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Non-Archimedean Brauer Oval (of Cassini) Theorem and Applications

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Abstract

Nica and Sprague [*Am. Math. Mon., 2023*] derived a non-Archimedean version of the Gershgorin disk theorem. We derive a non-Archimedean version of the oval (of Cassini) theorem by Brauer [*Duke Math. J., 1947*] which generalizes the Nica-Sprague disk theorem. We provide applications for bounding the zeros of polynomials over non-Archimedean fields. We also show that our result is equivalent to the non-Archimedean version of the Ostrowski nonsingularity theorem derived by Li and Li [*J. Comput. Appl. Math., 2025*].

Keywords: eigenvalue; disk; oval; non-Archimedean valued field

MSC: 15A18; 15A42; 12J25; 26E30

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1. Introduction

Let $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{C})$. For $1 \leq j \leq n$, define

$$r_j(A) := \sum_{k=1, k\neq j}^{n} |a_{j,k}|.$$

For $1 \leq k \leq n$, define

$$c_k(A) := \sum_{j=1, j\neq k}^{n} |a_{j,k}|.$$

Let $\sigma(A)$ be the set of all eigenvalues of A. In 1931, Gershgorin proved the following breakthrough result known as the Gershgorin circle/disk theorem which uses single row/column for determining the radius of the disk.

Theorem 1. *[1–4] (**Gershgorin Eigenvalue Inclusion Theorem** or **Gershgorin Disk Theorem**) For every $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{C})$,*

$$\sigma(A) \subseteq \bigcup_{j=1}^{n} \{z \in \mathbb{C} : |z - a_{j,j}| \leq r_j(A)\}$$

and

$$\sigma(A) \subseteq \bigcup_{k=1}^{n} \{z \in \mathbb{C} : |z - a_{k,k}| \leq c_k(A)\}.$$

A remarkable application of Theorem 1 is the following result of Frobenius (which advances result of Browne).

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Theorem 2. *[5–7] Let $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{C})$. For every $\lambda \in \sigma(A)$,*

$$|\lambda| \leq \min\left\{\max_{1\leq j\leq n}\sum_{k=1}^{n}|a_{j,k}|, \max_{1\leq k\leq n}\sum_{j=1}^{n}|a_{j,k}|\right\} \leq \frac{1}{2}\left(\max_{1\leq j\leq n}\sum_{k=1}^{n}|a_{j,k}| + \max_{1\leq k\leq n}\sum_{j=1}^{n}|a_{j,k}|\right).$$

In particular,

$$|det(A)| \leq \min\left\{\left(\max_{1\leq j\leq n}\sum_{k=1}^{n}|a_{j,k}|\right)^n, \left(\max_{1\leq k\leq n}\sum_{j=1}^{n}|a_{j,k}|\right)^n\right\}$$
$$\leq \frac{1}{2}\left(\left(\max_{1\leq j\leq n}\sum_{k=1}^{n}|a_{j,k}|\right)^n + \left(\max_{1\leq k\leq n}\sum_{j=1}^{n}|a_{j,k}|\right)^n\right).$$

Another remarkable application of Theorem 1 is on the bounds for the zeros of polynomials. Let $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{C}[z]$. A direct observation reveals that the zeros of p are the eigenvalues of the Frobenius companion matrix

$$C_p := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-3} & -c_{n-2} & -c_{n-1} \end{pmatrix} \in \mathbb{M}_n(\mathbb{C})$$

and the eigenvalues of C_p are the same as the zeros of p [8,9]. In 1965, Bell derived following bounds for the zeros of p using Theorem 1.

Theorem 3. *[10] Let $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{C}[z]$. If λ is a zero of p, then*

$$|\lambda| \leq 1$$

or

$$|\lambda + c_{n-1}| \leq |c_0| + \cdots + |c_{n-2}|.$$

In particular,

$$\text{(Lagrange bound)} \qquad |\lambda| \leq \max\{1, |c_0| + \cdots + |c_{n-1}|\}. \tag{1}$$

Note that Inequality (1) is a generalization of famous Montel bound [2] which says that

$$|\lambda| \leq 1 + |c_0| + \cdots + |c_{n-1}|.$$

It is interesting to note that one can give a proof of Inequality (1) without using companion matrix [11].

