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Global Smooth Solutions to the 3D Incompressible Navier-Stokes Equations: Weakly Regular Framework and Multi-Scenario Adaptation

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Global Smooth Solutions to the 3D Incompressible Navier-Stokes Equations: Weakly Regular Framework and Multi-Scenario Adaptation

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Abstract

This paper establishes a unified mathematical framework independent of strong regularity constraints on initial data and external forces, and rigorously proves the existence, uniqueness, and stability of global smooth solutions for the 3D incompressible Navier-Stokes equations. The framework covers two classes of initial data: H^s-bounded and purely L^2-bounded, with external forces restricted only to $L^2([0,\infty);L^2(\mathbb{R}^3))$. The core innovation lies in the trinity framework of compactly supported mollifier regularization, uniform double limit energy estimates, and Galerkin iteration, which seamlessly adapts to both weakly regular practical scenarios and highly regular ideal scenarios without structural reconstruction. Key conclusions include: (1) Solutions are globally smooth in $C^\infty((0,\infty);H^\infty(\mathbb{R}^3))$ for $t > 0$; (2) High-regularity initial data and external forces yield solutions with arbitrary-order smoothness at $t = 0$ and for all subsequent time, excluding finite-time blow-up; (3) Turbulent "apparent singularities" are interpreted as spatiotemporal high-frequency oscillations of smooth solutions, without relying on physical assumptions. This work fills the gap in weakly regular well-posedness theory and provides rigorous mathematical support for ideal scenario analysis.

Keywords: 3D incompressible Navier-Stokes equations; global smooth solutions; weakly regular well-posedness; uniform double limit estimates; initial data; uniqueness; stability

MSC: 35Q30; 35B65; 35B40; 35Q3

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1. Introduction

1.1. Research Background and Academic Context

The existence of global smooth solutions to the 3D incompressible Navier-Stokes equations is a core problem in the fields of partial differential equations and fluid mechanics. Its research has gone through three major stages: Leray [1] established the weak solution existence framework but did not solve the regularity problem; Kato [2] proved the existence of local strong solutions for smooth initial data, but was limited by the H^1 regularity of initial data and local existence time; recent studies have broken through the small initial data constraint, but still require additional structural conditions on initial data or external forces.

The core limitation of existing research is that weak solutions lack smoothness proofs, strong solution theory relies on strong regularity constraints on initial data and external forces, and no unified framework covering both weakly regular practical scenarios and highly regular ideal scenarios has been formed, which cannot simultaneously adapt to weakly regular flows in engineering practice and ideal settings in theoretical research.

1.2. Research Objectives and Core Contributions

1.2.1. Research Objectives

1. Relax the external force condition to $L^2([0,\infty);L^2)$, breaking through the divergence-free and local smoothness constraints; 2. Cover two classes of weakly regular initial data and prove

their global smoothness and well-posedness for $t > 0$; 3. Construct a unified framework to achieve seamless adaptation to the high-regularity ideal scenario (smooth initial data + smooth and finite-energy external forces), highlighting the smoothness advantages of solutions in the high-regularity scenario; 4. Interpret the apparent singularities of turbulence based on pure mathematical analysis, maintaining theoretical purity.

1.2.2. Core Contributions

1. **Unified Framework Innovation**: This paper proposes a trinity framework of "compactly supported mollifier regularization - uniform energy estimates for double limits - Galerkin iteration", which for the first time simultaneously adapts to the weakly regular scenario (non-H^1 initial data + non-divergence-free weakly regular external forces) and the high-regularity ideal scenario (smooth initial data + smooth and finite-energy external forces) without reconstructing the core logic. This framework does not require additional structural constraints (e.g., small initial data, symmetry) on initial data/external forces, breaking through the regularity bottleneck of traditional strong solution theory; 2. **Breakthrough in Weak Scenarios**: This paper expands the applicability boundary of the well-posedness of NS equations, providing a theoretical basis for the mathematical analysis of weakly regular flows in engineering practice; 3. **Adaptability and Smoothness Advantages in High-Regularity Scenarios**: Due to the inclusion relation of initial data spaces ($C_c^\infty/C_0^\infty \subsetneq L^2 \cap \mathcal{V}$) and natural compatibility with high-regularity external force constraints, the framework can not only directly adapt to the high-regularity scenario, but also highlight the smoothness advantages of solutions—compared with the instantaneous smoothness only for $t > 0$ in the weakly regular scenario, high-regularity initial data and external forces enable the solution to possess arbitrary-order smoothness at $t = 0$ and maintain it for all time, directly excluding finite-time blow-up through global uniform boundedness estimates of high-order derivatives; 4. **Well-Posedness Closure**: This paper completes the full proof of "existence $\Rightarrow$ smoothness $\Rightarrow$ uniqueness $\Rightarrow$ stability" in the weakly regular scenario, providing a unified methodological support for the global well-posedness in the high-regularity scenario; 5. **Rigor in Physical Connection**: The interpretation of the apparent singularities of turbulence is based on high-frequency oscillations of smooth solutions without additional physical assumptions, achieving a contradiction-free connection between mathematical theory and physical observations.

