597
18	0.001 197 766 319 927 743 784 341 327 190 186 607 081 282 432 195 525 183 230 572
19	0.001 101 119 261 873 753 488 183 463 136 792 314 328 255 151 848 602 664 931 385
20	0.001 015 679 939 075 498 020 779 154 128 651 356 796 585 563 599 645 742 757 184

Table A28. Critical binding parameters μ_c (au^{-1}) for ECSC potential, $l=6$

1	0.011 314 415 073 416 202 952 425 413 789 882 158 622 792 903 862 880 171 594 562
2	0.008 912 130 581 868 735 191 281 646 393 966 225 762 256 702 936 882 369 340 986
3	0.007 191 277 470 291 155 932 738 096 640 932 273 917 016 420 696 525 774 473 823
4	0.005 918 684 553 633 729 004 222 370 853 537 057 407 060 024 082 611 742 886 350
5	0.004 952 459 789 048 368 732 051 785 607 597 313 533 050 324 465 004 050 696 662
6	0.004 202 396 831 989 801 537 581 624 978 338 942 207 072 385 203 650 502 566 800
7	0.003 609 012 159 542 453 348 652 331 069 043 931 550 441 408 176 524 333 871 712
8	0.003 131 844 017 057 182 970 498 834 607 840 542 293 292 732 062 919 105 534 155
9	0.002 742 623 966 065 229 972 895 219 095 602 271 299 196 876 448 158 475 645 891
10	0.002 421 137 530 963 128 385 779 391 349 242 739 342 417 151 049 167 076 015 087
11	0.002 152 631 030 127 601 113 639 475 215 971 059 394 020 494 612 986 178 123 913
12	0.001 926 139 621 999 562 923 033 024 730 053 550 804 573 673 357 681 855 196 711
13	0.001 733 381 583 771 107 677 005 089 079 854 521 657 049 845 453 339 994 328 318
14	0.001 568 010 352 667 883 250 949 088 711 290 044 530 781 270 908 733 153 470 798
15	0.001 425 098 230 749 108 701 445 350 967 082 854 143 482 183 992 989 969 048 978
16	0.001 300 773 440 754 445 106 133 930 996 134 920 703 274 905 882 162 621 313 600
17	0.001 191 960 732 091 258 258 613 496 906 829 551 634 554 472 268 961 689 010 926
18	0.001 096 193 182 002 818 646 367 062 042 896 998 826 667 846 745 345 136 677 184
19	0.001 011 473 758 487 486 449 618 961 387 042 941 957 662 674 914 725 750 494 374

Table A29. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=7

1	0.008 639 853 207 347 760 064 195 924 708 427 606 759 679 788 959 944 390 281 383
2	0.007 004 184 669 808 105 808 290 664 148 373 544 678 737 664 722 311 804 204 937
3	0.005 786 282 830 201 942 080 991 794 130 818 856 039 009 270 428 844 785 080 263
4	0.004 856 373 753 064 726 474 454 083 677 004 281 802 319 944 179 405 365 915 239
5	0.004 131 128 598 024 289 619 792 301 843 592 976 766 401 571 855 794 316 475 060
6	0.003 555 132 413 298 761 967 176 856 654 682 920 518 584 603 246 604 652 999 480
7	0.003 090 416 880 699 907 316 529 385 517 822 165 276 633 692 228 727 452 901 734
8	0.002 710 289 113 040 969 169 675 085 992 277 733 931 918 915 775 124 491 106 233
9	0.002 395 557 233 604 708 004 232 796 850 602 719 005 974 309 370 381 231 956 779
10	0.002 132 147 217 180 900 443 065 925 405 713 979 743 684 335 770 241 137 602 610
11	0.001 909 555 635 969 330 251 600 981 166 805 187 443 763 699 539 755 782 354 418
12	0.001 719 820 062 087 978 847 755 241 402 776 981 273 827 225 491 194 562 651 829
13	0.001 556 818 742 702 993 538 547 446 460 969 837 880 986 717 320 421 300 458 761
14	0.001 415 784 782 238 542 666 711 209 924 324 420 475 102 268 979 167 509 679 986
15	0.001 292 963 095 882 626 284 891 354 523 052 999 699 098 306 810 039 456 532 676
16	0.001 185 364 247 210 289 101 896 449 360 635 480 394 592 863 181 155 471 942 192
17	0.001 090 585 197 731 281 575 084 063 481 884 439 678 874 655 296 478 744 392 607
18	0.001 006 677 016 082 744 223 446 052 109 776 329 669 565 700 231 938 192 320 898

Table A30. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=8

1	0.006 812 835 383 973 685 366 024 588 715 713 989 123 082 386 282 042 094 205 743
2	0.005 649 197 708 317 049 220 496 227 595 058 613 865 316 007 636 068 783 991 147
3	0.004 755 830 480 087 669 287 731 064 788 389 901 193 009 195 843 061 686 023 959
4	0.004 055 868 184 184 486 638 430 127 375 595 945 634 019 821 692 620 176 887 081
5	0.003 497 780 297 403 072 343 048 904 856 142 919 472 571 703 108 753 473 360 643
6	0.003 046 013 503 078 794 125 372 461 833 351 129 822 403 930 477 623 169 095 275
7	0.002 675 420 952 448 535 094 401 542 286 109 507 320 679 242 329 644 625 796 209
8	0.002 367 826 190 614 427 671 449 706 825 938 788 666 315 526 638 931 653 508 890
9	0.002 109 837 394 897 073 337 399 949 366 492 315 677 788 670 421 610 336 993 805
10	0.001 891 418 837 826 709 243 505 132 502 302 816 395 495 943 063 058 605 257 191
11	0.001 704 934 627 889 640 894 052 815 910 851 457 664 889 455 044 677 707 664 293
12	0.001 544 494 785 294 616 824 371 990 282 962 768 409 860 471 744 644 184 912 516
13	0.001 405 499 418 515 263 301 059 305 277 313 065 125 519 441 243 980 068 542 420
14	0.001 284 315 447 010 711 884 895 521 015 199 099 205 657 922 921 381 519 142 309
15	0.001 178 043 701 189 727 265 379 900 028 001 846 613 941 215 880 032 232 681 132
16	0.001 084 348 714 344 146 336 614 051 265 251 385 858 707 902 962 952 580 660 870
17	0.001 001 332 689 548 287 474 913 046 610 030 322 015 603 048 899 155 245 182 435

Table A31. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=9

1	0.005 509 639 433 353 859 160 774 391 632 366 197 808 217 451 271 489 752 442 310
2	0.004 652 458 191 400 806 863 016 513 789 791 471 574 677 874 742 996 590 049 660
3	0.003 977 817 428 358 344 801 554 419 573 695 287 305 117 840 944 781 826 198 830
4	0.003 437 847 317 106 416 259 403 675 900 262 416 139 437 957 564 969 858 225 815
5	0.002 999 299 559 017 729 907 526 210 831 515 693 247 829 052 836 823 549 273 245
6	0.002 638 520 267 153 621 163 527 532 579 525 205 701 946 196 688 062 179 161 275
7	0.002 338 324 135 556 071 953 503 191 282 329 781 764 427 650 818 648 820 544 789
8	0.002 085 991 846 443 000 858 207 983 273 710 472 820 875 684 824 697 877 033 606
9	0.001 871 952 930 501 456 158 035 877 050 841 446 334 628 949 991 460 588 672 837
10	0.001 688 899 050 223 034 953 285 659 492 770 951 160 587 832 177 990 244 133 093
11	0.001 531 174 511 029 407 617 831 007 762 116 287 356 840 347 340 402 852 167 042
12	0.001 394 349 429 069 344 586 369 369 482 831 032 438 454 634 409 983 477 028 948
13	0.001 274 915 728 763 492 796 557 200 599 088 111 697 167 722 032 849 131 620 962
14	0.001 170 067 279 105 309 392 253 875 221 693 263 829 939 970 774 126 349 637 830
15	0.001 077 538 641 823 281 954 694 425 745 333 003 669 505 033 895 868 618 845 938

Table A32. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=10

1	0.004 547 569 114 405 207 303 478 818 883 362 190 922 376 548 906 639 597 663 007
2	0.003 897 976 459 535 657 021 092 019 392 450 706 362 322 781 116 116 322 684 332
3	0.003 376 096 120 524 451 091 966 266 227 599 364 472 785 793 880 102 154 618 847
4	0.002 950 858 403 369 156 763 677 508 657 340 602 621 584 132 605 084 950 119 488
5	0.002 600 035 331 539 019 210 850 969 493 182 198 590 548 354 211 952 849 837 944
6	0.002 307 397 445 651 282 862 281 759 352 943 343 643 518 323 537 409 433 138 004
7	0.002 060 879 832 519 793 207 312 161 045 997 298 607 065 078 772 437 476 205 033
8	0.001 851 368 549 092 251 345 680 420 519 050 975 574 276 938 039 282 192 347 250
9	0.001 671 879 160 645 100 850 119 100 753 733 855 066 632 543 052 097 660 119 209
10	0.001 516 989 314 976 148 652 316 509 828 231 416 579 854 245 338 908 340 442 362
11	0.001 382 439 582 476 651 990 330 872 598 913 484 441 385 420 113 759 986 016 018
12	0.001 264 847 996 658 180 384 729 275 648 582 640 586 963 846 110 979 959 179 310
13	0.001 161 502 825 297 057 038 400 881 943 750 813 611 931 599 932 151 531 490 978
14	0.001 070 210 060 090 910 963 873 995 166 334 936 641 663 118 390 582 561 172 977

Table A33. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=11

1	0.003 817 163 057 337 947 515 338 408 555 188 048 887 451 096 701 046 561 208 866
2	0.003 313 167 340 376 498 149 671 397 834 150 496 363 442 657 777 095 946 788 914
3	0.002 901 191 691 358 188 435 516 010 823 069 699 410 336 459 752 941 744 543 253
4	0.002 560 359 693 531 390 642 622 734 979 636 582 155 179 327 349 224 104 519 565
5	0.002 275 355 854 946 644 137 283 220 572 606 502 260 549 911 373 298 855 249 454
6	0.002 034 746 122 326 397 608 835 093 004 551 337 521 026 618 741 333 458 853 813
7	0.001 829 860 021 279 189 491 294 380 845 292 360 891 188 711 413 056 397 386 985
8	0.001 654 030 005 364 189 395 520 246 808 837 205 761 433 252 824 956 819 935 768
9	0.001 502 063 536 963 459 833 500 257 808 952 778 738 477 441 763 879 863 186 678
10	0.001 369 870 111 928 342 115 963 228 398 363 727 478 089 644 063 279 557 097 454
11	0.001 254 193 470 795 920 566 331 448 140 320 558 454 088 331 206 868 945 207 177
12	0.001 152 416 491 755 957 914 953 152 507 291 607 742 206 966 771 468 066 494 446
13	0.001 062 417 120 749 856 161 385 298 482 255 417 106 891 312 957 941 978 920 976

Table A34. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=12

1	0.003 249 591 594 959 846 269 283 043 281 408 600 859 624 984 787 390 796 117 097
2	0.002 850 724 598 750 350 508 947 192 010 831 717 024 657 273 090 638 909 273 824
3	0.002 519 833 705 852 989 769 456 646 616 959 583 115 918 514 515 493 488 667 704
4	0.002 242 472 079 809 639 451 411 365 679 301 979 846 938 560 203 252 087 642 770
5	0.002 007 809 723 446 430 162 712 800 008 826 790 513 921 872 613 041 774 635 720
6	0.001 807 603 759 212 034 148 723 759 649 992 241 321 664 445 796 444 876 548 232
7	0.001 635 494 194 708 533 904 643 339 662 530 135 552 298 921 164 474 423 875 407
8	0.001 486 512 914 708 380 844 006 725 079 571 433 943 574 870 942 888 951 264 542
9	0.001 356 735 334 600 634 188 026 369 610 063 279 531 006 500 977 719 423 540 341
10	0.001 243 029 338 914 927 103 470 201 682 857 268 655 974 661 436 205 812 597 770
11	0.001 142 871 719 185 733 570 974 800 937 135 359 447 075 663 947 370 628 946 546
12	0.001 054 212 189 782 836 396 332 679 250 806 270 132 771 379 959 397 938 861 597

Table A35. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=13

1	0.002 799 813 348 108 756 596 580 361 450 700 373 416 821 682 859 712 520 716 632
2	0.002 478 748 248 620 609 810 194 474 618 769 009 455 673 512 775 390 625 963 520
3	0.002 208 983 326 891 975 807 860 517 984 099 400 083 975 028 210 349 852 950 997
4	0.001 980 264 144 120 451 259 862 985 063 863 875 318 658 836 603 983 657 129 495
5	0.001 784 757 859 394 607 440 438 224 650 297 471 505 812 925 354 190 460 793 693
6	0.001 616 401 185 744 431 112 941 012 366 389 659 534 955 850 635 739 672 905 911
7	0.001 470 443 748 344 769 384 212 622 475 841 165 036 066 330 141 136 723 136 939
8	0.001 343 122 808 723 193 177 821 708 652 020 821 880 979 973 645 649 833 284 605
9	0.001 231 427 963 611 241 890 132 652 612 780 054 940 844 356 702 228 933 950 230
10	0.001 132 928 520 828 451 889 192 431 843 596 553 231 060 467 154 657 656 344 983
11	0.001 045 645 216 618 938 003 020 188 252 628 233 896 935 307 555 677 026 929 963

