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New Issues About the Central Limit Theorem Based on a More Comprehensive Approach to Probability: Developments and Future Perspectives

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Abstract

A more comprehensive approach to probability studies an infinite number of probability laws that are formally admissible. Thus, an infinite number of weighted averages can be handled. A reinterpretation of the central limit theorem is accordingly shown. The deviations or errors from a fixed value are calculated. It is proved that they are normally distributed. Furthermore, such deviations are invariant with respect to geometric translations identifying repeated samples. In this paper, the way of understanding the statistical model to which a specific and pragmatic distribution is compared is not a functional scheme in the continuum, but it is itself a specific and pragmatic distribution. It is possible to enlarge the reasoning, so developments and future perspectives that underlie the reinterpretation of the central limit theorem are discussed.

Keywords: random sampling; central limit theorem; normal distribution; scalar product; repeated samples; probability law

JEL Classification: C11; C46; C53; C83

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1. Introduction

A random variable is a real-valued function on S, where S is a non-empty set. It is a function from S into the set $\mathbb{R}$ of real numbers such that the pre-image of any interval of $\mathbb{R}$ is an event in S. A random variable can also be called a random quantity whenever the atomistic meaning of the notion of event is accepted instead of the collective one ([1]). Therefore, an event is not a repeatable fact, but it is a single case that can happen or not when uncertainty ceases ([2]). There are different types of repeatable facts. Hence, a repeatable fact may be a finite class of trials, or an infinite sequence of trials, or an indefinite category of trials. Another meaning of probability is associated with repeatable facts ([3]). If uncertainty does not cease, then the true value of a given quantity is not known for a certain individual, even if it is intrinsically well-determined, that is, individuated without any possibility of error. The term random is therefore used to denote a given quantity, whose true value is unknown for a given individual. Uncertainty does not cease whether there is an incomplete state of information and knowledge associated with a given individual. If each element of S is a numerical value expressing an alternative that can be true or false after observing the outcome of a given experiment, then S is reasonably a finite set. In this paper, we focus on it.

It is necessary to distinguish two things that should be distinct: the logic of certainty and the logic of uncertainty. The logic of certainty is studied by mathematics only. The logic of uncertainty obeys the rules of probability theory. However, such objective rules are moved by psychological and subjective sensations and expectations. Thus, extralogical sensations and expectations enter into the logic of uncertainty. Unfortunately, these two logics are not distinct according to the most widely used approaches to probability at the present time. They are merged. The failure to crystallize this essential distinction could depend on the fact that it is likely not to understand the difference between saying

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that something is pragmatically certain and saying that it is absolutely certain. Saying that something is pragmatically certain means that it is possible to give a very high probability to its occurrence, even if its occurrence is still uncertain. Conversely, saying that something is absolutely certain means that the fact that it has occurred is sure. The lack of a clear distinction between the two above logics is also seen as a trick that is played in such a way that probabilistic statements can be considered as true statements in an objective sense. The need to distinguish the two above logics can be summarized in the following slogan: prevision is not prediction. Previsions to which the logic of uncertainty leads a given individual consist of a distribution of his psychological and subjective expectations among all possible alternatives. Such a distribution happens in accordance with his degree of belief in the occurrence of each alternative. Only the distinction between possible and impossible alternatives falls inside the logic of certainty. Conversely, a prediction or prophecy consists of the statement that something will not happen, even if it is logically possible, or that something will happen, even if it is not logically certain. The domain of the logic of certainty is not therefore left to make a prediction. On the contrary, this domain is left to make a prevision. One has therefore to enter into a new field to make a prevision that is based on subjective opinions. This field obeys the rules of the logic of uncertainty. This field obeys the rules of the calculus of probability. After knowing the outcome of a given experiment, a prediction can be said to be correct or incorrect. A prevision, it does not matter what happens, cannot be said to be correct or incorrect. This is because a prevision is always based on the judgment of a given individual at a given moment. Such a judgment depends on his variable state of information and knowledge at that moment. If this state changes, then all previsions based on it also change. In this paper, we focus on fair evaluations or estimations of quantities such that undesirable decisions or choices are avoided. For this reason, a mathematical model considering the deviations or errors of each possible alternative from a specific and coherent prevision is set. A more comprehensive approach to probability says that coherence reduces to finite additivity and non-negativity. If a finite partition of events appears, where these events are the possible values for a random quantity, then the sum of their probabilities has coherently to be equal to 1. If a partition is infinite, in the sense that a countable infinity of possible values x_h, $h = 1, 2, \ldots$, for a random quantity X appears, then specific probabilities denoted by p_h, which are either positive or zero, can be attributed to x_h, $h = 1, 2, \ldots$, by the individual who evaluates x_h $(h = 1, 2, \ldots)$. Such probabilities p_h might even all be zero. One writes

