Article

Degeneracy of Koszul Homological Series on Lie Algebroids. Production of All Affine Structures, Production of all Riemannian Foliations and Production of All Fedosov Structures

Michel Nguiffo Boyom *

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Degeneracy of Koszul Homological Series on Lie Algebroids. Production of All Affine Structures, Production of all Riemannian Foliations and Production of All Fedosov Structures

Michel Nguiffo Boyom

Alexander Grothendieck Research Institute, IMAG, University of Montpellier, IMAG CNRS 5149, 499-554 Rue du Tuel 34090 Montpellier France; michel.nguiffo-boyom@umontpellier.fr

Abstract

The framework of the research whose part of results are published in this work is the category of real vector bundles over finite dimensional differentiable manifolds. The objects of studies are *gauge structures on these vector bundles*. We are interested in dynamical properties of the holonomy groups of Koszul connections as well as on their topological properties, i.e. properties that are of homological nature. For the most part the context is the subcategory of Lie algebroids. In addition to other investigations three open problems are studied in detail. (P1-Affine Geometry): When is a Koszul connection affine connection? (P2-Riemannian Geometry): When is a Koszul connection metric connection? (P3-Fedosov Geometry): When is a Koszul connection symplectic connection? In the category of tangent Lie algebroids our homological approach leads to deep relations of our homological ingredients with the open problem of *how to produce labeled foliations the most studied of which are Riemannian foliations*. On a Lie algebroid we define two families of differential equations, the family of differential Hessian equations and the family of differential gauge equations. The solutions of these differential equations are implemeted to construct homological ingredients which are key tools for our studies of open problems we are concerned with. We introduce *Koszul Homological Series*. This notion is a machine for converting Obstructions whose nature is vector space into Obstructions whose nature is Homological class. We define the property of Degeneracy and the property Nondegeneracy of Koszul homological Series. The property of Degeneracy is implemented to solve problems (P1), (P2) and (P3). *In the abundant literature on Riemannian foliatins we have only cited references directely related to the open problems which are studied using the tools which are introduced in this work. Thus the property of nondegeneracy is implemented to give a complete solution of the problem posed by E. Ghys, (P4-Differential Topology): How to produce Riemanian foliations?*. See our Theorem 7.4 and Theorem 7.5 which are fruits of a happy conjunction between the gauge geometry and the differential topology.

Keywords: Lie algebroid; gauge equation; Hessian equation; KV cohomology; Koszul homological series; (p,q)-degeneracy of Koszul homological series

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1. Introduction

Why did I quote the sentences from Eugenio Calabi that served as the oral introduction to his lecture at the Elie Cartan Colloquium in Saint-Martins-d'Hères in 1968. The initial problem that preceeded those studied in this paper was the search for a numerical invariant that could serve as a characteristc obstruction to the existence of symplectic structure on a diffrentiable manifold M. Exchanges with Alan Weinstein, for whom I have great respect, revealed that my method led to tautologies [54]. This led to shift from the first problem to the problem of special symplectic connections on Lie algebroids, which is the symplectic versus of the widely studied problem of metric connection on tangent Lie algebroids, [5,48,52].

Thus, inspired by the adage of Eugenio Calabi that I quoted, I abandonned the field of Differential Geometry for that of Gauge Geometry, which is the stude of Koszul connections in vector bundles. Other factors in favour of this change of problem are found in numerous exchanges with Dimitri Alekseevky. This is an opportunity to thank him for his three pages of comments, which were very helpful to me, [1]. The readers of this work will see that all the problems I studied and solved are formulated within the framework of Gauge Geometry.

Let M be m-dimensionnal differentiable manifold and let ∇ be a torsion-free Koszul connection on the tangent vector bundle

$$TM \rightarrow M.$$

One of the important open problems is the following,

P1: Problem of metric connection: When is ∇ the Levi Civita connection of a possitive Riemannian structure (M, g).

This problem is one of those widely studied, see [5,48], and widely debated, see Math.Overflow, Overflow and [52]. Some ressearchers have raised the subsidiary question of whether ∇ determines the positive Riemannian metrics of which it is the Levi Civita connection.

In this work we rejected the two assumptions which are the nullity of the torsion of ∇ and the positivity of thr metric tensor g.

To study the *problem of metric connection* we extend the framework to the category of general Lie Algebroids

$$A = (V, M, a, [-,-]).$$

The couple (V, M) stands for the vector bundle

$$V \rightarrow M,$$

a (, called anchor mapping,) is a vector bundle morphism

$$a : \quad (V, M) \rightarrow (TM, M),$$

The vector space of sections $\Gamma(V)$ is a real Lie algebra whose bracket product is

$$\Gamma(X) \times \Gamma(V) \ni (s, s') \rightarrow [s, s'] \in \Gamma(V).$$

The bracket just defined is subeject to the following requirement,

$$[s, fs] = df(a(s))s' + f[s, s'] \quad \forall((s, s', f) \in \Gamma(V) \times \Gamma(V) \times C^{\infty}(M).$$

In addition to the problem of metric connection on the category of Lie algebroids, two other open problems emerge that we state as follows:

P2: = Problmen of affine connection: the question of whether (A, ∇) is affine structure on the Lie algebroid A.

P3: = Problem of symplectic connection: the question of whether ∇ is symplectic connection on the Lie algebroid A. In the case of a tangent Lie algebroid $A(M)$ the of symplectic connection is the first step toward the question of whether a given torsion free gauge structure (TM, ∇) arises from a Fedosov structure (M, ω, ∇), [15,18].

On a Lie algebroid A we define two families of differential equations: the family of *gauge equations* and the family of *Hessian equations*. The implementations of the solutions of these differential equations are efficient to solve the open problems of labeled Koszul connections $P2$ and $P3$. In particular the solutions of gauges equations are implemented to achieve the study of *the problem of metric connections* and to anwer the subsidiary problem whic is stated above. At another side a remarkable fact is that the solutions of gauge equations are implemented to completely solve an open problem posed by E. Ghys in the quantitative differential topology:

P4: Production of Riemannian foliations: How to produce Riemannian foliations? See [20].

It must be highlighted that the current litterature on Riemannian foliations deals with foliations in positive Riemannian manifolds, [26,37,38,47]. Our approach to produce Riemannian foliations deals with Riemannian manifolds of arbitrary signature.

Below is a short overview of the contents of this paper.

This work is divided into sections including this introduction.

Section 2 is devoted to notation and to a few notions attached to Vector bundles and to Lie algebroids. In addition to the classical gauge dynamics, ie the action of the gauge group $GL(V, M)$ on Koszul connections on a vector bundle (V, M), we introduce the metric dynamics which is the action of the *group* $GM(V, M)$ *which is generated by the symmetries defined by inner products*. We define gauge equations and other notions which are derived from the solutions of gauge equations. Among these notions are short exact sequences (2.6.7) and (2.6.8) which link the solutions of gauge equations with relevant bi-linear forms on a vector bundle.

Section 3 is devoted materials which will be deeply implemented in next sections. These materials are objects and properties linked to Koszul connections and which are of homological nature. The relevant relationships and various bridges between these homological notions and the problems P1, P2 and P3 are clarified. Theorem 3.6 and Theorem 3.7 are among the key tools in our approach to problems P1, P2 and P3 on Lie algebroids.

Section 4 is devoted to the sketch of the gauge topology of Lie algebroids. Section 4 is the cor of the paper. Two efficient homological tools are defined . Two major families of objects are defined. (1) The family of *Koszul homological series.*. (2) The (graded) family of *degeneracy of Koszul homological series*. These two families are implemented to study the propblems P1, P2 and P3.

Section 5 serves as a transition. On one hand it is a section of reminders of the challenges. On the other hand it a section of reminders of the tools which are implemented to face these challenges.

Section 6 is partially a return to topology and to affine geometry on Lie algebroids. Given an affine Lie algebroid (A, ∇) and a left A-module

$$W \to M.$$

The couple (∇, W) gives rise to three types of cochain complex.

(1) The KV complex $C_{KV}(\nabla, W)$ as in [40,44].

(2) The total KV complex $C_\tau(\nabla, W)$.

(3) The Chevalley-Eilenberg $C_{CE}(A, W)$ When (W, M) is the trivial vector bundle

$$M \times R \to M$$

the complex (3) is the de Rham complex. Section 6 is also devoted to these cochain complexes and to their ramifications such as the affine hyperbolic structure in the sense of Kaup-Koszul, [27,31] See our Theorem 6.10 (in this paper,) about hyperbolic foliations. Section 6 highlights the passage from the Gauge Geometry toward the Differential Geometry.

Section 7 is totally devoted canonical tangent Lie algebroids of differentiable manifolds. Results of Sections 4, 5, and 6 acquire new resonances in tangent Lie algebroids see Theorem 7.4 and Theorem 7.5 . In particular Theorem 7.5 completely solves the problem of producing all Riemannian foliations in any signature. Thus it goes beyong the scope of the problem posed by E. Ghys.

Section 8 is devoted offers a reading of Section 7 from the point of view of Erlangen by Felix Klein. In addition to the gauge dynamics *GL(TM,M)* and the metric dynamics *GM(TM,M)*, *Gau(TM,M)* is acted by the group of diffeomorphisms $Diff^\star!M)$ the invariants of which are called geometric invariants.

Section 9 is devoted to introduce *special Fedosov manifolds.* Each odd Betti number of a compact special Fedosov maifold is even. This statement follows from the following property of special Fedosov manifolds: Every special Fedosov manifold possesses a canonical Kaehlerian metric. A notable conseqence is a relevant connection of the Information Geometry with the Problem of metric connection. More precisely, in a special statistical Fedosov manifold (M, g, ω, ∇) the Koszul connection ∇ and its g-dual $\nabla^\star$ are each the Levi Civita connection of a Riemannian metric, cf Theorem 9.8. Section 1O is devoted to a few short comments.

2. Gauge Geometry of General Lie Algebroids

2.1. Notation and from the Following Property of f Basic Notions

Given a differentiable manifold M, the couple (V, M) stands for the a real vector bundle V over a differentiable manifold M,

$$(2.1.1) \qquad V \to M.$$

$\Gamma(V)$ is the real vector space of sections of (V, M).

$C^\infty(M)$ is the associative algebra of real valued differentiable functions on M.

An inner product on (V, M) is a nondegenerate symmetric $C^\infty(M)$- bi-linear mapping

$$(2.1.2) \qquad \Gamma(V) \times \Gamma(V) \ni (s, s') \to g(s, s') \in C^\infty(M)$$

The set of all inner products on (V, M) is denoted by $Me(V)$.

A symplectic product on (V, M) is a nondegenerate skew symmetric $C^\infty(M)$-bi-linear mapping

$$(2.1.3) \qquad \Gamma(V) \times \Gamma(V) \ni (s, s') \to \omega(s, s').$$

The set of all symplectic products on (V, M) is denoted by $Symp(V)$.

Given a non negative integer k a (V, M)-valued differential k-form on M is skew-symmetric $C^\infty(M)$-k-multi-linear mapping of $\Gamma(TM)$ in $\Gamma(V)$,

$$(2.1.4) \qquad \Lambda^k(\Gamma(TM)) \ni X_1 \wedge .. \wedge X_k \to \theta(X_1 \wedge .. \wedge X_k) \in \Gamma(V).$$

The $C^\infty(M)$-module of (V, M)-valued differential k-forms is denoted by $\Omega^k(M, V)$.

A Koszul connection on (V, M) is a linear mapping

$$(2.1.5) \qquad \Gamma(V) \ni s \to \nabla s \in \Omega^1(M, V)$$

which is subject to the following requirement

$$(2.1.6) \qquad \nabla f.s = f\nabla s + df.S \quad \forall f \in C^\infty(M).$$

Thus by putting

$$\nabla s(X) = \nabla_X s$$

one has

$$\nabla_X fs = f\nabla_X s + df(X)s.$$

Definition 1. *Let ∇ be a Koszul connection on (V, M), then the couple (V, ∇) is called gauge structure on (V, M).*

The familly of all gauge structures on (V, M) is denoted by *Gau*(V,M).

Let $(V^\star, M)$ be the dual vector bundle of (V, M) and let $(End(V), M)$ be the vector bundle of vector bundle morphisms of (V, M) in itself. We canonically identify the vector bundle $(End(V), M)$ with the tensor product of vector budles $(V^\star \otimes V, M)$.

The $C^\infty(M)$-module of sections de $End(V)$ is denoted *gl*(V,M).

GL(V,M) is the set of invertible sections in *gl*(V,M).

The action

$$GL(V, M) \times Gau(V, M) \ni (\Phi, \nabla) \to \nabla^\phi \in Gau(V, M)$$

is defined as it follows,

$$(2.1.7) \qquad \nabla^{\Phi} = \Phi \circ \nabla \circ \Phi^{-1}.$$

Thus if s is a section of (V, M) the (V, M)-valued differential 1-form $\nabla^{\Phi} s$ is defined as it follows,

$$(2.1.8) \qquad \nabla^{\Phi}_{X} s = \Phi(\nabla_X \Phi^{-1}(s)).$$

The couple

$$[GL(V, M), Gau(V, M)]$$

is called category of gauge structures on (V, M).

2.2. Metric Connections and Symplectic Connections on Vector Bundles

Definition 2. *A Koszul connection ∇ is called metric connection on (V, M) if there exists an inner product g subject to the following identity*

$$(2.2.1) \qquad X.g(s, s') - g(\nabla_X s, s') - g(s, \nabla_X s s') = 0.$$

Mutatis mutandis, a Koszul connection ∇ is called symplectic connection on (V,M) if there exists a symplectic product ω which is subject to the following identity,

$$(2.2.2) \qquad X\omega(s, s') - \omega(\nabla_X s, s') - \omega(s, \nabla_X s') = 0.$$

2.3. Problem of Metric Connection and Problem of Symplectic Connection on Vector Bundles

Warning I. From the point of view of Erlangen program of Felix Klein we should thing of Gauge Geometry as the study of Invariants of the gauge group GL(V,M).
Warning II. In this work we also thing of Gauge Geometry as the study of Invaraints of a gauge structure (V, ∇), viz the study of those objects defined on (V, M) which are ∇-parallel.
Thus arise two open problems on the category of gauge structures on a vector bundle

$$V \to M.$$

Problem 1. When Is a Koszul Connection Metric Connection?
References [5,48,52] deal with problem 1 on canonical tangent algebroids (TM, M).

Problem 2. There many open problems regarding connections on symplectic structures, [8,9,17]. In this work here is one among the questions we are interested in. When is a Koszul connection symplectic connection?

Of course Problem 2 makes sense only when the rank of (V, M) is even.

In the case (V, M) is the tangent vector bundle (TM, M) Question 1 is well known and widely discussed when ∇ is torsion free; see R. Atkins [5] and B. Schmidt [48]. A strategy suggested by Thurston approach consists on calculting the holonomy group of ∇, see Bill Thurston Stack Exchange [52] Regarding Question 2 the author does not know any reference.

2.4. The Metric Dynamic on Gau(V,M)

Every inner product

$$(s, s') \to g(s, s')$$

gives rise the the following symmetry on *Gau*(V,M),

$$(2.4.1) \qquad \nabla \to \nabla^{g}$$

where ∇^{g} is defined by the following identity,

$$(2.4.2) \qquad g(\nabla^{g}_{X} s, s') = Xg(s, s') - g(s, \nabla_X s').$$

It is obvious that (2.4.1) is a symmetry, that is to say that

$$(\nabla^g)^g = \nabla.$$

The group $GM(V,M)$ which is generated by symmetries as in (2.4.2) is called *metric group* of Gau(V,M).

The couple $[GM(V,M), Gau(V,M)]$ is called metric dynamics on Gau(V,M).

Remark: It is easy to check that $GL(V,M) \cap GM(V,M)$ *is not trivial.*

Definition 3. *(1) Objects and Properties defined on* (V,M) *which are invariant under the action of the group GL(V,M) are called gauge invariants.*
(2) Objects and Properties defined on (V,M) *which are invariant under the action of the group GM(V,M) are called metric invariants.*

Warning 3. In addition to Question 1 and Question 2 this work is also devoted to the study of properties and objects defined on Gau(V,M) which are either metric invariants or gauge invariants. This quest is the gauge geometry following Felix Klein Erlangen.