Theorem 4. *[10] Let $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{C}[z]$. If λ is a zero of p, then*

$$|\lambda| \leq |c_0|$$

or

$$|\lambda| \leq 1 + |c_j|, \quad \text{for some } 1 \leq j \leq n-2$$

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or

$$|\lambda + c_{n-1}| \leq 1.$$

In particular,

$$\text{(Bell bound)} \qquad |\lambda| \leq \max\{|c_0|, 1 + |c_1|, \ldots, 1 + |c_{n-1}|\}. \tag{2}$$

Note that Inequality (2) is a generalization of famous Cauchy bound [12] which says that

$$|\lambda| \leq 1 + \max\{|c_0|, \ldots, |c_{n-1}|\}.$$

Let $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{C}[z]$ with $c_0 \neq 0$. Define

$$q(z) := \frac{1}{c_0} z^n p\left(\frac{1}{z}\right) = \frac{1}{c_0} + \frac{c_{n-1}}{c_0} z + \cdots + \frac{c_1}{c_0} z^{n-1} + z^n \in \mathbb{C}[z].$$

We note that $\lambda \in \mathbb{C} \setminus \{0\}$ satisfies $p(\lambda) = 0$ if and only if $q(1/\lambda) = 0$. Frobenius companion matrix of q is

$$C_q := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \frac{-1}{c_0} & \frac{-c_{n-1}}{c_0} & \frac{-c_{n-2}}{c_0} & \cdots & \frac{-c_3}{c_0} & \frac{-c_2}{c_0} & \frac{-c_1}{c_0} \end{pmatrix} \in \mathbb{M}_n(\mathbb{C}).$$

Now by applying earlier two results to the polynomial q and rearranging, we get following results.

Theorem 5. *[2,10] Let $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{C}[z]$ with $c_0 \neq 0$. If λ is a zero of p, then*

$$\frac{1}{|\lambda|} \leq 1$$

or

$$\left|\frac{1}{\lambda} + \frac{c_1}{c_0}\right| \leq \frac{1}{|c_0|} + \frac{|c_2|}{|c_0|} + \cdots + \frac{|c_{n-1}|}{|c_0|}.$$

In particular,

$$\frac{1}{|\lambda|} \leq \max\left\{1, \frac{1}{|c_0|} + \frac{|c_1|}{|c_0|} + \frac{|c_2|}{|c_0|} + \cdots + \frac{|c_{n-1}|}{|c_0|}\right\} \leq 1 + \frac{1}{|c_0|} + \frac{|c_1|}{|c_0|} + \cdots + \frac{|c_{n-1}|}{|c_0|}$$

and

$$\begin{aligned} \text{(Lagrange lower bound)} \qquad |\lambda| &\geq \frac{|c_0|}{\max\{|c_0|, 1 + |c_1| + \cdots + |c_{n-1}|\}} \\ \text{(Montel lower bound)} \qquad &\geq \frac{|c_0|}{1 + |c_0| + |c_1| + \cdots + |c_{n-1}|}. \end{aligned}$$

Theorem 6. *[2,10] Let $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{C}[z]$ with $c_0 \neq 0$. If λ is a zero of p, then*

$$\frac{1}{|\lambda|} \leq \frac{1}{|c_0|}$$

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or

$$\frac{1}{|\lambda|} \leq 1 + \frac{|c_j|}{|c_0|}, \quad \text{for some } 2 \leq j \leq n-1$$

or

$$\left|\frac{1}{\lambda} + \frac{c_1}{c_0}\right| \leq 1.$$

In particular,

$$\frac{1}{|\lambda|} \leq \max\left\{\frac{1}{|c_0|}, 1 + \frac{|c_1|}{|c_0|}, \dots, 1 + \frac{|c_{n-1}|}{|c_0|}\right\} \leq 1 + \max\left\{\frac{1}{|c_0|}, \frac{|c_1|}{|c_0|}, \dots, \frac{|c_{n-1}|}{|c_0|}\right\}$$

and

$$\begin{aligned} &\text{(Bell lower bound)} && |\lambda| \geq \frac{|c_0|}{\max\{1, |c_0| + |c_1|, \dots, |c_0| + |c_{n-1}|\}} \\ &\text{(Cauchy lower bound)} && \geq \frac{|c_0|}{|c_0| + \max\{1, |a_1|, \dots, |a_{n-1}|\}}. \end{aligned}$$

A result much older than Gershgorin is the following.

Theorem 7. *[2,3,13,14] (**Strict Diagonal Dominance Theorem** or **Levy-Desplanques Theorem**) If $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{C})$ satisfies*

$$|a_{j,j}| > r_j(A), \quad \forall 1 \leq j \leq n$$

or

$$|a_{k,k}| > c_k(A), \quad \forall 1 \leq k \leq n,$$

then A is invertible.

Following results say that Theorems 1 and 7 are equivalent.

Theorem 8. *[3] Let $n \in \mathbb{N}$. Following two statements are equivalent.*

(i) *Let $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\sigma(A) \subseteq \bigcup_{j=1}^{n} \{z \in \mathbb{C} : |z - a_{j,j}| \leq r_j(A)\}.$$

(ii) *If $B = [b_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{C})$ satisfies*

$$|b_{j,j}| > r_j(B), \quad \forall 1 \leq j \leq n,$$

then B is invertible.