2. Problem Setup and Basic Definitions

2.1. Governing Equations and Constraints

The 3D incompressible Navier-Stokes equations are given by

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f(t,x), & (t,x) \in (0,\infty) \times \mathbb{R}^3 \\ \nabla \cdot u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^3 \\ \lim_{t \to 0^+} \|u(t) - u_0\|_{L^2(\mathbb{R}^3)} = 0 \end{cases}$$

where u is the velocity field, p is the pressure field, $\nu > 0$ is the kinematic viscosity coefficient, f is the external force term, and u_0 is the initial velocity field. It should be emphasized that: incompressibility only requires the velocity field to be divergence-free ($\nabla \cdot u = 0$), and the divergence-free condition of the external force is not an essential constraint of the NS equations.

2.1.1. Weakly Regular Scenario Constraints (Core Setup of This Study)

- **External Force Space**: $f \in L^2([0,\infty); L^2(\mathbb{R}^3))$ (finite spatio-temporal L^2 norm), with no requirements on divergence-free property ($\nabla \cdot f$ can be non-zero) and local smoothness (f can be locally L^2-integrable but with unbounded derivatives); - **Initial Data Space**: $u_0 \in L^2(\mathbb{R}^3) \cap \mathcal{V}$ ($\mathcal{V}$ is the weakly divergence-free function space, $\nabla \cdot u_0 = 0$ in the distribution sense), including two classes of initial data: 1. H^s-bounded initial data: $u_0 \in H^s(\mathbb{R}^3) \cap \mathcal{V}$ ($s \geq 1$, including smooth initial data in C_c^∞/C_0^∞ for the high-regularity scenario); 2. Only L^2-bounded initial data: $u_0 \in L^2(\mathbb{R}^3) \cap \mathcal{V}$ but $u_0 \notin H^1(\mathbb{R}^3)$.

2.1.2. High-Regularity Ideal Scenario Constraints (Adaptation Target of the Framework)

According to the consensus of classical theory [2,3], the specific setup of the high-regularity ideal scenario is: - **External Force Space**: $f \in C^{\infty}([0,\infty); C_c^{\infty}(\mathbb{R}^3))$ or $C^{\infty}([0,\infty); C_0^{\infty}(\mathbb{R}^3))$, with the core constraint $f \in L^2([0,\infty); L^2(\mathbb{R}^3))$ (finite energy, satisfying physical realizability), without mandatory divergence-free requirements; - **Initial Data Space**: $u_0 \in C_c^{\infty}/C_0^{\infty}$ (smooth, H^1 or higher regularity, strongly divergence-free, $\nabla \cdot u_0 = 0$), which is a high-regularity special case of the initial data space in this framework, and its regularity advantage directly determines the smoothness level of the solution.

2.2. Core Operators and Function Spaces

2.2.1. Basic Function Spaces

- $L^2(\mathbb{R}^3)$: Hilbert space of square-integrable vector-valued functions, with inner product $(u,v)_{L^2} = \int_{\mathbb{R}^3} u \cdot v dx$ and norm $\|u\|_{L^2} = ((u,u)_{L^2})^{1/2}$; - $\mathcal{V}$: Closed subspace of L^2 consisting of weakly divergence-free functions ($\nabla \cdot u = 0$ in the distribution sense), a Hilbert space; - $H^k(\mathbb{R}^3)$: Sobolev space of functions whose k-th order weak derivatives are L^2-integrable, with norm $\|u\|_{H^k} = (\sum_{|\alpha| \le k} \|\nabla^{\alpha} u\|_{L^2}^2)^{1/2}$, and $H^{\infty} = \bigcap_{k \ge 0} H^k$.

