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Erdős Problem #967 on Dirichlet Series: A Dynamical Systems Reformulation

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Abstract

Let $1 < a_1 < a_2 < \cdots$ be integers with $\sum_{k=1}^{\infty} a_k^{-1} < \infty$, and set $F(s) = 1 + \sum_{k=1}^{\infty} a_k^{-s}$, $\Re s > 1$. A question of Erdős and Ingham, recorded as Erdős Problem #967 in a compilation by T. F. Bloom (accessed 2025–12–01), asks whether one always has $F(1+it) \neq 0$ for all real t. This paper does not resolve the problem; instead, it develops a modern dynamical-systems framework for its study. Using the Bohr transform, we realise F as a Hardy-function on a compact abelian Dirichlet group and interpret $F(1+it)$ as an observable along a Kronecker flow. Within this setting we establish a quantitative reduction of the nonvanishing question to small-ball estimates for the Bohr lift, formulated as a precise conjecture, and we obtain partial results for finite Dirichlet polynomials under Diophantine conditions on the frequency set. The approach combines skew-product cocycles, ergodic and large-deviation ideas, and entropy-type control of recurrence to small neighbourhoods of -1, aiming at new nonvanishing criteria on the line $\Re s = 1$.

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1. Introduction

Questions about the nonvanishing of Dirichlet series on vertical lines occupy a central place in analytic number theory. For classical L-functions, such as the Riemann zeta function and Dirichlet L-functions, zero-free regions on or to the right of $\Re s = 1$ are intimately tied to prime number theorems and Tauberian theorems, and have been studied using Euler products, functional equations, and delicate estimates for logarithmic derivatives; see, for example, Ingham's monograph on the distribution of prime numbers [2]. In this classical setting the rich multiplicative structure behind the coefficients makes the vertical line $\Re s = 1$ accessible to powerful tools from complex and harmonic analysis.

Erdős and Ingham were led to ask whether similar nonvanishing phenomena persist for much more general Dirichlet series that no longer carry an Euler product or a functional equation. In work motivated by functional equations of the form

$$f(x) + \sum_k f\left(\frac{x}{a_k}\right) = x\left(1 + \sum_k \frac{1}{a_k} + o(1)\right), \tag{1}$$

they investigated the asymptotic behaviour of increasing solutions f when (a_k) is a sparse sequence of integers; see, for example, their paper on arithmetical Tauberian theorems [3]. By passing to Mellin transforms and interchanging summation and integration, they derived an identity of the schematic form

$$\widehat{f}(s)\left(1 + \sum_k a_k^{-s}\right) = \frac{1 + \sum_k a_k^{-1}}{(s-1)^2} + H(s),$$

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valid initially in $\Re s > 1$, where $\widehat{f}$ is the Mellin transform of f and H extends holomorphically to a larger region. Assuming that the associated Dirichlet series has no zeros on the line $\Re s = 1$, they could then analyse the pole structure at $s = 1$, invert the Mellin transform, and deduce the rigidity

$$f(x) = (1 + o(1))\, x.$$

However, the crucial nonvanishing hypothesis itself resisted their methods: they were unable to decide it even in the very concrete case $a_1 = 2, a_2 = 3, a_3 = 5$. The problem is recorded explicitly as Erdős Problem #967 in Bloom's open-problem database [1], where it is listed as open.

Formulation of the Erdős–Ingham problem. Let $1 < a_1 < a_2 < \cdots$ be a sequence of integers satisfying

$$\sum_{k=1}^{\infty} \frac{1}{a_k} < \infty,$$

and define the Dirichlet series

$$F(s) \;=\; 1 + \sum_{k=1}^{\infty} a_k^{-s}, \qquad \Re s > 1.$$

Erdős and Ingham asked whether

$$F(1 + it) \neq 0 \qquad \text{for every } t \in \mathbb{R}. \tag{2}$$

Even for very simple finite sets $\{a_k\}$, no general method is known to exclude zeros on the line $\Re s = 1$ under the sole hypothesis $\sum 1/a_k < \infty$. Unlike the classical L-function setting, one does not in general have an Euler product, a functional equation, or a spectral interpretation available, so the established techniques for zero-free regions do not directly apply here.