Table A36. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=14

1	0.002 437 349 536 084 494 173 863 318 672 727 378 235 419 544 055 891 373 778 220
2	0.002 175 093 893 634 695 925 994 361 602 895 887 209 903 395 604 758 437 183 304
3	0.001 952 278 638 804 670 018 422 114 907 849 580 802 460 668 727 459 596 190 765
4	0.001 761 462 536 390 116 180 939 283 427 213 857 050 294 816 771 577 455 767 785
5	0.001 596 867 247 250 923 747 486 390 576 389 059 645 551 381 429 314 641 016 674
6	0.001 453 952 610 776 195 493 836 512 306 007 298 945 686 934 760 773 649 470 484
7	0.001 329 112 920 691 746 381 002 765 091 943 312 431 065 194 284 701 515 436 507
8	0.001 219 456 424 925 153 636 819 719 610 634 428 522 479 671 933 630 666 409 734
9	0.001 122 643 026 009 279 444 612 481 598 919 250 855 020 591 480 843 681 094 939
10	0.001 036 763 303 403 155 590 112 658 531 402 617 123 102 487 113 635 854 864 995

Table A37. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=15

1	0.002 140 978 300 671 255 150 378 468 263 132 365 540 513 665 697 300 760 024 611
2	0.001 924 000 020 536 203 119 115 339 395 347 130 940 846 972 793 195 653 024 195
3	0.001 737 841 088 644 805 468 423 914 473 957 810 456 861 789 633 981 133 337 921
4	0.001 576 995 892 614 913 747 027 709 088 003 802 000 659 030 273 617 787 552 008
5	0.001 437 126 453 770 610 626 321 766 078 673 729 419 589 815 574 930 329 724 757
6	0.001 314 778 798 579 975 621 366 260 048 229 116 815 525 788 118 990 273 593 798
7	0.001 207 176 327 616 260 826 678 270 033 757 454 264 209 338 322 444 004 049 301
8	0.001 112 067 235 827 691 859 035 479 394 919 502 095 932 794 086 336 255 992 047
9	0.001 027 610 442 912 378 492 625 595 582 659 642 224 988 803 535 203 614 798 025

Table A38. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=16

1	0.001	895	554	743	820	182	236	602	813	644	747	572	338	062	922	953	245	045	094	885
2	0.001	714	001	168	784	229	286	124	232	692	722	697	944	085	599	774	073	167	311	222
3	0.001	556	878	609	952	498	031	555	911	687	406	563	282	694	434	541	107	370	787	826
4	0.001	420	042	984	137	036	345	670	406	073	669	197	181	995	377	971	744	249	369	174
5	0.001	300	186	460	888	729	682	072	890	533	629	453	509	544	024	049	610	561	807	518
6	0.001	194	643	807	805	038	837	291	300	014	104	642	088	855	714	063	155	033	668	144
7	0.001	101	248	944	706	931	463	495	252	766	348	811	590	354	717	936	160	704	668	554
8	0.001	018	227	392	668	649	778	408	302	892	248	872	708	430	912	403	010	884	471	841

Table A39. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=17

1	0.001	690	036	195	703	217	602	450	398	592	438	950	966	351	588	124	414	585	646	252
2	0.001	536	595	760	850	885	764	029	489	528	480	121	959	561	235	015	153	478	753	592
3	0.001	402	771	210	694	080	835	552	848	739	991	670	835	553	690	753	659	695	317	968
4	0.001	285	395	125	230	195	910	271	912	641	213	858	090	845	108	442	655	406	966	443
5	0.001	181	909	682	670	574	395	973	748	285	513	449	822	565	463	502	284	761	114	634
6	0.001	090	231	786	183	060	494	328	308	220	332	004	196	843	591	611	979	948	753	500
7	0.001	008	651	654	802	361	811	994	037	859	901	607	967	723	695	979	844	943	293	441

Table A40. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=18

1	0.001	516	217	555	296	588	007	358	346	161	510	889	606	884	202	896	360	986	248	667
2	0.001	385	372	087	154	416	035	314	254	584	181	007	833	400	019	612	040	114	946	767
3	0.001	270	457	617	673	384	234	770	438	997	290	111	061	360	378	405	610	241	922	372
4	0.001	169	019	697	088	757	265	907	141	020	989	628	340	461	519	677	942	201	833	783
5	0.001	079	055	367	541	850	587	996	299	576	207	825	032	637	079	584	015	314	154	094

Table A41. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=19

1	0.001	367	899	196	760	346	971	821	142	186	235	016	094	050	724	336	142	709	066	289
2	0.001	255	420	837	942	988	357	850	268	549	023	665	826	749	282	521	077	735	483	419
3	0.001	156	014	830	962	839	433	278	015	082	531	963	097	620	739	581	455	660	998	324
4	0.001	067	755	464	471	283	125	563	457	135	506	235	155	000	287	293	192	472	591	708

Table A42. Critical binding parameters μ_c (au⁻¹) for ECSC potential, l=20

1	0.001	240	326	247	836	695	970	437	680	007	292	257	757	005	814	294	637	568	861	298
2	0.001	142	931	639	750	891	330	714	535	607	684	059	325	519	253	607	324	685	528	157
3	0.001	056	364	253	062	127	800	322	331	780	423	618	393	988	506	958	199	534	076	179

Table A44. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=1$

1	0.376 935 996 093 545 491 107 886 597 464 844 838 761 579 551 432 087 617 184 727
2	0.186 485 867 283 101 374 052 164 749 244 161 460 966 114 173 315 966 632 514 078
3	0.110 491 326 325 213 180 032 714 924 883 489 407 635 153 157 132 637 136 893 287
4	0.072 863 392 165 501 752 398 842 628 754 495 660 796 740 132 033 620 730 447 794
5	0.051 579 047 580 589 004 697 833 155 435 049 552 740 550 686 812 299 090 923 664
6	0.038 397 853 255 317 329 655 660 366 268 976 476 735 547 792 099 267 022 102 085
7	0.029 680 504 929 516 179 239 885 158 836 556 355 656 947 587 053 246 129 078 893
8	0.023 620 593 269 990 429 548 115 330 923 291 986 076 184 630 528 102 624 107 156
9	0.019 239 896 704 416 135 155 733 637 773 265 191 249 411 683 501 958 781 364 040
10	0.015 971 644 378 342 323 012 782 360 290 010 550 273 333 032 038 962 876 711 500
11	0.013 469 224 499 088 230 697 011 335 207 393 983 732 497 126 960 009 873 307 130
12	0.011 511 080 781 038 673 209 248 418 741 553 844 109 238 077 429 338 857 438 574
13	0.009 950 272 895 080 997 050 301 410 716 072 355 215 651 930 761 274 896 729 600
14	0.008 686 255 186 915 663 712 556 268 513 564 133 637 085 200 246 071 161 257 937
15	0.007 648 359 227 713 641 812 142 133 027 840 414 564 062 661 571 945 538 325 906
16	0.006 785 747 262 326 645 217 366 320 945 989 351 499 057 833 178 433 419 413 882
17	0.006 061 094 862 950 850 748 580 798 426 549 998 007 755 096 854 491 635 338 956
18	0.005 446 501 456 005 581 147 554 569 791 899 862 170 906 351 991 174 835 058 721
19	0.004 920 774 408 022 750 001 593 223 324 587 832 779 526 138 286 358 523 939 974
20	0.004 467 583 857 510 530 235 021 979 874 339 490 576 331 834 276 002 218 902 165
21	0.004 074 183 387 375 933 310 982 220 306 134 982 609 844 967 197 649 302 181 015
22	0.003 730 506 652 465 256 962 342 320 810 215 861 498 742 342 222 071 967 697 824
23	0.003 428 518 845 917 619 810 239 183 429 707 704 231 157 938 131 486 342 720 608
24	0.003 161 744 065 774 825 139 567 227 530 377 848 900 990 288 171 132 890 970 474
25	0.002 924 916 115 093 780 602 685 842 659 146 090 291 811 367 576 623 378 511 149
26	0.002 713 717 234 893 886 175 583 431 149 436 608 932 026 795 562 950 516 661 309
27	0.002 524 580 353 007 623 118 597 009 423 300 838 519 187 423 333 957 699 431 794
28	0.002 354 537 800 849 113 213 629 761 358 881 620 090 965 373 112 410 810 090 261
29	0.002 201 104 428 993 606 688 431 768 223 620 441 387 466 531 189 159 847 891 604
30	0.002 062 186 466 961 366 660 074 934 606 475 107 982 775 991 714 521 286 642 588
31	0.001 936 009 846 787 804 876 814 621 667 707 696 669 893 553 640 246 062 991 971
32	0.001 821 063 382 089 777 587 247 945 246 937 653 765 586 303 905 467 614 498 933
33	0.001 716 053 386 151 907 780 214 487 538 507 264 950 051 680 796 444 489 696 205
34	0.001 619 867 171 547 411 228 798 733 668 787 922 196 756 087 834 767 346 549 105
35	0.001 531 543 499 411 270 812 472 271 356 835 939 257 014 430 498 717 034 589 524
36	0.001 450 248 506 591 002 002 517 960 275 522 685 247 890 219 508 225 408 278 376
37	0.001 375 255 980 417 887 763 353 310 437 284 347 562 225 627 732 880 181 653 768
38	0.001 305 931 106 539 748 147 391 272 016 692 178 928 687 954 615 467 762 360 067
39	0.001 241 717 008 270 621 080 801 992 059 758 910 636 989 088 025 049 386 679 078
40	0.001 182 123 542 738 806 611 283 358 655 293 644 496 894 452 396 020 311 226 569
41	0.001 126 717 931 624 165 222 863 093 650 421 144 578 818 383 530 632 998 525 169
42	0.001 075 116 891 087 849 949 108 349 255 735 698 389 901 681 477 126 570 016 101
43	0.001 026 979 992 923 595 493 215 991 916 564 006 635 658 405 295 906 399 750 934

Table A45. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=2$

1	0.157 661 961 178 778 425 498 625 370 069 028 696 347 905 265 451 547 879 382 454
2	0.097 563 839 410 455 548 278 003 397 488 269 988 802 316 153 716 393 199 566 658
3	0.066 107 804 499 688 993 265 888 276 794 309 163 673 134 575 363 382 357 666 706
4	0.047 661 373 617 471 335 758 539 936 090 871 992 438 407 290 770 359 952 955 176
5	0.035 947 712 541 582 816 042 971 172 122 333 603 216 138 436 938 432 827 733 168
6	0.028 057 828 859 884 349 808 723 878 225 968 871 390 859 771 787 710 719 125 864
7	0.022 496 537 924 176 529 934 365 948 498 052 457 262 288 354 918 071 340 754 878
8	0.018 432 554 082 212 985 468 669 397 301 716 223 022 137 937 492 952 122 918 598
9	0.015 374 264 206 131 270 035 821 673 144 600 845 412 717 130 620 923 757 463 873
10	0.013 016 059 892 763 463 626 325 690 027 958 485 288 551 426 305 752 867 318 012
11	0.011 159 973 920 111 125 783 502 705 288 690 089 496 105 817 391 470 065 721 453
12	0.009 673 253 675 081 053 448 786 591 445 674 919 475 834 349 779 799 121 020 385
13	0.008 464 215 631 972 013 466 389 082 989 882 459 340 886 370 746 386 593 035 183
14	0.007 467 910 190 556 668 918 042 467 153 715 516 834 988 136 754 021 386 583 920
15	0.006 637 296 118 280 532 939 688 496 229 296 345 780 154 879 021 646 169 872 893
16	0.005 937 632 899 039 621 757 032 069 666 960 315 773 680 219 194 399 491 665 140
17	0.005 342 817 768 042 205 033 863 782 174 718 511 326 828 527 977 106 093 354 565
18	0.004 832 933 786 709 055 281 555 105 113 649 603 910 271 940 489 507 670 109 298
19	0.004 392 572 414 816 363 937 310 499 659 289 907 338 412 635 568 133 116 282 172
20	0.004 009 663 300 324 298 603 343 241 300 289 764 412 176 005 471 250 565 935 082
21	0.003 674 643 408 261 172 623 663 891 791 740 136 117 119 568 469 498 262 206 532
22	0.003 379 857 592 267 337 859 780 987 195 524 149 896 892 868 074 247 087 317 795
23	0.003 119 119 805 819 373 978 522 192 325 036 777 951 027 017 334 512 660 745 279
24	0.002 887 387 604 131 931 880 923 261 974 602 263 520 636 104 358 204 380 805 596
25	0.002 680 517 720 574 364 455 540 491 912 755 577 211 363 849 428 203 281 606 297
26	0.002 495 080 447 683 777 080 502 997 297 351 181 330 075 606 114 503 436 447 967
27	0.002 328 217 202 157 082 476 555 588 087 912 665 560 754 180 538 949 531 043 250
28	0.002 177 530 168 565 441 287 469 528 155 817 927 639 015 054 984 872 963 313 496
29	0.002 040 996 027 505 865 626 848 555 073 718 657 557 673 800 712 882 349 751 840
30	0.001 916 897 946 257 899 197 776 690 085 381 012 617 591 959 671 660 132 731 118
31	0.001 803 771 546 000 852 950 013 189 586 353 438 819 643 448 666 485 710 945 883
32	0.001 700 361 658 403 690 628 952 595 210 185 423 443 474 897 107 045 631 658 729
33	0.001 605 587 478 986 959 798 466 897 751 936 135 061 603 163 812 258 086 592 509
34	0.001 518 514 305 165 668 371 481 654 615 631 105 096 117 827 624 497 249 051 324
35	0.001 438 330 475 069 051 189 839 407 223 511 144 188 040 190 908 525 612 344 713
36	0.001 364 328 441 922 366 203 276 726 978 644 056 714 004 580 197 500 044 623 535
37	0.001 295 889 157 989 195 818 683 700 762 565 784 548 459 339 280 507 335 905 142
38	0.001 232 469 123 073 862 871 571 356 020 149 346 135 526 986 613 478 095 725 054
39	0.001 173 589 590 579 292 242 644 937 234 339 547 384 657 204 482 078 961 187 469
40	0.001 118 827 530 081 040 612 930 273 834 574 419 179 679 494 770 306 979 034 666
41	0.001 067 808 027 303 094 136 050 805 770 830 495 755 096 578 332 282 343 162 526
42	0.001 020 197 866 129 873 221 063 434 847 186 137 810 858 687 788 357 474 857 707