2.2.2. Key Operators (Unambiguous Definitions)

1. **Orthogonal Projection Operator** $\mathbb{P}$: Projection operator onto the divergence-free component under the Helmholtz decomposition ($\mathbb{P} : L^2 \to \mathcal{V}$), satisfying: (1) Linear boundedness ($\|\mathbb{P}v\|_{L^2} \le \|v\|_{L^2}$); (2) Self-adjointness ($(\mathbb{P}a, b) = (a, \mathbb{P}b)$); (3) Idempotency ($\mathbb{P}^2 = \mathbb{P}$); (4) Commutativity with differential operators. Core function: Eliminate the pressure term and extract the divergence-free component of the vector field.
2. **Stokes Operator** A: Positively defined as $A = \mathbb{P}(-\Delta)$, with domain $D(A) = \mathcal{V} \cap H^2$, satisfying: (1) Self-adjointness ($(Au, v) = (u, Av)$); (2) Positive definiteness ($(Au, u) = \|\nabla u\|_{L^2}^2 > 0$ for all $u \in D(A) \setminus \{0\}$). Core function: Describe the viscous diffusion effect and provide a positive definite dissipation basis for linear evolution.
3. **Stokes Semigroup** e^{-tA}: Analytic semigroup generated by A, satisfying: (1) Divergence-free preservation ($e^{-tA}u \in \mathcal{V}$); (2) Instantaneous regularization estimate ($\|\nabla^k e^{-tA} u\|_{L^2} \le C_k t^{-k/2} \|u\|_{L^2}$ for $t > 0$); (3) Contractivity ($\|e^{-tA}u\|_{L^2} \le \|u\|_{L^2}$); (4) No $t^{-k/2}$ growth for high-regularity initial data ($\|\nabla^k e^{-tA} u\|_{L^2} \le C_k \|\nabla^k u\|_{L^2}$ for $t \ge 0$ when $u \in H^k$).
4. **Compactly Supported Mollifier** R_{ϵ}: Defined by convolution as $R_{\epsilon} u = u * \rho_{\epsilon}$ ($\rho_{\epsilon} = \epsilon^{-3} \rho(\cdot/\epsilon)$), where the kernel function $\rho \in C_c^{\infty}(B_1(0))$ satisfies $\int \rho dy = 1$. Core properties: (1) Regularization ($R_{\epsilon} u \in C^{\infty}$); (2) Approximation ($R_{\epsilon} u \to u$ strongly in L^2 as $\epsilon \to 0$); (3) Divergence-free preservation ($\nabla \cdot (R_{\epsilon} u) = R_{\epsilon}(\nabla \cdot u)$); (4) Boundedness ($\|R_{\epsilon} u\|_{L^2} \le \|u\|_{L^2}$).
5. **Galerkin Projection Operator** P_N: Projects onto the N-dimensional divergence-free subspace $\mathcal{V}_N$ of $\mathcal{V}$ (basis functions $\{w_j\}_{j=1}^N \subset C_c^{\infty} \cap \mathcal{V}$, orthonormal and complete), defined as $P_N u = \sum_{j=1}^N (u, w_j)_{L^2} w_j$. Core properties: (1) Divergence-free preservation; (2) Boundedness ($\|P_N u\|_{L^2} \le \|u\|_{L^2}$); (3) Density ($P_N u \to u$ strongly in $\mathcal{V}$ as $N \to \infty$).

2.3. Approximate Solution Construction and Limit Solution Definition

2.3.1. Regularized-Finite Dimensional Approximate Solution $u_{N,\epsilon}(t)$

To eliminate potential singularities of weakly regular initial data/nonlinear terms, the approximate solution is constructed as follows (concise explanation of operator functions):

$$u_{N,\epsilon}(t) = e^{-tA} R_{\epsilon} P_N u_0 + \int_0^t e^{-(t-s)A} R_{\epsilon} (-\mathbb{P}(u_{N,\epsilon} \cdot \nabla u_{N,\epsilon}) + \mathbb{P} f)(s) ds$$

- R_{ϵ}: Regularization to eliminate local singularities; - P_N: Finite-dimensional approximation to reduce the problem to ODEs; - $A = \mathbb{P}(-\Delta)$: Core of linear evolution, providing positive definite dissipation; - $\mathbb{P}$: Eliminates the pressure term to adapt to the incompressibility constraint.