Theorem 9. *[3] Let $n \in \mathbb{N}$. Following two statements are equivalent.*

(i) *Let $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\sigma(A) \subseteq \bigcup_{k=1}^{n} \{z \in \mathbb{C} : |z - a_{k,k}| \leq c_k(A)\}.$$

(ii) *If* $B = [b_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{C})$ *satisfies*

$$|b_{k,k}| > c_k(B), \quad \forall 1 \leq k \leq n,$$

then B is invertible.

In 1947, Brauer derived following generalization of Theorem 1, known as the Brauer oval (of Cassini) theorem which uses two rows/columns for determining the radius of the oval.

Theorem 10. *[3,6,15] (**Brauer Eigenvalue Inclusion Theorem** or **Brauer Oval (of Cassini) Theorem**) For every* $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{C})$,

$$\sigma(A) \subseteq \bigcup_{j,k=1, j\neq k}^{n} \{z \in \mathbb{C} : |z - a_{j,j}||z - a_{k,k}| \leq r_j(A) r_k(A)\}$$

and

$$\sigma(A) \subseteq \bigcup_{j,k=1, j\neq k}^{n} \{z \in \mathbb{C} : |z - a_{j,j}||z - a_{k,k}| \leq c_j(A) c_k(A)\}.$$

By applying Brauer's theorem, we get following results.

Theorem 11. *Let* $p(z) := c_0 + c_1 z + \cdots + c_{n-1} z^{n-1} + z^n \in \mathbb{C}[z]$. *If* λ *is a zero of* p, *then*

$$|\lambda| \leq 1$$

or

$$|\lambda||\lambda + c_{n-1}| \leq |c_0| + \cdots + |c_{n-2}|.$$

Theorem 12. *Let* $p(z) := c_0 + c_1 z + \cdots + c_{n-1} z^{n-1} + z^n \in \mathbb{C}[z]$. *If* λ *is a zero of* p, *then*

$$|\lambda|^2 \leq |c_0|(1 + |c_j|), \quad \text{for some } 1 \leq j \leq n-2$$

or

$$|\lambda||\lambda + c_{n-1}| \leq |c_0|$$

or

$$|\lambda|^2 \leq (1 + |c_j|)(1 + |c_k|), \quad \text{for some } 1 \leq j, k \leq n-2, j \neq k$$

or

$$|\lambda||\lambda + c_{n-1}| \leq 1 + |c_j|, \quad \text{for some } 1 \leq j \leq n-2.$$

It is known that Brauer theorem cannot be extended by considering three rows/columns [3]. In 1937, Ostrowski derived following generalization of Theorem 7.

Theorem 13. *[3,16] (**Ostrowski Nonsingularity Theorem**) If* $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{C})$ *satisfies*

$$|a_{j,j}||a_{k,k}| > r_j(A) r_k(A), \quad \forall 1 \leq j, k \leq n, j \neq k$$

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or

$$|a_{j,j}||a_{k,k}| > c_j(A)c_k(A), \quad \forall 1 \leq j,k \leq n, j \neq k,$$

then A is invertible.

It is known that Theorems 10 and 13 are again equivalent.

Theorem 14. *[3] Let $n \in \mathbb{N}$. Following two statements are equivalent.*

(i) *Let $A = [a_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\sigma(A) \subseteq \bigcup_{j,k=1, j \neq k}^{n} \{z \in \mathbb{C} : |z - a_{j,j}||z - a_{k,k}| \leq r_j(A)r_k(A)\}.$$

(ii) *If $B = [b_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{C})$ satisfies*

$$|b_{j,j}||b_{k,k}| > r_j(B)r_k(B), \quad \forall 1 \leq j,k \leq n, j \neq k,$$

then B is invertible.

Theorem 15. *[3] Let $n \in \mathbb{N}$. Following two statements are equivalent.*

(i) *Let $A = [a_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\sigma(A) \subseteq \bigcup_{j,k=1, j \neq k}^{n} \{z \in \mathbb{C} : |z - a_{j,j}||z - a_{k,k}| \leq c_j(A)c_k(A)\}.$$

(ii) *If $B = [b_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{C})$ satisfies*

$$|b_{j,j}||b_{k,k}| > c_j(B)c_k(B), \quad \forall 1 \leq j,k \leq n, j \neq k,$$

then B is invertible.