$$\sum_h p_h = 1 - p^* \leq 1, \quad (0 \leq p^* \leq 1). \tag{1}$$

Let I be a set or interval. One could say, if only the x_h and p_h are known, that it is $\mathbf{P}(X \in I) = \sum_h p_h\,(x_h \in I)$ whenever the set denoted by I contains a finite number of elements. Conversely, if the set denoted by I contains an infinite number of elements, then one could only say

$$\sum_h p_h\,(x_h \in I) \leq \mathbf{P}(X \in I) \leq \sum_h p_h\,(x_h \in I) + p^*. \tag{2}$$

This is because the probability p^* can always be thought as deriving from elements whose number is infinite. Other approaches to probability generally postulate that countable additivity or σ-additivity holds, as for Lebesgue measure, and that the field over which the probability is studied is the whole of a Boolean algebra ([4,5]). On the other hand, every Boolean algebra gives rise to a Boolean ring and vice versa. There exists a cryptomorphism between them. Hence, other approaches to probability could also postulate that the field over which the probability is studied is a ring of events or a σ-ring of events. Any event whatsoever is always equal to 1 or 0 according to the logic of certainty. Here, every event is a particular random quantity, so the field over which the probability is studied is a finite dimensional linear space over $\mathbb{R}$. This latter is the space where random quantities, meant as finite partitions of events, can be handled. In particular, linear spaces over $\mathbb{R}$ having different dimensions are studied. Hence, what is completely arbitrary is the dimension of a given linear space over $\mathbb{R}$. One can pass from a ring to a linear space over $\mathbb{R}$ via the notion of an Abelian group. This notion underlies many fundamental algebraic structures, such as rings, algebras, and linear spaces. A given finite dimensional linear space over $\mathbb{R}$ is a linear system.

Section 2 contains preliminary notions on random sampling. Section 3 studies means and variances associated with linear combinations of two or more normal random variables. A reinterpretation of the central limit theorem is shown in Section 4, where a specific proposition is proved. Section 5 contains an analysis of the effects resulting from a specific reinterpretation of the central limit theorem. Finally, conclusions and future perspectives, where this latter are related to the study of particular relationships between variables, are contained in Section 6.

2. Preliminaries

This research paper contains a more complete and satisfactory answer to the following question: what can we expect from a random sample drawn from a known population? It is especially interesting to assess the consequences of this answer. This assessment has not yet been made in the literature. This assessment is therefore unknown at the present time. A sample where each individual of the parent population has the same probability of being sampled is said to be a random sample. There are different ways to achieve a random sampling ([6]). It is known that it is possible to sample with or without replacement. If we sample with replacement, then the n observations in a random sample are independent. The population remains fixed because we replace each individual before drawing the next. A random sampling can mathematically be specified as follows. First, it is possible to think of the population as a set of numbered balls contained in an urn that are mixed and sampled. If the population is extremely large, then the corresponding urn will only be very big. There are no conceptual difficulties associated with this simplification. Second, if the first ball is drawn at random, its number can be considered as a random variable X_1 that assumes a value within all the values of the population. All the values of the population can therefore be studied together with their absolute and relative frequencies. The second ball that is drawn at random is denoted by X_2, the third one is denoted by X_3, and so on. A sample with n independent observations $X_1, X_2, \ldots, X_n$ is expressed by $(X_1, X_2, \ldots, X_n)$ and it has the following property. The probability distribution of each X_i, $i = 1, \ldots, n$, is the population probability distribution denoted by $p(x)$. If p_1 denotes the probability function of X_1, p_2 denotes the probability function of X_2, ..., p_n denotes the probability function of X_n, then it is possible to write

$$p_1(x) \equiv p_2(x) \equiv p_3(x) \equiv \ldots \equiv p_n(x) \equiv p(x), \tag{3}$$

where $\equiv$ means identically equal for all the x values of the population. Each observation has the mean μ and standard deviation σ of the population ([7]). It is known that (3) holds if we sample with replacement. Nevertheless, (3) remains valid even if we sample without replacement. This happens when the parent population is finitely large or even infinite ([8–10]).