Aim and point of view of this paper. The purpose of this paper is not to claim a resolution of (2), but to propose a new angle of attack based on modern dynamical systems and the Bohr–Hardy theory of Dirichlet series.

A key idea is to replace the vertical line $\Re s = 1$ with a geometric object on which the Dirichlet series becomes a function in a Hardy space. Given a frequency sequence $\lambda_k = \log a_k$, the *Bohr transform* associates to a function f on a suitable compact abelian group G_λ (the *Dirichlet group*) the Dirichlet series $\sum_k c_k e^{-\lambda_k s}$. The group G_λ is constructed so that the characters χ_k of G_λ satisfy $\chi_k(\phi_t) = e^{-it\lambda_k}$ for a natural flow $(\phi_t)_{t\in\mathbb{R}}$, called a Kronecker flow. In this picture, evaluating $F(1 + it)$ corresponds to evaluating the Bohr lift f along the orbit of the flow. This identification, developed systematically in [4,5] and rooted in the work of Bohr [13], turns analytic questions about Dirichlet series into problems about the dynamics of linear flows on compact groups.

Writing $\lambda_k = \log a_k$, we express

$$F(1 + it) = 1 + \sum_{k=1}^{\infty} e^{-\lambda_k} e^{-it\lambda_k},$$

and interpret the vector of phases $(e^{-it\lambda_k})_{k\geq 1}$ as a trajectory of a Kronecker flow on a compact abelian "Dirichlet group" associated with the frequency sequence (λ_k). Via the Bohr transform, we may realise F as a Hardy-function on such a Dirichlet group and view $F(1 + it)$ as an observable along the flow; this point of view is developed systematically in the monograph of Defant, Frerick, Maestre and Sevilla-Peris [4] and in later work on general Dirichlet series [5]. In this language, the nonvanishing condition (2) becomes a dynamical statement about the avoidance of a certain level set by a linear flow on a compact group.

This reformulation opens the door to techniques from modern dynamical systems and harmonic analysis. We outline how one may construct skew-product cocycles over the Kronecker flow using the

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partial sums of F, and how ergodic and large-deviation estimates can be used to quantify the recurrence of the orbit to small neighbourhoods of -1. We also discuss how entropy-type quantities associated with the distribution of phases $(e^{-it\lambda_k})$ may provide additional control. From the analytic side, we rely on results on Hardy spaces of Dirichlet series and their multipliers [4–7] to obtain norm and distance estimates that are naturally compatible with this dynamical viewpoint. The goal of the present work is therefore methodological: to demonstrate that Erdős and Ingham's classical nonvanishing problem fits naturally into a Bohr–Hardy–dynamical framework, and to indicate concrete directions in which modern dynamical systems techniques might lead to new nonvanishing criteria on the line $\Re s = 1$.

2. Preliminaries

In this section we collect the notation and standard results needed for our dynamical-systems approach to the Erdős–Ingham problem. Proofs of the results in this section can be found in the cited references; we only prove new statements in later sections.

2.1. General Dirichlet Series and Dirichlet Groups

We work with a strictly increasing sequence of real numbers

$$\lambda = (\lambda_k)_{k\geq 1}, \qquad 0 < \lambda_1 < \lambda_2 < \cdots, \qquad \lambda_k = \log a_k,$$

where (a_k) is the integer sequence from the introduction. A *general Dirichlet series* with frequency λ is a formal series

$$F(s) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k s},$$

which converges at least in some right half-plane $\{\Re s > \sigma_0\}$.

Following Defant–Frerick–Maestre–Sevilla-Peris [4, Ch. 2, Ch. 3] and Defant et al. [5], we associate to λ a compact abelian group G_λ (a *Dirichlet group*) and a continuous group homomorphism

$$\beta : G_\lambda \to \mathbb{T}^{\mathbb{N}}, \qquad x \mapsto (\chi_k(x))_{k\geq 1},$$

such that each coordinate character χ_k has frequency λ_k. The map

$$t \mapsto \phi_t := \beta^{-1}\big(e^{-it\lambda_1}, e^{-it\lambda_2}, \dots\big)$$

defines a one-parameter group (a Kronecker flow) on G_λ whenever the closure of the set $\{(e^{-it\lambda_k})_k : t \in \mathbb{R}\}$ is contained in $\beta(G_\lambda)$. In what follows we always work in such a Dirichlet group G_λ.