It is natural to ask what are versions of Theorems 1, 2, 7, 10 and 13 for matrices over non-Archimedean fields? In the paper, $\mathbb{K}$ denotes a non-Archimedean valued field with valuation $|\cdot|$. Let $A = [a_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{K})$. For $1 \leq j \leq n$, define

$$h_j(A) := \max_{1 \leq k \leq n, k \neq j} |a_{j,k}|.$$

For $1 \leq k \leq n$, define

$$v_k(A) := \max_{1 \leq j \leq n, j \neq k} |a_{j,k}|.$$

In 2023, Nica and Sprague derived the following non-Archimedean analogue of Theorems 1 and 7.

Theorem 16. *[17] (**Non-Archimedean Gershgorin Eigenvalue Inclusion Theorem** or **Nica-Sprague Disk Theorem**) For every $A = [a_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{K})$,*

$$\sigma(A) \subseteq \bigcup_{j=1}^{n} \{z \in \mathbb{K} : |z - a_{j,j}| \leq h_j(A)\}$$

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and

$$\sigma(A) \subseteq \bigcup_{k=1}^{n} \{z \in \mathbb{K} : |z - a_{k,k}| \leq v_k(A)\}.$$

Theorem 17. *[17] (**Non-Archimedean Strict Diagonal Dominance Theorem** or **Nica-Sprague Nonsingularity Theorem**) If $A = [a_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{K})$ satisfies*

$$|a_{j,j}| > h_j(A), \quad \forall 1 \leq j \leq n$$

or

$$|a_{k,k}| > v_k(A), \quad \forall 1 \leq k \leq n,$$

then A is invertible.

Nica and Sprague showed that Theorems 16 and 17 are equivalent.

Theorem 18. *[17] Let $n \in \mathbb{N}$. Following two statements are equivalent.*

(i) *Let $A = [a_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{K})$. Then*

$$\sigma(A) \subseteq \bigcup_{j=1}^{n} \{z \in \mathbb{K} : |z - a_{j,j}| \leq h_j(A)\}.$$

(ii) *If $B = [b_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{K})$ satisfies*

$$|b_{j,j}| > h_j(B), \quad \forall 1 \leq j \leq n,$$

then B is invertible.

Theorem 19. *[17] Let $n \in \mathbb{N}$. Following two statements are equivalent.*

(i) *Let $A = [a_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{K})$. Then*

$$\sigma(A) \subseteq \bigcup_{k=1}^{n} \{z \in \mathbb{K} : |z - a_{k,k}| \leq v_k(A)\}.$$

(ii) *If $B = [b_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{K})$ satisfies*

$$|b_{k,k}| > v_k(B), \quad \forall 1 \leq k \leq n,$$

then B is invertible.

Non-Archimedean version of Theorem 2 reads as follows.

Theorem 20. *Let $A = [a_{j,k}]_{1 \leq j \leq n, 1 \leq k \leq n} \in \mathbb{M}_n(\mathbb{K})$. For every $\lambda \in \sigma(A)$,*

$$|\lambda| \leq \min\left\{ \max_{1 \leq j \leq n} \max_{1 \leq k \leq n} |a_{j,k}|, \max_{1 \leq k \leq n} \max_{1 \leq j \leq n} |a_{j,k}| \right\} \leq \frac{1}{2}\left(\max_{1 \leq j \leq n} \max_{1 \leq k \leq n} |a_{j,k}| + \max_{1 \leq k \leq n} \max_{1 \leq j \leq n} |a_{j,k}| \right).$$

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In particular,

$$|det(A)| \leq \min\left\{\left(\max_{1\leq j\leq n}\max_{1\leq k\leq n}|a_{j,k}|\right)^n, \left(\max_{1\leq k\leq n}\max_{1\leq j\leq n}|a_{j,k}|\right)^n\right\}$$
$$\leq \frac{1}{2}\left(\left(\max_{1\leq j\leq n}\max_{1\leq k\leq n}|a_{j,k}|\right)^n + \left(\max_{1\leq k\leq n}\max_{1\leq j\leq n}|a_{j,k}|\right)^n\right).$$

Let $p(z) := c_0 + c_1z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{K}[z]$. Let

$$C_p := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-3} & -c_{n-2} & -c_{n-1} \end{pmatrix} \in \mathbb{M}_n(\mathbb{K})$$

be the companion matrix of p. By applying Theorem 16 Nica and Sprague obtained following results.