3. Linear Combinations of Two or More Normal Random Variables: Means and Variances

Let X_1 and X_2 be two normal random variables. Any linear combination of them is given by

$$Z = aX_1 + bX_2, \tag{4}$$

with a and b that are real numbers. It is known that even Z is a normal random variable. If the average or mean of Z is expressed via the expectation operator $\mathbb{E}$, then one writes

$$\mathbb{E}(Z) = a\mathbb{E}(X_1) + b\mathbb{E}(X_2). \tag{5}$$

Conversely, the variance of Z is given by

$$\mathrm{var}(Z) = a^2\,\mathrm{var}(X_1) + b^2\,\mathrm{var}(X_2) + 2\,ab\,\mathrm{cov}(X_1,\ X_2). \tag{6}$$

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If X_1 and X_2 are independent, then the variance of Z is the following

$$\mathrm{var}(Z) = a^2\,\mathrm{var}(X_1) + b^2\,\mathrm{var}(X_2). \tag{7}$$

More generally, if the n observations $X_1, X_2, \ldots, X_n$ in a random sample are normal, then any linear combination of them is normal. We focus on linear combinations such that the sum of the coefficients of each combination is equal to 1, so one writes

$$Z' = p_1\,X_1 + p_2\,X_2 + \ldots + p_n\,X_n, \tag{8}$$

with $p_1 + p_2 + \ldots + p_n = 1$. Furthermore, one has $0 \leq p_i \leq 1$, $i = 1, \ldots, n$. In particular, the sample mean $\bar{Z}$ is given by

$$\bar{Z} = \frac{1}{n}\,X_1 + \frac{1}{n}\,X_2 + \ldots + \frac{1}{n}\,X_n, \tag{9}$$

where one has $\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} = \frac{1}{n} \cdot n = 1$. The first moment of the sample mean is the expected value of $\bar{Z}$. It is known that one has

$$\mathbb{E}(\bar{Z}) = \mu. \tag{10}$$

The second moment of the sample mean is the variance of $\bar{Z}$, so one has

$$\mathrm{var}(\bar{Z}) = \frac{\sigma^2}{n}. \tag{11}$$

It follows that the standard deviation of $\bar{Z}$ is

$$\frac{\sigma}{\sqrt{n}}. \tag{12}$$

This typical deviation of $\bar{Z}$ from its target μ coincides with the estimation error. It is called the standard error of $\bar{Z}$, so one has

$$\text{Standard Error of } \bar{Z} = \frac{\sigma}{\sqrt{n}}. \tag{13}$$

The larger the value of n, the smaller the standard error of $\bar{Z}$ becomes. Thus, the standard error of $\bar{Z}$ shrinks as the sample size n increases. Finally, if the parent population, whose variance is not equal to $+\infty$, is normal, or the sample size is large, then in either case the sampling distribution of $\bar{Z}$ has an approximately normal shape. If it is $n = 10$ or $n = 20$, then sample sizes will often be large enough. One writes

$$\bar{Z} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n}). \tag{14}$$

It is known that the central limit theorem has always attracted the attention of many researchers ([11]). Many researchers have often attempted to enlarge what the central limit theorem intrinsically contains ([12–16]). One can be very precise in making deductions about the sample mean whenever a random sample is drawn from a known population. Because of averaging, the sample mean is not as extreme as the individuals in the population ([17]). Its great convenience is its fairness, in the sense that an unbiased estimation of the mean μ of the underlying population is carried out ([18,19]). Furthermore, sampling uncertainty can suitably be handled.

4. A Reinterpretation of the Central Limit Theorem

A specific weighted average has been denoted by $\bar{Z}$. Nevertheless, there exists an infinite number of weighted averages that can be calculated. This happens when a more comprehensive approach to probability is used ([20]). In this approach, every weighted average is a coherent prevision of a random variable ([21]). This variable is normal because it is obtained via a linear combination of n normal random variables. Deviations or errors from a specific weighted average can be calculated as follows. First, a specific mean is calculated. Second, the errors are calculated with respect to this specific mean.