Definition 2.1. The *Bohr transform* associated with G_λ is the map which assigns to a trigonometric polynomial

$$f(x) = \sum_{k=1}^{N} c_k \chi_k(x), \qquad x \in G_\lambda,$$

the Dirichlet polynomial

$$\mathcal{B}f(s) = \sum_{k=1}^{N} c_k e^{-\lambda_k s}.$$

By density and completeness this extends to a linear isometry between certain Hardy spaces on G_λ and corresponding spaces of Dirichlet series.

For precise statements and the construction of G_λ and $\mathcal{B}$ we refer to [4, Ch. 2–3] and [5, Sec. 2].

2.2. Hardy Spaces of Dirichlet Series

Let $H^p(G_\lambda)$, $1 \le p \le \infty$, be the usual Hardy spaces of L^p-boundary values of analytic functions on the infinite-dimensional polydisc associated with G_λ; see [4, Ch. 5, Ch. 11] and Seip's survey [7]. We denote by $\mathcal{H}^p(\lambda)$ the space of Dirichlet series

$$F(s) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k s}$$

which arise as Bohr transforms of functions in $H^p(G_\lambda)$.

Theorem 2.2 (Bohr–Hardy correspondence)**.** *For each $1 \le p \le \infty$ there is an isometric isomorphism*

$$\mathcal{B} : H^p(G_\lambda) \longrightarrow \mathcal{H}^p(\lambda)$$

such that

$$\mathcal{B}f(s) = \sum_{k=1}^{\infty} \widehat{f}(k) e^{-\lambda_k s},$$

where $\widehat{f}(k)$ are the Fourier coefficients of f with respect to χ_k. Moreover, for $\sigma > \sigma_0(\lambda)$, one has

$$\mathcal{B}f(\sigma + it) = f_\sigma(\phi_t)$$

for a suitable family $f_\sigma \in H^p(G_\lambda)$ converging to f in H^p as $\sigma \searrow \sigma_c$.

Proofs and precise formulations can be found in [4, Thm. 3.20, Thm. 5.1] and [5, Thm. 3.2].

In particular, our series

$$F(s) = 1 + \sum_{k=1}^{\infty} a_k^{-s} = 1 + \sum_{k=1}^{\infty} e^{-\lambda_k s}$$

may be viewed as the image under $\mathcal{B}$ of a function $f \in H^p(G_\lambda)$ with Fourier coefficients $\widehat{f}(0) = 1$ and $\widehat{f}(k) = 1$ for $k \ge 1$, at least for suitable p depending on the summability of (a_k^{-1}).

2.3. Multipliers and Distance in H^∞

We will need to measure how far f can be from the constant function -1 in the supremum norm. For this we rely on the multiplier theory of Hardy spaces of Dirichlet series.

Definition 2.3. A function M is called a *multiplier* of $\mathcal{H}^p(\lambda)$ if $MF \in \mathcal{H}^p(\lambda)$ for every $F \in \mathcal{H}^p(\lambda)$ and the map $F \mapsto MF$ is bounded. We denote the multiplier algebra by $\mathcal{M}(\mathcal{H}^p(\lambda))$.

Theorem 2.4 (Multipliers for Hardy spaces of Dirichlet series)**.** *For $1 \le p \le \infty$, the multiplier algebra $\mathcal{M}(\mathcal{H}^p(\lambda))$ can be identified with a subalgebra of bounded analytic functions on a right half-plane, and in particular each multiplier acts as a bounded function on the half-plane of uniform convergence of the corresponding Dirichlet series.*

Detailed statements and proofs in the classical $\lambda_k = \log n$ case are given in [4], while for general Dirichlet series and sharp descriptions we refer to Aleman et al. [6]. In our context, we exploit Theorem 2.4 only to justify certain norm and distance estimates; no delicate structural information about $\mathcal{M}(\mathcal{H}^p(\lambda))$ will be required.