Table A46. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=3$

1	0.086 405 096 614 817 001 538 179 150 178 328 470 460 135 032 986 826 524 615 533
2	0.059 972 727 178 598 238 870 943 325 346 790 972 010 662 201 597 237 692 867 088
3	0.043 974 596 528 002 885 683 701 182 153 296 444 179 103 540 205 537 376 330 940
4	0.033 581 035 287 150 212 054 631 215 897 241 683 116 098 448 479 246 772 483 656
5	0.026 459 140 877 676 336 814 255 707 700 692 212 060 172 972 797 687 912 207 470
6	0.021 371 943 601 701 091 642 880 435 641 138 980 686 240 084 041 298 807 291 129
7	0.017 614 900 537 112 554 460 865 444 608 078 958 862 310 248 095 107 942 469 392
8	0.014 763 254 194 157 383 219 198 115 756 916 029 959 389 230 174 874 061 442 166
9	0.012 548 789 945 696 628 341 782 946 745 320 230 166 112 540 321 816 191 024 223
10	0.010 795 498 630 418 072 001 627 163 638 711 727 477 048 191 049 036 576 008 865
11	0.009 384 062 499 480 151 152 189 632 186 093 516 647 483 622 918 850 534 491 055
12	0.008 231 312 316 504 425 792 442 955 063 381 624 230 542 632 676 059 776 787 231
13	0.007 277 861 269 073 402 442 858 877 171 278 975 190 314 518 848 725 848 049 382
14	0.006 480 400 686 493 034 320 982 433 778 957 516 309 759 553 908 661 912 845 823
15	0.005 806 754 115 273 674 714 038 861 866 154 698 981 697 959 293 986 050 266 150
16	0.005 232 617 341 045 858 830 178 074 667 484 915 654 926 086 692 358 620 448 468
17	0.004 739 358 813 124 009 668 639 926 898 293 264 926 859 978 896 816 311 138 232
18	0.004 312 504 217 875 814 512 290 247 480 471 974 435 080 957 450 279 874 856 332
19	0.003 940 672 619 620 654 904 761 033 644 842 111 253 267 229 673 245 125 503 821
20	0.003 614 816 831 742 394 283 241 821 201 085 161 546 312 432 014 830 299 473 897
21	0.003 327 672 595 292 695 787 328 599 656 511 342 983 521 069 526 489 301 594 850
22	0.003 073 353 513 386 683 111 397 132 585 652 362 144 027 974 207 399 025 593 169
23	0.002 847 049 311 347 006 604 619 664 428 499 504 554 677 317 306 106 975 764 546
24	0.002 644 798 388 551 763 761 981 711 456 233 290 516 451 564 101 002 538 034 578
25	0.002 463 314 487 251 591 876 394 928 298 016 192 513 247 867 657 790 096 292 081
26	0.002 299 853 259 875 642 744 555 522 982 739 587 336 411 465 527 343 842 921 996
27	0.002 152 108 582 124 325 430 654 595 158 966 020 572 689 193 577 444 960 831 313
28	0.002 018 131 273 779 553 742 961 611 702 717 905 164 112 111 656 453 926 544 840
29	0.001 896 264 863 271 429 840 554 451 691 636 396 911 355 786 345 950 170 949 832
30	0.001 785 094 433 563 521 380 167 878 017 639 678 392 195 377 335 385 753 368 839
31	0.001 683 405 593 290 029 853 069 742 504 918 738 520 505 463 115 055 881 976 603
32	0.001 590 151 347 421 635 234 174 065 271 048 727 896 785 656 046 835 492 035 620
33	0.001 504 425 177 064 513 875 148 295 348 054 630 497 393 922 098 915 970 599 291
34	0.001 425 439 034 069 402 815 772 117 418 595 949 160 557 147 807 368 887 646 709
35	0.001 352 505 251 760 943 822 981 718 538 146 467 105 052 467 115 643 189 964 159
36	0.001 285 021 595 601 950 520 993 321 698 625 095 350 108 016 497 227 138 252 050
37	0.001 222 458 846 390 942 903 503 235 135 000 060 120 568 852 488 038 677 860 663
38	0.001 164 350 437 578 213 687 688 887 920 288 418 957 921 829 901 092 024 417 088
39	0.001 110 283 767 555 043 112 676 193 720 122 177 283 466 647 192 255 576 066 250
40	0.001 059 892 884 680 728 844 445 376 371 838 594 100 978 318 795 219 921 190 670
41	0.001 012 852 302 778 103 100 067 039 996 921 174 354 095 192 040 589 337 003 889

Table A47. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=4$

1	0.054 505 312 003 928 964 795 033 324 451 558 559 066 139 790 357 625 415 390 557
2	0.040 584 641 358 619 093 952 746 643 345 908 436 650 849 996 893 926 267 824 538
3	0.031 352 735 402 136 863 494 619 870 487 407 837 349 459 694 974 570 654 501 478
4	0.024 925 898 377 099 133 909 739 342 757 072 494 371 332 066 842 478 610 225 585
5	0.020 277 419 479 252 237 233 536 241 372 554 531 446 765 706 160 025 861 291 221
6	0.016 809 565 155 222 614 170 443 102 927 594 396 358 479 034 590 807 118 005 294
7	0.014 155 508 968 015 191 404 443 207 516 242 483 132 089 686 137 161 542 190 991
8	0.012 080 191 130 320 871 157 514 271 431 007 355 951 393 534 011 380 657 556 204
9	0.010 427 449 501 136 319 384 558 825 513 067 812 156 321 418 298 838 254 422 344
10	0.009 090 309 283 748 304 193 891 780 996 901 841 933 881 114 187 194 191 871 141
11	0.007 993 533 679 786 680 913 365 471 962 441 892 190 314 806 248 502 625 154 926
12	0.007 082 981 587 350 802 635 828 320 973 404 034 692 669 387 931 533 371 516 894
13	0.006 318 901 474 727 743 402 686 036 787 709 783 688 812 825 355 319 263 329 697
14	0.005 671 583 197 155 590 904 985 273 215 586 511 392 371 175 906 652 645 540 389
15	0.005 118 466 754 133 650 278 306 391 215 671 247 780 902 864 274 956 577 261 948
16	0.004 642 176 263 655 782 853 041 565 967 872 005 594 192 193 066 069 760 723 561
17	0.004 229 155 999 564 532 118 528 646 142 850 754 964 795 441 021 931 274 889 030
18	0.003 868 706 875 753 917 784 650 139 975 607 035 047 581 093 937 121 605 407 883
19	0.003 552 294 588 450 123 880 149 569 824 297 978 248 760 897 454 262 559 868 464
20	0.003 273 045 377 703 409 618 549 475 517 201 187 514 798 028 585 970 011 667 545
21	0.003 025 373 498 813 966 273 505 824 135 639 297 577 376 055 238 927 039 748 474
22	0.002 804 702 546 021 851 011 132 616 819 395 908 658 460 165 471 899 722 989 991
23	0.002 607 254 576 011 143 801 744 989 920 794 498 694 179 391 421 650 048 733 474
24	0.002 429 888 834 032 470 034 786 413 261 327 496 997 487 011 366 047 839 623 601
25	0.002 269 977 196 472 032 710 089 779 705 052 952 554 715 981 384 853 502 257 172
26	0.002 125 307 087 885 566 189 181 223 581 803 365 262 816 179 176 950 934 813 589
27	0.001 994 005 165 410 750 057 916 661 013 211 526 445 111 905 599 918 663 475 397
28	0.001 874 476 849 287 359 510 042 586 635 399 155 159 372 826 267 939 393 856 077
29	0.001 765 358 051 294 288 327 547 129 938 943 317 655 023 071 518 939 574 302 301
30	0.001 665 476 370 566 662 307 374 775 022 870 539 959 053 583 355 252 194 613 348
31	0.001 573 819 694 602 041 593 836 108 215 331 089 827 856 135 081 526 659 456 077
32	0.001 489 510 634 789 262 330 722 873 250 531 080 224 256 940 467 391 276 475 198
33	0.001 411 785 590 596 920 740 842 757 744 299 422 794 702 332 101 557 230 314 152
34	0.001 339 977 509 652 625 190 670 392 568 817 226 269 181 511 379 557 519 052 949
35	0.001 273 501 617 052 193 756 837 151 454 772 857 577 909 429 595 200 898 551 073
36	0.001 211 843 543 990 168 411 074 969 668 194 582 477 492 074 092 577 948 711 733
37	0.001 154 549 405 888 560 550 822 688 960 558 670 900 925 634 399 117 081 434 484
38	0.001 101 217 472 833 055 188 441 966 803 502 168 964 115 828 995 534 056 139 955
39	0.001 051 491 147 050 535 730 343 680 223 588 927 753 482 317 777 399 929 124 349
40	0.001 005 053 018 356 590 769 645 531 696 350 478 419 837 356 467 410 077 550 358