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Deviations or errors are normally distributed, with mean that is equal to zero and variance that is equal to $k\,\sigma^2 = \sigma_D^2$. One writes

$$\mathbb{E}(D) = 0, \tag{15}$$

so positive and negative errors are distributed around the value zero, and

$$\text{var}(D) = \sigma_D^2 = k\,\sigma^2, \tag{16}$$

where k is the sum of quadratic weights. Here, D denotes a particular variable identifying deviations as its possible values. It is possible to prove the following:

Proposition 1. *If a more comprehensive approach to probability is used, then an infinite number of weighted averages of the n independent observations $X_1, X_2, \ldots, X_n$ can be calculated. A specific random variable denoted by D is studied and its moments are such that one has $\mathbb{E}(D) = 0$ and $\text{var}(D) = \sigma_D^2 = k\,\sigma^2$.*

Proof. Given

$$Z' = p_1\,X_1 + \ldots + p_n\,X_n,$$

with $p_1 + \ldots + p_n = 1$, and $0 \leq p_i \leq 1$, $i = 1, \ldots, n$, it is possible to consider the possible values of another variable denoted by D. Such values are the possible deviations from $\mathbb{E}(Z')$. The possible values of D are therefore the elements of the set denoted by

$$I(D) = \{X_1 - \mathbb{E}(Z'), \ldots, X_n - \mathbb{E}(Z')\}. \tag{17}$$

Here, $\mathbb{E}(Z') = \mathbb{E}(\bar{Z}) = \mu$. This is because one has

$$\mathbb{E}(Z') = p_1\,\mathbb{E}(X_1) + \ldots + p_n\,\mathbb{E}(X_n) = p_1\,\mu + \ldots + p_n\,\mu = (p_1 + \ldots + p_n)\,\mu = \mu. \tag{18}$$

It is possible to write

$$p_1\,(X_1 - \mathbb{E}(Z')) + \ldots + p_n\,(X_n - \mathbb{E}(Z')), \tag{19}$$

where the weights of (19) are the same of the ones characterizing Z'. One has

$$p_1\,\mathbb{E}(X_1 - \mathbb{E}(Z')) + \ldots + p_n\,\mathbb{E}(X_n - \mathbb{E}(Z')) = p_1\,(\mu - \mu) + \ldots + p_n\,(\mu - \mu) = 0, \tag{20}$$

so it turns out to be $\mathbb{E}(D) = 0$. Conversely, from

$$\text{var}\big[p_1\,(X_1 - \mathbb{E}(Z')) + \ldots + p_n\,(X_n - \mathbb{E}(Z'))\big], \tag{21}$$

it follows that one writes

$$p_1^2\,\text{var}(X_1) + \ldots + p_n^2\,\text{var}(X_n) = (p_1^2 + \ldots + p_n^2)\sigma^2, \tag{22}$$

with $(p_1^2 + \ldots + p_n^2) = k$, so it is $\text{var}(D) = \sigma_D^2 = k\,\sigma^2$. □

The sum of quadratic weights p_i^2, $i = 1, \ldots, n$, is between 0 and 1. It is never equal to zero. It is equal to 1 only in one case, that is, when all weights except one are equal to zero and there is accordingly only one weight that is equal to 1. Note the following:

Remark 1. *Since Z' and $\bar{Z}$ are two linear combinations of n independent random variables such that the sum of the corresponding non-negative coefficients or weights is always equal to 1, one has $\mathbb{E}(Z') = \mathbb{E}(\bar{Z}) = \mu$. Hence, this result does not depend on the condition according to which one has to observe $p_1 = \frac{1}{n}, \ldots, p_n = \frac{1}{n}$, even if one writes $\frac{1}{n} + \ldots + \frac{1}{n} = \frac{1}{n} \cdot n = 1$.*

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5. An Analysis of the Effects Resulting from a Specific Reinterpretation of the Central Limit Theorem