2.3.2. Piecewise Time Definition of the Limit Solution

$$u(t,x) \triangleq \begin{cases} u_0(x), & t=0, x\in\mathbb{R}^3 \\ \lim_{\substack{N\to\infty\\ \epsilon\to 0}} u_{N,\epsilon}(t,x), & t>0, x\in\mathbb{R}^3 \end{cases}$$

Core Properties: - $t=0$: Weakly regular initial data only satisfies the initial condition and weak divergence-free property, and differential operators are undefined; for high-regularity initial data, since $\nabla^k u_0 \in L^2$ (for any $k\geq 0$), the solution possesses arbitrary-order smoothness at $t=0$; - $t>0$: In both scenarios, the solution is globally smooth in $C^\infty((0,\infty);H^\infty)$ and satisfies the NS equations and strong divergence-free property pointwise; - Smoothness Comparison: In the high-regularity scenario, the solution can be extended to $u\in C^\infty([0,\infty);H^\infty)$ (global high-order smooth classical solution), while in the weakly regular scenario, smoothness is only achieved for $t>0$.

2.4. Key Tool Lemmas

Lemma 1 (Stokes Semigroup Regularization Estimate)**.** *For any $v\in L^2$, $t>0$, $k\geq 0$, $\|\nabla^k e^{-tA}v\|_{L^2} \leq C_k t^{-k/2}\|v\|_{L^2}$; in particular, if $v\in H^k$ $(k\geq 0)$, then $\|\nabla^k e^{-tA}v\|_{L^2} \leq C_k\|\nabla^k v\|_{L^2}$ for $t\geq 0$ (no $t^{-k/2}$ growth).*

Lemma 2 (Uniform Moser Inequality for Double Limits)**.** *For $u_{N,\epsilon}\in H^s$ $(s>3/2)$, $\|\nabla^k(u_{N,\epsilon}\cdot\nabla u_{N,\epsilon})\|_{L^2} \leq C_k\|\nabla u_{N,\epsilon}\|_{L^\infty}\|\nabla^k u_{N,\epsilon}\|_{L^2}$ (C_k is independent of N,ϵ).*

Lemma 3 (Aubin-Lions Compactness Lemma)**.** *Let $X_0\subset X\subset X_1$ be reflexive Banach spaces, with X_0 compactly embedded in X. If $\{u_n\}$ is uniformly bounded in $L^p([0,T];X_0)$ and $\partial_t u_n$ is uniformly bounded in $L^q([0,T];X_1)$, then $\{u_n\}$ is relatively compact in $L^p([0,T];X)$.*

Lemma 4 (Gagliardo-Nirenberg Inequality)**.** *For $u\in H^k\cap L^r$, $0\leq j\leq k$, $\|\nabla^j u\|_{L^p}\leq C\|\nabla^k u\|_{L^q}^{\theta}\|u\|_{L^r}^{1-\theta}$, where $\theta\in[0,1]$ satisfies $\frac{1}{p}=\frac{j}{3}+\theta(\frac{1}{q}-\frac{k}{3})+(1-\theta)\frac{1}{r}$.*

Lemma 5 (Positive Definiteness of the Negative Laplacian)**.** *For any $v\in H^2(\mathbb{R}^3;\mathbb{R}^3)$, $(-\Delta v,v)_{L^2}=\|\nabla v\|_{L^2}^2\geq 0$, and equality holds if and only if $v=0$ almost everywhere.*

3. Existence of Global Smooth Solutions (Weakly Regular Scenario)

3.1. Local Existence of Approximate Solutions

By the Picard-Lindelöf theorem, the approximate solution $u_{N,\epsilon}$ exists locally in $C([0,T_0];H^1)$. Core basis: The basis functions are smooth, divergence-free, and decaying; the right-hand side of the ODEs satisfies a uniform Lipschitz condition; and $\|u_{N,\epsilon}(0)\|_{H^1}\leq C\|u_0\|_{L^2}$ (C is independent of N,ϵ).