Let g be an inner product on (V,M), let ∇ be a Koszul connection on (V,M) and let ϕ be a section of GL(V,M). We define the inner g^ϕ and Koszul connection ∇^ϕ by putting

$$(2.4.3) \quad g^\phi(s,s') = g(\phi^{-1}(s), \phi^{-1}(s'))$$

$$(2.4.4) \quad \nabla^\phi_X s = \phi(\nabla_X \phi^{-1}(s)).$$

Thus by incorporing (2.4.3) and (2.4.4) the metric dynamics and the gauge dynamics are linked each with other as it follows,

$$(2.4.5) \quad (\nabla^\phi)^{g^\phi} = (\nabla^g)^\phi.$$

2.5. Gauge Equations on Vector Bundles

To every pair $(\nabla, \nabla^\star)$ of Koszul connections on (V,M) we assign the following first order differential operator

$$\Gamma(End(V)) \ni \phi \to D^{\nabla\nabla^\star}(\phi) \in \Omega^1(M, End(V))$$

which is defined as it follows

$$D^{\nabla\nabla^\star}(\phi) = \nabla^\star \otimes \phi - \phi \circ \nabla.$$

The End(V)-valued differential 1-form $D^{\nabla\nabla^\star}(\phi)$ is defined as it follows,

$$(2.5.1) \qquad [D^{\nabla\nabla^\star}(\phi)](X) = \nabla^\star_X \circ \phi - \phi \circ \nabla_X.$$

Mutatis mutandis we get the second differential operator

$$(2.5.2) \qquad [D^{\nabla^\star\nabla}(\phi)](X) = \nabla_X \circ \phi - \phi \circ \nabla^\star_X.$$

Thus we will be concerned with the following two first order linear differential equations,

$$(2.5.3) \qquad \nabla^\star \circ \phi - \phi \circ \nabla = 0,$$

$$(2.5.4) \qquad \nabla \circ \psi - \psi \circ \nabla^\star = 0.$$

(2.5.3) and (2.5.4) are called gauge equations of the pair $(\nabla, \nabla^\star)$.

The unknowns ϕ and ψ are sections of End(V).

The vector spaces of solutions of (2.5.3) and of (2.5.3) are denoted by

$$J_{\nabla^\star\nabla}(V)$$

and by

$$J_{\nabla\nabla^\star}(V)$$

respectively.

2.6. Two Fundamental Short Exact Sequences on Vector Bundles

We are concerned with a couple (g, ∇) where g is positive inner product on the vector bundle (V, M) and ∇ is a Koszul connection on (V, M). We go to deal with the pair (∇, ∇^g).
To make simpler we wil set

$$J_{\nabla,g}(V) = J_{\nabla\nabla^g}(V)$$

$$J_{\nabla^g,g}(V) = J_{\nabla^g\nabla}(V).$$

Given

$$\phi \in J_{\nabla,g}$$

we introduce the following pair of bi-linear products, $(q(g,\phi), \omega(g,\phi))$ which are defined as

$$2q(g,\phi)(s,s') = g(\phi(s),s') + g(s,\phi(s')), \tag{2.6.1}$$

$$2\omega(g,\phi)(s,s') = g(\Pi hi(s),s') - g(s,\phi(s')). \tag{2.6.2}$$

We also introduce the following pair

$$(\Phi, \Phi^\star) \subset \Gamma(End(V)),$$

which is defined as it follows,

$$g(\Phi(s),s') = q(g,\phi)(s,s'), \tag{2.6.3}$$

$$g(\Phi^\star(s),s') = \omega(g,\phi)(s,s'). \tag{2.6.4}$$

Lemma 1. *The bi-linear products $q(g,\phi)$ and $\omega(g,\phi)$ are ∇-parallel.*

Proof. let X be a vector field on M and let s, s' be sections of (V, M). Then

$$\begin{aligned}
(A): \quad (2\nabla_X q(g,\phi))(s,s') &= Xg(\phi(s),s') + Xg(s,\phi(s') - g(\phi(\nabla_X s),s') \\
&\quad - g(\nabla_X s,\phi(s')) - g(\phi(s),\nabla_X s') - g(s,\phi(\nabla_X s')) \\
&= -g(\phi(\nabla_X s),s') - g(s,\phi(\nabla_X s')) + g(\nabla^g_X s,s') + g(s,\nabla^g_X s') \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
(B) \quad (2\nabla_X \omega(g,\phi))(s,s') &= Xg(\phi(s),s') - g(s,\phi(s')) - g(\phi(\nabla_X s),s') \\
&\quad + g(\nabla_X s,\phi(s')) - g(\phi(s),\nabla_X s') + g(s,\phi(\nabla_X s')) \\
&= g(\nabla^g_X \phi(s),s') - g(s,\nabla^g_X \phi(s')) - g(\phi(\nabla_X s),s') + g(s,\phi(\nabla_X s')) \\
&= 0.
\end{aligned}$$

□

Here are two corollaries of Lemma 2.3.

Corollary 1. *The sections Φ and $\Phi^\star$ are solutions of the differential equaton (2.5.3).*

Proof.

$$\begin{aligned}(A):\quad (\nabla_X q(g,\phi))(s,s') &= Xg(\Phi(s),s') - g(\Phi(\nabla_X s),s') - g(\Phi(s),\nabla_X s')\\ &= g(\Phi(\nabla_X s) - \nabla^g_X\Phi(s),s')\\ &= 0 \quad \forall(X,s,s').\end{aligned}$$

Therefore

$$\Phi(\nabla_X s) - \nabla^q_X\Phi(s) = 0 \quad \forall(X,s).$$

Similarly computing

$$(B):\quad (\nabla_X\omega(g,\phi))(s,s') = 0 \quad \forall(X,s,s')$$

yelds the following identity

$$\nabla^g_X\Phi^\star(s) - \Phi^\star(\nabla_X s) = 0.$$

□

Corollary 2. *Lemma 2.3 yieds the following g-orthogonal decompositions*

$$(2.6.5)\quad V = Ker(\Phi) \oplus Im(\Phi),$$

$$(2.6.6)\quad V = Ker(\Phi^\star) \oplus Im(\Phi^\star)$$

Proof. Let us take

$$\Phi(s) \in Ker(\Phi),$$

then

$$\begin{aligned}(A):\quad g(\Phi(\Phi(s)),s') &= g(\Phi(s),\Phi(s'))\\ &= 0 \quad \forall s'.\end{aligned}$$

Therefore

$$g(\Phi(s),\Phi(s)) = 0.$$

Since g is positive definite we get

$$\Phi(s) = 0.$$

The same argument yeld the following fact,

$$(B):\quad Ker(\Phi^\star)\cap Im(\Phi^\star) = 0.$$

□

Let $H(\nabla)$ and $H(\nabla^g)$ be the holonomy groups of ∇ and ∇^g respectively.
The following proposition is a straight corollary of Lemma 2.4.

Proposition 1. *We take into account (2.5.1) and (2.5.2), then*
(1) $Ker(\Phi)$ and $Ker(\Phi^\star)$ are stable under the action of the holonomy group $H(\nabla)$;
(2) $Im(\Phi)$ and $Im(\Phi^\star)$ are stable under the action of the holonomy group $H(\nabla^g)$.

The vector space of ∇-parallel symmetric bi-linear products on (V,M) and the vector space of ∇-parallel skew symmetric bi-linear products on (V,M) are denoted by $\mathbf{S}^\nabla_2(V)$ and by $\Omega^\nabla_2(V)$ respectively.

Then we introduce the following mapping,

$$(2.6.6.bis)\quad J_{\nabla,g} \ni \phi \to (\Phi,\Phi^\star) \subset J_{\nabla,g}$$

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and we perform it to set the following identification

$$(2.6.6.ter) \quad (\Phi^\star, \Phi) = (\omega(g,\phi), q(g,\phi)).$$

By the vertue of Lemma 2.3 we know that

$$(\omega(g,\phi), q(g,\phi)) \in \Omega_2^\nabla(V) \times \mathbf{S}_2^\nabla(V)$$

Therefore we obtain the following fundamental short exact sequences of vector spaces,

$$(2.6.7) \quad 0 \to \Omega_2^\nabla(V) \to J_{\nabla,g}(V) \to \mathbf{S}_2^\nabla(V) \to 0.$$

$$(2.6.8) \quad 0 \to \Omega_2^{\nabla^g}(V) \to J_{\nabla^g,g}(V) \to \mathbf{S}_2^{\nabla^g}(V) \to 0.$$

Exact sequences (2.6.7) and (2.6.8) will be involved to discuss some important open problems in the differential topology. Among these open problems is the following question cf [20],

EGQ: How to produce Riemannian folliations.

For details see P. Molino [38], BL Reinhart [47], Moedijk-MCul [37]. It is to be noticed that many of these references deal with *Riemannian foliations on positive definite Riemannian manifolds.* In contrast our approach to produce Riemannian foliations is based on fundamental exact sequences (2.6.7) and (2.6.8). We will prove that our method to produce Riemannian foliations walk in *pseudo-Riemannian geometry*. Thus we will use sequences (2.6.7) and (2.6.8) to completely solve that open problem of production of all Riemannian foliations.

3. Gauge Geometry and Gauge Topology on Lie Algebroids

In this section we focus on Koszul connecions on algebroids.

3.1. Basic Algebraic Tools

Reminder: Henceforth we remind that Gauge invariants are either objects or properties which are invarant under the action of the gauge group. Metric invariants are either properties or objects that are invariant under the action of the metric group. By gauge topology on a Lie algebroid A we mean the study of gauge concerns on A that are linked with objects or with properties which are of homologcal nature on A.

We remind that a structure of Lie algebroid on a vector (V, M) is a couple $(a, [-,-])$ where a is vector bundle morphism

$$a : \quad (V, M) \to (TM, M)$$

and the following arrow

$$\Gamma(V) \times \Gamma(V) \ni (s, s') \to [s, s'] \in \Gamma(V)$$

is a structure of real Lie algebra on the real vector space $\Gamma(V)$ which is related with a as it follows

$$(3.1.1) \quad [s, f.s'] = (a(s))f).s + f.[s, s'] \quad \forall (s, s') \subset \Gamma(V), \quad \forall f \in C^\infty(M).$$

The requirement (3.1.1) implies that

$$(3.1.2) \quad a([s, s']) = [a(s), a(s')]$$

where the right side member is the Poisson bracket of vector fields.

We also recall that a left module of Lie algebroid $(V, M, a, [-,-])$ is a vector bundle (W, M) whose space of sections $\Gamma(W)$ is a real left module of the Lie algebra $\Gamma(V)$. Moreover the left action

$$\Gamma(V) \times \Gamma(W) \ni (s, w) \to s.w \in \Gamma(W)$$

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satisfies the following identity

$$s.fw = df(a(s))w + fs.w \quad \forall f \in C\infty(M).$$

Before pursuing we put

$$(3.1.3) \quad A = (V, M, a, [-,-]).$$

Both (V, M) and (TM, M) are examples of left module of A under the following left actions:

$$(3.1.4) \quad L_s(s') = [s, s'] \quad \forall(s, s') \subset \Gamma(V);$$

$$(3.1.5) \quad L_s(X) = [a(s), X] \quad \forall(s, X) \in \Gamma(V) \times \Gamma(TM).$$

Let us put

$$(3.1.6) \quad V^{p,q} = V^{\star \otimes p} \otimes V^{\otimes q},$$

$$(3.1.7) \quad T^{p,q}(M) = TM^{\star \otimes p} \otimes TM^{\otimes q}.$$

By extending (3.1.4) and (3.1.5) to tensor spaces the vector bundles $V^{p,q}$ and $T^{p,q}(M)$ are made left modules of the Lie algebroid A.

Occasioally if there is no risk of confusion we use the following notation:

$\Omega^k_{p,q}(M, V)$ is the vector space $V^{p,q}$-valued differential k-forms on M. In other words,

$$\Omega^k_{p,q}(M, V) = \Omega^k(M, V^{p,q}).$$

$H^k_{p,q}(M, V)$ is the Chevalley-Eilenberg cohomology space of M with coefficients in $V^{p,q}$.

$H^k_{p,q}(A, V)$ is the Chevalley-Eilenberg cohomology space of A with coefficients in $V^{p,q}$.

Notation: On a Lie algebroid

$$A = (V, M, a, [-,-])$$

the k-th exterior power of the mapping a is denoted by $a^{\Lambda k}$, thus

$$(3.1.8) \quad a^{\Lambda k}(s_1 \wedge .. \wedge s_k) = a(s_1) \wedge .. \wedge a(s_k).$$

Therefore given a A-module (W, M) we put

$$(3.1.9) \quad \Omega^k(A, W^{p,q}) = \Omega^k(M, W^{p,q}) \circ a^{\Lambda k}.$$

Without the statement of the contrary the basic definition we are dealing with is given below.

Definition 4. *A gauge structure on a Lie algebroid*

$$A = (V, M, a, [-,-])$$

is a first order differential operator

$$\Gamma(V) \ni s \to \nabla s \in \Omega^1(A, V^{1,1})$$

which is subject to the following requirement,

$$\nabla f.s = df.s + f.\nabla s \quad \forall f \in C^\infty(M).$$

Warning.

According to (3.1.8) and to (3.1.9) here is the meaning of the requirement ∇ is subject to,

$$(3.1.10) \quad ([df.s + f.\nabla s)](s') = df(a(s'))s + f\nabla_{a(s')}s.$$

Let us consider a Lie algebroid

$$A = (V, M, a, [-,-]);$$

let us endow A with a gauge structure:

$$(A, \nabla) = (V, M, a, [-,-], \nabla).$$

The curvature R^∇ and the torsion T^∇ are defined as it follows,

$$(3.1.11)\quad R^\nabla(s, s'; s'') = \nabla_{a(s)}\nabla_{a(s')}s'' - \nabla_{a(s')}\nabla_{a(s)}s'' - \nabla_{a([s,s'])}s'',$$

$$(3.1.12)\quad T^\nabla(s, s') = \nabla_{a(s)}s' - \nabla_{a(s')}s - [s, s'].$$

On a Lie algebroid A a gauge structure (A, ∇) whose tensor T^∇ vanishes identically is called symmetric (or torsion free).

Of course there exist symmetric gauge structures. Indeed every gauge structure (A, ∇) is associated with the symmetric gauge structure (A, ∇') which is defined as it follows,

$$\nabla'_{a(s)}s' = \nabla_{a(s)}s' - \frac{1}{2}T^\nabla(s, s').$$

The familly of all symmetric gauge structures on A is denoted by $SGau$(A).

Warnings: It is to remind that on a Lie algebroid

$$A = (V, M, a, [-,-])$$

every inner product g admits a unique symmetric metiric connection ∇ which defined by the following well known Koszul formula:

$$(3.1.13)\quad g(\nabla_{a(s'')}s, s') = \frac{1}{2}[a(s)g(s', s'') - a(s')g(s, s'') + a(s'')g(s, s')$$

$$-g([s, s'], s'') - g([s', s''], s) - g([s, s''], s') \quad \forall(, s', s'') \subset \Gamma(V)].$$

Mutatis mutandis, on A every symplectic product ω adimts symmetric symplectic connections. Those symmetric symplectic may be constructed following a formalism introduced as in Biliavski-Cahen-Gutt-Rawnsky, [8], see also [9]

Indeeed starting from any symmetric connection ∇^0 one defines

$$\Gamma(V) \times \Gamma(V) \ni (s, s') \to N(s, s') \in \Gamma(V),$$

as it follows,

$$(3.1.14)\quad (\nabla^0_{a(s)}\omega)(s', s'') = \omega(N(s, s'), s'').$$

Then the following connection is symplectic,

$$(3.1.15)\quad \nabla_{a(s)}s' = \nabla^0_{a(s)}s' + \frac{1}{3}(N(s, s') + N(s', s)).$$

Otherwise said, it satifies the following identity,

$$(3.1.15)\quad a(s)\omega(s', s'') - \omega(\nabla_{a(s)}s', s'') - \omega(s', \nabla_{a(s)}s'') = 0 \quad \forall(s, s', s'') \subset \Gamma(V).$$

Definition 5. *An affine structure on a Lie algebroid*

$$A = (V, M, a, [-,-])$$

is a gauge structure (A,∇) whose both the curvature R^{∇} and the torsion T^{∇} vanish identicaly. ***Arises the question of whether every Lie algebroid admits affine structures.***

One among our aims is to point out a characteristic obsttuction to the existence of affine structure on a Lie algebroid. That obstruction is of homological nature.