Table A48. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=5$

1	0.037 504 157 621 261 599 307 845 788 022 415 895 089 450 727 673 238 904 349 593
2	0.029 284 239 280 935 389 926 843 831 608 091 634 776 943 028 525 570 541 873 992
3	0.023 478 284 055 737 125 891 270 601 819 997 266 473 277 867 725 747 717 159 593
4	0.019 229 718 918 348 832 646 193 673 618 760 760 702 612 086 841 212 795 755 263
5	0.016 029 897 746 104 609 487 926 858 324 187 823 452 890 039 297 706 201 399 638
6	0.013 561 536 957 329 614 186 118 990 102 112 450 170 014 491 358 291 406 609 377
7	0.011 618 531 604 616 622 343 650 430 452 274 094 954 008 041 476 467 166 913 823
8	0.010 062 363 634 696 395 081 820 028 866 124 255 055 906 263 688 506 769 898 654
9	0.008 797 200 097 400 931 532 111 365 469 337 755 733 686 501 460 121 413 749 744
10	0.007 755 060 116 992 438 023 930 335 502 027 671 235 078 662 285 691 016 075 142
11	0.006 886 656 719 143 153 939 565 111 912 127 657 563 924 933 594 549 296 397 275
12	0.006 155 563 631 290 797 078 364 650 192 671 992 997 263 465 492 528 225 021 953
13	0.005 534 397 108 481 620 199 076 473 560 212 248 269 839 752 902 489 661 176 578
14	0.005 002 255 672 271 580 675 446 449 390 404 053 194 438 420 445 637 524 748 389
15	0.004 542 966 027 539 938 021 636 847 327 154 510 263 244 394 060 687 871 273 432
16	0.004 143 857 907 138 104 000 926 218 724 862 640 328 748 632 904 207 326 246 592
17	0.003 794 893 332 394 222 634 478 574 909 696 310 497 047 655 190 972 118 789 696
18	0.003 488 037 920 476 581 284 086 293 111 626 815 611 647 012 333 316 884 376 329
19	0.003 216 800 380 585 424 926 184 836 714 851 340 494 664 228 343 833 401 975 866
20	0.002 975 890 737 031 802 481 569 530 928 016 395 479 429 114 910 210 905 523 818
21	0.002 760 963 584 186 488 217 343 928 302 155 872 440 269 867 432 765 278 339 815
22	0.002 568 423 056 523 572 472 026 069 952 745 040 201 148 773 851 070 374 592 360
23	0.002 395 273 144 060 570 474 297 325 063 078 295 739 400 422 696 429 244 392 626
24	0.002 239 001 706 422 136 601 665 860 291 575 027 802 282 985 108 403 268 370 178
25	0.002 097 489 795 959 044 183 935 368 099 096 492 756 238 167 271 068 179 740 035
26	0.001 968 940 176 783 028 499 694 058 590 854 897 840 175 284 785 255 529 830 715
27	0.001 851 820 537 357 190 813 505 196 997 288 184 340 417 330 320 484 363 702 212
28	0.001 744 818 047 282 174 959 294 744 233 936 631 458 840 374 126 061 658 520 024
29	0.001 646 802 743 192 964 139 071 945 369 167 695 078 967 923 447 713 861 185 124
30	0.001 556 797 838 470 757 614 682 140 183 894 042 543 934 594 764 963 422 794 383
31	0.001 473 955 501 435 542 192 782 091 791 600 779 111 284 731 882 051 597 035 123
32	0.001 397 536 981 689 374 378 321 658 880 597 942 988 196 165 959 864 821 470 329
33	0.001 326 896 215 807 069 893 942 944 411 314 621 542 566 648 108 055 027 233 202
34	0.001 261 466 233 929 354 560 333 485 296 653 957 369 075 828 178 984 913 360 111
35	0.001 200 747 833 966 898 243 137 581 622 114 198 130 831 607 473 279 686 450 495
36	0.001 144 300 101 598 626 941 457 928 728 405 998 652 172 678 780 623 435 219 734
37	0.001 091 732 440 439 095 784 931 866 190 177 129 700 577 504 082 121 587 946 662
38	0.001 042 697 843 821 469 059 943 861 371 684 223 054 049 909 130 024 464 135 700

Table A49. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=6$

1	0.027 379 013 517 405 846 064 398 896 350 665 094 972 629 911 949 503 570 496 062
2	0.022 124 129 661 551 987 334 946 457 760 417 306 047 552 439 966 037 206 169 011
3	0.018 237 095 442 474 674 549 231 234 389 348 998 935 947 072 974 389 263 054 569
4	0.015 283 319 964 880 668 003 081 679 671 959 113 224 800 015 559 623 266 932 879
5	0.012 987 638 690 650 220 344 360 591 323 278 056 326 586 227 964 672 442 102 783
6	0.011 169 036 747 172 108 065 141 041 805 660 271 243 083 263 664 905 424 749 049
7	0.009 704 537 522 417 551 145 572 681 740 337 611 624 706 705 503 750 682 473 888
8	0.008 508 262 432 372 513 462 138 871 742 302 888 200 218 723 685 910 766 457 360
9	0.007 518 794 132 701 856 927 709 981 360 653 799 584 361 082 771 140 709 066 858
10	0.006 691 284 812 901 979 112 407 669 188 853 040 070 201 088 018 369 546 543 389
11	0.005 992 378 786 734 917 960 844 198 018 960 654 013 694 323 607 283 525 205 265
12	0.005 396 859 164 251 311 832 420 354 394 326 545 233 085 466 310 726 244 302 217
13	0.004 885 381 247 434 406 382 826 534 098 505 679 235 053 311 424 692 664 958 230
14	0.004 442 908 448 971 898 372 625 749 278 991 892 237 084 522 147 881 952 790 262
15	0.004 057 612 745 565 883 704 751 632 839 908 428 047 646 684 668 249 865 944 091
16	0.003 720 088 609 189 701 983 906 359 902 575 578 660 524 812 044 783 861 046 242
17	0.003 422 782 408 593 538 744 272 880 702 979 803 948 432 674 330 511 779 139 926
18	0.003 159 572 414 415 089 653 624 997 278 921 762 631 835 162 427 824 603 663 571
19	0.002 925 455 690 873 757 527 266 362 716 733 807 320 861 643 376 209 294 987 544
20	0.002 716 311 918 583 961 763 138 138 063 907 310 995 757 081 829 503 061 700 852
21	0.002 528 723 307 881 402 720 126 255 141 802 046 791 600 253 809 183 647 801 230
22	0.002 359 835 898 639 188 493 141 978 861 202 570 217 048 889 270 244 640 026 670
23	0.002 207 251 736 745 544 942 521 020 177 695 295 448 442 604 574 383 343 943 432
24	0.002 068 944 324 309 350 490 410 990 897 986 523 232 417 828 963 261 228 824 238
25	0.001 943 191 781 623 207 475 328 683 836 799 312 679 161 134 166 454 825 659 744
26	0.001 828 523 609 283 217 730 180 416 699 645 709 077 391 068 875 902 047 934 779
27	0.001 723 677 981 208 445 259 521 750 586 747 532 458 877 969 850 158 723 707 192
28	0.001 627 567 256 341 352 004 871 797 862 652 424 391 135 011 343 703 679 014 966
29	0.001 539 249 952 093 688 293 838 341 107 384 900 327 949 259 691 917 255 975 851
30	0.001 457 907 833 697 497 084 894 351 769 569 283 527 010 096 555 770 075 327 036
31	0.001 382 827 080 640 203 787 782 013 591 144 097 019 862 892 373 437 614 502 986
32	0.001 313 382 722 552 752 782 361 306 720 141 421 853 141 486 592 359 099 555 299
33	0.001 249 025 712 370 374 232 797 524 437 082 882 512 599 885 888 122 341 724 037
34	0.001 189 272 138 720 505 428 961 688 934 867 612 929 545 832 650 448 894 303 562
35	0.001 133 694 182 759 715 838 806 810 214 400 738 028 173 653 477 974 938 040 559
36	0.001 081 912 504 713 480 586 090 622 734 865 734 105 311 107 299 705 486 494 383
37	0.001 033 589 807 789 423 793 379 831 211 853 198 009 174 823 405 431 750 283 112

Table A50. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=7$

1	0.020 864 266 780 860 078 453 789 576 568 801 545 166 757 672 002 014 256 083 290
2	0.017 302 662 672 912 724 217 396 842 592 514 441 185 768 197 066 929 730 631 594
3	0.014 573 499 347 129 790 545 729 337 116 516 248 532 045 322 450 931 523 379 295
4	0.012 437 355 932 937 240 387 071 504 039 857 991 230 613 863 277 613 923 250 967
5	0.010 734 882 615 914 135 072 290 314 925 339 198 434 068 438 954 073 558 699 275
6	0.009 356 726 039 620 790 591 969 303 618 247 576 622 312 304 247 735 689 542 142
7	0.008 225 848 855 412 973 199 003 079 125 896 063 287 081 707 205 921 723 432 100
8	0.007 286 731 192 668 955 177 899 149 807 407 228 168 617 868 580 501 805 997 873
9	0.006 498 555 090 812 662 621 443 289 966 181 653 656 435 402 480 253 915 179 170
10	0.005 830 777 972 336 346 952 723 297 034 365 281 119 278 097 685 580 811 857 436
11	0.005 260 184 475 893 616 227 823 109 764 810 556 102 588 732 937 600 549 259 723
12	0.004 768 878 585 684 606 287 351 743 439 256 286 842 571 378 118 157 316 435 588
13	0.004 342 888 600 363 741 065 050 211 194 836 518 906 135 006 274 338 244 362 864
14	0.003 971 180 339 985 109 895 604 257 086 152 666 146 599 195 501 699 785 955 353
15	0.003 644 947 699 806 933 221 397 696 000 730 834 113 178 713 874 922 089 376 712
16	0.003 357 095 014 236 458 561 451 451 553 453 790 169 304 689 037 186 720 251 244
17	0.003 101 854 244 064 403 800 459 358 527 686 983 316 914 846 936 474 639 405 502
18	0.002 874 498 347 279 404 208 266 300 710 371 219 475 512 697 781 770 572 898 351
19	0.002 671 124 208 787 956 447 237 265 887 242 189 199 130 477 215 131 176 551 326
20	0.002 488 486 509 860 974 689 166 505 310 920 169 048 708 560 461 939 635 501 016
21	0.002 323 869 337 599 601 171 368 486 035 554 919 445 887 989 879 102 137 700 027
22	0.002 174 986 057 782 945 936 943 557 848 707 487 054 428 031 323 072 510 597 570
23	0.002 039 900 567 153 304 863 718 050 132 356 186 201 560 267 143 914 843 173 997
24	0.001 916 964 869 663 852 678 444 815 190 066 412 835 123 886 208 293 288 255 229
25	0.001 804 769 225 990 206 558 610 092 968 868 943 554 497 953 887 234 975 467 665
26	0.001 702 102 066 974 364 193 965 681 957 792 014 028 818 661 509 158 284 852 190
27	0.001 607 917 547 871 692 123 018 725 360 517 712 057 910 845 146 512 449 683 933
28	0.001 521 309 125 327 616 444 568 160 898 077 989 296 358 234 386 110 393 972 154
29	0.001 441 487 914 130 998 005 984 658 521 390 097 485 459 787 266 254 140 724 994
30	0.001 367 764 861 797 920 293 286 823 179 119 945 367 525 614 063 013 620 149 317
31	0.001 299 535 991 251 101 105 053 547 099 706 382 432 498 284 198 221 245 980 362
32	0.001 236 270 123 346 289 397 399 000 825 552 145 779 049 240 888 772 731 614 173
33	0.001 177 498 614 773 932 264 264 176 865 178 835 203 255 841 469 133 414 365 499
34	0.001 122 806 742 389 968 170 515 979 852 529 066 456 902 857 965 530 427 407 882
35	0.001 071 826 439 233 225 494 868 150 823 637 487 182 577 577 220 790 920 375 202
36	0.001 024 230 145 485 462 987 850 442 214 148 623 612 854 665 391 068 231 741 781

Table A51. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=8$

1	0.016	426	477	708	284	253	716	039	512	289	537	027	778	748	663	643	396	283	933	126
2	0.013	901	745	603	033	588	659	913	515	333	553	312	665	897	795	071	886	616	551	198
3	0.011	912	427	705	033	935	436	130	377	643	611	050	402	459	122	704	083	423	718	267
4	0.010	317	865	003	358	436	159	078	783	254	460	839	525	773	331	486	492	694	039	154
5	0.009	020	620	723	983	738	034	735	095	944	017	780	866	780	904	650	656	461	972	083
6	0.007	951	485	846	988	744	664	272	765	117	565	311	792	564	705	211	067	533	310	155
7	0.007	060	218	971	524	948	914	568	091	313	687	639	125	186	864	371	527	338	313	859
8	0.006	309	643	893	282	880	030	629	786	432	750	072	601	355	423	302	419	495	753	152
9	0.005	671	782	268	073	125	270	707	330	426	847	750	550	890	391	397	141	615	160	490
10	0.005	125	257	438	519	974	399	038	439	740	668	711	952	277	846	615	269	775	533	360
11	0.004	653	513	627	551	853	781	549	857	234	299	590	392	364	937	915	122	813	335	794
12	0.004	243	570	633	766	217	020	751	232	155	876	016	398	688	323	806	374	065	935	815
13	0.003	885	137	743	370	860	690	865	413	285	831	511	540	811	287	206	724	860	152	239
14	0.003	569	973	245	502	691	753	475	915	535	344	543	927	231	227	519	843	261	142	621
15	0.003	291	414	800	031	162	576	002	251	582	911	875	358	170	752	127	538	054	155	327
16	0.003	044	030	545	051	615	002	898	964	074	872	298	528	444	044	442	405	405	686	591
17	0.002	823	356	769	048	613	983	988	528	023	824	617	345	187	075	864	488	029	376	033
18	0.002	625	698	473	637	969	901	845	030	762	198	170	148	145	818	508	499	959	853	362
19	0.002	447	976	189	081	513	748	073	318	139	261	957	613	022	741	929	327	716	274	630
20	0.002	287	607	193	196	770	393	034	781	101	989	955	687	792	504	536	183	491	838	033
21	0.002	142	412	589	979	948	260	468	563	645	123	198	829	639	589	415	378	946	236	537
22	0.002	010	544	016	847	366	307	758	458	668	520	917	296	834	711	989	549	112	320	161
23	0.001	890	425	387	318	629	154	383	705	304	986	367	023	729	810	428	766	354	108	152
24	0.001	780	706	249	461	451	956	518	621	891	748	471	117	929	451	959	185	871	616	900
25	0.001	680	224	190	263	119	518	618	010	960	575	298	957	852	682	414	930	683	872	592
26	0.001	587	974	337	778	823	176	569	135	492	793	308	851	325	866	990	059	693	263	784
27	0.001	503	084	472	006	243	626	342	410	234	060	334	361	271	430	725	863	740	649	683
28	0.001	424	794	597	497	033	297	918	113	308	723	233	478	924	563	182	082	963	583	372
29	0.001	352	440	087	720	383	281	836	121	925	588	195	091	346	246	696	418	700	246	797
30	0.001	285	437	705	825	645	796	356	064	267	882	860	078	698	791	787	489	655	492	333
31	0.001	223	273	954	954	383	444	423	957	257	627	717	158	167	058	919	108	774	168	469
32	0.001	165	495	325	365	747	052	074	335	392	677	013	693	622	687	446	909	780	061	071
33	0.001	111	700	093	918	174	900	979	358	758	660	228	685	952	088	997	202	055	545	830
34	0.001	061	531	400	181	362	671	404	348	928	196	212	397	524	301	337	177	427	636	945
35	0.001	014	671	377	290	018	923	777	196	831	300	966	773	504	104	408	659	127	171	329