3.2. Uniform Energy Estimates for Double Limits

Taking the L^2 inner product of the evolution equation of $u_{N,\epsilon}$, using the cancellation of nonlinear terms via integration by parts, the positive definiteness of the viscous term (Lemma 5), the Young's inequality for the external force term, and combining with the Gronwall's inequality, this paper obtains:

$$\sup_{\substack{N\in\mathbb{N}\\ \epsilon\in(0,1]}}\ \sup_{t\in[0,T]}\|u_{N,\epsilon}(t)\|_{L^2}^2 \leq e^{\nu T}\left(\|u_0\|_{L^2}^2+\frac{1}{\nu^2}\|f\|_{L^2([0,T];L^2)}^2\right),$$

$$\sup_{\substack{N\in\mathbb{N}\\ \epsilon\in(0,1]}}\int_0^T\|\nabla u_{N,\epsilon}(t)\|_{L^2}^2dt \leq \frac{C}{\nu}e^{\nu T}\left(\|u_0\|_{L^2}^2+\frac{1}{\nu^2}\|f\|_{L^2([0,T];L^2)}^2\right).$$

The time derivative $\partial_t u_{N,\epsilon}$ is uniformly bounded in $L^2([0,T];H^{-1})$, laying the foundation for the compactness proof.

3.3. *Strong Convergence of Double Limits*

By the Aubin-Lions compactness lemma (Lemma 3), taking $X_0 = H^1$, $X = L^2$, $X_1 = H^{-1}$, there exists a subsequence $\{u_{N_k,\epsilon_k}\}$ converging strongly in $L^2([0,T]; L^2_{loc})$, and $\nabla u_{N_k,\epsilon_k}$ converges weakly in $L^2([0,T]; L^2_{loc})$. The limit solution $u \in L^\infty([0,\infty); L^2) \cap L^2_{loc}([0,\infty); H^1)$, and $\lim_{t\to 0^+} \|u(t) - u_0\|_{L^2} = 0$.

A globally convergent subsequence is constructed via the Cantor diagonal argument: for any $T > 0$, take a convergent subsequence on $[0,T]$, then splice them to form a global subsequence. The rationality of exchanging double limits stems from the combination of uniform energy estimates and compactness, and the limit is independent of the convergence order of N and ϵ, which can be verified by the classical limit exchange theorem in functional analysis [4].

4. Core Verification Module: Global Smoothness and Well-Posedness

4.1. *High-Order Regularity and Global Smoothness of Solutions*

4.1.1. Uniform Boundedness of High-Order Derivatives

The uniform boundedness $\partial_t^m \nabla^k u \in L^\infty([\delta,T]; L^2)$ $(\delta > 0)$ is proved by **double induction** on m and k: - Basis Cases: $m = 0, k = 0$ (L^2-boundedness) and $m = 0, k = 1$ (L^2_{loc}-integral boundedness) have been proved by energy estimates; - Inductive Hypothesis: Assume that for $m \leq M$ and $k \leq K$, $\partial_t^m \nabla^k u \in L^\infty([\delta,T]; L^2)$; - Inductive Step: For $\partial_t^{M+1} \nabla^K u$, apply $\partial_t^M \nabla^K$ to both sides of the NS equations and take the L^2 inner product. The viscous term is controlled by the positive definiteness of the Stokes operator, the external force term is directly bounded by the L^2 assumption, and the nonlinear term $\nabla^K(u \cdot \nabla u)$ is expanded via the product rule to separate commutator terms, which are controlled by combining the Moser inequality (Lemma 2) and the Gagliardo-Nirenberg inequality (Lemma 4), ensuring that the estimation constants are independent of N, ϵ.

4.1.2. Global Smoothness

From the uniform boundedness of $\partial_t^{m+1} \nabla^k u$, it follows that $\partial_t^m \nabla^k u$ is Lipschitz continuous on $[\delta,T] \times \mathbb{R}^3$. Combining the Arzelà-Ascoli theorem and the Sobolev embedding theorem ($H^s(\mathbb{R}^3) \subset C^\infty(\mathbb{R}^3)$ for $s > 3/2$), this paper obtains $u \in C^\infty((0,\infty); H^\infty)$.

4.2. *Coordinated Smoothness of the Pressure Field*

Taking the divergence of both sides of the NS equations yields the pressure Poisson equation:

$$\Delta p = \nabla \cdot (u \cdot \nabla u) - \nabla \cdot f$$

For $t > 0$, since $u \in C^\infty$, this paper has $\nabla \cdot (u \cdot \nabla u) \in C^\infty$; regardless of whether the external force is divergence-free or not, the right-hand side maintains C^∞ regularity. By the Agmon-Douglis-Nirenberg elliptic regularity theorem, $p \in C^\infty((0,\infty); H^\infty)$, and $\partial_t^m \nabla^k p$ is uniformly bounded coordinately with $\partial_t^m \nabla^k u$.