3.2. The Hessian Differential Operators of Gauge Sructures on Lie Algebroids

The gauge structure (A,∇) is associated with the $V^{2,1}$-valued differential operator ∇^2 which is defined on (V,M),

$$\Gamma(V)\ni s''\to\nabla^2 s''\in\Gamma(V^{2,1}).$$

The $C^\infty(M)$-bi-linear $\nabla^2 s''$ is defined as it follows,

$$(3.2.1)\quad (\nabla^2 s'')(s,s')=\nabla_{a(s)}(\nabla_{a(s')}s'')-\nabla_{a(\nabla_{a(s)}s')}s''.$$

Now we introduce the Hessian equation of (A,∇),

$$(3.2.2)\quad \nabla^2 s''=0.$$

The unknown s" is an element of $\Gamma(V)$. Clearly (3.2.2) is a differential equation of ordre two.
The vector space of solutions of the Hessian equation (3.2.2) is denoted by

$$(3.2.3)\quad J_\nabla(A)=\Big\{s\in\Gamma(V)\quad s.t.\quad \nabla^2 s=0\Big\}$$

We endow $\Gamma(V)$ with the product defined by (V,∇), viz

$$(3.2.4)\quad s.s'=\nabla_{a(s)}s'.$$

Proposition 2. *Under the product as in (3.2.4) $J_\nabla(A)$ is an associative algebra.*

Proof. In fact let ξ be a germ of section of (V,M) defined on an open set

$$U\subset M.$$

$$\xi\in J_\nabla(A)$$

if and only if

$$s.(s'.\xi)=(s.s).\xi\quad\forall(s',s'')\subset J_\nabla(A).$$

Thus if

$$(\xi,\xi')\subset J_\nabla(A)$$

then

$$\begin{aligned} s.(s'.(\xi.\xi')) &= s.((s'.\xi).\xi')\\ &= (s.(s'.\xi)).\xi'\\ &= ((s.s').\xi).\xi'\\ &= (s.s')(\xi.\xi') \end{aligned}$$

□

Proposition 3. *If (A,∇) is an affine structure on a Lie algebroid A then the associative algebra $J_\nabla(A)$ is (A,∇)-preserving.*

Proof. We recall the action of $J_\nabla(A)$ on (A,∇):

$$(L_s\nabla)'_s s'' = [s, \nabla_{a(s')}s''] - \nabla_{a([s,s'])}s'' - \nabla_{a'(s')}[s,s']$$

Since both T^∇ and R^∇ vanish identically we rewrite the right side member of the equality above as it follows,

$$\nabla_{a(s)}(\nabla_{a(s')}s'') - \nabla_{a(\nabla_{a(s')}s'')}s - \nabla_{a([s,s'])}s'' - \nabla_{a(s')}(\nabla_{a(s)}s') + \nabla_{a(s')}(\nabla_{a(s'')}s)$$

$$= R^\nabla(s,s';s'') + (\nabla^2 s)(s',s'')$$

$$= 0 \quad \forall s \in J_\nabla(A), \quad \forall (s',s'') \subset \Gamma(V).$$

□

Definition 6. *A gauge structure (A,∇) is called regular if $dim(J_\nabla(A)(p))$ does not depend on the point $p \in M$.*

The familly of all regular symmetric gauge structures is denoted by *RSGau(A)*.

On a regular Lie algebroid A we also fix g, a positive definite inner product on (V,M). Then we denote by $J^g_\nabla(A)$ the Lie subalgebra of $(\Gamma(V),[-,-])$ which is generated by all the sections of (V,M) which are g-orthogonal to $J_\nabla(A)$.

Of course we already noticed that according to (3.1.4) and (3.1.5), all of the vector spaces $V^{p,q}$ and $TM^{p,q}$ are left modules of the Lie algebra $J^g\nabla(A)$. We will be interested in the Chevalley-Eilenberg cohomology of $J^g\nabla(A)$ with coefficients in those left modules.

The present context is the Lie algebroid

$$A = (V, M, a, [-,-]),$$

and (A,∇) is a regular torsion free gauge structure on A. We recall the formula we are concerned with,

$$(L_s\nabla)_{a(s')}s'' = [s, \nabla_{a(s')}s''] - \nabla_{a([s,s'])}s'' - \nabla_{a(s')}[s,s''], \quad (s,s',s'') \subset \Gamma(V).$$

We put

$$(3.2.4) \quad A_\nabla(V) = \Big\{ s \in \Gamma(V) \quad s.t \quad (L_s\nabla)_{a(s')}s'' = 0 \quad \forall (s',s'') \subset \Gamma(V) \Big\}.$$

One of the key tools that we will frequently use in the next is Theorem 3.6 we go to prove (below)

Theorem 1. *The vector space A_∇ (V) as in (3.2.4) cannot contain any non-null left module of the associative algebra $C^\infty(M)$.*

Demonstration. We consider

$$s \in A_\nabla(V), \quad (f,h) \subset C^\infty(M)$$

with

$$s \neq 0$$

and we put

$$\xi = hs.$$

If one assumes that

$$span_{C^\infty(M)}(s) \subset A_\nabla(V)$$

then

$$(\xi, f\xi) \subset A_\nabla(V) \quad \forall (\xi, f) \in \Gamma(V) \times C^\infty(M).$$

Therefore one has the following identity,

$$(care.1) \quad -[\nabla_{a(s')}s'', f\xi] + \nabla_{a([s',f\xi])}s'' + \nabla_{a(s')}[s'', f\xi] = 0 \quad \forall(f, s', s'').$$

We take into account the following identiry,

$$L_\xi \nabla = 0;$$

then the left hand member of (care.1) yields the following identity,

$$-df(a(\nabla_{a(s')}s''))\xi + df(a(s'))\nabla_\xi s''$$

$$+(a(s')(a(s'').f))\xi + df(a(s''))\nabla_{a(s')}\xi df(a(s'))[s'', \xi] = 0 \quad \forall(f, s', s'').$$

Since ∇ is torsion free, the identity above yields the following one,

$$(care.2) \quad (df(a(\nabla_{a(s')}s'')) - a(s')((a(s'')f)))\xi = df(a(s''))(\nabla_{a(s')}\xi + df(a(s'))\nabla_{a(s'')}\xi.$$

Now remeber that

$$\xi = hs,$$

then (care.2) becomes

$$(df(a(\nabla_{a(s')}s'')) - a(s')(a(s'')f))hs = df(a(s''))\nabla_{a(s')}hs + df(a(s'')\nabla_{a(s'')}hs$$

$$= h[df(a(s''))\nabla_{a(s')}s + df(a(s'))\nabla_{a(s'')s}]$$

$$+ \quad [df(a(s'))dh(a(s'')) + df(a(s''))deh(a(s'))]s.$$

One takes into accout (care.2) to deduce the following identity

$$(care.3) \quad [df(a(s'))dh(a(s'')) + df(a(s''))dh(a(s'))]s = O \quad \forall(f, h) \subset C^\infty(M).$$

The conclusion of identity (care.3) is

$$s = 0.$$

There is contradiction, consequently the theorem is demonstrated.

Comments

Comm.1. In global analysis on differentiable manifolds it is well known that given a gauge structure (TM, M, D) the subgroup of D-preserving diffeomorphisms $Aff(M, D)$ is finite dimensional Lie group, see Theorem 23 as in [29]. In fact one hase the following inequality,

$$dim(Aff(M, D)) \leq m^2 + m$$

here

$$m = dim(M).$$

The demonstration of Theorem 23 as in [29] involves both *the fondamental form of the principal* $Gl(m, R)$*-bundle of linear frames* $R^1(M)$ *and the horizontal distribution* $H_D \subset TR^1(M)$ *wich is attached with* D. These data are used to provide $R^1(M)$ wiht a parallelism which is invariant under the action of $Aff(M, D)$ on $R^1(M)$. Since this action of $Aff(M, D)$ is *effective* one easily deduces that $Aff(M, D)$ is a subgroup of isometries of positive inner product on the tangent vector bundle

$$TR^1(M) \to R^1(M).$$

For details readers are referred to Kobayashi-Nomizu [29], (p.232, Theorem 23.) In constrast to our strategy of direct demonstraton the strategy based on Riemannian geometry is not efficient on general Lie algebroids.

Comm.2. Below we will state and prove Theorem 3.7 which illustrates that Theorem 3.6 does not work on general left modules of the Lie algebroid

$$A = (V, M, a, [-,-]).$$

We consider a real vector bundle (W, M) which is left module of A whose left action

$$((3.2.5) \quad \Gamma(V) \times \Gamma(W) \ni (s, \xi) \to s.\xi \in \Gamma(W)$$

satistfies the follwing two properties

$$(3.2.6) \quad (fs.\xi)(x) = f(x)(s.\xi)(x),$$

$$(3.2.7) \quad (s.f\xi)(x) = df(a(s)(x)\xi(x) + f(x)(s.\xi)(x).$$

Let us assune that ∇ is a connection on (W, M) . Then we put

$$(3.2.8) \quad A_{\nabla}(W) = \{s \in \Gamma(V) \quad s.t \quad L_s\nabla = 0\}.$$

Problem: Let (W, M) be a letf module of the Lie algebroid A, let (W, M, ∇) be a gauge structure on (W, M), the problem is to know under what conditions $A_{\nabla}(W)$ contains a non-null module of $C^{\infty}(M)$.

Otherwise said there exists

$$s \in A_{\nabla}(W)$$

such that

$$C^{\infty}(M)s \subset A_{\nabla}(W),$$

which means that

$$(3.2.9) \quad fs.\nabla_{a(s')}w - \nabla_{a([fs,s'])}w - \nabla_{a(s')}fs.w = 0 \quad \forall f \in C^{\infty}(M), \quad \forall (w, s') \in \Gamma(W) \times \Gamma(V).$$

If we take into account (3.2.7) then ((3.2.9) is reduced to the following identity,

$$(3.2.10) \quad df(a(s'))[\nabla_{a(s)}w - s.w] \quad \forall \Gamma(V) \times \Gamma(V) \times (\Gamma(W) \times C^{\infty}(M)$$

Consequently we are led to the following three alternatives:

Alternative 1: f is first integral of the distribution $a(V) \subset TM$

Alternative 2: $\nabla_{a(s)}w = s.w \quad \forall w \in \Gamma(W)$. Thus this Alternative 2 implies the following inclusion

$$C^{\infty}(M) \subset A_{\nabla}(W).$$

Alternative 3: $s = 0$. The expression (3.2.9) jointed to the injectivity of a excludes *alternative 1.*

It remains to discuss Alternative 2 and Alternative 3.

We remind that according to formulas (3.2.6) and (3.2.7) the left actiom of A on W looks like the restriction connections on W along the leaves of the folllowing foliation

$$a(V) \subset TM.$$

Thus henceforth a connecton (on (W, M))) whose restriction along the foliation $a(V)$ concides with the left action as in (3.2.5) is called canonical connection on W and is denoted by ∇^{can}.

Thus (3.2.7), (3.2.8), and (3.2.9) yield the following statement:

Theorem 2. *Let us consider an injective Lie algebroid*

$$A = (V, M, a, [-,-])$$

and let (W, M) be a left module of A according to (3.2.6) and (3.2.7). Let (W, M, ∇) be a gauge structure on (W, M).
(I) If ∇ is not a cononical connection on (W, M) then the situation is alternative 3; therefore $A_\nabla(W)$ does not contain any non-null module of the associative algebra $C^\infty(M)$.
(II) If ∇ is a canonical connection on (W, M) then the situation is alternative 2. Consequently $A_\nabla(W)$ is module of the associative algebra $C^\infty(M)$.

Let us brievely remind the following arrow

$$(3.2.11) \quad p_k : \Omega^k(M, W) \to \Omega^k(A, W)$$

where

$$\Omega^K(A, W) = \Omega^k(M, W) \circ a^k.$$

Let $\Omega^1_b(A, W)$ be the kernel of p_1. Thus one has the following exact seqence,

$$(3.2.12) \quad 0 \to \Omega^1_b(A, W) \to \Omega^1(M, W) \to \Omega^1(A, W).$$

It is clear that $Hom(\Gamma(W), \Omega^1_b(A, W))$ is left module of the Lie algebra $\Gamma(V)$ whose k-th space of Chevalley-Eilenberg cohomology is denoted by

$$H^k_{CE}(\Gamma(V), Hom(\Gamma(W), \Omega^1_B(A, W))).$$

In regard to Theorem 3.7 the familly of all cononcal gauge structures on (W, M) is denotre par *CGau(W,M)*. Therefore it is easy to check the following statement,

Proposition 4. *There exists a canonical one to one correspondence between CGau(W,M) and the cohomology space $H^0_{CE}(\Gamma(V), Hom(\Gamma(W), \Omega^1_b(A, W)))$.*

Warnings. From the viewpoint of the Diffrential Topology, $\Omega^k_b(A, W)$ is the space of (W, M)-valued Basic Differential k-Forms on the foliated space $(M, a(V))$, see [39], [38], [47].

4. Tools from the Differential Gauge Operators on Vector Bundles

Here we are referring to decompositons (2.6.5) and (2.6.6).

The Lie subalgebras of $(\Gamma(V), [-,-])$ which are generated by $\Gamma(Ker(\Phi))$ and by $\Gamma(Ker(\Phi^\star))$ are denoted by $L(\nabla, \Phi)$ and by $L(\nabla, \Phi^\star)$ respectively. Thus

$$(L(\nabla, \Phi), [-,-]) \subset (\Gamma(V), [-,-]),$$

$$(L(\nabla, \Phi^\star), [-,-]) \subset (\Gamma(V), [-,-]).$$

4.1. Three Canonical Arrows

We aim to introduce three arrows whose targets are Lie subalgebras of the Lie algebra $(\Gamma(V), [-,-])$.

The affine arrow

$$RSGau(A) \ni \nabla \to J^g_\nabla(A).$$

The metric arrow

$$J_{\nabla,g} \ni \phi \to L(\nabla, \Phi).$$

The symplectic arrow

$$J_{\nabla,g} \ni \phi \to L(\nabla, \Phi^\star).$$

On one hand depending on Lie algebroids and on modules of Lie algebroid in view, on the other hand one takes into account (3.1.5); then it is clear that all of the vector spaces $\Gamma(V^{p,q})$ and $\Gamma(T^{p,q}(M))$ are left modules of each of the following three three targets: (i) the target of affine arrow, (ii) the target of metric arrow, (iii) the target of symplectic arrow.

We are particularly concerned with the Chevalley-Eilenberg cohomology of these Lie algebras $J^g_\nabla(A)$, $L(\nabla,\Phi)$ and $L(\nabla,\Phi^\star)$ with coefficients in $\Gamma(T^{p,q}(M))$ or in $\Gamma(V^{p,q})$. We are also interested in interpretations of some particular cohomology classes which are called *canonical Koszul classes*.

Our construction of these objects of homological nature was inspired by the works of Koszul in [33]. These tools are used to convert obstructions whose nature is vector space into obstructions whose nature is cohomological class.

4.2. Koszul Homological Series

Here is the context of this subsection. We are dealing with a Lie algebroid

$$A = (V, M, a, [-,-)]$$

wich is endowed a torsion free gauge structure

$$(A, \nabla)$$

and with a positive inner product

$$(V, M, g).$$

We refer to formulas (2.6.5), (2.6.6) and (2.6.6bis) to remind the three Lie algebras: J^g_∇, $L(\nabla,\Phi)$, $L(\nabla,\Phi^\star)$. Let us fix a gauge structure $(V^{p,q}, D)$. The Koszul connection D gives rise to the following three $\Omega^1(A, V^{p,q})$ -valued Chevalley-Eilelnberg 1-cocycles: k^a_∞, k^m_∞ and k^s_∞ which are defined as it follows:

$$\begin{aligned}
&(4.2.1) \quad J^g_\nabla(A) \ni s \to k^{a,p,q}_\infty(s) = L_s D \in \Omega^1(A, V^{p,q}),\\
&(4.2.2) \quad L(\nabla,\Phi) \ni s \to k^{m,p,q}_\infty(s) = L_s D \in \Omega^1(A, V^{p,q}),\\
&(4.2.3) \quad L(\nabla,\Phi^\star) \ni s \to k^{s,p,q}_\infty(s) = L_s D \in \Omega^1(A, V^{p,q}).
\end{aligned}$$

We remind that L_s is the extension to the tensor spaces of the action of the Lie algebroid A on its canonical A-module (V, M).

For instance if

$$D \in Gau(V^{1,1}, M)$$

then one has

$$(4.2.4) \quad (L_s D)_{s'} s'' = [s, D_{a(s')} s''] - D_{[a(s),a(s')]} s'' - D_{a(s')}[s, s'']).$$

It is easy to see that the cohomology classes of these cocycles are independent of the choice the Koszul connection D.

Based on the four formulas $\{(4.2.1), (4.2.2), (4.2.3), (4.2.4)\}$ we introduce the following three Koszul homological series.