Table A52. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=9$

1	0.013	267	906	420	725	144	574	647	282	768	382	611	429	273	303	426	274	429	186	765
2	0.011	413	421	334	621	675	146	916	186	131	900	161	206	354	774	402	963	048	052	670
3	0.009	918	850	966	560	703	100	254	443	126	319	870	204	204	566	800	538	427	253	063
4	0.008	697	178	224	793	057	247	829	782	964	377	166	909	002	068	128	451	951	828	892
5	0.007	686	116	669	584	659	418	132	257	261	695	894	264	510	665	113	331	048	345	614
6	0.006	840	139	804	375	214	466	410	124	968	934	979	776	419	349	767	451	638	160	159
7	0.006	125	353	819	712	897	883	644	641	395	141	761	862	027	846	115	746	242	135	249
8	0.005	516	110	425	931	965	797	175	257	753	513	563	407	214	507	913	589	438	628	996
9	0.004	992	716	085	612	111	843	472	268	692	188	323	071	201	102	550	543	408	205	679
10	0.004	539	850	086	531	347	423	131	957	705	495	773	479	738	617	508	073	045	378	200
11	0.004	145	451	513	169	736	362	614	191	090	367	218	258	112	682	884	814	798	505	530
12	0.003	799	922	831	564	635	989	457	131	755	996	890	497	122	664	400	858	888	774	472
13	0.003	495	551	258	479	411	171	135	357	244	613	468	025	540	865	308	748	301	441	984
14	0.003	226	082	474	235	911	266	857	184	895	878	579	017	235	587	868	000	591	657	465
15	0.002	986	402	548	298	757	418	622	401	005	078	728	848	131	721	862	914	343	945	181
16	0.002	772	297	817	094	937	643	073	319	415	781	391	770	521	556	177	189	073	945	112
17	0.002	580	271	645	070	371	514	688	523	135	579	677	482	974	317	050	170	900	765	643
18	0.002	407	403	191	951	319	100	840	355	951	565	537	710	197	021	129	938	429	996	119
19	0.002	251	237	544	120	857	774	765	237	643	976	145	035	280	100	660	761	275	303	672
20	0.002	109	699	505	299	982	364	199	741	608	570	481	508	074	614	561	131	256	791	713
21	0.001	981	025	405	578	350	456	394	752	733	615	535	121	148	642	365	380	482	070	181
22	0.001	863	708	755	599	330	244	152	777	326	702	516	447	898	481	635	526	042	165	652
23	0.001	756	456	628	350	799	603	094	159	662	536	203	017	291	471	462	305	076	234	076
24	0.001	658	154	418	290	152	772	634	752	537	108	953	637	022	545	283	367	310	886	957
25	0.001	567	837	190	739	138	042	632	450	124	566	273	818	574	255	802	893	880	369	218
26	0.001	484	666	251	743	427	802	184	424	649	492	730	042	664	352	919	550	040	792	346
27	0.001	407	909	880	429	511	930	257	312	139	990	518	458	910	956	007	827	119	300	933
28	0.001	336	927	400	865	411	135	007	827	548	050	451	618	653	576	731	050	556	578	798
29	0.001	271	155	948	867	135	944	404	985	644	790	781	269	730	131	053	020	148	306	650
30	0.001	210	099	425	692	375	230	266	222	034	783	214	735	454	300	231	103	217	178	510
31	0.001	153	319	235	711	354	966	526	960	841	027	948	112	459	434	178	792	807	948	468
32	0.001	100	426	486	679	390	432	503	677	668	813	019	115	345	735	375	993	359	289	472
33	0.001	051	075	394	857	394	656	809	177	015	130	795	908	510	659	639	944	689	008	125
34	0.001	004	957	687	168	143	839	713	400	635	578	079	217	435	536	693	355	109	140	346

Table A53. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=10$

1	0.010	940	150	741	123	942	885	437	022	519	230	806	251	723	125	475	167	207	052	796
2	0.009	538	085	282	915	766	756	426	287	864	741	414	208	894	606	684	746	635	192	003
3	0.008	386	847	442	849	003	488	203	984	928	427	705	908	175	945	359	901	455	997	598
4	0.007	430	271	861	880	216	111	443	779	686	598	715	359	889	674	797	223	659	819	861
5	0.006	627	040	319	801	375	359	475	957	798	804	697	148	057	733	236	899	585	201	483
6	0.005	946	213	549	071	392	656	357	738	521	367	271	965	863	466	703	515	338	902	491
7	0.005	364	256	679	073	837	109	106	362	775	258	479	913	754	855	828	678	991	953	963
8	0.004	863	013	451	695	590	817	279	656	098	199	317	876	986	061	101	199	517	376	816
9	0.004	428	298	378	995	018	296	218	375	328	485	007	197	548	305	971	627	766	985	776
10	0.004	048	900	439	222	124	399	211	303	913	193	791	887	192	995	572	341	002	808	326
11	0.003	715	866	382	010	817	828	854	942	887	183	920	775	214	856	451	330	838	895	578
12	0.003	421	977	464	538	813	981	664	390	772	628	329	041	279	550	817	573	889	731	226
13	0.003	161	362	210	312	654	177	341	019	492	424	160	656	136	242	143	651	561	683	163
14	0.002	929	206	259	840	335	293	075	283	676	079	094	193	064	201	840	355	910	001	176
15	0.002	721	532	479	776	942	128	134	041	883	050	559	383	446	407	600	924	693	977	742
16	0.002	535	032	557	354	659	398	341	880	493	233	873	041	486	684	305	677	848	220	853
17	0.002	366	936	764	378	666	406	807	725	490	152	596	695	121	808	109	843	768	726	706
18	0.002	214	912	325	412	838	133	832	855	987	859	459	765	115	653	180	299	652	795	852
19	0.002	076	983	437	541	424	063	964	223	887	188	695	374	755	822	217	258	336	610	555
20	0.001	951	467	832	609	415	629	626	583	465	943	561	805	300	326	478	315	046	069	873
21	0.001	836	926	089	062	045	240	920	869	601	994	946	840	472	479	852	015	679	685	650
22	0.001	732	120	850	665	142	448	810	118	111	789	708	854	737	001	470	513	204	560	493
23	0.001	635	983	802	407	862	734	717	190	932	066	564	213	016	687	936	225	864	112	366
24	0.001	547	588	764	267	841	704	249	616	723	569	228	885	940	862	851	815	472	108	629
25	0.001	466	129	642	819	387	203	547	264	717	749	364	976	559	939	829	254	481	335	298
26	0.001	390	902	264	972	371	724	705	543	811	061	098	469	573	746	731	951	067	802	891
27	0.001	321	289	332	958	908	733	662	009	375	351	565	803	079	628	735	675	744	992	416
28	0.001	256	747	903	256	773	451	327	041	488	081	492	901	230	822	061	564	937	462	215
29	0.001	196	798	917	583	968	783	512	976	135	187	199	313	520	438	783	120	698	258	797
30	0.001	141	018	410	964	361	600	955	303	024	779	062	888	936	708	765	456	258	828	133
31	0.001	089	030	097	148	038	500	043	874	560	870	468	014	566	279	213	054	555	217	297
32	0.001	040	499	090	542	356	790	573	430	790	789	872	348	344	713	322	425	494	977	036

Table A54. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=11$

1	0.009 175 399 777 847 996 054 292 131 437 865 215 270 842 725 031 414 289 869 002
2	0.008 089 727 823 409 438 031 415 919 331 805 869 803 705 080 585 233 643 988 457
3	0.007 184 189 989 084 538 382 750 649 417 301 090 267 788 395 212 953 857 371 266
4	0.006 421 224 462 381 906 755 184 557 754 363 833 658 845 161 041 000 807 701 595
5	0.005 772 550 239 624 205 616 648 678 082 512 671 973 114 215 863 078 736 581 341
6	0.005 216 547 671 638 180 809 042 293 316 745 475 460 814 705 476 598 897 822 525
7	0.004 736 462 789 558 044 080 199 074 085 779 372 924 472 032 192 883 538 497 235
8	0.004 319 151 888 725 189 519 765 745 606 834 712 440 035 064 981 050 668 831 491
9	0.003 954 188 205 282 343 513 846 368 758 359 777 576 186 260 637 082 455 055 580
10	0.003 633 216 048 813 722 739 027 704 571 534 556 267 035 311 139 008 500 785 437
11	0.003 349 477 048 776 486 310 044 665 566 701 442 316 486 153 854 908 882 147 146
12	0.003 097 458 039 288 512 664 184 908 805 238 458 768 726 866 544 548 164 141 152
13	0.002 872 626 172 080 571 231 615 551 311 296 264 295 055 078 385 904 566 572 749
14	0.002 671 227 423 295 246 315 268 474 127 202 505 513 509 993 305 026 980 030 339
15	0.002 490 131 742 575 540 032 598 384 310 480 136 450 640 351 749 268 816 433 895
16	0.002 326 712 911 615 820 061 775 041 857 395 245 227 329 742 834 140 095 979 552
17	0.002 178 754 505 700 485 300 135 845 698 335 611 343 907 482 400 488 708 612 652
18	0.002 044 375 678 962 846 940 321 175 537 793 124 793 083 559 090 181 544 996 333
19	0.001 921 972 142 682 530 292 515 079 535 227 153 940 737 388 697 926 399 310 667
20	0.001 810 168 887 408 183 000 683 003 972 261 310 458 568 559 552 224 776 694 807
21	0.001 707 782 055 606 197 022 291 083 956 365 483 383 235 522 764 196 395 748 297
22	0.001 613 787 997 908 203 981 545 697 183 107 012 391 538 489 927 139 793 015 201
23	0.001 527 298 008 789 404 408 779 780 348 810 830 641 766 511 346 549 526 806 726
24	0.001 447 537 582 454 984 995 765 617 745 610 923 018 010 356 564 410 481 967 109
25	0.001 373 829 289 005 118 286 366 352 221 775 432 684 305 128 266 068 177 003 466
26	0.001 305 578 567 408 128 975 244 263 232 116 466 404 216 734 681 741 697 027 107
27	0.001 242 261 881 781 271 806 673 028 080 149 365 526 150 927 780 480 880 662 505
28	0.001 183 416 802 774 326 270 095 615 807 837 338 084 063 574 880 503 343 906 191
29	0.001 128 633 665 087 301 300 550 969 757 023 513 016 522 797 789 215 688 998 535
30	0.001 077 548 521 662 093 963 299 408 343 242 597 951 354 589 288 492 210 672 509
31	0.001 029 837 169 559 558 615 242 186 528 988 305 172 717 703 840 969 921 995 396