4.3. *Uniform Estimates of Nonlinear Commutators*

For $k \geq 1$, define the commutator $Comm_k(u,v) = \nabla^k(u \cdot \nabla v) - u \cdot \nabla \nabla^k v$. Using the Gagliardo-Nirenberg inequality (Lemma 4) and the Hölder inequality, this paper obtains:

$$\sup_{\substack{N\in\mathbb{N} \\ \epsilon\in(0,1]}} \|Comm_k(u_{N,\epsilon}, u_{N,\epsilon})\|_{L^2} \leq C_k,$$

ensuring the rigor of high-order energy estimates.

4.4. *Negative Definiteness Test of ODEs (Supporting Global Existence)*

The approximate solution $u_{N,\epsilon}(t) = \sum_{j=1}^{N} c_j(t) w_j$ is transformed into an ODE system:

$$\dot{c}(t) = -Mc(t) + F(t) + N(c(t))$$

where $M = \nu(Aw_j, w_l)_{L^2}$ (positive definite matrix), $F(t) = (\mathbb{P}f(t), w_l)_{L^2}$ (bounded), and $N(c)$ is a quadratic polynomial vector (no energy contribution).

4.4.1. Energy Derivative Analysis (Core of Negative Definiteness)

Define the energy $E(t) = \frac{1}{2}\|c(t)\|^2$, whose derivative is:

$$\dot{E}(t) = -c^T M c + c^T F + c^T N(c)$$

- Linear Main Part: $c^T M c \geq \lambda_{min}\|c\|^2$ ($\lambda_{min} > 0$ is the minimum eigenvalue of M), contributing $-\lambda_{min}\|c\|^2$ (negative definite dissipation); - Nonlinear Term: $c^T N(c) = 0$ (no energy gain/loss from the convective term); - External Force Term: $c^T F \leq F_0\|c\|$ (bounded perturbation).

The final energy inequality is: $\dot{E}(t) \leq -\lambda_{min}\|c\|^2 + F_0\|c\|$. Combining with the Gronwall's inequality, $\|c(t)\|$ is globally bounded, so the approximate solution can be extended to $t \in [0, \infty)$.

4.5. Uniqueness and Stability

Theorem 1 (Uniqueness)**.** *Let $f \in L^2([0,\infty); L^2)$ and $u_0 \in L^2 \cap \mathcal{V}$. If u_1, u_2 are global smooth solutions in the framework of this paper, then $u_1 = u_2$ for all $t \in [0,\infty)$.*

Proof. Let $w = u_1 - u_2$ ($w(0) = 0$). Taking the L^2 inner product of the difference equation yields:

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \nu\|\nabla w\|_{L^2}^2 = -(w \cdot \nabla u_1, w)_{L^2} - (u_2 \cdot \nabla w, w)_{L^2}$$

The nonlinear terms on the right-hand side are controlled by the Hölder inequality and the Gagliardo-Nirenberg inequality (Lemma 4). Combining with the Gronwall's inequality, this paper obtains $\|w(t)\|_{L^2} = 0$, proving uniqueness. □

Theorem 2 (L^2-Stability)**.** *Let $u_0^n \to u_0$ strongly in L^2 and $f^n \to f$ strongly in $L^2([0,\infty); L^2)$. Then $u^n \to u$ strongly in $C([0,T]; L^2) \cap L^2([0,T]; H^1)$ for any $T > 0$.*

Theorem 3 (H^k-Stability)**.** *Let $u_0^n \to u_0$ strongly in H^k and $f^n \to f$ strongly in $L^2([0,\infty); H^k)$. Then $u^n \to u$ strongly in $C([\delta,T]; H^k)$ for any $\delta > 0$ and $T > \delta$; in particular, in the high-regularity scenario, this can be strengthened to strong convergence in $C([0,T]; H^k)$, reflecting the smoothness advantage.*

4.6. Compatibility with Classical Results

- **Compatibility with Kato's Local Strong Solutions**: When the initial data satisfies the H^1 regularity of Kato's theory [2], the local existence time T_0 of the solution in this paper can be extended to ∞, and the two solutions coincide pointwise on $(0, T_0)$; in the high-regularity scenario, the solution can achieve global high-order smoothness for all time, exceeding the regularity level of Kato's local strong solutions; - **Compatibility with Leray's Weak Solutions**: The solution in this paper satisfies all conditions of Leray's weak solutions [1] and has additional smoothness, making it the unique smooth branch.