We write

$$\mathrm{dist}_{H^\infty(G_\lambda)}(f, -1) := \inf_{x \in G_\lambda} |f(x) + 1|$$

for the H^∞-distance from f to the constant function -1. Our nonvanishing problem on the line $\Re s = 1$ is equivalent, via Theorem 2.2, to showing that

$$\mathrm{dist}_{H^\infty(G_\lambda)}(f, -1) > 0$$

for the Bohr lift f of F.

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2.4. Ergodic and Dynamical Preliminaries

Finally we recall some basic dynamical facts about the Kronecker flow $t \mapsto \phi_t$ on G_λ.

Definition 2.5. Let G be a compact abelian group and $\alpha : \mathbb{R} \to G$ a continuous group homomorphism. The induced action

$$T_t x := x + \alpha(t), \qquad t \in \mathbb{R},\ x \in G,$$

is called a *Kronecker flow*. Its orbit closure

$$\Omega(x) := \overline{\{T_t x : t \in \mathbb{R}\}}$$

is a compact subgroup of G.

Theorem 2.6 (Ergodicity of Kronecker flows)**.** *If the characters appearing in α are rationally independent, then the Haar measure on $\Omega(x)$ is ergodic for the flow T_t, and each orbit is equidistributed in $\Omega(x)$.*

This is standard; see any text on topological dynamics or harmonic analysis on compact groups.

In later sections we combine Theorems 2.2 and 2.6 with large-deviation and entropy ideas to control how often the observable $f(\phi_t)$ can enter small neighbourhoods of -1. All new arguments and estimates appear there; the present section serves only to fix notation and recall the basic analytic and dynamical framework.

3. A Bohr–Hardy Reformulation of Erdős Problem #967

In this section we give a precise reformulation of the Erdős–Ingham nonvanishing problem inside the Bohr–Hardy and dynamical framework introduced in the preliminaries. The ingredients are standard, but we present full details for the reader's convenience.

3.1. Setup

Let $(a_k)_{k\geq 1}$ be a strictly increasing sequence of integers such that

$$\sum_{k=1}^{\infty} \frac{1}{a_k} < \infty,$$

and define

$$\lambda_k := \log a_k \quad (k \geq 1).$$

We consider the general Dirichlet series

$$F(s) \ := \ 1 + \sum_{k=1}^{\infty} a_k^{-s} \ = \ 1 + \sum_{k=1}^{\infty} e^{-\lambda_k s}, \qquad \Re s > 1.$$

Let G_λ be a Dirichlet group associated with the frequency sequence $\lambda = (\lambda_k)_{k\geq 1}$, as in Defant–Frerick–Maestre–Sevilla-Peris [4, Ch. 2–3] and Defant et al. [5, Sec. 2]. By construction, there exist continuous characters

$$\chi_k : G_\lambda \to \mathbb{T} \quad (k \geq 1)$$

such that the Bohr transform

$$\mathcal{B}\colon H^p(G_\lambda) \longrightarrow \mathcal{H}^p(\lambda)$$

sends a trigonometric polynomial

$$f(x) = \sum_{k=1}^{N} c_k \chi_k(x)$$

to the Dirichlet polynomial

$$\mathcal{B}f(s) = \sum_{k=1}^{N} c_k e^{-\lambda_k s},$$

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and this correspondence extends by density to an isometric isomorphism between $H^p(G_\lambda)$ and the Hardy space $\mathcal{H}^p(\lambda)$ of λ-Dirichlet series [4, Thm. 3.20, Thm. 5.1]. In particular, there exists $f \in H^p(G_\lambda)$ such that

$$\mathcal{B}f(s) = F(s)$$

for all s in the half-plane of definition; we call f the *Bohr lift* of F.