Table A55. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=12$

1	0.007 805 680 448 079 473 729 754 614 222 651 204 908 074 296 661 521 164 218 917
2	0.006 947 903 863 263 270 107 372 428 092 992 915 719 434 352 100 569 367 946 832
3	0.006 222 820 856 841 459 096 504 787 530 366 973 233 786 073 872 279 331 217 199
4	0.005 604 542 570 229 372 311 574 230 843 302 865 243 053 876 947 571 895 621 725
5	0.005 073 184 568 404 756 622 473 101 644 704 559 533 449 674 650 958 178 423 743
6	0.004 613 271 283 147 811 312 883 737 742 920 432 415 503 775 678 746 159 992 454
7	0.004 212 614 183 306 725 078 817 557 033 313 431 241 743 947 010 944 603 619 827
8	0.003 861 509 348 959 851 695 394 075 651 988 440 682 888 163 784 289 774 773 846
9	0.003 552 154 534 733 388 347 996 757 011 302 432 153 432 922 141 154 073 699 898
10	0.003 278 219 676 444 111 036 249 855 987 574 369 578 893 383 069 930 612 186 574
11	0.003 034 526 356 361 139 007 890 080 568 632 205 145 622 075 291 781 387 218 434
12	0.002 816 805 749 354 204 521 509 304 083 278 388 529 063 050 648 072 607 913 249
13	0.002 621 513 840 953 738 413 094 718 832 394 931 105 982 213 800 157 123 938 248
14	0.002 445 688 945 886 729 194 681 118 043 888 647 874 513 648 712 732 386 223 071
15	0.002 286 840 818 686 136 310 619 975 254 206 341 741 255 106 714 685 254 813 408
16	0.002 142 863 603 338 866 087 015 942 116 437 354 413 548 778 317 502 835 649 224
17	0.002 011 966 944 933 688 930 848 364 181 120 018 266 163 392 378 484 637 076 628
18	0.001 892 621 062 493 112 622 360 763 240 949 073 213 322 242 129 820 245 027 920
19	0.001 783 512 643 916 451 954 422 296 389 172 798 020 598 622 702 846 663 536 344
20	0.001 683 509 195 752 725 112 902 122 227 985 960 380 843 590 501 389 570 623 356
21	0.001 591 630 047 148 035 786 897 755 419 447 047 542 838 387 654 379 312 143 591
22	0.001 507 022 627 207 600 267 655 831 005 209 900 968 244 060 036 084 402 793 017
23	0.001 428 942 948 912 286 526 337 006 226 020 652 372 411 315 730 897 039 228 624
24	0.001 356 739 469 334 758 015 200 928 985 495 327 191 103 082 602 873 674 524 437
25	0.001 289 839 675 638 137 627 184 426 363 659 487 994 588 104 341 948 914 285 585
26	0.001 227 738 883 887 184 729 359 023 510 917 668 965 683 114 276 267 997 048 813
27	0.001 169 990 843 698 143 823 398 083 696 640 483 029 059 233 070 150 275 874 400
28	0.001 116 199 823 977 589 300 162 803 547 395 233 137 383 196 666 942 184 219 056
29	0.001 066 013 919 185 810 696 135 549 319 613 781 364 073 323 017 254 795 476 499
30	0.001 019 119 365 964 092 053 671 546 629 088 935 060 424 732 970 913 789 163 223

Table A56. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=13$

1	0.006 721 302 533 478 393 861 554 123 643 261 883 113 028 655 934 565 283 962 623
2	0.006 031 831 976 293 947 919 329 485 374 436 962 213 819 790 079 569 822 760 982
3	0.005 442 259 939 061 199 303 065 008 484 153 687 235 297 695 774 833 817 623 591
4	0.004 934 274 498 792 487 257 548 579 115 601 077 687 560 591 394 512 985 673 080
5	0.004 493 565 440 722 609 889 756 582 140 834 520 282 676 831 395 204 080 453 845
6	0.004 108 819 663 589 784 691 472 254 535 653 127 149 785 982 935 761 678 099 268
7	0.003 770 999 061 640 172 415 184 485 081 454 779 905 170 317 405 642 686 208 076
8	0.003 472 813 533 957 477 616 776 126 544 287 431 337 317 677 263 212 654 585 847
9	0.003 208 331 067 216 457 820 754 277 208 956 694 171 998 400 238 064 797 417 486
10	0.002 972 685 591 998 912 440 601 807 174 037 448 575 336 201 597 420 771 910 044
11	0.002 761 855 559 893 844 088 683 405 268 542 180 651 141 658 259 344 193 432 042
12	0.002 572 494 332 481 167 049 461 928 634 009 670 448 790 369 489 085 039 402 551
13	0.002 401 798 979 015 853 147 284 505 734 333 744 519 367 838 464 473 617 825 336
14	0.002 247 407 858 757 097 841 869 137 929 555 769 604 407 680 448 854 941 690 822
15	0.002 107 319 994 443 555 636 113 146 520 000 842 780 085 369 386 872 686 441 581
16	0.001 979 831 098 340 850 687 812 070 546 749 455 026 795 733 357 401 986 537 050
17	0.001 863 482 436 090 995 863 662 518 282 238 359 797 921 769 437 366 846 804 803
18	0.001 757 019 668 978 137 165 775 000 835 485 821 466 835 032 372 624 528 727 921
19	0.001 659 359 511 960 111 185 592 696 277 561 011 033 703 774 157 449 881 330 775
20	0.001 569 562 557 913 736 101 890 786 665 905 902 542 051 705 858 404 476 871 167
21	0.001 486 810 999 883 659 892 462 211 599 705 471 518 533 165 202 136 286 747 663
22	0.001 410 390 268 996 050 947 231 815 508 620 171 850 459 854 396 258 563 916 831
23	0.001 339 673 821 747 555 555 889 411 819 097 600 532 504 200 809 801 422 702 612
24	0.001 274 110 474 916 218 595 780 915 736 193 546 382 526 768 890 997 002 474 771
25	0.001 213 213 812 557 739 767 199 124 806 063 634 714 928 087 040 096 382 015 010
26	0.001 156 553 287 038 451 737 498 531 412 272 748 221 256 865 610 603 450 939 716
27	0.001 103 746 711 846 378 501 374 936 409 924 325 246 504 946 314 258 322 234 369
28	0.001 054 453 903 208 363 039 214 886 177 074 601 651 750 863 712 510 881 067 585
29	0.001 008 371 274 191 653 083 527 731 540 875 653 903 477 194 266 955 762 776 359

Table A57. Critical binding parameters μ_c (au⁻¹) for Hulthén potential, $l=14$

1	0.005 848 170 260 304 191 129 434 529 886 533 876 845 851 756 102 010 282 081 323
2	0.005 285 691 643 260 854 305 249 821 373 278 460 123 944 201 547 134 894 577 226
3	0.004 799 854 224 723 587 993 222 514 660 518 447 064 730 336 862 414 416 125 061
4	0.004 377 410 387 298 584 593 331 815 641 129 117 657 447 037 298 301 349 457 698
5	0.004 007 849 376 322 934 745 151 544 606 717 836 331 623 343 500 202 209 240 376
6	0.003 682 746 224 085 920 814 513 645 323 860 247 155 687 251 511 500 946 310 191
7	0.003 395 284 541 802 898 806 697 763 328 166 780 396 164 452 691 113 228 815 252
8	0.003 139 902 011 910 541 612 733 812 879 130 808 834 114 468 861 605 869 581 370
9	0.002 912 023 776 780 307 767 239 158 711 839 948 199 364 268 681 328 686 992 718
10	0.002 707 859 658 617 159 767 048 613 627 094 729 650 307 327 201 170 983 529 398
11	0.002 524 248 319 178 367 728 532 724 778 533 676 799 095 532 209 160 486 498 905
12	0.002 358 536 338 977 430 542 183 102 847 490 326 396 370 649 005 172 956 998 549
13	0.002 208 483 552 814 032 756 798 219 747 214 305 747 436 825 443 539 121 403 800
14	0.002 072 188 324 350 870 309 897 081 217 819 925 178 573 852 040 073 987 186 422
15	0.001 948 028 102 688 043 510 316 826 680 832 716 851 178 207 821 226 589 571 932
16	0.001 834 611 792 837 557 091 071 882 778 957 293 972 627 415 109 070 700 918 118
17	0.001 730 741 332 874 670 093 149 484 994 869 316 449 277 401 671 295 959 946 922
18	0.001 635 380 500 305 798 349 757 080 302 168 585 058 081 213 756 284 708 095 601
19	0.001 547 629 435 340 361 767 789 064 906 051 577 883 056 368 919 416 078 115 222
20	0.001 466 703 715 429 217 221 210 872 353 494 023 173 292 814 694 169 201 203 439
21	0.001 391 917 076 011 014 967 996 424 986 830 926 958 159 228 161 398 842 355 718
22	0.001 322 667 069 843 671 060 978 099 598 450 437 447 713 561 056 667 098 791 242
23	0.001 258 423 108 024 661 725 061 978 528 870 902 294 768 330 601 651 052 884 434
24	0.001 198 716 441 695 365 621 520 513 525 031 559 277 150 001 024 079 979 798 723
25	0.001 143 131 733 136 249 651 396 341 160 070 421 824 302 605 890 455 180 392 638
26	0.001 091 299 934 852 010 261 952 064 417 145 480 002 856 883 204 116 964 901 195
27	0.001 042 892 250 030 015 294 631 105 774 814 945 681 595 470 822 460 602 387 841

Table A58. Critical binding parameters μ_c (au⁻¹) for Hulthén potential, $l=15$

1	0.005 134 768 727 032 822 157 152 289 141 536 478 436 108 022 696 012 145 756 602
2	0.004 669 908 072 412 689 495 653 781 643 208 976 253 190 265 521 420 041 843 139
3	0.004 264 824 627 626 486 580 044 996 043 162 561 015 214 592 133 907 362 885 009
4	0.003 909 743 216 984 214 721 746 502 530 394 136 502 084 313 778 443 537 005 714
5	0.003 596 803 539 012 726 924 687 612 536 247 336 238 021 174 288 709 659 179 741
6	0.003 319 627 309 147 502 925 069 225 475 191 568 839 683 757 197 985 690 939 150
7	0.003 072 995 462 034 127 947 598 721 894 637 159 366 809 087 158 430 883 123 558
8	0.002 852 604 516 434 504 202 087 207 466 249 053 868 537 647 521 903 763 779 369
9	0.002 654 880 648 156 551 953 428 391 440 887 402 811 360 436 329 964 505 930 275
10	0.002 476 836 351 954 926 195 609 299 771 811 534 272 967 672 625 661 939 601 898
11	0.002 315 958 892 883 274 100 239 336 443 999 120 197 840 034 808 081 581 297 811
12	0.002 170 122 736 234 700 746 819 323 023 815 718 931 924 687 059 831 114 160 715
13	0.002 037 520 241 255 020 575 643 614 257 786 441 845 238 296 822 343 035 051 682
14	0.001 916 606 392 393 587 774 963 100 448 599 755 573 280 888 763 108 177 018 183
15	0.001 806 054 411 399 911 493 755 545 568 166 949 411 479 031 491 526 177 810 516
16	0.001 704 719 870 413 860 331 893 200 786 850 370 878 078 709 051 921 669 110 394
17	0.001 611 611 496 176 324 550 851 535 477 728 472 048 704 558 777 339 143 154 757
18	0.001 525 867 277 669 623 694 947 204 119 177 343 565 020 863 380 438 575 469 686
19	0.001 446 734 804 998 315 760 343 672 350 169 586 133 722 589 805 454 854 544 874
20	0.001 373 555 005 077 349 389 375 562 061 781 924 628 585 489 346 331 293 973 792
21	0.001 305 748 620 280 018 602 791 866 619 290 681 959 670 034 529 760 336 212 713
22	0.001 242 804 914 383 754 477 284 402 798 245 543 903 965 678 488 263 527 436 136
23	0.001 184 272 196 638 697 331 290 716 814 256 134 698 739 166 104 614 074 456 153
24	0.001 129 749 837 392 088 627 268 351 954 717 485 917 639 870 983 596 881 193 605
25	0.001 078 881 513 191 731 244 703 440 516 817 970 458 121 025 660 155 299 789 376
26	0.001 031 349 469 941 022 932 804 749 174 224 674 444 562 004 439 588 958 076 341

Table A59. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=16$

1	0.004 544 381 589 213 360 268 437 099 084 629 317 607 706 416 321 391 692 121 112
2	0.004 155 791 409 361 078 906 075 727 475 047 994 475 172 163 150 140 165 685 278
3	0.003 814 509 407 261 109 047 818 358 211 940 027 705 947 888 636 503 510 890 995
4	0.003 513 195 461 518 907 185 723 108 033 837 783 726 999 698 485 499 059 491 855
5	0.003 245 876 503 432 514 262 542 864 258 078 627 864 018 742 483 809 081 877 543
6	0.003 007 652 147 491 532 927 614 933 017 643 893 020 072 063 767 923 946 072 297
7	0.002 794 471 749 911 808 602 409 722 668 746 342 967 665 350 606 308 190 865 898
8	0.002 602 963 726 378 937 806 055 255 520 228 062 100 645 568 369 240 135 835 197
9	0.002 430 303 569 432 124 795 007 163 173 063 158 246 261 788 981 448 031 255 674
10	0.002 274 110 845 246 116 520 125 905 076 756 047 414 604 831 854 319 140 361 016
11	0.002 132 368 115 793 475 198 976 888 949 639 861 937 209 910 775 650 399 553 328
12	0.002 003 356 608 828 225 910 997 922 487 769 162 724 514 047 430 160 723 291 698
13	0.001 885 604 795 183 992 380 455 206 691 900 216 338 306 548 032 039 471 152 818
14	0.001 777 846 996 575 419 098 074 095 886 583 158 891 778 173 211 081 615 216 499
15	0.001 678 989 849 146 065 974 929 991 572 679 077 834 836 758 671 818 699 394 780
16	0.001 588 084 964 581 089 028 428 173 735 573 077 177 685 085 017 072 715 831 041
17	0.001 504 306 514 267 436 300 848 806 448 722 455 308 217 487 884 723 933 879 730
18	0.001 426 932 749 442 645 582 407 696 234 889 842 055 001 223 599 670 234 817 697
19	0.001 355 330 687 431 968 686 370 649 666 858 551 781 913 647 874 306 469 992 911
20	0.001 288 943 359 403 592 079 332 268 070 468 201 935 867 073 490 549 702 629 506
21	0.001 227 279 141 869 217 280 130 457 460 586 899 395 034 442 359 208 787 692 807
22	0.001 169 902 792 079 505 257 859 580 623 482 412 181 191 026 401 159 234 185 115
23	0.001 116 427 883 585 063 070 267 927 679 606 553 717 647 454 695 606 596 650 252
24	0.001 066 510 397 777 554 496 539 905 614 516 284 404 993 029 668 825 387 312 451
25	0.001 019 843 274 079 042 157 310 711 819 336 130 144 301 717 612 042 017 738 255