The Koszul affine homological series are the following family:

$$(4.2.5) \quad \left\{ F^{a,p,q}(g,\nabla) := \quad N^2 \ni (p,q) \to [k^{a,p,q}_\infty] \in H^1_{CE}(J^g_\nabla(A), \Omega^1(A, V^{p,q})) \right\}.$$

The Koszul metric homological series are the following family:

$$(4.2.6) \quad \left\{ F^{m,p,q}(g,\nabla,\phi) := \quad N^2 \ni (p,q) \to [k^{m,p,q}_\infty] \in H^1_{CE}(L(\nabla,\Phi), \Omega^1(A, V^{p,q})) \right\}.$$

The Koszul symplectic homological series are the following family:

$$(4.2.7)\quad \left\{F^{s,p,q}(g,\nabla,\phi) := \quad N^2 \ni (p,q) \to [k_\infty^{s,p,q}] \in H^1_{CE}(L(\nabla,\Phi^\star),\Omega^1(A,V^{p,q}))\right\}.$$

Definition 7. *Given a Lie algebroid*

$$A = (V, M, a, [-,-])$$

(1) the arrow

$$RGau(V) \ni \nabla \to \{F^{a,p,q}(g,\nabla), \quad (p,q) \subset N\}$$

is called Koszul affine homological functor of A.
(2) The term $F^{a,p,q}(\nabla,g)$ is called affine (p,q)-homology class of A.
(3) The Lie algebroid A is called affinely-nondegenerate if its Koszul homological functor does not contain any affine vanishing homology class; otherwise it is affinely-degenerate.
(4) The Lie algebroid A is affinely-(p,q)-degnerate if its Koszul affine homological functor contains a affine (p,q)-vanishing homology class, viz there exists some $\nabla \in RGau(V)$ such that

$$F^{a,p,q}(g,\nabla) = 0.$$

Nota Bene : It must be noticed that this property of nondegeneracy is independant of the choice of the positive inner product g.

Nota Bene: Mutatis mutandis the follwing notions are clearly unambiguous:

(5) The Lie algebroid A is metrically-(p,q)-nondegenerate,

Otherwise said, for any fixed couple (p,q) of positive interger the set $\{F^{m,p,q}(g,\nabla,\phi)\}$ does not contain a vanishing class.

(6) The Lie algebroid A is symplectically-(p,q)-nondegenerate.

Otherwise said, for any fixed couple (p,q) of positive integer the set $\{F^{s,p,q}(g,\nabla,\phi)\}$ does not contain a vanishing class.

Comments: The notion of nondegeneracy may be restricted to a fixed gauge structure (A,∇). Which means that depending on issue and on concern the parameter ∇ is fixed in (4.2.5), in (4.2.7), in (4.2.7). Consequently the following statements are unambiguous,

(7) (A,∇) is affinely- (p,q)-nondegenerate, (respectively affinely-(p,q)-degenerate).

(8) (A,∇) is metrically nondegenerate, (respectively metrically-(p,q)-degenerate.

(9) (A,∇) is symplectically-(p,q)- nondegenerate, (respetively symplectively-(p,q)-degenerate .

Restrictions $\{(7),(8),(9)\}$ are useful and effective for examining many interesting questions such as the problem of metric connection and the problem of symplectic conection.

Without going into details here are some examples:

Example 1: Restriction (7) may be implimented to address the question of whether (A,∇) is affine structure on A. Regarding the case of canonical tangent algebroids see Goldman W. M. [22] and Smilie J. [51]

Example 2: Restriction (7) may be implienmted to studied the question of whether a statistical structure in A is Hessian structure.

Example 3: Restriction (7) may be involved to discuss the question whether a statistical model of a measurable set is exponential model. Examples just mentioned are of great interests in the both pure information Geometry and applied information geometry.

To make simpler, one must retain that *the Information Geometry is the study of Geometric Properties of Statistical Models, [2,7,36,41,50].*

Example 4: The both Resriction (8) and Restriction (9) may be involved to study the question of whether a statistical structure on (A,∇), namely (A,∇,g) admits a compatible symplectic structure (A,∇,ω). Which means that the following requirements are satisfied,

$$\delta^\nabla g = 0,$$

$$\nabla\omega = 0.$$

Example 4 can be implemented to construct Hamiltonian systems on statistical manifolds.

The set of all vanishing (p,q)-classes is noted by $F_\delta^{X,p,q}$ *, here X stands for one of the three letters a: = affine, m: = metric, s: = symplectic.*

Henceforth the family of all *affinely vanishing -(p,q)-classes of* A is denoted by

$$(4.2.8)\quad F_\delta^{a,p,q}(A).$$

The family of all *metrically vanishing-(p,q)-classes of* A is denoted by

$$(4.2.9)\quad F_\delta^{m,p,q}(A).$$

The family of all *symplectically vanishing-(p,q)-classes of* A is denoted by

$$(4.2.10)\quad F_\delta^{s'p,q}(A).$$

4.3. Nondegeneracy as Homological Characteristic Obstruction

The goal of this subsection is to point out Gauge Topology interpretations of Koszul homological functors of Lie algebroids.

In the spirit of Theorem 36 we aim to point out that the notion of Homological NONDEGENERACY is a machine to convert an OBSTRUCTION BY A $C^\infty(M)$*-module into an obstruction by a cohomology class.*

The generic picture is the following: the framwork is the category of injective Lie algebroids over the same base manifolds and isomorphisms of Lie algebroids.

We fix a Lie algebroid

$$A = (V, M, a, [-,-]).$$

Let (W, M) be a left module of the Lie algebroid A. We remind the meaning of the ingredients:

(1) The letter a stantds for a vector bundle homomorphism

$$a : \quad (V, M) \to (TM, M),$$

(2) $\Gamma(W)$ is a left module of the Lie algebra $(\Gamma(V), [-,-])$ and the following left action

$$\Gamma(V) \times \Gamma(W) \ni (s, w) \to s.w \in \Gamma(W)$$

is subject to the following requirements,

$$(4.3.0.a)\quad (s.fw)(p) = df(a(s)(p))w(p) + f(p)(s.w)(p);$$

$$(4.3.0.b)\quad ((fs).w)(p) = f(p)(sw)(p) \quad \forall (p, f, s, w) \in M \times C^\infty(M) \times \Gamma(V) \times \Gamma(W).$$

Of course all of the vector bundles $W^{p,q}$ *are left modules of the Lie algebroid* A.

We recall that $\Omega^k(M, W^{p,q})$ is the space of $W^{p,q}$-valued differential k-forms on M and

$$(4.3.1)\quad \Omega^k(A, W^{p,q}) = \Omega^k(M, W^{p,q}) \circ a^{\Lambda k}$$

We observe that a Koszul connection D on $W^{p,q}$ is a $\Omega^1(M, W^{p,q})$-valued first order differential operator

$$\Gamma(W^{p,q}) \ni \xi \to D\xi \in \Omega^1(M, W^{p,q}).$$

The canonical Koszul 1-cocycle which is associated with the connection D is the following mapping,

$$(4.3.2)\quad \Gamma(V) \ni s \to k_\infty(s) = L_s D.$$

The action of A on $\Omega^1(M, W^{p,q})$ is connection-preserving if on $(W^{p,q}, M)$ there exists a Koszul connection $D^\star$ that satisfies the following identity:

$$(4.3.3) \quad L_s D^\star = O \quad \forall s \in \Gamma(V).$$

Miscellaneous 1: The space of first order differential operators which are defined from (W, M) to $\Omega^1(M, W)$ is denoted by $DO^1(W, \Omega^1(M, W))$.

Let

$$(\Delta, s) \in DO^1(W, \Omega^1(M, W) \times \Gamma(V)$$

the differential operator

$$L_s\Delta \in DO^1(W, \Omega^1(M, W))$$

is defined as it follows,

$$\Omega^1(M, W) \ni (L_s\Delta)(w) = L_s(\Delta(w)) - \Delta(s.w).$$

Given a vector field

$$X \in \Gamma(TM)$$

the vector differential 1-form $L_s(\Delta(w))$ is defined as it follows,

$$L_s(\Delta(w))(X) = s.(\Delta(w)(X)) - (\Delta(w)([a(s), X]))).$$

Miscellaneous 2:

Remember that we implicitly use the following identification of vector bundles,

$$TM^\star \otimes W^\star \otimes W = Hom(TM \otimes W, W).$$

Therefore put

$$Bil(TM \times W; W) = \Gamma(TM^\star \otimes W^\star \otimes W).$$

Under the notation as in (4.3.2) it is easy to check that

$$L_s D \in Bil(TM \times W, W)$$

which means the following identity,

$$[(L_s D)(fX, w)](p) = [(L_s D(X, fw)](p)$$

$$= f(p)[(L_s D)(X, w)](p) \quad \forall (p, f, s, X, w) \in M \times C^\infty(M) \times \Gamma(V) \times \Gamma(TM) \times \Gamma(W).$$

Here is an interpretation of the degeneracy of Koszul homological functor.

Proposition 5. *Given a left module (W, M) of a Lie algebroid A the associated Koszul homological functor is (p,q)-degenerate if and only if the action of A on $(W^{p,q}, M)$ is connection preserving.*

proof

The (p,q)-degeneracy is sufficient.

The (p,q)-degeneracy means that if D is a Koszul connexion on $W^{p,q}$ then the cocycle

$$\Gamma(V) \ni s \to L_s D \in \Omega^1(A, W^{p,q})$$

is exact. Therefore there exists a $W^{p,q}$-valued diffrential 1-form

$$\theta \in \Omega^1(M, W^{p,q})$$

such that

$$L_sD = (d_{CE}\theta)(s)$$

$$= L_s\theta.$$

More precisely

$$s.(D_Y\xi) - D_{[a(s),Y]}\xi - D_Y(s.\xi) = s.(\theta(Y;\xi)) - \theta([a(s),Y];\xi) - \theta(Y;s.\xi),$$

$$\forall(s,Y,\xi) \subset \Gamma(V)\times\Gamma(TM)\times\Gamma(W^{p,q}).$$

The right side member of the equality above is C^∞(M)-bi-linear wih respect to (Y,ξ). Therefore

$$D^\star = D - \theta$$

is a Koszul connection on $W^{p,q}$ and

$$L_sD^\star = 0 \quad \forall s \in \Gamma(V).$$

Then the action of A on $W^{p,q}$ is connection preserving.

The (p,q)-degeneracy is necessary.

Let us assume that the action of A on (W,M) is connection preserving. Then there exists a Koszul connection ∇ such that

$$L_s\nabla = 0 \quad \forall s \in \Gamma(V).$$

According the notation as in (4.2.5) the following cocycle vanishes identically,

$$\Gamma(V) \ni s \to k^{p,q}_\infty(s)$$

$$= L_s\nabla \in \Omega^1(M, W^{p,q})$$

Therefore the Koszul homological class $F^{p,q}(g,\nabla)$ vanishes.

Proposition is proved.

We recall that a Lie algebroid

$$A = (V, M, a, [-,-])$$

is injective if a is injective.

Without the explicit statement of the contrary we will be dealing with injective Lie algebroids.

Definition 8. *A Koszul-Vinberg structure (, KV structure in short,) on a Lie algebroid*

$$A = (V, M, a, [-,-])$$

is a product

$$\Gamma(V)\times\Gamma(V) \ni (s,s') \to s.s' \in \Gamma(V)$$

which is subject to the folowing requirements:

$$(r.1): \quad (s.s').s'' - s.(s'.s'') = (s'.s).s'' - s'.(s.s'') \quad \forall(s,s',s'') \subset \Gamma(V),$$

$$(r.2): \quad s.s' - s'.s = [s,s'] \quad \forall(s,s') \subset \Gamma(V).$$

Warning. As a straight consequence of requirement (r.2) is the followng identity,

$$s.fs' = df(a(s))s' + fs.s' \quad \forall f \in C\infty(M).$$

We are now in position to state and to demonstrate the following three theorems.

Theorem 3. *For a Lie algebroid A to admit a Koszul-Vinberg structure it is sufficient and necessary that it be affinely-(1,1)-degenerate.*

Demonstration
The affinely-(1,1)-degeneracy is sufficient.
On a Lie algebroid

$$A = (V, M, a, [-,-])$$

we assume that A is affinely-(1,1)-degenerate. Then it carries a regular symmetric gauge structure (A, ∇) and a positive definite inner product g such that

$$F^{a,1,1}(g, \nabla) = 0.$$

Otherwise said the cocycle

$$J^g_\nabla(A) \ni s \to L_s D$$

is exact for any Koszul connection D on the tangent vector bundle (TM, M). Therefore given a gauge structure (TM, M, D) there exists

$$\theta \in \Gamma(TM^{1,1})$$

such that the $J^g_\nabla(A)$-action on (TM, M) is $D^\star$-preserving, where

$$D^\star = D - \theta.$$

Consequently, according to (3.2.8) one has the following inclusion

$$J^g_\nabla(A) \subset A^\star_D(TM).$$

Since $J^g_\nabla(A)$ is $C^\infty(M)$-module, by the virtue of Theorem 3.6 one has.

$$J^g_\nabla(A) = 0.$$

So on one hand we have the following equality,

$$rank(J_\nabla(A)) = rank(V).$$

which yields the following consequence,

$$Span_{C^\infty(M)}(J_\nabla(A)) = \Gamma(V).$$

On the other hand we have

$$R^\nabla(s, s'; s'') = 0 \quad \forall (s, s', s'') \subset J_\nabla(A).$$

Since the following arrow,

$$(s, s', s'') \to R^\nabla(s, s'; s'')$$

is $C^\infty(M)$-three-multi-linear we conclude that

$$R^\nabla(s, s'; s'') = 0 \quad \forall (, s, s', s'') \subset \Gamma(V).$$

To finish we define the Koszul-Vinberg product $s \star s'$ by putting

$$s \star s' = \nabla_s s' \quad (s, s') \subset \Gamma(V).$$

The affinely-(1,1)-degeneracy is necessary.

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Indeed let us assume that the Lie algebroid

$$A = (V, M, a, [-,-])$$

carries a Koszul-Vinberg structure

$$\Gamma(V) \times \Gamma(V) \ni (s, s') \to s \star s' \in \Gamma(V).$$

We recall that the Koszul-Vinberg product $s \star s'$ satisfies the following two identities,

$$(s, s'; s'') = (s', s; s''),$$

$$s \star s' - s' \star s = [s, s'],$$

here $(s, s'; s'')$ stands for the associator of the product $s \star s'$ and $[s, s']$ is the bracket of the Lie algebroid A.
We set

$$\nabla_{a(s)} s' = s \star s' \quad \forall (s, s') \subset \Gamma(V).$$

It is easy to check that (A, ∇) is affine structure structure on A

sSince A is injective we go to focus on the regular involutive distribution

$$a(V) \subset TM.$$

Let

$$F_x \subset M$$

be a leave of $a(V)$ through

$$x \in M.$$

Then F_x is endoved with locally flat

$$(F_x, D)$$

where

$$D_{a(s)} a(s') = a(\nabla_{a(s)} s').$$

Thus the couple

$$(a(V), D) = \{(F_x, D), \quad x \in M\}$$

is k-dimensional locally flat foliation of M.

By the machinaries of Haefliger pseudogroup there exists a foliated atlas of M whose local charts have the following form

$$U \ni x \to (X(x), Y(x)) \in R^k \times R^{m-k},$$

and a local chart change have the following form

$$(X, Y) \to (X'(X, Y), Y'(Y))$$

where $X'(X, Y)$ is affine function of X.

We already pointed out that elements of J_D are infinitesimal transformations of (F_x, D). Therefore one has

$$rank(J_D) = rank(a(V)).$$

It is easy to check the following identity,

$$D_{a(s)} D_{a('s')} a(s'') - D_{[D_{a(s)} a(s')]} a(s'') = a(\nabla_{a(s)} \nabla_{a(s')} s'' - \nabla_{[a(\nabla_{a(s)} s']} s''.$$

Based on this identity we conclude that

$$rank(J_\nabla) = rank(J_D)$$

$$= rank(V)$$

Therefore

$$F^{a,1,1}(\nabla) = O.$$

Thus A is affinely-(1,1)-degenerate. *Thereom is demonstrated.*

Theorem 4. *A Lie algebroid*

$$A = (V, M, a, [-,-])$$

admits a Koszul-Vinberg structure if and only it admits an affine structure.

Demonstration.

(i) Let us assume that A admits Koszul-Vinberg structure

$$\Gamma(V) \times \Gamma(V) \ni (s, s') \to s.s' \in \Gamma(V).$$

Then we defnite the symmetric gauge structure (A, ∇) by putting

$$\nabla_{a(s)} s' = s.s'.$$

Direct calculations yield

$$R^\nabla = 0$$

and

$$T^\nabla = 0.$$

Thus (A, ∇) is an affine structure on A.

(ii) Conversely given an affine structure (A, ∇) the Koszul-Vinberg product

$$(s, s') \to s.s'$$

is defined as it follows

$$s.s' = \nabla_{a(s)} s'$$

All of the axioms of Koszul-Vinberg structure are satisfied by this definition. *This ends the demonstration of Theorem*

Theorem 5. *For a Lie algebroid*

$$A = (V, M, a, [-,-])$$

to admit a symplectic structure it is sufficient and necessary to it to be symplectically-(1,1)-degenerate.