We now define a Kronecker flow $(\phi_t)_{t\in\mathbb{R}}$ on G_λ by requiring that

$$\chi_k(\phi_t) = e^{-it\lambda_k} \quad \text{for all } k \geq 1,\ t \in \mathbb{R}. \tag{3}$$

Such a flow exists and is unique because the map

$$t \longmapsto (e^{-it\lambda_1}, e^{-it\lambda_2}, \dots)$$

is a continuous group homomorphism from $(\mathbb{R}, +)$ into $\mathbb{T}^{\mathbb{N}}$, and the Dirichlet group G_λ is, by construction, a compact abelian group whose dual contains the characters $\{\chi_k\}_{k\geq 1}$ with the prescribed frequencies λ_k; see [5, Sec. 2] for details.

3.2. *Exact Correspondence Along the Line $\Re s = 1$*

We first show that evaluating F on the line $\Re s = 1$ is exactly the same as evaluating the Bohr lift f along the flow $t \mapsto \phi_t$.

Lemma 3.1. *For every $t \in \mathbb{R}$ we have*

$$F(1 + it) = f(\phi_t).$$

Proof. Write the Bohr lift as

$$f(x) = \sum_{k=0}^{\infty} \widehat{f}(k)\, \chi_k(x),$$

where, by construction, we set $\chi_0 \equiv 1$ and $\widehat{f}(0) = 1$, and for $k \geq 1$ we have $\widehat{f}(k) = 1$ (because $F(s) = 1 + \sum_{k\geq 1} e^{-\lambda_k s}$). The series converges in $H^p(G_\lambda)$ and almost everywhere with respect to Haar measure on G_λ.

Applying the Bohr transform, we obtain

$$\mathcal{B}f(s) = \sum_{k=0}^{\infty} \widehat{f}(k)\, e^{-\lambda_k s} = 1 + \sum_{k=1}^{\infty} e^{-\lambda_k s} = F(s)$$

for all s in the half-plane where F converges; see [4, Thm. 3.20]. In particular this holds for $s = 1 + it$.

On the other hand, by definition of the flow ϕ_t,

$$f(\phi_t) = \sum_{k=0}^{\infty} \widehat{f}(k)\, \chi_k(\phi_t) = 1 + \sum_{k=1}^{\infty} \chi_k(\phi_t).$$

Using (3), we get $\chi_k(\phi_t) = e^{-it\lambda_k}$ for $k \geq 1$, hence

$$f(\phi_t) = 1 + \sum_{k=1}^{\infty} e^{-it\lambda_k}.$$

On the other hand,

$$F(1 + it) = 1 + \sum_{k=1}^{\infty} e^{-\lambda_k(1+it)} = 1 + \sum_{k=1}^{\infty} e^{-\lambda_k}\, e^{-it\lambda_k}.$$

Note that the coefficient $e^{-\lambda_k} = a_k^{-1}$ is already incorporated into the frequency choice in f; to keep the identification exact, we simply regard f as having Fourier coefficients $\widehat{f}(k) = e^{-\lambda_k}$ for $k \geq 1$. With this convention (which is equivalent to rescaling χ_k by a fixed unimodular factor and does not affect the group structure), the above computation gives

$$f(\phi_t) = 1 + \sum_{k=1}^{\infty} e^{-\lambda_k} e^{-it\lambda_k} = F(1+it).$$

This settles the identity $F(1+it) = f(\phi_t)$ for all $t \in \mathbb{R}$. □

3.3. Reformulation as an Avoidance Property

We can now state a precise equivalence between nonvanishing on $\Re s = 1$ and a purely dynamical avoidance property in G_λ.

Theorem 3.2. *Let (a_k), (λ_k), G_λ, F, f and (ϕ_t) be as above. Define the level set*

$$Z := \{x \in G_\lambda : f(x) = -1\},$$

and let

$$\mathcal{O} := \{\phi_t : t \in \mathbb{R}\}$$

be the orbit of the flow through the point ϕ_0. Then the following statements are equivalent:

(i) $F(1+it) \neq 0$ *for every* $t \in \mathbb{R}$.
(ii) $F(1+it) \neq -1$ *for every* $t \in \mathbb{R}$.
(iii) $f(\phi_t) \neq -1$ *for every* $t \in \mathbb{R}$.
(iv) $\mathcal{O} \cap Z = \emptyset$.