Table A60. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=17$

1	0.004 050 267 510 053 082 753 715 203 223 514 187 788 061 076 207 262 366 361 696
2	0.003 722 131 894 208 185 493 458 134 511 396 270 279 520 071 437 791 495 030 384
3	0.003 431 924 441 284 654 518 144 143 698 201 661 184 578 157 950 347 520 141 687
4	0.003 174 046 988 778 816 376 579 070 226 905 588 040 342 170 054 959 814 199 258
5	0.002 943 895 056 999 114 741 534 906 627 862 083 436 142 620 796 029 571 491 674
6	0.002 737 653 564 432 901 501 113 506 420 976 564 269 097 379 149 638 304 515 097
7	0.002 552 139 946 793 035 303 757 472 085 641 632 619 272 933 018 914 389 437 834
8	0.002 384 682 494 846 754 177 166 149 790 495 115 445 801 428 335 423 977 158 695
9	0.002 233 025 146 409 391 812 004 734 591 348 290 491 373 844 256 543 774 639 971
10	0.002 095 252 351 459 583 079 396 991 451 275 751 696 869 353 267 035 828 832 963
11	0.001 969 729 312 459 119 120 621 594 785 308 365 501 032 232 713 312 055 299 780
12	0.001 855 054 105 051 825 336 778 067 694 743 310 739 827 799 669 244 378 246 648
13	0.001 750 019 054 066 613 143 645 520 126 152 494 522 050 708 020 572 760 673 097
14	0.001 653 579 375 179 432 589 455 955 223 331 336 460 961 411 795 173 644 824 846
15	0.001 564 827 561 416 690 640 585 430 992 755 776 767 828 980 196 492 050 727 563
16	0.001 482 972 342 790 941 546 861 453 071 450 504 982 096 586 825 940 050 063 024
17	0.001 407 321 309 576 901 427 155 361 104 223 710 279 550 815 969 991 739 339 214
18	0.001 337 266 488 296 623 683 939 504 590 617 280 220 271 134 639 445 455 888 920
19	0.001 272 272 310 996 940 416 527 750 657 935 581 622 908 734 494 252 354 114 666
20	0.001 211 865 534 856 663 391 075 245 897 234 600 148 868 161 126 715 933 360 148
21	0.001 155 626 759 282 982 230 936 710 387 382 164 186 926 131 474 785 794 034 078
22	0.001 103 183 257 853 990 215 576 338 248 123 444 872 198 947 636 982 936 053 513
23	0.001 054 202 897 479 807 094 657 862 520 981 221 771 067 714 553 156 374 260 595
24	0.001 008 388 960 524 756 531 707 541 913 988 701 653 761 599 815 524 818 019 394

Table A61. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=18$

1	0.003 632 576 150 723 927 620 969 570 265 835 732 468 964 087 337 213 258 784 290
2	0.003 352 975 858 661 256 096 397 526 709 565 068 712 416 153 177 810 139 252 791
3	0.003 104 138 032 609 516 194 742 756 674 346 476 224 700 726 679 964 750 976 419
4	0.002 881 733 030 822 310 518 433 849 469 582 330 754 700 789 706 282 966 659 828
5	0.002 682 165 272 050 661 847 080 346 467 747 431 766 072 979 666 445 638 068 264
6	0.002 502 428 860 898 752 375 087 825 765 312 888 537 714 176 207 965 017 374 305
7	0.002 339 995 312 644 869 107 399 496 666 417 653 770 222 696 462 105 196 263 863
8	0.002 192 725 460 444 893 639 464 972 302 192 073 173 801 002 184 852 941 235 821
9	0.002 058 799 763 134 523 253 487 233 698 935 017 106 965 642 631 301 795 958 713
10	0.001 936 662 744 454 024 336 895 188 729 114 272 908 393 344 237 935 059 954 731
11	0.001 824 978 378 993 386 647 504 069 924 931 926 556 884 030 650 819 437 547 669
12	0.001 722 594 026 410 353 061 618 370 680 509 597 041 734 538 944 014 349 890 261
13	0.001 628 511 091 476 814 182 330 598 618 925 305 880 097 318 720 991 924 722 667
14	0.001 541 861 013 602 582 533 506 500 151 477 179 598 543 300 417 718 295 057 277
15	0.001 461 885 507 564 405 076 955 251 004 644 660 381 830 652 570 161 535 039 646
16	0.001 387 920 216 649 215 939 028 019 193 478 028 106 207 872 826 496 538 118 611
17	0.001 319 381 121 178 915 131 055 866 448 071 375 033 461 176 027 605 491 069 079
18	0.001 255 753 184 380 171 274 366 305 255 130 667 159 653 055 038 950 099 019 182
19	0.001 196 580 824 619 619 582 095 623 204 576 335 092 629 120 716 990 888 912 686
20	0.001 141 459 886 040 749 692 271 728 924 824 502 464 521 520 491 851 595 571 528
21	0.001 090 030 844 426 041 656 010 023 791 862 375 491 323 900 048 623 301 573 022
22	0.001 041 973 035 977 424 263 230 840 965 230 596 045 910 550 967 993 275 663 971

Table A62. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=19$

1	0.003 276 321 942 295 284 268 927 346 710 599 961 692 559 938 887 886 267 507 461
2	0.003 036 136 670 807 694 535 338 788 641 143 946 192 629 911 499 134 992 765 490
3	0.002 821 163 654 617 121 035 696 050 680 549 029 110 800 693 838 222 453 595 830
4	0.002 628 011 902 612 244 952 584 533 370 548 634 167 720 319 612 145 316 406 449
5	0.002 453 840 670 588 978 023 266 328 479 526 857 345 357 626 878 931 195 695 195
6	0.002 296 255 731 555 489 270 712 307 005 295 473 942 882 740 427 160 037 589 697
7	0.002 153 227 782 202 233 169 259 832 716 810 923 242 365 232 799 292 725 639 642
8	0.002 023 027 738 184 566 249 701 371 629 941 711 992 731 659 278 401 743 938 036
9	0.001 904 175 032 797 214 495 334 956 453 299 862 196 189 374 365 538 395 891 571
10	0.001 795 396 012 888 863 426 655 957 308 212 880 406 437 434 718 646 566 285 265
11	0.001 695 590 237 800 424 707 198 050 240 777 420 878 888 282 501 774 579 937 687
12	0.001 603 803 010 037 974 396 995 753 558 182 128 636 886 073 895 831 395 239 853
13	0.001 519 202 854 192 996 958 669 310 094 524 280 510 838 974 995 501 941 211 027
14	0.001 441 062 950 810 662 933 982 426 012 127 980 947 127 906 014 059 320 559 732
15	0.001 368 745 750 890 362 764 804 165 133 741 860 180 455 709 531 997 376 302 042
16	0.001 301 690 163 269 152 074 253 243 316 383 716 537 444 335 129 828 536 471 414
17	0.001 239 400 834 785 573 998 130 549 269 020 973 090 273 419 204 135 893 708 504
18	0.001 181 439 141 635 368 560 084 213 495 510 605 364 378 308 438 518 473 216 062
19	0.001 127 415 586 870 252 853 498 479 702 304 814 172 608 648 446 585 062 238 641
20	0.001 076 983 358 835 347 217 584 477 485 999 835 635 406 755 616 405 848 600 841
21	0.001 029 832 852 413 496 719 956 585 166 272 963 894 189 146 192 669 287 771 666

Table A63. Critical binding parameters μ_c (au^{-1}) for Hulthén potential, $l=20$

1	0.002 970 020 408 554 037 412 595 886 756 347 445 583 998 031 122 194 854 341 056
2	0.002 762 175 598 866 848 723 052 246 494 686 428 430 597 632 245 439 068 551 174
3	0.002 575 190 231 886 158 542 169 863 335 592 657 160 425 870 907 184 188 193 083
4	0.002 406 377 988 957 387 743 659 673 172 133 758 068 494 798 132 614 005 957 287
5	0.002 253 470 516 189 183 645 371 211 050 510 437 890 025 746 620 142 485 284 388
6	0.002 114 541 772 432 972 158 064 328 425 995 874 829 227 660 433 453 847 537 377
7	0.001 987 947 898 060 277 720 433 566 352 656 948 737 202 153 564 880 560 532 202
8	0.001 872 279 063 091 162 566 685 730 643 217 623 236 768 040 185 610 387 928 412
9	0.001 766 320 639 027 081 507 299 055 838 942 042 369 512 776 297 895 750 415 159
10	0.001 669 021 684 423 584 148 862 618 667 866 216 977 136 225 604 293 335 168 663
11	0.001 579 469 209 696 681 426 086 692 616 120 732 150 623 880 236 376 351 741 418
12	0.001 496 867 040 104 491 113 449 075 300 222 702 646 001 438 348 698 549 725 952
13	0.001 420 518 360 936 608 288 296 992 592 932 352 631 713 800 430 211 669 867 232
14	0.001 349 811 229 429 515 597 816 964 557 984 284 441 458 983 060 469 828 344 629
15	0.001 284 206 490 745 413 847 337 290 823 995 785 147 121 539 225 050 499 196 747
16	0.001 223 227 652 701 146 367 700 489 525 874 971 845 912 183 906 393 328 923 880
17	0.001 166 452 364 677 088 982 853 353 739 447 739 241 974 398 703 047 912 343 888
18	0.001 113 505 216 770 061 999 473 959 932 355 228 449 065 840 459 560 276 029 007
19	0.001 064 051 630 580 795 466 747 954 538 691 099 456 141 086 134 275 226 083 800
20	0.001 017 792 656 620 823 989 279 679 497 927 270 798 619 696 685 319 233 148 552

Appendix A.4. Pseudo-Hulthén Potential μ_c vs l for $\mathcal{D} = 1/\mu \leq 1000 \text{ au}$

Table A64. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=0$

1	2.000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
2	0.500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
3	0.222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
4	0.125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
5	0.080 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
6	0.055 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 556
7	0.040 816 326 530 612 244 897 959 183 673 469 387 755 102 040 816 326 530 612 245
8	0.031 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
9	0.024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691
10	0.020 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
11	0.016 528 925 619 834 710 743 801 652 892 561 983 471 074 380 165 289 256 198 347
12	0.013 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
13	0.011 834 319 526 627 218 934 911 242 603 550 295 857 988 165 680 473 372 781 065
14	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
15	0.008 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
16	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
17	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
18	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
19	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
20	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
21	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
22	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
23	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
24	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
25	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
26	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
27	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
28	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
29	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
30	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
31	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
32	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
33	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
34	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
35	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
36	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
37	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
38	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
39	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
40	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
41	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
42	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
43	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
44	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A67. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=3$

1	0.125 000
2	0.080 000
3	0.055 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555
4	0.040 816 326 530 612 244 897 959 183 673 469 387 755 102 040 816 326 530 612 245
5	0.031 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
6	0.024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691
7	0.020 000
8	0.016 528 925 619 834 710 743 801 652 892 561 983 471 074 380 165 289 256 198 347
9	0.013 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888
10	0.011 834 319 526 627 218 934 911 242 603 550 295 857 988 165 680 473 372 781 065
11	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
12	0.008 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888
13	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
14	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
15	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
16	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
17	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
18	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
19	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
20	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
21	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
22	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
23	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
24	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
25	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
26	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
27	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
28	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
29	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
30	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
31	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
32	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
33	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
34	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
35	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
36	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
37	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
38	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
39	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
40	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
41	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A68. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=4$

1	0.080 000
2	0.055 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555 555
3	0.040 816 326 530 612 244 897 959 183 673 469 387 755 102 040 816 326 530 612 245
4	0.031 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
5	0.024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691
6	0.020 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
7	0.016 528 925 619 834 710 743 801 652 892 561 983 471 074 380 165 289 256 198 347
8	0.013 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
9	0.011 834 319 526 627 218 934 911 242 603 550 295 857 988 165 680 473 372 781 065
10	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
11	0.008 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
12	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
13	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
14	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
15	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
16	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
17	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
18	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
19	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
20	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
21	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
22	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
23	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
24	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
25	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
26	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
27	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
28	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
29	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
30	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
31	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
32	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
33	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
34	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
35	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
36	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
37	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
38	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
39	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
40	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A69. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=5$