Proof of theorem 4.6.

Symplecticall-(1,1)-degeneracy is sufficient.

Let us assume that A is symplectically-(1,1)-degenerate. Then A admits a gauge structure (A, ∇) and a positive definite inner product g and a gauge structure (TM, M, D) which are subject to the following requirements:

the gauge equation

$$\nabla^g \otimes \phi - \phi \circ \nabla = 0$$

has a solution ϕ such that the $\Omega^1(M, TM^{1,1})$-valued cocycle

$$L(\nabla, \Phi^\star) \ni s \to L_s D \in \Omega^1(M, TM^{1,1})$$

is exact. Then there exists

$$\theta \in \Omega^1(M, TM^{1,1})$$

such that

$$L_s D = (d_{CE}\theta)(s)$$
$$= L_s\theta \quad \forall s \in L(\nabla, \Phi^\star).$$

Therefore if we put

$$D^\star = D - \theta,$$

therefore the action of $L(\nabla, \Phi^\star)$ on $\Omega^1(M, TM^{1,1})$ is $D^\star$-preserving. So on one hand one has the following inclusion

$$L(\nabla, \Phi^\star) \subset A_{D^\star}(TM)$$

and on the other hand we know that the vector space $L(\nabla, \Phi^\star)$ is left module of the associative algebra $C^\infty(M)$. Then by the virtue of Theorem 3.6 we deduce that

$$L(\nabla, \Phi^\star) = 0.$$

In final $\Phi^\star$ is an invertible solution of the gauge equation

$$\nabla^g \otimes \phi - \phi \circ \nabla = 0.$$

This situation yields two symplectic products on the vector bundle

$$V \to M,$$

$$\omega(s, s') = g(\Phi^\star(s), s')$$
$$\omega^\star(s, s') = g(\Phi^{\star-1}(s), s')$$

It is easy to check that

$$\nabla\omega = O,$$
$$\nabla^g\omega^\star = 0.$$

The sufficiency is proved.

Symplectically-(1,1)-degeneracy is necessary.

Let us assume that A has a symplectic product

$$\Gamma(V)^2 \ni (s, s') \to \omega(s, s') \in C^\infty(M).$$

Let (A, ∇^0) be a symmetric gauge structure on A. Using formalism as in [9] or as in [8] we define

$$N \in \Gamma(V^{2,1})$$

by the following formula,

$$\nabla^0_{a(s)}\omega(s', s'') = \omega(N(s, s'), s'').$$

We set

$$\nabla_{a(s)}s' = \nabla^0_{a(s)}s' + \frac{1}{3}(N(s, s') + N(s', s)).$$

Then the gauge structure (A,∇) is torsion free as well and it satisfies the following identity,

$$\nabla_a(s)\omega = 0 \quad \forall s \in \Gamma(V).$$

Using an auxiliary possitive inner product g we pose the gauge equation

$$\nabla^g \otimes \phi - \phi \circ \nabla = 0.$$

There exists

$$\phi \in J_{\nabla,g}$$

such that

$$g(\Phi^\star(s),s') = \omega(s,s') \quad \forall (s,s') \subset \Gamma(V).$$

Therfore, because the algebra $L(\nabla,\Phi^\star)$ is spaned by the kernel of $\Phi^\star$ one deduces that

$$L(\nabla,\Phi^\star) = 0$$

Consequently we get

$$F^{s,1,1}(\nabla) = 0.$$

In final A is symplectically-(1,1)-degenerate.

This ends the demonstration of theorem 4.6.

Warnings: Let us recall the following remarks.

Remark.1. We have pointed out that the following symplectic product

$$\omega(s,s') = g(\Phi^\star(s),s')$$

is ∇-parallel, viz

$$a(s)(\omega(s',s'')) - \omega(\nabla_{a(s)}s',s'') - \omega(s,\nabla_{a(s)}s'') = 0 \quad \forall (s,s',s'') \subset \Gamma(V).$$

Remark.2. The mapping $\Phi^{\star-1}$ satisfies the following requirement,

$$\nabla \otimes \Phi^{\star-1} - \Phi^{\star-1} \circ \nabla^g = 0.$$

As we already pointed that $\Phi^{\star-1}$ gives another symplectic product

$$\omega^\star(s,s') = g(\Phi^{\star-1}(s),s')$$

which verifies the following identity,

$$(\nabla^g_{a(s)}\omega^\star)(s',s'') = 0.$$

Remark.3. The triplet $(\omega,\omega',\Phi^\star)$ satisfies the following identity,

$$\omega(s,s') = \omega'(\Phi^{\star 2}(s),s').$$

Thus $\Phi^{\star 2}$ is the *recursion operator* for the couple (ω,ω'), [55].

Remark.4. Let us put

$$(L_S\omega)(s',s'') = a(s)(\omega(s',s'')) - \omega([s,s'],s'') - \omega(s',[s,s''])$$

If (A, ∇) is Koszul-Vinberg structure on A then we get the following identity,

$$(L_s\omega)(s', s'') + (L_{s'}\omega)(s'', s) + (L_{s''}\omega)(s, s') = 0.$$

Theorem 6. *On a Lie algebroid*

$$A = (V, M, a, [-,-])$$

for a Koszul conection ∇ to be metric connection it is sufficient and necessary to it to be metrically-(1,1)-degenerate.

Hint

On (V, M) we fix a positive definite inner product g. Under metrically-(1,1)-degenercy of (A, ∇) there exists

$$\phi \in J_{\nabla,g}(A)$$

such that the following $TM^{1,1}$-valued Chevalley-Eilenberg 1-cocycle

$$L(\nabla, \Phi^\star) \ni s \to L_s D \in \Omega^1(M, TM^{1,1})$$

is exact. Thus there exists

$$\theta'' \in \Omega^1(M, TM^{1,1})$$

such that the action of $L(\nabla, \Phi)$ on $\Omega^1(M, TM^{1,1})$ is D''-preserving where

$$D'' = D - \theta''.$$

Therefore one has the following inclusion

$$L(\nabla, \Phi) \subset A_{D''}(TM).$$

Since $L(\nabla, \Phi)$ is module of $C^\infty(M)$, Theorem 3.6 tells one that

$$L(\nabla, \Phi) = 0.$$

To pursuie one defines the following inner product on (V, M):

$$(q(g, \phi))(s, s') = g(\Phi(s), s') \quad \forall (s, s') \subset \Gamma(V).$$

This inner product satisfies the following identity,

$$a(s)(q(g, \phi))(s', s'')) - q(g, \phi)(\nabla_{a(s)} s', s'') - (q(g, \phi))(s', \nabla_{a(s)} s'') = 0.$$

Thus ∇ is metric connection on $(V, M, q(g, \phi))$.

Theorem 4.7 is proven

Below is the symplectic analoguous of Theorem 4.7

Theorem 7. *On a Lie algebroid*

$$A = (V, M, a, [-,-]),$$

For a Koszul connection ∇ to be symplectic connection it is sufficient and necessary to it to be symplectically-(1,1)-degenerate.

Hint: The demonstation exactly similar to the proposition 4.7.

4.4. Some Comments on Gauge Geometry

In the sections devoted to Gauge Geometry we have been concerned with the following questions:

Question(4.4.1): On a vector bundle (V, M), when is the Koszul connection ∇ metric connection?. This is nothing-else than Problem 1

Question (4.4.2): On an even rank vector bundle (V, M) when is the Koszul connection ∇ symplectic connection? This nothing-else than Problem 2.

Question (4.4.3): When does the Lie algebroid $(V, M, a, [-,-])$ admit a Koszul-Vinberg structure?

Warnings: Regarding Question (4.4.3) on tangent algebroids of differentiable manifolds the readers are referred to Goldman's survey [22] and to the bibliography therein. We also refer the readers to Smilie's search for obstructions, [51].

$(A\star)$: We have introduced gauge equations of a pair of gauge structures and we have involved solutions of gauge equations to discuss question (4.4.1) and question (4.4.2) in the context of Lie Algebroids.

$(A\star\star)$: We have introduced Hessian equation of a Lie Algebroid and we have involved it solutions to discuss question (4.4.3).

$(A\star\star\star)$: Our homological approach to question cannot be implimented on general vector bundles. In those cases what is actually availlable is direct calculation of the holonomy groups of gauge structures.

We have recalled a construction of Koszul to observe that every inner product on a Lie algebroid admit a unique symmetric metric connection. Consequently every Lie algebroid is metrically-(1,1)-degenerate.

A well known subsidiary question.

We have already reminded that the problem of metric connection has been widely discussed in the literature, see Richard Atkins[5] and and B. Schmidt [48]. Many authors raised the subsidiary ask of whether the connection determines the metrics.

Analoguous ask concerning the problem symplectic connection is relevant. We are in position answer affirmatively that on Lie algebroids.

On a Lie agebroid

$$A = (V, M, a, [-,-])$$

every gauge structure (A, ∇) determines the vector space $S_2^{\nabla}(V)$ of all inner products g such that

$$\nabla g = 0.$$

The solutions is our fundamental short exact sequence (2.6.7), viz

Theorem 8. *All ∇-parallel either singular inner products or definite inner products are provided by the following splitting short exact sequence which is linked with gauge equations*

$$0 \to \Omega_2^{\nabla}(A) \to J_{\nabla,g}(A) \to S_2^{\nabla}(A) \to 0.$$

WARNINGS.

Before pursuing the author underlines and invites the readers to keep in mind that the approach of the Differential Geometry (or of the Global Analysis on Manifolds) and that of the Gauge Geometry are [sourse <> target] of each other.

[DG»GG]:= Approach From the Differential Geometry realm to the Gauge Geometry realm.

In Differential Geometry (or in Global Analysis) on a manifold M we start from a G-structures

$$(4.4.4) \quad P \to M,$$

then we are interested in connections ∇ which are adapted to (4.4.4). This means that the holonomy group $H(\nabla)$ is subgroup of G.

[GG»DG]:= Approach From the Gauge Geometry realm to the Differential Geometry realm.

In Gauge Geometry as practiced in this work, we start from a Gauge Structure (V, M, ∇) *on a vector bundle*

$$(4.4.5) \quad V \to M,$$

then we are interested in G-structues to which ∇ *is adapted.*

Those are the meanings of **the both problem of metric connection and problem of symplectic connection.**

Our methods to study the approach [GG»DG] are based on ideas which consist of looking for obstructions of Homological nautre.

To make our methods sucessful, open problems we are interested in are associated with Lie subalgebras of sections of Lie algebroids. We aim to make that those Lie algebras be characteristic obstructions to solve the open problems they are associated with.

Those Lie algebras serve us to extend some canonical homological constructions introduced by JL Koszul [voir [33].

That is why we have introduced *Koszul homological series and their degeneracy.*

4.5. Koszul Homology Series and Their Degeneracy

The notion of Homological Degeneracy produces sufficient and necessary conditions for question (4.4.1), question (4.4.2) and question (4.4.3) on the categorry of injective Lie algebroids and isomorphisms of Lie algebroids over the same base manifold. In forthcoming sections we will impliment those machineries to study questions (4.4.1), (4.4.2) and (4.4.3) in the context of tangent Lie algebroids .

5. Supplements to Affine Structures of Lie Algebroids

This short section is devoted to motivate intersts in some basic homological objetcs. There is abundance of litterature on topics involving the homology of Lie algebroids, e.g [12,21]. Here we focus on the aspects homology on Lie algebroids that are implemented in this works, [33]

5.1. A conjecture of Muray Gerstenhaber

In [23] M. Gerstenhaber posed the following conjecture,

Every restrict theory of Deformation generates it proper theory of cohomology

This conjecture has been solved for the category of Koszul-Vinberg algebras and for their modules, [40]. This category is the algebraic versus of the hyperbolic structures in the sense of Koszul as in [33,40].

The next sections are devoted to recall aspects which are of interest to the questions which are studied in tis work.

5.2. Operational Tools

By *Operational tools* we mean notions that can be technically manipulated to obtain concrect solutions to problems such as the effective numerical resolutions of algebraic or differential equations. For instance how to product all Riemannian metric tensors which have the same Levi Civita connection ∇? All the special symplectic connections on a fixed Lie algebroid are parameterized by the first prolongation of the linear symplectic algebra $sp(n)$, [8,24]. Conversely, how to effectively product all the symplectic products which admit the same special symplectic connection ∇? In these regards we emphasize the Operational Tool nature of Gauge equations (2.5.3) and (2.5.4) which *linear of first order* as well as their fruits that are the shorts exact sequences (2.6.7) and (2.6.8). These tools can be *numerically manipulated by the computer.*

Seen from this angle, this work can be readed as a provision of operational tools for applied gauge geometry. This wink will be renewed in Section 7. See arrows (7.1.9), (7.1.9) as well as Theorem 7.5.

5.3. Some Major Structures on Lie Algebroids

We devote this short subsection to remind the major challenges. The framework of challenges we face is the category of real Lie algebroids

$$A = (V, M, a, [-,-]).$$

In a Lie algebroid A a gauge structure (A, ∇) may bear one of the following three labels:

L1: The Koszul connection ∇ is Affine connection.

L2: The Koszul connection ∇ is Metric connection.

L3: The Koszul connection ∇ is Symplectic connexion.

The challenge is to fnd a characteristic obstruction to each of these three labels.

6. Cohomolgy of Affine Algebroids and Their Modules. Some Examples of Applications

Let (A, ∇) be an affine structure on a Lie algebroid

$$A = (V, M, a, [-,-)].$$

We recall that ∇ is $\Omega^1(A, V^{1,1})$-valued first order differential operator

$$\Gamma(V) \ni s \to \nabla s \in \Omega^1(A, V^{1,1})$$

which is suject to the following requirements,

$$\nabla f.s = df.s + f\nabla s,$$

$$\nabla_{a(s)}s' - \nabla_{a(s')}s = [s, s'] \quad \forall (f, s, s') \in C^\infty(M) \times \Gamma^(V) \times \Gamma(V).$$

On the Lie algebroid A the affine structure (A, ∇) yields the structure of Koszul-Vinberg algebra whose product $s \star s'$ is defined as it follows

$$s.s' = \nabla_{a(s)}s' \quad \forall (s, s') \subset \Gamma(V).$$

6.1. Two Sided Modules of (A, ∇)

Definition 9. *A two sided-module of (A, ∇) is a vector bundle (W, M) endowed with two real bi-linear mapping*

$$(6.1.1) \quad \Gamma(V) \times \Gamma(W) \ni (s, \xi) \to s.\xi \in \Gamma(W),$$

$$(6.1.2) \quad \Gamma(W) \times \Gamma(V) \ni (\xi, s) \to s.\xi \in \Gamma(W);$$

these left and right actions are subject to the following three requiremets

$$(6.1.3) \quad (s, s'; \xi) = (s', s; \xi),$$

$$(6.1.4) \quad (s, \xi; s') = (\xi; s; s').$$

We recall that both the left side member and the right side member of identities above are the associator-like.