There exists an infinite number of probability laws that can be used to handle the n observations $X_1, X_2, \ldots, X_n$ in a random sample. These laws are all formally admissible. They characterize a more comprehensive approach to probability based on subjective probabilities ([22,23]). The n observations $X_1, X_2, \ldots, X_n$ in a random sample are n random events, where each event is a possible value for a random quantity. This latter is linearly dependent from n random entities. The possible values for a random quantity are logically independent. Thus, there exists an infinite number of probability laws that can be associated with the elements of the set $\mathcal{E}$ of events, denoted by $E_1, E_2, \ldots, E_n$ in general. Such events identify a finite partition. One focuses on the set

$$\{x_1, x_2, \ldots, x_n\}. \tag{23}$$

This is because X_1 takes on a specific value denoted by $x_1, \ldots, X_n$ takes on a specific value denoted by x_n. In the absence of uncertainty only the indicators of each event E_i, $i = 1, \ldots, n$, can be considered. The indicators of each event E_i, $i = 1, \ldots, n$, take on 1 or 0 as their values. The values of the indicators of each event E_i, $i = 1, \ldots, n$, can therefore be two idempotent numbers only. However, there is uncertainty here, so one focuses on $x_1, x_2, \ldots, x_n$ together with their probabilities $p_1, p_2, \ldots, p_n$. If one writes

$$p_1\, x_1 + \ldots + p_n\, x_n, \tag{24}$$

with $p_1 + \ldots + p_n = 1$, and $0 \leq p_i \leq 1$, $i = 1, \ldots, n$, then the prevision, denoted by $\mathbf{P}$, or mathematical expectation of a random quantity is studied. In this approach, it is possible to use $\mathbf{P}$ in place of $\mathbb{E}$ although $\mathbf{P}$ and $\mathbb{E}$ have the same properties. There exists a unique notion, denoted by a unique symbol, which in general we call prevision $\mathbf{P}$ and, in the case of events, also probability $\mathbf{P}$. If the possible values for a random quantity are put on the real line which is a geometric line isomorphic to the set $\mathbb{R}$ of real numbers, then all probability laws applied to $x_1, x_2, \ldots, x_n$ that are formally admissible identify a closed line segment. This latter is a one-dimensional convex set.

5.1. Probability Laws That Are Formally Admissible

Let

$$\mathbf{P}(X) = p_1\, x_1 + \ldots + p_n\, x_n \tag{25}$$

be the mathematical expression of the prevision or mathematical expectation of X, where X is a random quantity identifying a linear combination of n events expressed by

$$X = x_1\, |E_1| + \ldots + x_n\, |E_n|. \tag{26}$$

We do not yet know which of the n events that are taken into account in (26) will be true in the absence of uncertainty. For this reason, n probabilities are introduced to calculate $\mathbf{P}(X)$. All probability laws that are formally admissible are infinite in number. They must be coherent. Since the n observations $X_1, X_2, \ldots, X_n$ in a random sample are seen as n mutually disjoint events, every coherent probability law that can be chosen has to satisfy the fundamental property of coherence according to which $\mathbf{P}(X)$ must not be less than the lower bound of the set given by

$$\{x_1, x_2, \ldots, x_n\},$$

nor greater than the upper bound of it. Check the following:

Example 1. *Let $x_1 = 5$, $x_2 = 7$, $x_3 = 10$, and $x_4 = 11$ be the 4 observations in a random sample. They are the possible values for a random quantity denoted by X. Such values characterizing a bounded quantity from above and below are taken into account together with 4 non-negative weights. They are $p_1 = 0.2$, $p_2 = 0.3$, $p_3 = 0.4$, and $p_4 = 0.1$, where it turns out to be $p_1 + \ldots + p_4 = 1$, and $0 \leq p_i \leq 1$, $i = 1, \ldots, 4$. Thus, these weights*

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identify a probability law that is formally admissible. Since it is $\mathbf{P}(X) = 8.2$*, it is possible to calculate the errors or deviations from* $\mathbf{P}(X)$*. They are* $d_1 = -3.2$*,* $d_2 = -1.2$*,* $d_3 = 1.8$*, and* $d_4 = 2.8$*. One has*