1	0.055 555
2	0.040 816 326 530 612 244 897 959 183 673 469 387 755 102 040 816 326 530 612 245
3	0.031 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
4	0.024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691
5	0.020 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
6	0.016 528 925 619 834 710 743 801 652 892 561 983 471 074 380 165 289 256 198 347
7	0.013 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888
8	0.011 834 319 526 627 218 934 911 242 603 550 295 857 988 165 680 473 372 781 065
9	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
10	0.008 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888
11	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
12	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
13	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
14	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
15	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
16	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
17	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
18	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
19	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
20	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
21	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
22	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
23	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
24	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
25	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
26	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
27	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
28	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
29	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
30	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
31	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
32	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
33	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
34	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
35	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
36	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
37	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
38	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
39	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A70. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=6$

1	0.040 816 326 530 612 244 897 959 183 673 469 387 755 102 040 816 326 530 612 245
2	0.031 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
3	0.024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691
4	0.020 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
5	0.016 528 925 619 834 710 743 801 652 892 561 983 471 074 380 165 289 256 198 347
6	0.013 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
7	0.011 834 319 526 627 218 934 911 242 603 550 295 857 988 165 680 473 372 781 065
8	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
9	0.008 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
10	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
11	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
12	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
13	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
14	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
15	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
16	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
17	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
18	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
19	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
20	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
21	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
22	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
23	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
24	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
25	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
26	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
27	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
28	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
29	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
30	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
31	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
32	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
33	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
34	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
35	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
36	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
37	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
38	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A71. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=7$

1	0.031 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
2	0.024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691
3	0.020 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
4	0.016 528 925 619 834 710 743 801 652 892 561 983 471 074 380 165 289 256 198 347
5	0.013 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
6	0.011 834 319 526 627 218 934 911 242 603 550 295 857 988 165 680 473 372 781 065
7	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
8	0.008 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
9	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
10	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
11	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
12	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
13	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
14	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
15	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
16	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
17	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
18	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
19	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
20	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
21	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
22	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
23	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
24	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
25	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
26	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
27	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
28	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
29	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
30	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
31	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
32	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
33	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
34	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
35	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
36	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
37	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A72. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=8$

1	0.024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691 358 024 691
2	0.020 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
3	0.016 528 925 619 834 710 743 801 652 892 561 983 471 074 380 165 289 256 198 347
4	0.013 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888
5	0.011 834 319 526 627 218 934 911 242 603 550 295 857 988 165 680 473 372 781 065
6	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
7	0.008 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888
8	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
9	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
10	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
11	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
12	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
13	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
14	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
15	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
16	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
17	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
18	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
19	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
20	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
21	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
22	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
23	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
24	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
25	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
26	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
27	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
28	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
29	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
30	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
31	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
32	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
33	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
34	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
35	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
36	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A73. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=9$

1	0.020 000
2	0.016 528 925 619 834 710 743 801 652 892 561 983 471 074 380 165 289 256 198 347
3	0.013 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
4	0.011 834 319 526 627 218 934 911 242 603 550 295 857 988 165 680 473 372 781 065
5	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
6	0.008 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
7	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
8	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
9	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
10	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
11	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
12	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
13	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
14	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
15	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
16	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
17	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
18	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
19	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
20	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
21	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
22	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
23	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
24	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
25	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
26	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
27	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
28	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
29	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
30	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
31	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
32	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
33	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
34	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
35	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A74. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=10$

1	0.016 528 925 619 834 710 743 801 652 892 561 983 471 074 380 165 289 256 198 347
2	0.013 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
3	0.011 834 319 526 627 218 934 911 242 603 550 295 857 988 165 680 473 372 781 065
4	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
5	0.008 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 889
6	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
7	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
8	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
9	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
10	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
11	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
12	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
13	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
14	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
15	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
16	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
17	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
18	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
19	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
20	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
21	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
22	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
23	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
24	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
25	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
26	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
27	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
28	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
29	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
30	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
31	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
32	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
33	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
34	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A76. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=12$

1	0.011 834 319 526 627 218 934 911 242 603 550 295 857 988 165 680 473 372 781 065
2	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
3	0.008 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888 888
4	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
5	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
6	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
7	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
8	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
9	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
10	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
11	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
12	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
13	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
14	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
15	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
16	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
17	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
18	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
19	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
20	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
21	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
22	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
23	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
24	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
25	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
26	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
27	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
28	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
29	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
30	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
31	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
32	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A77. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=13$

1	0.010 204 081 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061
2	0.008 888
3	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
4	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
5	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
6	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
7	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
8	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
9	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
10	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
11	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
12	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
13	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
14	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
15	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
16	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
17	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
18	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
19	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
20	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
21	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
22	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
23	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
24	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
25	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
26	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
27	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
28	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
29	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
30	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
31	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A79. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=15$

1	0.007 812 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
2	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
3	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
4	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
5	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
6	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
7	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
8	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
9	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
10	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
11	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
12	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
13	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
14	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
15	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
16	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
17	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
18	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
19	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
20	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
21	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
22	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
23	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
24	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
25	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
26	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
27	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
28	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
29	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A80. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=16$

1	0.006 920 415 224 913 494 809 688 581 314 878 892 733 564 013 840 830 449 826 990
2	0.006 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 172 839 506 173
3	0.005 540 166 204 986 149 584 487 534 626 038 781 163 434 903 047 091 412 742 382
4	0.005 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
5	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
6	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
7	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
8	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
9	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
10	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
11	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
12	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
13	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
14	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
15	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
16	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
17	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
18	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
19	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
20	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
21	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
22	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
23	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
24	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
25	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
26	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
27	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
28	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A81. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=17$

1	0.006	172	839	506	172	839	506	172	839	506	172	839	506	172	839	506	173			
2	0.005	540	166	204	986	149	584	487	534	626	038	781	163	434	903	047	091	412	742	382
3	0.005	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000
4	0.004	535	147	392	290	249	433	106	575	963	718	820	861	678	004	535	147	392	290	249
5	0.004	132	231	404	958	677	685	950	413	223	140	495	867	768	595	041	322	314	049	587
6	0.003	780	718	336	483	931	947	069	943	289	224	952	741	020	793	950	850	661	625	709
7	0.003	472	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222
8	0.003	200	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000
9	0.002	958	579	881	656	804	733	727	810	650	887	573	964	497	041	420	118	343	195	266
10	0.002	743	484	224	965	706	447	187	928	669	410	150	891	632	373	113	854	595	336	077
11	0.002	551	020	408	163	265	306	122	448	979	591	836	734	693	877	551	020	408	163	265
12	0.002	378	121	284	185	493	460	166	468	489	892	984	542	211	652	794	292	508	917	955
13	0.002	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222
14	0.002	081	165	452	653	485	952	133	194	588	969	823	100	936	524	453	694	068	678	460
15	0.001	953	125	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000
16	0.001	836	547	291	092	745	638	200	183	654	729	109	274	563	820	018	365	472	910	927
17	0.001	730	103	806	228	373	702	422	145	328	719	723	183	391	003	460	207	612	456	747
18	0.001	632	653	061	224	489	795	918	367	346	938	775	510	204	081	632	653	061	224	490
19	0.001	543	209	876	543	209	876	543	209	876	543	209	876	543	209	876	543	209	876	543
20	0.001	460	920	379	839	298	758	217	677	136	596	055	514	974	433	893	352	812	271	731
21	0.001	385	041	551	246	537	396	121	883	656	509	695	290	858	725	761	772	853	185	596
22	0.001	314	924	391	847	468	770	545	693	622	616	699	539	776	462	853	385	930	309	007
23	0.001	250	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000
24	0.001	189	767	995	240	928	019	036	287	923	854	848	304	580	606	781	677	572	873	290
25	0.001	133	786	848	072	562	358	276	643	990	929	705	215	419	501	133	786	848	072	562
26	0.001	081	665	765	278	528	934	559	221	200	648	999	459	167	117	360	735	532	720	389
27	0.001	033	057	851	239	669	421	487	603	305	785	123	966	942	148	760	330	578	512	397

Table A82. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=18$

1	0.005	540	166	204	986	149	584	487	534	626	038	781	163	434	903	047	091	412	742	382
2	0.005	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000
3	0.004	535	147	392	290	249	433	106	575	963	718	820	861	678	004	535	147	392	290	249
4	0.004	132	231	404	958	677	685	950	413	223	140	495	867	768	595	041	322	314	049	587
5	0.003	780	718	336	483	931	947	069	943	289	224	952	741	020	793	950	850	661	625	709
6	0.003	472	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222
7	0.003	200	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000
8	0.002	958	579	881	656	804	733	727	810	650	887	573	964	497	041	420	118	343	195	266
9	0.002	743	484	224	965	706	447	187	928	669	410	150	891	632	373	113	854	595	336	077
10	0.002	551	020	408	163	265	306	122	448	979	591	836	734	693	877	551	020	408	163	265
11	0.002	378	121	284	185	493	460	166	468	489	892	984	542	211	652	794	292	508	917	955
12	0.002	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222	222
13	0.002	081	165	452	653	485	952	133	194	588	969	823	100	936	524	453	694	068	678	460
14	0.001	953	125	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000
15	0.001	836	547	291	092	745	638	200	183	654	729	109	274	563	820	018	365	472	910	927
16	0.001	730	103	806	228	373	702	422	145	328	719	723	183	391	003	460	207	612	456	747
17	0.001	632	653	061	224	489	795	918	367	346	938	775	510	204	081	632	653	061	224	490
18	0.001	543	209	876	543	209	876	543	209	876	543	209	876	543	209	876	543	209	876	543
19	0.001	460	920	379	839	298	758	217	677	136	596	055	514	974	433	893	352	812	271	731
20	0.001	385	041	551	246	537	396	121	883	656	509	695	290	858	725	761	772	853	185	596
21	0.001	314	924	391	847	468	770	545	693	622	616	699	539	776	462	853	385	930	309	007
22	0.001	250	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000
23	0.001	189	767	995	240	928	019	036	287	923	854	848	304	580	606	781	677	572	873	290
24	0.001	133	786	848	072	562	358	276	643	990	929	705	215	419	501	133	786	848	072	562
25	0.001	081	665	765	278	528	934	559	221	200	648	999	459	167	117	360	735	532	720	389
26	0.001	033	057	851	239	669	421	487	603	305	785	123	966	942	148	760	330	578	512	397

Table A83. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=19$

1	0.005 000
2	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
3	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
4	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
5	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
6	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
7	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
8	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
9	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
10	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
11	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
12	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
13	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
14	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
15	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
16	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
17	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
18	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
19	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
20	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
21	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
22	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
23	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
24	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
25	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

Table A84. Critical binding parameters μ_c (au^{-1}) for PseudoHulthén potential, $l=20$

1	0.004 535 147 392 290 249 433 106 575 963 718 820 861 678 004 535 147 392 290 249
2	0.004 132 231 404 958 677 685 950 413 223 140 495 867 768 595 041 322 314 049 587
3	0.003 780 718 336 483 931 947 069 943 289 224 952 741 020 793 950 850 661 625 709
4	0.003 472 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
5	0.003 200 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
6	0.002 958 579 881 656 804 733 727 810 650 887 573 964 497 041 420 118 343 195 266
7	0.002 743 484 224 965 706 447 187 928 669 410 150 891 632 373 113 854 595 336 077
8	0.002 551 020 408 163 265 306 122 448 979 591 836 734 693 877 551 020 408 163 265
9	0.002 378 121 284 185 493 460 166 468 489 892 984 542 211 652 794 292 508 917 955
10	0.002 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222 222
11	0.002 081 165 452 653 485 952 133 194 588 969 823 100 936 524 453 694 068 678 460
12	0.001 953 125 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
13	0.001 836 547 291 092 745 638 200 183 654 729 109 274 563 820 018 365 472 910 927
14	0.001 730 103 806 228 373 702 422 145 328 719 723 183 391 003 460 207 612 456 747
15	0.001 632 653 061 224 489 795 918 367 346 938 775 510 204 081 632 653 061 224 490
16	0.001 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543 209 876 543
17	0.001 460 920 379 839 298 758 217 677 136 596 055 514 974 433 893 352 812 271 731
18	0.001 385 041 551 246 537 396 121 883 656 509 695 290 858 725 761 772 853 185 596
19	0.001 314 924 391 847 468 770 545 693 622 616 699 539 776 462 853 385 930 309 007
20	0.001 250 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
21	0.001 189 767 995 240 928 019 036 287 923 854 848 304 580 606 781 677 572 873 290
22	0.001 133 786 848 072 562 358 276 643 990 929 705 215 419 501 133 786 848 072 562
23	0.001 081 665 765 278 528 934 559 221 200 648 999 459 167 117 360 735 532 720 389
24	0.001 033 057 851 239 669 421 487 603 305 785 123 966 942 148 760 330 578 512 397

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