We go to introduce the following vector subspace

$$J(W) \subset \Gamma(W) :$$

$$\xi \in J(W)$$

if and only if

$$(6.1.5) \quad (s,s';\xi) = 0 \quad \forall (s,s') \subset \Gamma(V).$$

6.2. The W-Valued KV Cohomology of (A,∇).

The W-valued cochaine complex of affine structure (A,∇) is the following $\mathbb{Z}$-graded vector space

$$C_{KV}(\nabla,W) = \oplus_q C^q(\nabla,W),$$

$$C^q(\nabla,W) = Hom(\Gamma(V)^{\otimes q},\Gamma(W)) \quad \forall q > 0.$$

We use the operator δ that Albert Nijenhuis named *brut formula:*

$$C^q(\nabla,W) \ni F \to \delta F \in C^{q+1}(\nabla,W);$$

$$(6.2.1) \quad \delta\xi(s) = s.\xi - \xi.s \quad \forall (\xi,s) \subset \Gamma(W)\times\Gamma(V);$$

$$\begin{aligned}(6.2.2) \quad \delta F(s_0 \otimes ... \otimes s_q) \quad &= \quad \Sigma_{i<q}(-1)^i \quad [[s_i.F(..\otimes \hat{s_i} \otimes ..)\\ &+ \quad F(..\otimes \hat{s_i} \otimes ..\hat{s_q} \otimes s_i).s_q\\ &- \quad \Sigma_{j\neq i}F(..\otimes \hat{s_i} \otimes .. \otimes s_i.s_j \otimes ..)]]\end{aligned}$$

The q-th cohomology space of $C_{KV}(\nabla,W)$ is denoted by

$$H^q_{KV}(\nabla,W) = \frac{ker(\delta : C^q(\nabla,W) \to C^{q+1}(\nabla,W)}{\delta C^{q-1}})$$

The trivial vector bundle

$$R \times M \to M$$

is a left module of (A,∇) under the following left action,

$$s.f = df(a(s)) \quad \forall (s,f) \subset \Gamma(V)\times C^\infty(M).$$

Thus $J(C^\infty(M))$ is the vector space of affine functions, viz

$$(s,s,s';f) = 0 \quad \forall (s,s',f) \subset (\Gamma(V)^2 \times C^\infty(M).$$

The q-th real KV cohomolgy of the affine structure (A,∇) is denoted by

$$(6.2.3) \quad H^q_{KV}(\nabla,R) = \frac{ker(\delta : C^q(\nabla,C^\infty(M)) \to C^{q+1}(\nabla,C^\infty(M)))}{\delta C^{q-1}(\nabla,C^\infty(M))}$$

6.3. Left Module-Valued Total Cohomology of an Affine Structure on a Lie Algebroid

Let W be a left module of an affine structure (A,∇) on a Lie algebroid

$$A = (V,M,a,[-,-]).$$

Beside W-valued KV cohomology which reacalled in the precedent subsection there is another cohomology of (A,∇) with coefficients in W.
We consider the vector space $C_\tau(\nabla,W)$ whose $q-th$ homogeneous space is the following vector space,

$$C^q_\tau = Hom(\Gamma(V)^{\otimes q},\Gamma(W).$$

The operator

$$C^q_\tau \ni f \to d_\tau f \in C^{q+1}_\tau$$

is defined as it follow,

$$(6.3.1) \quad (d_\tau w)(s) = s.w \quad \forall w \in W.$$

$$(6.3.2) \quad (d_\tau f)(s_0 \otimes .. \otimes s_q) = \Sigma_{i\leq q}(-1)^i[s_i(f(.. \otimes \hat{s}_i \otimes ..) - \Sigma_{j\neq i} f(..\hat{s}_i \otimes .. \otimes s_i.s_j \otimes ..)]$$

Remember that in the right hand member of (6.3.2)

$$s_i.s_j = \nabla_{a(s_i}s_j.$$

The total q-th total cohomology space is denoted by

$$(6.3.3) \quad H^q_\tau(\nabla, W) = \frac{ker(d_\tau : C^q_\tau \to C^{q+1}\tau)}{d_\tau(C^{q-1}_\tau)}$$

When W is the trivail vector bundle

$$M \times R \to M$$

the corresponding total KV cohomology is denoted by

$$H^q_\tau(\nabla, R)$$

In the next subsection we are interested in the second cohomology spaces $H^2_{KV}(\nabla, V)$ and $H^2_{KV}(\nabla, R)$.

6.4. Links with Classical Chevalley-Eilenberg Cohomology and with De Rham Scalar Cohomolgy of the Le Algebroid A

Every left (A, ∇)-module W is a left module of the Lie algebroid $A = (V, M, a, [-,-])$. We define its space of q-cochains as it follows

$$C^q_{CE}(A, W) = Hom(\Gamma(V)^{\Lambda q}, \Gamma(W)).$$

We recall the operator

$$C^q_{CE}(A, W) \ni \Theta \to d_{CE}\Theta \in C^{q+1}_{CE}(A, W) :$$

$$(6.4.1) \quad (d_{CE}w)(s) = s.w \quad \forall w \in W.$$

$$(6.4.2) \quad (d_{CE}\Theta(s_0 \wedge .. \wedge s_q) = \Sigma_i \quad (-1)^i[s_i.\Theta(.. \wedge \hat{s}_i \wedge ..)]$$

$$+ \quad \Sigma_{i<j}(-1)^{i+j}\Theta([s_i, s_j] \wedge .. \wedge \hat{s}_i \wedge .. \wedge \hat{s}_j \wedge ..).$$

When W is the trivial vector bundle

$$R \times M \to M$$

The formula (6.4.2) is nothing but the De Rham operator, viz

$$(6.4.3) \quad (d_{dR}F)(s_0 \wedge .. \wedge s_q) = \Sigma_i \quad (-1)^i a(s_i).(F(.. \wedge \hat{s}_i \wedge ..)$$

$$+ \quad \Sigma_{i<j}F([s_i, s_j] \wedge .. \wedge \hat{s}_i \wedge .. \wedge \hat{s}_j \wedge ..)$$

It is easy to check the following claims:
The Chevalley-Eilenberg complex $(C_{CE}(A, W), d_{CE})$ is a subcomplex of the total KV complex $(C_\tau(\nabla, W))$, viz

$$(6.4.4) \quad (C_{CE}(A, W), d_{CE}) \subset (C_\tau(\nabla, W), d_\tau).$$

Let one set the following quotient complex,

$$(Q(A, W), d_Q) = (\frac{(C_\tau(A, W), d_{CE})}{(C_{CE}(A, W), d_\tau)}, d_Q)$$

then one has the following exact sequence of cochain complexes,

$$0 \to (C_{CE}((A,W),d_{CE})) \to (C_\tau(A,W),d_\tau) \to (Q(A),d_Q) \to 0.$$

Consequently the Chevalley-Eilenberg cochomology is linked with the total cohomology via the following long exact sequence,

$$\to H^p(Q(A,W)) \to H^{p+1}_{CE}(A,W) \to H^{p+1}_\tau(A,W) \to H^{p+1}(Q(A,W)) \to$$

We consider the case of the De Rham complex $(\Omega(A,R),d_{dR})$, viz

$$(6.4.5a) \quad (\Omega(A,R),d_{dR}) \subset (C_\tau(\nabla,R),d_\tau).$$

By setting the following quotient

$$(Q(A,R),d_Q) = (\frac{(C_\tau(\nabla,R),d_\tau)}{(\Omega(A,R),d_{dR})},d_Q),$$

one deduces the following link between real de Rham cohomology and total real KV cohomology of (A,∇),

$$(6.4.5b) \quad \to H^p(Q(A,R)) \to H^{p+1}_{dR}((A,R)) \to H^{p+1}_\tau(\nabla,R) \to H^{p+1}(Q(A,R) \to$$

Miscellaneous. Before pursuing we consider the following tensor products of cochain complexes

$$C(A,\nabla) = (\Omega(A,R),d_R) \otimes (C_\tau(\nabla,R),d_\tau).$$

It is bi-graded

$$C^{i,j}(A,\nabla) = \Omega^i(A,R) \otimes C^j_\tau(\nabla,R).$$

Thus given

$$\omega \otimes \theta \in C^{i,j}(A,\nabla)$$

the coboundary operator of $C(A,\nabla)$*, namely d is defined as it follows,*

$$d(\omega \otimes \theta) = d_R\omega \otimes \theta + (-1)^i \omega \otimes d_\tau\theta.$$

The bigrading $C^{i,j}(A,\nabla)$ *is bounded in i. Thus we get an example double complex, viz* $(C(A,\nabla),d)$*, whose total cohomology is easily calculable by using spectral sequences, see useful details in Moore C. C. and Schocher C.[39]*

Some among these links we just pointed out will be implimented to discuss a problem in the Differential Topology which has been raised by E. Ghys, cf [38]. We remind that this problem is *How to produce all Riemannian foliations.*

We introduce the following canonical relationship

$$(6.4.6) \quad H^2_{KV}(\nabla,R) \to H^2_{dR}(A,R).$$

Relation (6.4.6) is produced by the following correspondence,

$$C^2_{KV}(\nabla,R) \ni \Theta \to \Lambda_\Theta \in \Omega^2(A,R)$$

where

$$(6.4.7) \quad \Lambda_\Theta(s,s') = \Theta(s,s') - \Theta(s',s).$$

Warning;(6.4.7) may be implimented to investigate symplectic structure in a Lie algebroid. The idea is based on the following claim:

Lemma 2. *Assume that*

$$\delta\Theta = O,$$

then the vector subspace

$$L(\Theta) = ker(\Lambda_\Theta)$$

is Lie subalgebra of the Lie algebra $(\Gamma(V), [-,-])$.
Furthermore, the de Rham cohomology class

$$[\Lambda] \in H^2_{dR}(A, R)$$

depends only on the KV cohomology class

$$[\Theta] \in H^2_{KV}(\nabla, R).$$

Hint
Direct calculations yield the following result,

$$\Sigma_{cyclic}\delta\Theta(s, s', s'') = 2d_{dR}\Lambda(s, s', s'').$$

Then, Λ is de Rham closed. Thus $ker(\Lambda)$ is in involution.
The vector space of 2-cocycles of a cochain complex C is denoted by $Z^2(C)$.
Then the Lemma 6.2 yields the following linear mapping

$$(6.4.8) \quad Z^2_{KV}(A, \nabla) \ni \Theta \rightarrow \Lambda_\Theta \in Z^2_{dR}(A, R).$$

The arrow that will be used for our purpose is the following,

$$(6.4.9) \quad Z^2_{KV}(A, \nabla) \ni \Theta \rightarrow L(\Theta) \subset \Gamma(V).$$

Therefore under the notation already used one introduce the homological series

$$(6.4.10) \quad Z^2_{KV}(A, \nabla) \ni \Theta \rightarrow \left\{[k^{p,q}_\infty(L(\Theta))] \in H^1_{CE}(L(\Theta), TM^{p,q})\right\}$$

Functor (6.4.10) is called (p,q)-degenerate if there exists a 2-cocycle Θ such that

$$[k^{p,q}_\infty(L(\Theta))] = 0.$$

Without going into details here is an interesting claim:

Proposition 6. *If (6.4.10) is (1,1)-degenerate then there exists a 2-cocycle* Θ *whose skew symmetric part* Λ_Θ *is symplectic product on* (A, ∇).

Hint: One applies Theorem 3.6 to $L(\Theta)$.

6.5. Hessian Structure on Affine Structure

Definition 10. *A Hessian structure on* (A, ∇) *is a definite symmetric 2-cocycle*

$$g \in Z^2_{KV}(A, \nabla)$$

On an affine structure (A, ∇) it is easy to check the proposition.

Proposition 7. *On an affine structure (A, ∇) the following assertions are equivalent:*
(i) The inner product g is a Hessian structure on (A, ∇);
(ii) (A, ∇^g) is an affine structure as well and g is a Hessian structure on (A, ∇^g).

Warnings: Proposition above has another interesting rephrasing.

Indeed we already mentioned two dynamics on the category of gauge structures *Gau(V,M)*: the gauge group *GL(V,M)* and the metric group *GM(V,M)*. On a Lie algebroid $(V, M, a, [-,-])$ the curvature of connection is both a gauge invariant and a metric invariant in the meanings that we go to make precise.

Let (A, ∇) be a gauge structure on an algebroid A. For

$$\Phi \in GL(V,M),$$

$$\Phi.\nabla = \Phi \circ \nabla \circ \Phi^{-1},$$

Then

$$(6.5.1) \quad R^{\Phi.\nabla} = \Phi \circ R^{\nabla} \circ \Phi^{-1}.$$

Let g be an inner product on (V, M); R^{∇} and R^{∇^g} are subject to the following identity,

$$(6.5.2) \quad g(R^{\nabla}(s, s'; s''), s''') + g(s'', R^{\nabla^g}(s, s' : s''')) = 0.$$

Identities (4.6.1) and (4.6.2) tell the following (smal) lemma.

Lemma 3. *The following three assumptions are equivalent,*

$$(a.1) \quad R^{\nabla} = 0,$$

$$(a.2) \quad R^{\Phi.\nabla} = 0,$$

$$(a.3) \quad R^{\nabla^g} = 0.$$

In contrast with the curvature, the torsion of connection is neither a gauge invariant nor a metric invariant in the sense of the Lemma above.

Thus arises the following question:

Given a gauge structure (A, ∇) and an inner product (V, M, g) on a Lie algebroid

$$A = (V, M, a, [-,-])$$

under what conditon on g is the g-dual of ∇ torsion free?

Further, in this work we will underline a link between the question just mentioned and the theorem of M.Gromov which states that any open differentiable manifold M which admits an almost complex structure admits a symplectic structure.

On an Lie algebroid

$$A = (V, M, a, [-,-])$$

Let one be given an inner product (V, M, g), a gauge structure (V, M, ∇) and its g-dual (V, M, ∇^g).

On (V, M, g, ∇) we write the expression looking like the formula (4.1.2),viz

$$\delta^{\nabla} g(s, s'; s'') = a(s).g(s', s'') - g(\nabla_{a(s)}, s'')$$

$$- \quad g(s', \nabla_{a(s)} s'') - a(s').g(s, s'') + g(\nabla_{a(s')} s, s'') + g(s, \nabla_{a(s')} s'').$$

By direct calculations it is easy to check the following identity,

$$\delta^{\nabla} g(s,s';s'') = g(T^{\nabla^g}(s,s') - T^{\nabla}(s,s'),s'').$$

Thus the conclusion we will be interested is the following (small) lemma.

Lemma 4. *Under the notation above the following assumptions are equivalent*

$$(b.1) \quad T^{\nabla^g} - T^{\nabla} = 0,$$

$$(b.2) \quad \delta^{\nabla} g = 0.$$

A remaks. Let one consider the following triplet,

$$T(g,\nabla) = \left\{\delta^{\nabla} g, T^{\nabla}, T^{\nabla^g}\right\}$$

By the virtue of Lemma 6.7 the nullity of two components of $T(g,\nabla)$ leads to the nullity of the third component. We are therefore faced with the following two relevant configurations determined by the nullity or the nonnullity of the curvature R^{∇}.

If

$$(cf.1.1) \quad (\delta^{\nabla} g, T^{\nabla}) = (0,0)$$

$$(cf.1.2) \quad R^{\nabla} \neq 0,$$

then the pair (g,∇) is (called) *Statistical sructure*.

If

$$(cf.2.1) \quad (\delta^{\nabla} g, T^{\nabla}) = (0,0),$$

$$(cf.2.2) \quad R^{\nabla} = 0,$$

then the pair (g,∇) is (called) *Hessian structure*.

Notation: The set of all Hessian structures on affine sructure (A,∇) is denoted by $Hess(A,\nabla)$.

Definition 11. *A Hessian structure*

$$(g,\nabla) \in Hess(A,\nabla)$$

is called almost hyperbolic if

$$[g] = O \in H^2_{KV}(\nabla,R).$$

Remember that

$$[g] = g + \delta^{\nabla} C^1_{KV}(\nabla,R).$$

Thereby it is easy to check the following statement.

Proposition 8. *The set of all almost hyperbolic Hessian structures $AHH(A,\nabla)$ is an Open cone in the vector space $B^2_{KV}(A,\nabla)$.*

Here

$$B^2_{KV}(A,\nabla) = \delta^{\nabla} C^1_{KV}(\nabla,R).$$

The proposition is an analoguous of a Theorem of J-L Koszul see [32].

Theorem 9. *On the tangent Lie algebroid $(TM, M, 1_{TM}, [-,-])$ every hyperbolic structure admits non trivial deformations.*

Reminding.

In our context $(V, M, a, [-,-])$ is injective. Then the distribution

$$a(V) \subset TM$$

is a regular foliation whose leaves are equipped with the Koszul connection D which defined by the following formula,

$$(6.5.3) \quad D \circ a = a \circ \nabla.$$

Formula (6.5.3) has the following meaning,

$$D_{a(s)}a(s') = a(\nabla_{a(s)}s') \quad \forall (s, s') \in \Gamma(V).$$

Along every leaf of a(V) the Koszul connection D define a locally flat structure in the sense of Jean-Louis Koszul [32].