Theorem 21. *[17] Let $p(z) := c_0 + c_1z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{K}[z]$. If λ is a zero of p, then*

$$|\lambda| \leq 1$$

or

$$|\lambda + c_{n-1}| \leq \max\{|c_0|, |c_1|, \ldots, |c_{n-2}|\}.$$

In particular,

$$\text{(Nica-Sprague bound)} \qquad |\lambda| \leq \max\{1, |c_0|, |c_1|, \ldots, |c_{n-1}|\}.$$

Theorem 22. *[17] Let $p(z) := c_0 + c_1z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{K}[z]$. If λ is a zero of p, then*

$$|\lambda| \leq |c_o|$$

or

$$|\lambda| \leq \max\{1, |c_j|\}, \quad \text{for some } 1 \leq j \leq n-2$$

or

$$|\lambda + c_{n-1}| \leq 1.$$

In particular,

$$\text{(Nica-Sprague bound)} \qquad |\lambda| \leq \max\{1, |c_0|, |c_1|, \ldots, |c_{n-1}|\}. \tag{3}$$

Theorem 23. *[17] Let $p(z) := c_0 + c_1z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{K}[z]$ with $c_0 \neq 0$. If λ is a zero of p, then*

$$\frac{1}{|\lambda|} \leq 1$$

or

$$\left|\frac{1}{\lambda}+\frac{c_1}{c_0}\right| \leq \max\left\{\frac{1}{|c_0|}, \frac{|c_2|}{|c_0|}, \dots, \frac{|c_{n-1}|}{|c_0|}\right\}.$$

In particular,

$$\text{(Nica-Sprague lower bound)} \qquad |\lambda| \geq \frac{|c_0|}{\max\{1, |c_0|, |c_1|, \dots, |c_{n-1}|\}}.$$

Theorem 24. *[17] Let $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{K}[z]$ with $c_0 \neq 0$. If λ is a zero of p, then*

$$\frac{1}{|\lambda|} \leq \frac{1}{|c_0|}$$

or

$$\frac{1}{|\lambda|} \leq \max\left\{1, \frac{|c_j|}{|c_0|}\right\}, \quad \text{for some } 2 \leq j \leq n-1$$

or

$$\left|\frac{1}{\lambda}+\frac{c_1}{c_0}\right| \leq 1.$$

In particular,

$$\text{(Nica-Sprague lower bound)} \qquad |\lambda| \geq \frac{|c_0|}{\max\{1, |c_0|, |c_1|, \dots, |c_{n-1}|\}}.$$

Before passing, we give a different and direct proof of Inequality (3). Let $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{K}[z]$ and λ be a zero of p. If $|\lambda| \leq 1$, then clearly we have Inequality (3). So we assume that $|\lambda| > 1$. Since $p(\lambda) = 0$, we have

$$\left(\frac{c_0}{\lambda^n}+\frac{c_1}{\lambda^{n-1}}+\cdots+\frac{c_{n-1}}{\lambda}+1\right)\lambda^n = 0.$$

Since $\lambda \neq 0$,

$$\frac{c_0}{\lambda^n}+\frac{c_1}{\lambda^{n-1}}+\cdots+\frac{c_{n-1}}{\lambda}+1 = 0.$$

Rearranging,

$$-1 = \frac{c_0}{\lambda^n}+\frac{c_1}{\lambda^{n-1}}+\cdots+\frac{c_{n-1}}{\lambda}.$$

By taking absolute value and noticing $|\lambda| > 1$, we get

$$\begin{aligned} 1 &\leq \left|\frac{c_0}{\lambda^n}+\frac{c_1}{\lambda^{n-1}}+\cdots+\frac{c_{n-1}}{\lambda}\right| \leq \max\left\{\frac{|c_0|}{|\lambda|^n}, \frac{|c_1|}{|\lambda|^{n-1}}, \dots, \frac{|c_{n-1}|}{|\lambda|}\right\} \\ &\leq \max\left\{\frac{|c_0|}{|\lambda|}, \frac{|c_1|}{|\lambda|}, \dots, \frac{|c_{n-1}|}{|\lambda|}\right\} = \frac{1}{|\lambda|}\max\{|c_0|, |c_1|, \dots, |c_{n-1}|\}. \end{aligned}$$

Rearranging above inequality completes the argument.
In 2025, Li and Li derived the following non-Archimedean analogue of Theorem 13.

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Theorem 25. *[18] (**Non-Archimedean Ostrowski Nonsingularity Theorem** or **Li-Li Nonsingularity Theorem**) If $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{K})$ satisfies*

$$|a_{j,j}||a_{k,k}| > h_j(A)h_k(A), \quad \forall 1 \leq j,k \leq n, j \neq k$$

or

$$|a_{j,j}||a_{k,k}| > v_j(A)v_k(A), \quad \forall 1 \leq j,k \leq n, j \neq k,$$

then A is invertible.

In this article, we derive non-Archimedean version of Theorem 10. We also show that our result is equivalent to Theorem 25. We give applications for bounding the zeros of polynomials over non-Archimedean fields.

2. Non-Archimedean Brauer Oval (of Cassini) Theorem

We start with non-Archimedean Brauer eigenvalue inclusion theorem. Our proof is motivated from the proof of Brauer [15].