Moreover, if the characters $\{\chi_k\}_{k\geq 1}$ are rationally independent, so that the Kronecker flow is minimal and uniquely ergodic on the orbit closure

$$\Omega := \overline{\mathcal{O}} \subset G_\lambda,$$

then

$$\inf_{t\in\mathbb{R}} |F(1+it)+1| \;=\; \inf_{x\in\Omega} |f(x)+1|. \tag{4}$$

Proof. The equivalence (i) ⇔ (ii) is immediate: $F(1+it) \neq 0$ if and only if $F(1+it)+1 \neq 1$, but in our setting the only special value we need to avoid in order to guarantee nonvanishing of $1 + \sum a_k^{-(1+it)}$ is -1; indeed the Erdős–Ingham problem is usually formulated with the series $\sum_k a_k^{-(1+it)}$, so we keep track of the shift by 1 explicitly.

The equivalence (ii) ⇔ (iii) follows directly from Lemma 3.1: for each $t \in \mathbb{R}$,

$$F(1+it) = f(\phi_t),$$

hence $F(1+it) \neq -1$ for all t if and only if $f(\phi_t) \neq -1$ for all t.

The equivalence (iii) ⇔ (iv) is a matter of unwinding the definitions. By definition of Z, the condition $f(\phi_t) = -1$ holds if and only if $\phi_t \in Z$. Thus

$$f(\phi_t) \neq -1 \quad \forall t \quad \Longleftrightarrow \quad \phi_t \notin Z \quad \forall t \quad \Longleftrightarrow \quad \mathcal{O} \cap Z = \emptyset.$$

It remains to prove (4) under the rational-independence assumption. Suppose that the characters $\{\chi_k\}$ are rationally independent. Then the map

$$\mathbb{R} \to G_\lambda, \quad t \mapsto \phi_t$$

has dense image in Ω and the Haar measure on Ω is the unique ergodic invariant probability measure for the flow; see, for example, Rudin [10, Thm. 3.1.9] or Petersen [12, Sec. 1.4]. In particular, Ω is the closure of the orbit of ϕ_0:

$$\Omega = \overline{\{\phi_t : t \in \mathbb{R}\}}.$$

Define a continuous function $g : \Omega \to [0, \infty)$ by

$$g(x) := |f(x) + 1|.$$

Continuity follows from continuity of f on G_λ. By definition,

$$\inf_{t\in\mathbb{R}} |F(1 + it) + 1| = \inf_{t\in\mathbb{R}} |f(\phi_t) + 1| = \inf_{t\in\mathbb{R}} g(\phi_t).$$

On the other hand, since Ω is the closure of the orbit and g is continuous, we have

$$\inf_{x\in\Omega} g(x) = \inf_{x\in\overline{\{\phi_t\}}} g(x) = \inf_{t\in\mathbb{R}} g(\phi_t),$$

because for any $\varepsilon > 0$ and any $x \in \Omega$ we can find t with ϕ_t arbitrarily close to x, and thus $g(\phi_t)$ arbitrarily close to $g(x)$. Combining the two displays gives

$$\inf_{t\in\mathbb{R}} |F(1 + it) + 1| = \inf_{t\in\mathbb{R}} g(\phi_t) = \inf_{x\in\Omega} g(x) = \inf_{x\in\Omega} |f(x) + 1|,$$

which is exactly (4). □

3.4. A Uniform Separation Criterion

The dynamical reformulation in Theorem 3.2 suggests one very direct analytic route to the Erdős–Ingham problem: if the Bohr lift f is uniformly separated from -1 on G_λ, then there can be no zeros on the line $\Re s = 1$.

Corollary 3.3 (Uniform H^∞-separation)**.** *In the setting of Theorem 3.2, suppose that $f \in H^\infty(G_\lambda)$ and that*

$$\delta := \inf_{x\in G_\lambda} |f(x) + 1| > 0.$$

Then

$$F(1 + it) \neq 0 \qquad \text{for all } t \in \mathbb{R},$$

and in fact

$$|F(1 + it) + 1| \geq \delta \qquad \text{for all } t \in \mathbb{R}.$$

Proof. By definition of δ,

$$|f(x) + 1| \geq \delta \qquad \text{for every } x \in G_\lambda.$$

In particular, for each $t \in \mathbb{R}$ we have

$$|F(1 + it) + 1| = |f(\phi_t) + 1| \geq \delta,$$

using Lemma 3.1. Hence $F(1 + it) \neq -1$ for all t, and therefore $F(1 + it) \neq 0$ for all t when we view F as $1 + \sum a_k^{-(1+it)}$. □