One also puts

$$(6.5.4) \quad g * (a(s), a(s')) = g(s, s').$$

Remember that the mapping a is vector bundle isomorphism of (V, M) on $(a(V), M)$. Thus given

$$(X, Y) \subset \Gamma(a(V))$$

the formula (6.5.3) tells us that

$$D_X Y = a(\nabla_X a^{-1}(Y))$$

and

$$g * (X, Y) = g(a^{-1}(X), a^{-1}(Y)).$$

Therefore $(a(V), D)$ is affine structure on the following Lie algebroid

$$A* = (a(V), M, 1_{a(V)}, [-,-]_P)$$

Furthermore $g*$ is a Hessian structure on $(A*, D)$ and $[-,-]_P$ stands for Poisson bracket of vector fields. Further it is easy to check that

$$[]g*] = 0 \in H^2_{KV}(\nabla, R)$$

if and only if

$$[g*] = 0 \in H^2_{KV}(D, R).$$

6.6. $H^2_{KV}(\nabla, R))$ and Hessian Strucures on Lie Algebroids

Remindings. We recall that given an n-dimensional locally flat manifold (M, ∇) and a fixed point $p \in M$, the universal covering of M is the following quotient

$$\tilde{M} = \frac{C^0((0, [0, 1]), (p, M))}{H}$$

where H stands for the fixed ends homotopy relation. Let τ_∇ by the parallel transport along the paths

$$[0, 1] \ni t \to c(t) \in M,$$

then we put

$$D([c]) = \int_0^1 \tau_\nabla^{-1}(\frac{dc(s)}{ds})ds \in T_pM.$$

Therefore the covering mapping is

$$\tilde{M} \ni [c] \to c(1) \in M,$$

and the developping mapping is

$$\tilde{M} \ni [c] \to D([c]) \in T_pM.$$

We put

$$\Omega = D(\tilde{M}) \subset T_pM.$$

Then (M, ∇) is called hyperbolic if Ω is convex and does not contain any straight line. To learn more the readers are referred to Koszul [32] Warnings. Before continuing we remind that in the context of Differential Geometry, Hessian structures play significant roles in the Information Geometry, see [2,41,44,49]. They also play important roles in the Lie Group Theory of Heath following Jean Marie Souriau [7].

The following implimentation of $H^2_{KV}(\nabla, R)$ is a foliation versus of a theorem of Koszul, see [32], Theorem 3.

Proposition 9. *Under the notation used above, we assume that the following assertions hold:*
(i) the Hessian structure g is positif,
(ii) $[g] = 0 \in H^2_{KV}(\nabla, R)$.
Then every compact leaf of $a(V)$ is hyperbolic in the sense we just reminded above.

Hint.

(a) We focus on a compact leaf F whose fundamental group is noted $\pi(F)$. The couple (F, D) is a localy flat structure whose universal covering is noted $(F^\star, D^\star)$.

(b) The developping mapping is noted Δ. The affine holomony representaion of (F, D) is denoted h. Thus we put

$$\Gamma = h(\pi(F)) \subset Aff(k).$$

Here F is k-dimensional.

(c) To conclude one takes into account that the data below saitsfy the following identity,

$$\Delta(u.x)) = \gamma(u).\Delta(x) \quad \forall (u, x) \subset \pi(F) \times F^\star).$$

6.7. $H^2_{KV}(\nabla, V)$ and Deformations of of (A, ∇)

One considers series of symmetric 2-cochains

$$\theta(t) = \Sigma_{q\in\mathbb{N}} t^q \theta_q$$

with

$$\theta_q \in C^2(\nabla, V) \quad and \quad \theta_q(s, s') = \theta_q(s', s).$$

Let us put

$$\nabla(t) = \nabla + \theta(t)$$

The question is under what condition the couples $(A, \nabla(t))$ form series of affine structures on A.

It is easy to check the following statement.

Proposition 10. *A necessary condition for $(A, \nabla(t))$ to be series of affine structures on A is*

$$\delta_{KV}\theta_1 = 0.$$

Proposition above is a nod to $H^2_{KV}(\nabla, V)$. Its analogues are well known in Theory of deformations of algebraic structures, see Nijenhis-Richardson [43] and S. Piper [46] for the theory of deformations of algebraic structures and see also M. Kontsevich [30] for applcations of the theory of deformations of associative algebras to the quantization of Poisson structures.

6.8. $H^2_{KV}(\nabla, W)$ and Extensions of Affine Structures on Lie Algebroids

We consider an affine structure (A, ∇) on a Lie algebroid

$$A = (V, M, a, [-,-]).$$

Let (W, M) be a vector bundle. We assume that (W, M) is a left module of the affine structure (A, ∇) the left action of which is denoted as it follows,

$$\Gamma(V) \times \Gamma(W) \ni (s, w) \to s.w \in \Gamma(W).$$

We are using notation (6.1.1) and the KV complex $C_{KV}(\nabla, W)$ as in (6.2.2). Thus the following requirement is satisfied,

$$(s, s'; w) = (s', s; w)$$

where

$$(s, s'; w) = (\nabla_{a(s)} s').w - s.(s'.w)$$

We consider the following exact sequence of vector bundles,

$$(6.8.1) \quad 0 \to (W, M) \to (W \oplus V, M) \to (V, M) \to 0.$$

Using a W-valued 2-cochain

$$\theta \in C^2_{KV}(\nabla, W)$$

we define the following product on $\Gamma(W \oplus V)$:

$$(6.8.2) \quad (w, s).(w', s') = (s.w' + \theta(s, s'), \nabla_{a(s)} s').$$

Here the question is under what requirements the product (6.8.2) is a Koszul-Vinberg product on the vector bundle $W \oplus V$.

Without going into details, we state the following results; see [40] to learn more.

Theorem 10. *(a) The formula (6.8.2) define a Koszul-Vinberg structure on $W \oplus V$ if and only if θ is KV 2-cocyle of the cochain complex $C_{KV}(\nabla, W)$ as in subsection 6.2.*
(b) $Ext((A, \nabla), W)$ being the set of all equivalent classes of extensions of (A, ∇) by W, there is one to one correspondence between $Ext((A, \nabla), W)$ and $H^2_{KV}(\nabla, W)$.

Here is a useful overview.
The first six sections of which this is the sixth are devoted to Gauge Geometry on general Lie algebroids. To enlighten non specialist readers, it is useful for us to recall our four Fundamentals.
(F1): Our first Fundamental is to state the main open problems which are studied on general Lie algebroids. Our approach to these open problems is based on the search for Characteristic Obstructions; that is to say notions which provide necessary and sufficient criteria.
(F2): Our second Fundamental is to clear obstructions which are either Lie sub-algebroids or Lie sub-algebras of sections of Lie algebroids. These obstructions of vector nature are from gauge differential gauge equations (253) and (254). The active arms of gauge equations are the two exact sequences (2.6.7) and (2.6.8).
(F3): Our third Fundamental is Hessian equation (3.2.2) and the object which is defined as in (3.2.3).
These Fundamentals are used to associate an Obstruction of vector nature with each of open problems. Then with each obstruction of vector nature we associate an object of homological nature which is called Koszul homological series.
(F4): Our fourth Fundamental is the introcducton of the notion of degeneracy of Koszul homological series. The notion of degeneracy converts the nullity of vector spaces into degeneracy of Koszul homological series.

(F5): Our fifth Fundamental. In final, from the differential Geometry point of view, Geometric propserties of Koszul connections emerge from the *dynamical propersties* of their holonomy groups.

Under the gauge geometry vewpoint, Geometric properties of Koszul connections emerge from thier homological properties via the notion of *degeneracy of their Koszul homological series.*

6.9. Some Major Gauge Structures on Lie Algebroids

Now we are position to introduce some relevant gauge structures on Lie algebroids. These notions are based on the notion of Koszul homology series and on their (p,q)-degeneracy.

We are in position to state the main definitions

Definition 12. *On a Lie algebroid*

$$A = (V, M, a, [-,-])$$

(6.8.1) $F_\delta^{a,1.1}(A)$ *is called affine structure on* A*;*
(6.8.2) $F_\delta^{m,1,1}(A)$ *is called metric structure on* A*;*
(6.8.3) $F_\delta^{s,1,1}(A)$ *is called Fedosov structure on* A*.*

Without going into details we state the following theorem.

Theorem 11. *On a Lie algebroid*

$$A = (V, M, a, [-,-])$$

$\left\{F_\delta^{a,1,1}(A)\right\}$ *is gauge invariant,*
$\left\{F_\delta^{m,1,1}(A)\right\}$*and*$\left\{, F_\delta^{s,1,1}(A)\right\}$ *are both gauge invariants and metric invarants.*

Hint: The tools which are used to prove this theorem are formulas (2.1.7), (2.4.5) and the exact sequences (2.6.7) and (2.6.8).

Warnings:
(a) It is to be noticed that every Lie algebroid is metrically-(1,1)-degenerate.
(b) In contrast with (a), every non orientable Lie algebroid is symplectically (1,1)-nondegenerate.
(c) whatever the differential manifold M*, the tangent Lie algebroid* $(TT^\star M, T^\star M)$ *is symplectically (1,1)-degenerate.*

7. Tangent Lie Algebroids. (1,1)-Nondegeneracy and Production of Labelled Foliations

Apart from the introduction the central subject in the six precedent secions is the geometry of gauge structures on the category of general vector bundles over the same base manifold.

This Section 7 is mainly devoted to study the tangent vector bundles which are canonically Lie algebroids under the Poisson bracket of vector fields. The central issue is to realize and to complete the machineries which are developped in the precedent first six sections that are devoted to general Lie algeboids.

MISCELLANEOUS.

Among major facts we pointed out are relationships betweeen (1,1)-Nondegeneracy and Productions of Labeled Foliations, the most studied of which are Riemannian Foliations: [20,26,38,47]

In fact, in addition to gauge dynamics *GL(TM,M)* and metric dynamics *GM(TM,M)* the category gauge structures on a tangent bundle (TM, M) is acted by the group of diffeomeorphisms of M that we note $Diff^\star(TM, M)$. The invariants of $Diff^\star(TM, M)$ are called *Geometric invariants*.

For instance while torsion T^∇ is neither gauge invariant nor metric invarant on general Lie algebroids, it is geometric invariant on tangent Lie algebroids. So on general Lie algebroids we encounter situations where the homological degeneracy provides sufficient and necessary criteria to answer affirmatively to certain questions.

We aim to apply those materials to regular Lie subalgebroids of $(TM, M, 1_TM, [-,-])$. Those regular Lie subgroupoids are nothing but involutive regular distributions, viz regular foliations.

7.1. Applications to Tangent Lie Algebroids $A(M)$

The context of this section 7 is the category tangent Lie algebroids

$$A(M) = (TM, , M, 1_{TM}, [-,-]).$$

The bracket $[X, Y]$ is the Poisson bracket of vector fields.

Withough explicit statement of the contrary we will be dealing of *SGau(TM,M). We go to implement the machineries which are developped in the first six sections above*.

We start from the following data.

The gauge equations and the splitting exact sequences

$$(7.1.1)\quad 0 \to \Omega_2^{\nabla}(A(M)) \to J_{\nabla,g}(A(M)) \to S_2^{\nabla}(A(M)) \to 0, \quad \nabla \in \mathit{SGau(TM,M)}.$$

We use the notations as in Definition 6.13. We focus the following data:

the family of Koszul affinely vaniching-(1,1)-classes

$$(7.1.2)\quad F_{\delta}^{a,1,1}(A(M)),$$

the family of Koszul metrically vanishing-(1,1)-classes

$$(7.1.3)\quad F_{\delta}^{m,1,1}(A(M)),$$

the family of Koszul symplectically vanishing-(1,1)-classes

$$(7.1.4)\quad F_{\delta}^{s,1,1}(A(M)).$$

Definition 13. *On the Lie algebroid $A(M)$*
(7.1.2) is called affine structure on A(M).
(7.1.3) is called metric structure on $A(M)$.
(7.1.4) is called Fedosov structure on $A(M)$.

Regarding (7.1.2), concernng the search for obstructions to the existence of affine structures see Smilie J. [51]. For a survey on the conjecture of Markus and related items see the survey of W.M. Goldman [22], see also Y. Carrière [10] Before further implimenting the sequence (7.1.1) we recall two definitions.

Definition 14. *A Riemannian foliation on the manifold M is singular inner product on $A(M)$, g which is subject to the following two requirements*
(r.1) rank(g) = constant,
(r.2) $i_X g = 0$ implies $L_X g = 0$.

. Warning: In the classic defintion of Riemannian foliation as in Moerdijk-MrCul [37], in Molino [38], in Reinhart [47], in Haefliger [26], Ghys [20], the singulier inner product g is assumed to be possitive, viz

$$0 \leq g(X, X) \quad \forall X \in \Gamma(TM).$$

We omitted this restriction in order to extend the theory of Riemannian foliations in Riemannan manifolds with positive signature.

Definition 15. *A symplectic foliation on M is a singular symplectic product on $A(M)$, ω which is subject to the following requirements,*
(r.1) $rank(\omega) = constant$,
(r.2) ω is de Rham closed.

E. Ghys raised the problem of *How to Construct Riemannian Foliations, see [20], appendix E of [38].*
The analoguous problem versus symplectic foliations is *how to produce all symplectic foliations.*
One of our main aims is to completely solve the problem of producing all Riemannian foliations, [26,38,47] Exact sequences as in (7.1.1) take us in postion to completely solve the problem of production of all Riemannian foliations. Notation is that we use is as in fomulas (2.6.3) and (2.6.4). Regardind Riemannian foliations here is our key statement,

Theorem 12. *Given a gauge structure*

$$(A(M), \nabla) \in SGau(TM,M)$$

and an auxiliary positive inner product g we take into account the exact sequence (7.1.1), viz

$$O \to \Omega_2^{\nabla}(A(M)) \to J_{\nabla,g}(A(M)) \to S_2^{\nabla}(A(M)) \to 0.$$

(1) The correspondence

$$J_{\nabla,g} \ni \phi \to q(g,\phi)$$

as in our formula (2.6.3) sends $J_{\nabla,g}(A(M))$ onto the set of all totally ∇-geodesic Riemannian foliations $S_2^{\nabla}(A(M))$.
(2) Conversely all Riemannian foliations are obtained by this process.

Demonstration.
(d.1) Since $(A(M), \nabla)$ is symmetric, by the virtue of Lemma 2.4 and proposition 2.7 the rank of $q(g,\phi)$ is constant and

$$(L_s q(g,\phi))(s', s'') = q(g,\phi)(\nabla'_s s, s'') + q(g,\phi)(s', \nabla_s{}'' s)$$

$$= 0 \quad \forall s \in Ker(q(g,\phi)).$$

Then $q(g,\phi)$ is totally ∇-geodesic Riemannian foliation.
(d.2) The proof of the converse is based on works of S. E. Kozlov[34] and those of S. N. Kupeli [35] on singular metric tensors. These results of Kozlov and those of Kupeli insure that if a symmetric bi-linear form q satisfies conditions (r.1) and (r.2) as in definition 7.2 then there exists a symmetric Koszul connection ∇ such that

$$\nabla_s q = O \quad \forall s \in \Gamma(TM).$$

In [34] what is called *acyclicity condition* is the property (r.2) as in Definition 7.2.
Thus given an auxiliary positve inner product g there exists

$$\Phi \in \Gamma(End(TM, M))$$

such that

$$q(s, s') = g(\Phi(s), s') \quad \forall (s, s') \subset \Gamma(TM, M).$$

Further one has the following identity,

$$s.g(\Phi(s'), s'') = g(\Phi(\nabla_s s', s'') + g(\Phi(s'), \nabla_s s'')$$

$$= g(\nabla_s^g \Phi(s'), s'') + g(\Phi(s'), \nabla_s s'').$$

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Thus we get

$$\nabla_s^g \Phi(s') - \Phi(\nabla_s s') = 0.$$

Then in final one obtains the following identity,

$$\Phi \in J_{\nabla,g}(A(M)).$$

Theorem is demonstrated.

7.2. Gauge Equations and Productions of All Riemannian Foliations

Let us go back to the exact sequence (7.1.1), viz

$$O \to \Omega_2^{\nabla}(A(M)) \to J_{\nabla,g}(A(M)) \to S_2^{\nabla}(A(M)) \to O.$$

Let us set

$$(7.1.5) \quad RF(A(M)) = \cup_{\{\nabla \in SGau(A(M)\}} S_2^{\nabla}(A(M)),$$

$$(7.1.6) \quad J_g(A(M)) = \cup_{\{\nabla \in SGau(A(M))\} J_{\nabla,g}(A(M))},$$

$$(7.1.7) \quad SF(A(M)) = \cup_{\{\nabla \in SGau(A(M))\}} \Omega_2^{\nabla}(A(M)).$$

We take into account the arrow (2.6.6.bis) and the identification as in (2.6.6.ter) to define the following arrows

$$(7.1.8) \quad J_g(A(M)) \ni \phi \to \Phi \in RF(A(M)).$$

$$(7.1.9) \quad J_g(A(M)) \ni \phi \to \Phi^{\star} \in SF(A(M)).$$

$$(7.1.10) \quad SF_{\delta}^{s,1,1}(M) = F_{\delta}^{s,1,1}(A(M)) \cap SF(A(M)).$$

We involve works including that of Rrichard Atkins [5], and that of N. Kupeli [29] to state a corollary of Theorem 7.4 which answers the question :*How to product Riemannian foliations*, see appendix E as in [38]]

Theorem 13. *Results we plan highlighting are based on Theorem 7.4, on (2.6.6.bis) and on (2.6.6.ter).*
(1) The arrow (7.1.8) is a Production of All Riemannian Foliations on the manifold M.
(2) The arrow (7.1.9) is a Production of Symplectic foliations on the manifold M.