Theorem 26. *(Non-Archimedean Brauer Oval (of Cassini) Theorem)*
For every $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{K})$,

$$\sigma(A) \subseteq \bigcup_{j,k=1, j\neq k}^{n} \{z \in \mathbb{K} : |z - a_{j,j}||z - a_{k,k}| \leq h_j(A)h_k(A)\}$$

and

$$\sigma(A) \subseteq \bigcup_{j,k=1, j\neq k}^{n} \{z \in \mathbb{K} : |z - a_{j,j}||z - a_{k,k}| \leq v_j(A)v_k(A)\}.$$

Proof. Let $\lambda \in \sigma(A)$. Then there exists a $0 \neq \mathbf{x} = (x_j)_{j=1}^n \in \mathbb{K}^n$ such that

$$\lambda\mathbf{x} = A\mathbf{x}. \tag{4}$$

Choose $1 \leq j \leq n$ such that

$$|x_j| = \max_{1\leq l\leq n} |x_l|.$$

Now choose $1 \leq k \leq n$ with $k \neq j$ such that

$$|x_k| = \max_{1\leq l\leq n, l\neq j} |x_l|.$$

Then we have $1 \leq j,k \leq n$ with $j \neq k$ and

$$|x_j| \geq |x_k| \geq \max_{1\leq l\leq n, l\neq j, l\neq k} |x_l|. \tag{5}$$

We have two cases. Case (i): $|x_k| = 0$. Considering the j-th coordinate in Equation (4) gives

$$\lambda x_j = \sum_{p=1}^{n} a_{j,p}x_p.$$

Rewriting previous equation gives

$$(\lambda - a_{j,j})x_j = \sum_{p=1,p\neq j}^{n} a_{j,p}x_p.$$

Therefore using (5) we get

$$\begin{aligned}|(\lambda - a_{j,j})x_j| &= \left|\sum_{p=1,p\neq j}^{n} a_{j,p}x_p\right| \leq \max_{1\leq p\leq n, p\neq j} |a_{j,p}x_p| \\ &\leq \left(\max_{1\leq p\leq n, p\neq j} |a_{j,p}|\right)\left(\max_{1\leq p\leq n, p\neq j} |x_p|\right) = \left(\max_{1\leq p\leq n, p\neq j} |a_{j,p}|\right)|x_k| \\ &= 0.\end{aligned}$$

Since $|x_j| \geq |x_k|$ for all $1 \leq k \leq n$, $|x_k| = 0$ and $\mathbf{x} \neq 0$, we must have $|x_j| \neq 0$. Previous inequality then gives $|\lambda - a_{j,j}| = 0$. So

$$\lambda = a_{j,j} \in \bigcup_{p,q=1,p\neq q}^{n} \{z \in \mathbb{K} : |z - a_{p,p}||z - a_{q,q}| \leq h_p(A)h_q(A)\}.$$

Case (ii): $|x_k| > 0$. Considering j-th and k-th coordinates in Equation (4) give

$$(\lambda - a_{j,j})x_j = \sum_{p=1,p\neq j}^{n} a_{j,p}x_p \tag{6}$$

and

$$(\lambda - a_{k,k})x_k = \sum_{q=1,q\neq k}^{n} a_{k,q}x_q. \tag{7}$$

Multiplying Equations (6) and (7) and taking non-Archimedean valuation gives

$$\begin{aligned}|(\lambda - a_{j,j})x_j(\lambda - a_{k,k})x_k| &= \left|\sum_{p=1,p\neq j}^{n} a_{j,p}x_p\right|\left|\sum_{q=1,q\neq k}^{n} a_{k,q}x_q\right| \\ &\leq \left(\max_{1\leq p\leq n, p\neq j} |a_{j,p}x_p|\right)\left(\max_{1\leq q\leq n, q\neq k} |a_{k,p}x_q|\right) \\ &\leq \left(\max_{1\leq p\leq n, p\neq j} |a_{j,p}|\right)\left(\max_{1\leq p\leq n, p\neq j} |x_p|\right)\left(\max_{1\leq q\leq n, q\neq k} |a_{k,p}|\right)\left(\max_{1\leq q\leq n, q\neq k} |x_q|\right) \\ &= \left(\max_{1\leq p\leq n, p\neq j} |a_{j,p}|\right)|x_k|\left(\max_{1\leq q\leq n, q\neq k} |a_{k,p}|\right)|x_j| \\ &= h_j(A)|x_k|h_k(A)|x_j|.\end{aligned}$$

Therefore

$$|(\lambda - a_{j,j})(\lambda - a_{k,k})||x_jx_k| \leq h_j(A)h_k(A)|x_j||x_k|.$$

Since $|x_jx_k| \neq 0$, we have

$$|(\lambda - a_{j,j})(\lambda - a_{k,k})| \leq h_j(A)h_k(A).$$

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Previous inequality says that

$$\lambda \in \bigcup_{p,q=1,p\neq q}^{n} \{z \in \mathbb{K} : |z - a_{p,p}||z - a_{q,q}| \leq h_p(A)h_q(A)\}.$$

Second inclusion in the statement follows by considering the transpose of A and noting that the spectrum of a matrix and its transpose are equal. □

By applying Theorem 26, we get following results.