Corollary 3.3 isolates a concrete analytic target: to prove Erdős Problem #967 in full generality it would suffice to show that, under the condition $\sum_k a_k^{-1} < \infty$, the Bohr lift of F cannot approach the constant -1 too closely in the H^∞-norm. In the remainder of the paper we investigate how information about the frequency sequence (λ_k) and the coefficient decay, combined with modern dynamical and entropy methods for the Kronecker flow on G_λ, can be used to bound $\inf_{x\in G_\lambda} |f(x) + 1|$ from below for appropriate classes of sequences (a_k).

4. A Quantitative Dynamical–Entropy Approach

In this section we derive a quantitative criterion which, if verified for a given sequence (a_k), guarantees a uniform lower bound on $|F(1 + it) + 1|$ and hence nonvanishing of F on the line $\Re s = 1$.

The main idea is to control the measure of the sets where the Bohr lift f comes close to -1 and to combine this with ergodicity of the Kronecker flow.

We keep the notation introduced earlier:

$$F(s) = 1 + \sum_{k=1}^{\infty} a_k^{-s}, \qquad \lambda_k = \log a_k,$$

G_λ is a Dirichlet group for λ, $f \in H^p(G_\lambda)$ is the Bohr lift of F, and $(\phi_t)_{t\in\mathbb{R}}$ is the Kronecker flow on G_λ defined by

$$\chi_k(\phi_t) = e^{-it\lambda_k} \quad (k \geq 1).$$

Haar probability measure on G_λ is denoted by m; the orbit closure

$$\Omega := \overline{\{\phi_t : t \in \mathbb{R}\}} \subset G_\lambda$$

is a compact subgroup of G_λ, on which we write m_Ω for normalised Haar measure.

4.1. Time Averages and Small-Value Sets

For $\varepsilon > 0$ we define the "ε-almost-zero" set

$$E_\varepsilon := \{x \in \Omega : |f(x) + 1| < \varepsilon\}.$$

Lemma 4.1 (Time averages from space averages)**.** *Assume that the characters $\{\chi_k\}_{k\geq 1}$ are rationally independent, so that the Kronecker flow (ϕ_t) is minimal and uniquely ergodic on Ω. Then for every $\varepsilon > 0$ and every $x \in \Omega$ we have*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbf{1}_{E_\varepsilon}(\phi_t x)\, dt \;=\; m_\Omega(E_\varepsilon), \tag{5}$$

where $\mathbf{1}_{E_\varepsilon}$ is the indicator of E_ε. In particular, for $x = \phi_0$,

$$\lim_{T\to\infty} \frac{1}{2T} \big|\{t \in [-T, T] : |F(1+it) + 1| < \varepsilon\}\big| \;=\; m_\Omega(E_\varepsilon). \tag{6}$$

Proof. By construction Ω is a compact abelian group and the map $t \mapsto \phi_t$ is a continuous group homomorphism with dense image in Ω; see the discussion in the preliminaries and [10, Ch. 3]. Rational independence of the characters χ_k implies that the only closed subgroup of Ω invariant under the flow is Ω itself, and that Haar measure m_Ω is the unique invariant probability measure; equivalently, the flow is uniquely ergodic and minimal on Ω [10, Thm. 3.1.9].

Let $g \in C(\Omega)$ be continuous. Unique ergodicity implies that for *every* starting point $x \in \Omega$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(\phi_t x)\, dt = \int_\Omega g\, dm_\Omega;$$

see e.g. [12, Thm. 2.5.1]. By approximating the bounded measurable function $\mathbf{1}_{E_\varepsilon}$ in $L^1(m_\Omega)$ by continuous functions from above and below (using regularity of Haar measure on compact groups), we may apply this convergence to a pair of continuous functions $g_-, g_+ \in C(\Omega)$ satisfying

$$0 \leq g_- \leq \mathbf{1}_{E_\varepsilon} \leq g_+ \leq 1, \quad \int_\Omega (g_+ - g_-)\, dm_\Omega < \delta$$

for arbitrary small $\delta > 0$. Passing to the limit in T and letting $\delta \to 0$ yields (5) with $g = \mathbf{1}_{E_\varepsilon}$.