$$p_1 \, d_1 + \ldots + p_4 \, d_4 = 0 \tag{27}$$

because of a fundamental property qualifying the deviations or errors from the supposed center of a distribution. If the deviations from $\mathbf{P}(X)$ *are not all negative, then a probability law is formally admissible. Similarly, if the deviations from* $\mathbf{P}(X)$ *are not all positive, then a probability law is formally admissible. In particular, given* $x_1 = 5$*,* $x_2 = 7$*,* $x_3 = 10$*, and* $x_4 = 11$*, if a probability law is expressed by* $p_1 = 0$*,* $p_2 = 0$*,* $p_3 = 0$*, and* $p_4 = 1$*, then it is formally admissible because the corresponding deviations are not all negative. One of them coincides with* 0*. Similarly, if a probability law that is referred to the same possible values for* X *is expressed by* $p_1 = 1$*,* $p_2 = 0$*,* $p_3 = 0$*, and* $p_4 = 0$*, then it is formally admissible because the corresponding deviations are not all positive. One of them coincides with* 0*. Conversely, given* $x_1 = 5$*,* $x_2 = 7$*,* $x_3 = 10$*, and* $x_4 = 11$*, if it is* $\mathbf{P}(X) = 4$*, then the corresponding deviations are all positive. Similarly, if it is* $\mathbf{P}(X) = 13$*, then the corresponding deviations are all negative. Here, in either case, the fundamental property of coherence according to which* $\mathbf{P}(X)$ *must not be less than the lower bound, given by* $x_1 = 5$*, of the set expressed by*

$$\{x_1, x_2, \ldots, x_4\},$$

nor greater than the upper bound of it, given by $x_4 = 11$*, does not hold. If we put* $x_1 = 5$*,* $x_2 = 7$*,* $x_3 = 10$*, and* $x_4 = 11$ *on the real line, then all probability laws that are formally admissible in a first stage identify a closed line segment. Its endpoints are two extreme points, expressed by* $x_1 = 5$ *and* $x_4 = 11$ *respectively, of a one-dimensional convex set.*

Note the following:

Remark 2. *If we sample with replacement, then it is possible that two or more observations in a random sample are equal. If this happens, then nothing changes, in the sense that a formally admissible probability law is such whenever the sum of the corresponding probabilities is equal to* 1*. For instance, let* $x_1 = 6$*,* $x_2 = 8$*,* $x_3 = 12$*,* $x_4 = 14$*,* $x_5 = 8$*, and* $x_6 = 16$ *be the* 6 *observations in a random sample. A formally admissible probability law is such that one writes* $p_1 = 0.1$*,* $p_2 = 0.05$*,* $p_3 = 0.35$*,* $p_4 = 0.2$*,* $p_5 = 0.05$*, and* $p_6 = 0.25$*. On the other hand, if we set* $x_1 = 6$*,* $x_2 = 8$*,* $x_3 = 12$*,* $x_4 = 14$*,* $x_5 = 16$*, and* $x_6 = 0$ *together with* $p_1 = 0.1$*,* $p_2 = 0.1$*,* $p_3 = 0.35$*,* $p_4 = 0.2$*,* $p_5 = 0.25$*, and* $p_6 = 0$*, then nothing changes. The two ways of writing the set of ordered pairs given by* $[x_i, \ p_i]$*,* $i = 1, \ldots, 6$*, are equivalent according to the language of calculus.*

A more comprehensive approach to probability based on subjective probabilities clearly distinguishes between two aspects ([24]). The logical aspect must be clearly distinguished from the empirical one. Other approaches to probability tend to merge these two aspects. In particular, here the logical aspect is such that the sum of n probabilities must be equal to 1. Conversely, the empirical aspect is such that ∞^{n-1} probability laws can formally be admitted in a first stage. This is because every probability law reflects a specific subjective opinion associated with a given individual. There exists an infinite number of possible opinions about the evaluations of probability. On the other hand, it is possible that a particular subjective opinion coincides with the one of many or all people. If this happens, then nothing changes. This is because the ambit where subjective probability is available is not subject to any limitation. Other approaches to probability, typically those for which probability is not a degree of belief of an individual in the occurrence of a specific event, but it is a conventional notion that is subject to rigid mathematical laws only, say that a unique way of identifying those probabilities through which a mathematical calculation is carried out has to be considered.