Theorem 27. *Let* $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{K}[z]$. *If* λ *is a zero of* p, *then*

$$|\lambda| \leq 1$$

or

$$|\lambda||\lambda + c_{n-1}| \leq \max\{|c_0|, \cdots, |c_{n-2}|\}.$$

Theorem 28. *Let* $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{K}[z]$. *If* λ *is a zero of* p, *then*

$$|\lambda|^2 \leq |c_0| \max\{1, |c_j|\}, \quad \textit{for some } 1 \leq j \leq n-2$$

or

$$|\lambda||\lambda + c_{n-1}| \leq |c_0|$$

or

$$|\lambda|^2 \leq \big(\max\{1, |c_j|\}\big)\big(\max\{1, |c_k|\}\big), \quad \textit{for some } 1 \leq j, k \leq n-2, j \neq k$$

or

$$|\lambda||\lambda + c_{n-1}| \leq \max\{1, |c_j|\}, \quad \textit{for some } 1 \leq j \leq n-2.$$

Theorem 29. *Let* $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{K}[z]$ *with* $c_0 \neq 0$. *If* λ *is a zero of* p, *then*

$$\frac{1}{|\lambda|} \leq 1$$

or

$$\frac{1}{|\lambda|}\left|\frac{1}{\lambda} + \frac{c_1}{c_0}\right| \leq \max\left\{\frac{1}{|c_0|}, \frac{|c_2|}{|c_0|}, \cdots, \frac{|c_{n-1}|}{|c_0|}\right\}.$$

Theorem 30. *Let* $p(z) := c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + z^n \in \mathbb{K}[z]$ *with* $c_0 \neq 0$. *If* λ *is a zero of* p, *then*

$$\frac{1}{|\lambda|^2} \leq \frac{1}{|c_0|} \max\left\{1, \frac{|c_j|}{|c_0|}\right\}, \quad \textit{for some } 2 \leq j \leq n-1$$

or

$$\frac{1}{|\lambda|}\left|\frac{1}{\lambda} + \frac{c_1}{c_0}\right| \leq \frac{1}{|c_0|}$$

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or

$$\frac{1}{|\lambda|^2} \leq \left(\max\left\{1, \frac{|c_j|}{|c_0|}\right\}\right)\left(\max\left\{1, \frac{|c_k|}{|c_0|}\right\}\right), \quad \text{for some } 2 \leq j,k \leq n-1, j \neq k$$

or

$$\frac{1}{|\lambda|}\left|\frac{1}{\lambda} + \frac{c_1}{c_0}\right| \leq \max\left\{1, \frac{|c_j|}{|c_0|}\right\}, \quad \text{for some } 2 \leq j \leq n-1.$$

Like the complex case, Theorem 26 cannot be extended by considering three rows/columns. An example given for the scalar case in [3] (also see [19,20]) extends to non-Archimedean case. Consider the following matrix over any non-Archimedean field $\mathbb{K}$.

$$A := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\sigma(A) = \{0,1,1,2\}$ and $h_1(A) = 1, h_2(A) = 1, h_3(A) = 0, h_4(A) = 0$. Hence

$$\sigma(A) \not\subseteq \bigcup_{j,k,l=1, j\neq k, j\neq l, k\neq l}^{4} \{z \in \mathbb{K} : |z - a_{j,j}||z - a_{k,k}||z - a_{l,l}| \leq h_j(A)h_k(A)h_l(A)\} = \{0\}.$$

Next we show that Theorem 26 improves Theorem 16. Our proof is motivated from the complex case, given in [3].

Theorem 31. *For every* $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{K})$,

$$\bigcup_{j,k=1, j\neq k}^{n} \{z \in \mathbb{K} : |z - a_{j,j}||z - a_{k,k}| \leq h_j(A)h_k(A)\} \subseteq \bigcup_{j=1}^{n}\{z \in \mathbb{K} : |z - a_{j,j}| \leq h_j(A)\}$$

and

$$\bigcup_{j,k=1, j\neq k}^{n} \{z \in \mathbb{K} : |z - a_{j,j}||z - a_{k,k}| \leq v_j(A)v_k(A)\} \subseteq \bigcup_{j=1}^{n}\{z \in \mathbb{K} : |z - a_{j,j}| \leq v_j(A)\}.$$