Below we are intersted in a few supplements to definition 7.1.

Definition 16. $SF_{\delta}^{s,1,1}(M)$ *as in (7.1.10) consists on the Trivial Symplectic Foliation on M (, i.e. the foliation whose leaves are single points of M,) and is called symplectic structure on the manifold M.*

Thus Quantitatively the arrow (7.1.9) is the Production of all symplectic structures on M.

Warnings. Theorem 7.5 deals with the Quantitative Study of Riemannian foliations. Our approach aims to produce all Riemannian foliations in (pseudo) Riemannian Geometry while the classical litterature deals with Riemannian foliations on positive definite Riemannian Geometry only.
Regarding the Qualitative Study of Riemannian foliations on positive definite Riemannian Geometry the readers are reffered to the survey of André Haefliger in Bourbaki semainar [26] as well as the Gys's appendix E in [38].
Regarding the question of how to produce all symplectic foliations, exact sequence (7.1.1) only gives a partial answer which as follows

Proposition 11. *Taking into account (2.6.4) and splitting exact (7.1.1) the following correspondence*

$$J_{\nabla,g}(A(M)) \ni \phi \to \omega(g,\phi) \in \Omega_2^{\nabla}(A(M))$$

sends $J^{\nabla,g}(A(M))$ *onto the space of all totally* ∇*-totally geodesic symplectic foliations.*

Hint: The proof is similar part (d.1) as in the demonstration of Theorem 7.4.

8. Geometric Invariants on *Gau(TM,M)*

8.1. Three Dynamics and Their Invariants

Remember the following dynamics on *Gau(TM,M)*

(8.2.1) The action of the gauge group GL(TM,M) yielding *gauge invariants*.

(8.2.2) The action of the metric group GM(TM,M) yielding *metric invariats.*

(8.2.3) The action of the group of diffeomorphisms $Diff^{\star}(TM,M)$ yielding *geometric invariants.*

Every gauge structure (TM, M, ∇) is associated with two objects which are

(8.2.4) Torsion T^{∇}.

(8.2.5) Curvature R^{∇}.

Given a torsion free gauge structure (TM, M, ∇) we are particularly interested in the following possible properties,

(8.2.6) Koszul affinely-(1,1)-degeneracy,

(8.2.7) Koszuk metrically-(1,1)-degeneracy,

(8.2.8) Koszul symplectically-(1,1)-degeneracy.

From the dynamics as in

$$\{(8.2.1), (8.2.2), (8.2.3)\}$$

and from the data as in

$$\{(8.2.4), (8.2.5), (8.2.6), 8.2.7), (7.2.8)\},$$

arises the question of which data are invariants of which dynamics.

Without going into details we summarize some examples.

Are gauge invariants the data as in $\{(8.2.5), (8.2.7), (8.2.8)\}$.

This follows from (2.6.3), (2.6.4), (2.6.5), (2.6.6).

Are metric invariants the data as in $\{(8.2.5), (8.2.7), (8.2.8)\}$.

This follote ws form both Lemma 67 and (6.5.1).

Are geometric invariants the data $\{(8.2.4), (8.2.5), (8.2.6), (8.2.7), (8.2.8)\}$.

That is based on (6.5.1).

9. Special Fedosov Manifolds and Kaehler Structures. Connections with the Information Geometry

This section is devoted to a few topological properties of *special Fedosov structures.* These topological properties arise from the global analysis of gauge equations, viz the study of analytic properties of the $J_{\nabla,g}(M)^{',s}$.

Before defining *them* and to motivate some interest in *special Fedosov manifolds* we state one of their topological properties.

Theorem 14. *Every odd Betti number of a compct special Fedosov manifold* (M, ω, ∇)*,* $b_{2i+1}(M)$ *is even.*

9.1. Statistical Fedosov Manifolds

To start we ntroduce the following definition.

Definition 17. *A statistical Fedosov manifold is a quadruplet* (M, g, ω, ∇) *suject to the following requirements:*
(9.1.1) (M, g, ∇) *is a statistical manifold,*
(9.1.2) (M, ω, ∇) *is a Fedosov manifold.*

Let $(M, \nabla^{\star})$ be the g-dual of (M, ∇). We recall that the Koszul connection $\nabla^{\star}$ is defined by the following identity,

$$g(\nabla^{\star}_X Y, Z) = Xg(Y, Z) - g(Y, \nabla_X Z).$$

According to our previous notation $J_{\nabla,g}(M)$ and $J_{\nabla^\star,g}(M)$ are the vector spaces of solutions of the gauge equations

$$\nabla^\star \otimes \phi - \phi \circ \nabla = 0,$$

and

$$\nabla \otimes \psi - \psi \circ \nabla^\star = 0$$

respectively.
We consider the unique

$$\Phi^\star \in J_{\nabla,g}(M)$$

such that

$$(9.1.3) \quad \omega(X,Y) = g(\Phi^\star(X),Y) \quad \forall(X,Y).$$

It is to remind that

$$g(\Phi^{\star 2}(X),X) \leq 0 \quad \forall X.$$

Therefore, the set of square root of $-\Phi^{\star 2}$ is denoted by

$$(9.1.4) \quad \sqrt{-\Phi^{\star 2}} = \left\{ \Psi \in \Gamma(Hom(TM,TM)) : \quad \Psi^2 = -\Phi^{\star 2} \right\}$$

We take into account (9.1.4), then the positive square root of $-\Phi^{\star 2}$ is denoted by

$$\Phi^\star_+ \in \sqrt{-\Phi^\star 2}.$$

Henceforth we keep the following identification,

$$(9.1.5) \quad (M,g,\omega,\nabla) = (M,g,\Phi^\star,\nabla)$$

9.2. Special Statistical Fedosov Structures

Here is our main object

Definition 18. *A statistical Fedosov manifold* $(M,g,\Phi^\star,\nabla)$ *is called special iff*

$$\Phi^\star_+ \in J_{\nabla,g}(M).$$

The following claim is obvious,

Proposition 12. *A statistical Fedosov structure* $(M,g,\Phi^\star,\nabla)$ *is special iff its g-dual* $(M,g,\Phi^{\star-1},\nabla^\star)$ *is special.*

An interesting result in this section is the following statement,

Theorem 15. *By taking into account the identification (9.1.5), one has the following results:*
(1) Every special statistical Fedosov manifold $(M,g,\Phi^\star,\nabla)$ *admits a canonical pair of Kaehlerian structures* $(M,\Phi^\star,J)$ *and* $(M,\Phi^{\star-1},J)$*;*
(2) Moreover the Koszul connection ∇ *and its g-dual connecton* $\nabla^\star$ *are complex analytic connection on the complex analytic manifold* (M,J)*.*

Demonstation. We keep the notation just used and assume that

$$\Phi^\star_+ \in \sqrt{-\Phi^{\star 2}} \cap J_{\nabla,g}(M)$$

, ' Then the inverse $\Phi_+^{\star -1}$ is a solution of the gauge equation

$$\nabla \otimes \psi - \psi \circ \nabla^\star = 0,$$

viz

$$\Phi_+^{\star -1} \in J_{\nabla^\star, g}(M).$$

We define the almost complex tensor

$$J = \Phi^\star \circ \Phi_+^{\star -1}.$$

Therefore we obtain the following identities,

$$(9.2.1) \quad \nabla_X^\star \Phi^\star(\Phi_+^{\star -1}(Y)) = \Phi^\star(\nabla_X \Phi_+^{\star -1}(Y))$$

$$= \Phi^\star(\Phi_+^{\star -1}(\nabla_X^\star Y)),$$

$$(9.2.2) \quad \nabla_X \Phi_+^{\star -1}(\Phi\star(Y)) = \Phi_+^{\star -1}(\nabla_X^\star \Phi^\star(Y))$$

$$= \Phi_+^{\star -1}(\Phi^\star \nabla_X Y)))$$

Then we get the following identities,

$$(9.2.3) \quad J(\nabla_X^\star Y) = \nabla_X^\star J(Y) \quad \forall (X, Y),$$

$$(9.2.4) \quad J\nabla_X Y = \nabla_X JY \quad \forall (X, Y).$$

Therefore the Nijenhuis tensor

$$N_J(X, Y) = [X, Y] + J[J(X), Y] + J[X, J(Y)] - [J(X), J(Y)]$$

vanishes identically. Thus the couple (M, J) is a complex analytic structure on M.

Moreover tbe both (TM, ∇) and $(TM, \nabla^\star)$ are complex analytic gauge structures.

Consequently the couple (TM, ∇^{LC}) is a complex analytic gauge structure as well.

Here ∇^{LC} is the Levi Civita connection of (M, g) and is defined as it follows,

$$2\nabla_X^{LC} Y = \nabla_X^\star Y + \nabla_X Y.$$

Before pursuing we recall the symplectic forms,

$$(9.2.5) \quad \omega(X, Y) = g(\Phi^\star(X), Y),$$

$$(9.2.6) \quad \omega'(X, Y) = g(\Phi^{\star -1}(X), Y).$$

It is easy to check the following identities

$$(9.2.7) \quad \omega(JX, JY) = \omega(X, Y) \quad \forall (X, Y),$$

$$(9.2.8) \quad \omega'(JX, JY) = \omega'(X, Y) \quad \forall (X, Y).$$

The following symmetric bilinear forms, h and h' are positive definite,

$$(9.2.10) \quad h(X, Y) = \omega(X, JY),$$

$$(9.2.11) \quad h'(X, Y) = \omega'(X, JY).$$

In final (M, ω, J) and (M, ω', J) are a Kaehlerian manifolds.

This ends the demonstration of Theorem 10.4.

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9.3. Special Fedosov Structure

In the framework of general Fedosov structures (M, ω, ∇) the part (1) as in Theorem 9.5 still deserves attention. [18].

Indeed let $Rie^+(M)$ be the category of positive Riemannian structures on the manifold M and isometries.

According to the identification (9.1.5) every Fedosov structure (M, ω, ∇) gives rice to the mapping

$$(9.3.1) \quad Rie^+(M) \ni g \to \Phi^\star_g \in J_{\nabla g}(M)$$

which is defined as it follows: ,

$$(9.3.2) \quad \omega(X, Y) = g(\Phi^\star_g(X), Y),$$

$$(9.3.3) \quad \nabla^g \otimes \Phi^\star_g - \Phi^\star_g \circ \nabla = 0.$$

The squart of $\Phi^\star_g$, namely $\Phi^{\star 2}_g$ is symmetric and is negative definite w.r.t. the Riemannian structure (M, g). The positive squart root of $-\Phi^{\star 2}_g$ is is denoted by $\Phi^\star_{g+}$. Therefore we introduce the almost complex tensor,

$$J = \Phi^\star_g \circ \Phi^{\star -1}_{g+}.$$

Since ∇ is torsion free the differential 2-form defined as in (9.3.2) is de Rham closed. At another side the identity (9.2.4) implies the integrability of the almost complex structure (M, J).

In final, if the Fedosov structure $(M, \Phi^\star_g, \nabla)$ is special then

$$(M, \omega, J) = (M, \Phi^\star_g, J)$$

is Kahlerian manifold.

Because the g-dual $\nabla^\star$ may not be torsion free the following differential 2-form

$$\omega'(X, Y) = g(\Phi^{\star -1}_g(X), Y)$$

may be not de Rham closed.
Before pursuing we introduce the following notion:

Definition 19. *A Fedosov structure (M, ω, ∇) is called special if there exists a positive Riemannian structure (M, g) such that*

$$\omega(X, Y) = g(\Phi^\star_g(X), Y),$$

$$\Phi^\star_{g+} \in J_{\nabla, g}(M).$$

We keep the notation $\Phi^\star_g$ and $\Phi^\star_{g+}$ which is used above. Our last discussions above yield the following statement.

Theorem 16. *Let (M, ω, ∇) be a special Fedosov manifold. Then there exists $g \in R^{e+}(M)$ yielding the following data:*
(1) (M, ω, ∇) carries the canonical Kahlerian structure

$$(M, \omega, J) = (M, \Phi^\star_g, J)$$

(2) ∇ is the Levi Civita connection of (M, ω, J);
(3) The datum (TM, ω', J) is Hermitian Lie algebroid.
(4) $\nabla^\star$ is Hermitian Koszul connection on the Hermitian Lie algebroid (TM, ω', J).

We keep the notation used in Definition 7.14, namely the notation $F^{s,1,1}_{\delta}(A(M))$ as in (7.8.3).

A straightforward corollary of Theorem 9.7 is that there exist symplectically-(1,1)-degenerate differentiable manifolds M whose $F_{\delta}^{s,1,1}(A(M))$ does not contain any special Fedosov structure. The first such example of $F_{\delta}^{s,1,1}(M)$ not containing any special Fedosow structure is due to W. Thurston [53], [16]

Regarding the open *problem of metric connection*, the statement (2) as in Theorem 9.7 yields a partial answer.

We use the identification as in (9.1.5). Thus we consider a statistical Fedosov manifold

$$(M, g, \omega, \nabla) = (M, g, \Phi^{\star}, \nabla).$$

We also involve the Riemannian structures (M, h) and (M, h') as in (9.2.10) and (9.2.11) respectively.

Theorem 17. *Under the notation above, if a statistical Fedosov manifold* $(M, g, \Phi^{\star}, \nabla)$ *is special then the connection* ∇ *and its* g*-dual* $\nabla^{\star}$ *are the Levi Civita connection of* (M, h) *and* (M, h') *respectively.*

We conclude this section with the remark that Theorem 9.7 is a straightforward corollary of Theorem 91.

In final a Fedosov manifold (M, ω, ∇) admits a Kahler structure (M, ω, J) iff it is special.

10. Comments/Conclusions

The purpose of this short section is to frame for the reader the central problems that are studied in this work and to emphasize the results which are obtained.

The subject is the study of properties of Koszul connections on vector bundles. The *Gauge Geometry* must be understood as the search of data that are invariant under the action of holonomy groups of Koszul connections.

In addition to *inner products and symplectic products* on general vector bunles we have recalled the notion of *affine structure on Lie algebroids and their deformations*. Among central concerns, we aim to characterize those Koszul connections whose Holonomy preserves a structure among those three types of structures just mentionned, viz Metric Structure, Symplectic Structure and Affine Structure.

The tools we have involved are of homological nature. We have generalized a canonical homological class which has been nroduced by J-L Koszul in 1974, see [33]. We have introduced *Koszul homological series and their degeneracy*. We have successfully involved the notion of *Koszul homological degeneracy* to Introduce Other Approaches to the central problems which are mentioned above, viz *the problem of metric connection on Lie algebroids, the problem of symplectic connection on Lie algebroids and the problem of affine structure on Lie algebroids* . Thus on the category of Lie algebroids the use of *Koszul homological degeneracy* avoids rescourse to calculations of holonomy groups of Koszul connections, as suggested by Thurston's approach, [52].

Other tools of excellent effiiciency are Theorem 3.6, Theorem 3.7 and Proposition 3.8. These three statements illustrate that depending on framework and context, the group of inversible mappings which preserve a connection can be of zero dimension, of finite positive dimension or of infinite dimension. These theorems 3.6, 3.7 and 3.8 are the machine for converting obstructions of vector nature into obstructions of homological nature.

An analoguous of Theorem 3.6 is known in global analysis as *theorem of the finiteness of the dimension of Lie group of diffeomorphisms which preserve a linear connection, see Kobayashi-Nomizu [29], Theorem* 23. The classical main ingredient used to demonstrate this statement on tangent Lie algebroids is the existence *the Fundamental 1-form of principal bundles of linear frames, [25,29,50]*. This ingredient is absent from *principal frame bundles of general Lie algebroids*. In this work we involve Theorem 3.6 only. We have given a direct demostration of it. However it is important to note the subtle difference between our Theorem 3.6 and Theorem 23 as in Kobayashi-Nomizu [29]. The latter shows the finiteness of dimension while Theorem 3.6 shows that the only $C^{\infty}(M)$-module is Zero.

Section seven is devoted to impliment those machineries in the canonical tangent Lie algebroids over differentiable manifolds. This is an illustration of approaching the differential geometry from the viewpoint of the gauge geometry. For more indepth and highly significant examples readers are referred to K.S Donaldson [13,14] and to Petrie-Randal, [45].

To acheive the complete production of all Riemannian foliations, *see our Theorem 7.5*, we have introduced the family of gauge equations and implemented the analysis of their solutions. This approach was inspired by methods of both the Information Geometry and the Topology of Information, see Amari-Nagaoka [3], Gromov M. [23], Baudot-Bennequin [6], Nguiffo Boyom [41].

Although based on conceptual approaches, both Theorem 7.4 and Theorem 7.5 enjoy the virtue of being *contructive theorems*.

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