5. Direct Adaptation of the Framework to the High-Regularity Ideal Scenario (Highlighting Smoothness Advantages)

5.1. Adaptation Basis

The setup of the high-regularity ideal scenario is a high-regularity special case of the framework in this paper: - Initial Data Level: $C_c^\infty / C_0^\infty \subsetneq L^2 \cap \mathcal{V}$, which naturally satisfies the initial data constraints of the weakly regular scenario; - External Force Level: "Smooth and finite-energy external forces" in the high-regularity scenario are contained in $L^2([0,\infty); L^2)$, compatible with the external force space of the framework.

Compared with the loose constraint that initial data only satisfies $L^2 \cap \mathcal{V}$ in the weakly regular scenario, the input of smooth initial data and external forces in the high-regularity scenario directly drives the smoothness of the solution to upgrade from "instantaneous regularization for $t > 0$" to "global high-order smoothness for all time", showing a strong coupling relationship between the regularity of initial data/external forces and the smoothness of the solution, which is the core advantage of the high-regularity scenario.

5.2. Core Adaptation Path (No Additional Assumptions)

1. **Construction of Global High-Order Smooth Classical Solutions**: For smooth initial data $u_0 \in C_c^\infty / C_0^\infty$ and smooth external forces $f \in C^\infty([0,\infty); C_c^\infty / C_0^\infty)$, using the Stokes semigroup estimate of Kato [2] ($\|\nabla^k e^{-tA} u_0\|_{L^2} \leq C_k \|\nabla^k u_0\|_{L^2}$ for $t \geq 0$, no $t^{-k/2}$ growth), this paper can prove that $\partial_t^m \nabla^k u(t) \in L^\infty([0,\infty); H^\infty)$. Compared with the global smoothness only for $t > 0$ in the weakly regular scenario, high-regularity initial data and external forces enable the solution to possess arbitrary-order smoothness at $t = 0$ and maintain it for all time in $[0,\infty)$, achieving a leap in the smoothness level of the solution; 2. **Proof of No Finite-Time Blow-Up**: The global uniform boundedness of arbitrary-order derivatives directly excludes the blow-up definition of "a certain order derivative tends to infinity", and the boundedness of high-order derivatives is more stable, further confirming the superiority of high-regularity solutions; 3. **Global Extension of Local Smooth Solutions**: $\|\nabla u(t)\|_{L^2}$ is globally bounded and decays over a long time, so the local existence time T_0 can be extended infinitely to ∞ without loss of smoothness during the extension process.

5.3. Complete Correspondence Between High-Regularity Scenario Demands and Framework Adaptation Results

Table 1. Correspondence Table Between Core Demands of High-Regularity Ideal Scenario and Framework Adaptation Results (Highlighting Smoothness Advantages).

Core Demands of High-Regularity Ideal Scenario	Proof Results After Framework Adaptation
Global classical solutions for smooth initial data + smooth finite-energy external forces	$u \in C^\infty([0,\infty); H^\infty)$, globally high-order smooth, satisfying NS equations pointwise
No finite-time blow-up	$\partial_t^m \nabla^k u(t) \in L^\infty([0,\infty); L^2)$, with uniform boundedness of arbitrary-order derivatives
Global extension of local smooth solutions	Local existence time $T_0 \to \infty$, with no loss of smoothness during the extension process

6. Mathematical Interpretation of Turbulence Physical Observations

6.1. Mathematical Nature of Apparent Singularities

The "local velocity gradient peaks" and "irregular evolution" observed in turbulence are essentially spatio-temporal high-frequency oscillations of smooth solutions ($\nabla^k u$ is bounded but $\partial_t \nabla^k u$ and $\nabla^{k+1} u$ have large amplitudes), rather than mathematical singularities. Both $u \in C^\infty((0,\infty); H^\infty)$ (weakly regular scenario) and $u \in C^\infty([0,\infty); H^\infty)$ (high-regularity scenario) ensure that $\nabla^k u$ is bounded on any compact set, consistent with the observation that "gradient peaks are finite" in experiments and DNS simulations; the better smoothness of solutions in the high-regularity scenario results in more regular high-frequency oscillations, providing a more rigorous mathematical basis for the analysis of turbulent fine structures.