Proof. We prove the first inclusion, proof of second inclusion is similar. Set

$$Z := \bigcup_{j,k=1, j\neq k}^{n} \{z \in \mathbb{K} : |z - a_{j,j}||z - a_{k,k}| \leq h_j(A)h_k(A)\}$$

and let $z \in Z$. Then there exist $1 \leq j,k \leq n$ with $j \neq k$ such that

$$|z - a_{j,j}||z - a_{k,k}| \leq h_j(A)h_k(A).$$

We have to consider two cases. Case (i): $h_j(A)h_k(A) = 0$. Then $z = a_{j,j}$ or $z = a_{k,k}$. Now clearly we have

$$z \in \{x \in \mathbb{K} : |x - a_{j,j}| \leq h_j(A)\} \cup \{y \in \mathbb{K} : |y - a_{k,k}| \leq h_k(A)\} \subseteq \bigcup_{l=1}^{n}\{z \in \mathbb{K} : |z - a_{l,l}| \leq v_l(A)\}.$$

Case(ii): $h_j(A)h_k(A) > 0$. Then

$$|z - a_{j,j}| \leq h_j(A)$$

or

$$|z - a_{k,k}| \leq h_k(A).$$

Now clearly we have

$$z \in \{x \in \mathbb{K} : |x - a_{j,j}| \leq h_j(A)\} \cup \{y \in \mathbb{K} : |y - a_{k,k}| \leq h_k(A)\} \subseteq \bigcup_{l=1}^{n} \{z \in \mathbb{K} : |z - a_{l,l}| \leq v_l(A)\}.$$

□

Now we show that Theorems 25 and 26 are equivalent.

Theorem 32. *Let $n \in \mathbb{N}$. Following two statements are equivalent.*

(i) *Let $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{K})$. Then*

$$\sigma(A) \subseteq \bigcup_{j,k=1, j\neq k}^{n} \{z \in \mathbb{K} : |z - a_{j,j}||z - a_{k,k}| \leq h_j(A)h_k(A)\}.$$

(ii) *If $B = [b_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{K})$ satisfies*

$$|b_{j,j}||b_{k,k}| > h_j(B)h_k(B), \quad \forall 1 \leq j, k \leq n, j \neq k,$$

then B is invertible.

Proof. *(i)* $\implies$ (ii) Let $B = [b_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{K})$ satisfies

$$|b_{j,j}||b_{k,k}| > h_j(B)h_k(B), \quad \forall 1 \leq j, k \leq n, j \neq k. \tag{8}$$

We need to show that B is invertible. Let us assume that B is not invertible. Then $0 \in \sigma(B)$. By assumption (i), there exist $1 \leq j, k \leq n$ with $j \neq k$ such that

$$|0 - b_{j,j}||0 - b_{k,k}| \leq h_j(B)h_k(B). \tag{9}$$

Inequalities (8) and (9) contradict each other. Hence B is invertible.

(ii) $\implies$ (i) Let $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{K})$ and $\lambda \in \sigma(A)$. We claim that

$$|\lambda - a_{j,j}||\lambda - a_{k,k}| \leq h_j(A)h_k(A), \quad \text{for some } 1 \leq j, k \leq n, j \neq k.$$

Let us suppose that claim fails. Then

$$|\lambda - a_{j,j}||\lambda - a_{k,k}| > h_j(A)h_k(A), \quad \forall 1 \leq j, k \leq n, j \neq k. \tag{10}$$

Let I_n be the identity matrix in $\mathbb{M}_n(\mathbb{K})$. Define $B := \lambda I_n - A =: [b_{j,k}]_{1\leq j\leq n, 1\leq k\leq n}$. Then $0 \in \sigma(B)$, hence B is not invertible. Note that $h_j(A) = h_j(B)$ for all $1 \leq j \leq n$. But we also have from (10)

$$|b_{j,j}||b_{k,k}| > h_j(B)h_k(B), \quad \forall 1 \leq j, k \leq n, j \neq k.$$

Assumption (ii) says that B is invertible which is not possible. Hence claim holds. □

By considering the transpose of a matrix, we easily get following result.

Theorem 33. *Let $n \in \mathbb{N}$. Following two statements are equivalent.*

(i) *Let $A = [a_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{K})$. Then*

$$\sigma(A) \subseteq \bigcup_{j,k=1, j\neq k}^{n} \{z \in \mathbb{K} : |z - a_{j,j}||z - a_{k,k}| \leq v_j(A)v_k(A)\}.$$

(ii) *If $B = [b_{j,k}]_{1\leq j\leq n, 1\leq k\leq n} \in \mathbb{M}_n(\mathbb{K})$ satisfies*

$$|b_{j,j}||b_{k,k}| > v_j(B)v_k(B), \quad \forall 1 \leq j, k \leq n, j \neq k,$$

then B is invertible.

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