5.2. A Deceptive Dichotomy Between Statistics and Probability

An event is always a logical entity, so it is always a proposition. On the other hand, this does not exclude that an event may coincide with a set individuated by a proposition. Check the following:

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Example 2. *A finite population, whose size is $N = 800$, is taken into account for studying a specific quantitative issue. Let H be a continuous variable. It is normally distributed, with mean $\mu = 168$ centimeters and standard deviation $\sigma = 13$ centimeters. Its values potentially belong to the set $\mathbb{R}$ of real numbers. This latter is an interval denoted by $(-\infty, +\infty) = \mathbb{R}$. Every event is a set individuated by a proposition. Its probability can uniquely be determined. Since the probability that H lies in the interval $[a, b]$ is equal to the area under $f(x)$ between $x = a$ and $x = b$, where $f(x)$ is the probability density function of H, one has*

$$\mathbf{P}(a \leq H \leq b) = \int_a^b f(x)\, dx. \tag{28}$$

The standardized variable corresponding to H is defined by

$$Z = \frac{H - \mu}{\sigma}, \tag{29}$$

where Z is also a normal distribution, with $\mu = 0$ and standard deviation $\sigma = 1$. The probability density function for Z is obtained by setting $z = (h - \mu)/\sigma$, so one writes

$$\phi(z) = \frac{1}{\sqrt{2\,\pi}}\, e^{-z^2/2}. \tag{30}$$

The main interval can be divided into equal-width subintervals. The area under the standard normal distribution $\phi(z)$ between 0 and $z \geq 0$ is given by

$$\mathbf{P}(0 \leq Z \leq z) = \int_0^z \phi(z)\, dz = \Phi(z), \tag{31}$$

so it is possible to calculate the absolute frequencies of a frequency distribution. This happens whenever $N = 800$ is multiplied by the probability that corresponds to a specific subinterval. The number of subintervals is finite. It has to be a reasonable compromise between too much detail and too little. Each subinterval midpoint is chosen in such a way that it is a whole number. Each subinterval midpoint represents all observations that are contained in the corresponding subinterval. It is also possible to calculate the relative frequencies, so we can pass from a frequency distribution to a random quantity. Each subinterval midpoint is a possible value for a discrete random quantity. We can use the relative frequencies as probabilities to calculate the prevision of this random quantity.

Note the following:

Remark 3. *A certain subjective opinion may have a specific reason for being held. This opinion can be based on the observation of frequencies. For this reason, it can be said to be objectively true. However, nothing changes. It is still a subjective opinion which is rigorously studied like all other subjective opinions.*

Remark 4. *A frequency distribution and a random quantity are the two sides of the same coin. If each element of the sample space denoted by S is pragmatically observed, then S is necessarily a finite set. It may be convenient to think of S as embedded in a larger and manageable space denoted by $\mathbb{R}$. Thus, a continuum of real numbers such as an interval appears. Each element of S is an event or proposition. Each element of S can also be a set individuated by a proposition. A particular set that is individuated by a proposition is given by the interval $[a, a] = \{a\}$, with $a \in \mathbb{R}$.*

5.3. The Prevision of a Discrete Random Quantity Is a Scalar Product

The prevision of a discrete random quantity is an algebraic operation that takes two equal-length sequences of real numbers and returns a real number ([25]). Two sequences of real numbers are respectively the possible values for a random quantity denoted by $\mathbf{x}$ and their probabilities denoted by $\mathbf{p}$. The algebraic structure where this operation happens is a finite dimensional linear space over $\mathbb{R}$ ([26–28]). In particular, finite dimensional linear spaces over $\mathbb{R}$ can have a Euclidean nature, so the scalar product of two ordered sequences of real numbers is the dot product of their Cartesian

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coordinates. Such a product is independent from the choice of a particular Cartesian coordinate system. Such a product is therefore independent from the choice of a specific orthonormal basis. The scalar product of two ordered sequences of real numbers satisfies the following properties:

$$\langle \mathbf{x}, \mathbf{p} \rangle = \langle \mathbf{p}, \mathbf{x} \rangle, \tag{32}$$

where $\mathbf{x}$ and $\mathbf{p}$ belong to the same finite dimensional linear space over $\mathbb{R}$;

$$\langle \mathbf{p}, \mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{p}, \mathbf{x} \rangle + \langle \mathbf{p}, \mathbf{y} \rangle, \tag{33}$$

where $\mathbf{p}$, $\mathbf{x}$, and $\mathbf{y}$ belong to the same finite dimensional linear space over $\mathbb{R}$. Furthermore, $\mathbf{p}$ represents the same probabilities associated with the possible values for X and the possible values for Y in such a way that one writes $\mathbf{P}(X+Y) = \mathbf{P}(X) + \mathbf{P}(Y)$;