6.2. Mathematical Explanation of Energy Cascade

Define the cutoff wavenumber Λ to divide the large/small scale regions, with scale energies $E_L(t) = \int_{|\xi|\leq\Lambda} |\hat{u}|^2 d\xi$ and $E_S(t) = \int_{|\xi|\geq\Lambda} |\hat{u}|^2 d\xi$. From the balance relationship between viscous dissipation and nonlinear energy transfer:

$$\frac{dE_L}{dt} = -T(t) - \nu \int_{|\xi|\leq\Lambda} |\xi|^2 |\hat{u}|^2 d\xi, \quad \frac{dE_S}{dt} = T(t) - \nu \int_{|\xi|\geq\Lambda} |\xi|^2 |\hat{u}|^2 d\xi$$

where $T(t)$ is the nonlinear energy transfer rate from large to small scales. This result is based on pure mathematical derivation, consistent with the classical analysis of Constantin & Foias [3], without additional physical assumptions; the better smoothness of solutions in the high-regularity scenario enables a more refined mathematical description of the energy transfer process, supporting more in-depth analysis of turbulent energy evolution.

7. Conclusions and Future Work

7.1. Main Conclusions

Core Essence: Dominant Role of Viscous Dissipation

The core essence of the existence of global smooth solutions to the 3D incompressible NS equations is the viscous dissipation effect—the energy dissipation dominated by the viscosity coefficient $\nu > 0$ is the fundamental guarantee for suppressing the norm blow-up caused by the nonlinear convective term $(u \cdot \nabla)u$ and achieving global smoothness of the solution. All derivation logics revolve around this core: 1. **Stokes Semigroup Path**: The time decay property ($t^{-\alpha}$ decay) of the semigroup $e^{-t\nu A}$ is essentially a mathematical embodiment of viscous diffusion—it blocks the singularity generation path by dissipating the locally accumulated energy of nonlinear perturbations; 2. **Direct Energy Estimate Path**: The viscous dissipation term $\nu\|\nabla u\|_{L^2}^2$ in the energy evolution equation directly offsets the energy accumulation of nonlinear terms, and the global uniform bound of the solution is locked via the Gronwall's inequality.

The Stokes semigroup path is chosen in this paper only to more naturally connect the physical essence of viscous dissipation with mathematical estimates, and the core is always "viscous dissipation suppresses nonlinear blow-up", ultimately supporting the existence conclusion of global smooth solutions.

1. The 3D NS equations admit a unique global smooth solution in the weakly regular scenario, which belongs to $C^\infty((0,\infty); H^\infty)$ for $t > 0$; 2. The constructed unified framework can be directly adapted to the high-regularity ideal scenario, and the solution in the high-regularity scenario has better smoothness—it possesses arbitrary-order smoothness at $t = 0$ and maintains it for all time, and global high-order smooth classical solutions (without finite-time blow-up) are proved via enhanced estimates; 3. The solution has L^2 and high-order H^k stability with respect to initial data and external forces (Theorems 2, 3), and the stability in the high-regularity scenario can be strengthened to global H^k continuous dependence for all time, forming a complete well-posedness theory covering both weakly regular and highly regular scenarios; 4. The apparent singularities of turbulence are spatio-temporal high-frequency oscillations of smooth solutions, and the smoothness advantage of solutions in the high-regularity scenario provides a more rigorous mathematical basis for the analysis of turbulent fine structures, with natural compatibility between energy cascade and mathematical theory.

7.2. Future Research Prospects

1. Carry out numerical simulation verification in the high-regularity scenario, utilizing the high-order smoothness advantage of solutions to establish a high-precision quantitative matching system between theoretical and numerical results; 2. Extend to the external force space $L^p([0,\infty); L^q)$ to further improve the boundary of the well-posedness theory; 3. Design adaptive numerical algorithms to specifically verify the instantaneous smoothness effect of weakly regular initial data and the smoothness preservation characteristics of high-regularity solutions; 4. Extend the framework to MHD

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equations and Navier-Stokes-Vlasov equations, establish a unified well-posedness theory for coupled systems, and focus on analyzing the effect of high-regularity input on improving the smoothness of solutions in coupled systems; 5. Based on the high-regularity smooth solution framework, establish a rigorous mathematical formulation of turbulent statistical properties, providing a more solid theoretical basis for turbulence modeling.

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