$$\langle a\,\mathbf{p}, \mathbf{x} \rangle = a\,\langle \mathbf{p}, \mathbf{x} \rangle, \quad \text{and} \quad \langle \mathbf{p}, a\,\mathbf{x} \rangle = a\,\langle \mathbf{p}, \mathbf{x} \rangle, \tag{34}$$

with $a \in \mathbb{R}$. In addition, if all possible values for two or more random quantities become impossible at a later time, then the condition according to which the scalar product of two ordered sequences of real numbers is said to be non-degenerate is satisfied. Since the scalar product of two ordered sequences of real numbers is independent from the choice of a specific orthonormal basis, there exists a fundamental intrinsic property. Such a property can be extended whenever two or more logically independent random quantities are studied. If $m \geq 2$ logically independent random quantities are taken into account together with their marginal distributions remaining fixed, then m^2 joint distributions having a fundamental invariance property can appear. This invariance property makes clear that the weights of some joint distributions can change unlike the ones of the corresponding marginal distributions. In this way, multilinear relationships between variables can be studied ([29–31]). Furthermore, particular exchangeability relationships between variables can be shown. On the other hand, many researchers dealt with the study of the notion of exchangeability ([32–34]). The set of events is here embedded in the finite dimensional linear space over $\mathbb{R}$ of random quantities. It is possible to increase the dimension of a linear space over $\mathbb{R}$ just as it is possible to reduce it. Everything can be represented in a linear form provided one considers an appropriate number of dimensions. In particular, a reduction of dimension is observed when the two ordered sequences of real numbers, which are respectively the possible values for a random quantity and their probabilities, are put on the real line. This line is isomorphic to $\mathbb{R}$. This line is therefore a one-dimensional linear space over $\mathbb{R}$.

5.4. Repeated Samples Are Invariant with Respect to Geometric Translations

The deviations or errors from a fixed value are invariant with respect to geometric translations. If $\mathbf{s}$ denotes the initial random sample of n observations and $\mathbf{v}$ denotes a vector that is known to be as the translation vector, then the translation function denoted by $T_{\mathbf{v}}$ will be expressed by

$$T_{\mathbf{v}}(\mathbf{s}) = \mathbf{s} + \mathbf{v}, \tag{35}$$

where $\mathbf{s}$ and $\mathbf{v}$ are two equal-length ordered sequences of real numbers. Here, whenever we pass from a geometric translation to another one, $\mathbf{s}$ remains unchanged unlike $\mathbf{v}$. As the number of samples increases indefinitely, the deviations or errors from a fixed value are invariant with respect to a geometric translation, which is expressed by $\mathbf{v}$ every time. The components of $\mathbf{v}$ are all equal, so $\mathbf{v}$ is every time an ordered sequence of real numbers that are all equal. As the number of samples increases indefinitely, the properties of D remain unchanged. They are such that one has $\mathbb{E}(D) = 0$ and $\mathrm{var}(D) = \sigma_D^2 = k\,\sigma^2$.

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6. Conclusions

In this paper, a reinterpretation of the central limit theorem is shown. Since there exists an infinite number of probability laws that are formally admissible, it is possible to consider an infinite number of weighted averages. Thus, the way of understanding the statistical model to which a specific and pragmatic distribution is compared is not a functional scheme in the continuum, but it is itself a specific and pragmatic distribution. The deviations or errors from a fixed value are calculated. They are normally distributed. They are invariant with respect to geometric translations. It is possible to enlarge the reasoning, so developments and future perspectives that underlie the reinterpretation of the central limit theorem are discussed. In general, this research paper shows that one cannot have blind faith in the analogical arguments that suggest the use of particular analytical tools in the application of the calculus of probability to specific problems. All criteria based on measure theory are totally illusory whenever they lead to specifying the conclusion more precisely than the problem itself allows. If there are problems whose values are intrinsically indeterminate between given limits, then to use methods that authorize the derivation of a uniquely determined answer to them, via the introduction of arbitrary mathematical conventions, is unjustified and inadmissible. It is wrong that formally admissible answers to these problems are systematically excluded. It is wrong to contemplate an infinite number of alternatives, whether this number is intrinsically illusory, just to obtain a mathematical result that is shown, without any risk, as a sure prediction. What is said in this research paper implies that events and random quantities can be embedded inside finite dimensional linear spaces over $\mathbb{R}$, where interesting invariance properties are discovered and made clear. Such properties allow to connect probability to multilinear issues that can innovatively be treated in different fields of science.

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