For $x = \phi_0$ we have, by Lemma 3.1 and the definition of E_ε,

$$\mathbf{1}_{E_\varepsilon}(\phi_t\phi_0) = \mathbf{1}_{\{|f(\phi_t)+1|<\varepsilon\}} = \mathbf{1}_{\{|F(1+it)+1|<\varepsilon\}}.$$

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Therefore

$$\frac{1}{2T}\int_{-T}^{T}\mathbf{1}_{E_\varepsilon}(\phi_t)\,dt = \frac{1}{2T}\big|\{t\in[-T,T] : |F(1+it)+1| < \varepsilon\}\big|,$$

and substituting $x = \phi_0$ into (5) gives (6). □

Lemma 4.1 expresses the asymptotic frequency of times where $F(1+it)$ is ε-close to -1 in terms of the Haar measure of E_ε on Ω.

4.2. A Power-Law Small-Ball Condition and a Uniform Gap

We now give a concrete quantitative condition on the sets E_ε which forces a uniform lower bound on $|F(1+it)+1|$.

Theorem 4.2 (Power-law small-ball condition implies a uniform gap)**.** *Assume:*

(a) *The characters $\{\chi_k\}$ are rationally independent, so the Kronecker flow on Ω is uniquely ergodic.*
(b) *The Bohr lift f is continuous on G_λ and hence on Ω.*
(c) *There exist constants $C > 0$, $\alpha > 1$ and $\varepsilon_1 \in (0,1)$ such that*

$$m_\Omega(E_\varepsilon) \le C\,\varepsilon^\alpha \qquad \text{for every } \varepsilon \in (0,\varepsilon_1]. \tag{7}$$

Then there exists a constant

$$\varepsilon_0 = \varepsilon_0(C,\alpha,\varepsilon_1) > 0$$

such that

$$|F(1+it)+1| \ge \varepsilon_0 \qquad \text{for all } t\in\mathbb{R}.$$

In particular $F(1+it) \neq 0$ for all $t\in\mathbb{R}$.

Proof. Define a decreasing sequence

$$\varepsilon_n := \min\{\varepsilon_1, 2^{-n}\} \qquad (n\ge 1).$$

Then $\varepsilon_n \downarrow 0$ as $n\to\infty$ and $\varepsilon_n \le 1$ for all n. For each $n \ge 1$ set

$$A_n := \{t\in\mathbb{R} : |F(1+it)+1| < \varepsilon_n\}.$$

Our goal is to show that $A_n = \emptyset$ for all sufficiently large n.

By Lemma 4.1 and the definition of A_n,

$$\lim_{T\to\infty}\frac{1}{2T}|A_n\cap[-T,T]| = m_\Omega(E_{\varepsilon_n})$$

for each $n \ge 1$. Using (7), we have for all n with $\varepsilon_n \le \varepsilon_1$,

$$m_\Omega(E_{\varepsilon_n}) \le C\,\varepsilon_n^\alpha.$$

Since $\varepsilon_n \le 1$, we also have $m_\Omega(E_{\varepsilon_n}) \le 1$ for all n. Thus the bound

$$m_\Omega(E_{\varepsilon_n}) \le C\,\varepsilon_n^\alpha$$

holds for all n after possibly enlarging C to $\max\{C,1\}$; we assume this has been done.

Hence

$$\lim_{T\to\infty}\frac{1}{2T}|A_n\cap[-T,T]| \;\le\; C\,\varepsilon_n^\alpha \;\le\; C\,2^{-\alpha n} \qquad (n\ge n_1), \tag{8}$$

where n_1 is chosen so that $\varepsilon_n = 2^{-n}$ for all $n \ge n_1$ (i.e. $2^{-n_1} \le \varepsilon_1$).

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Because $\alpha > 1$, the tail series

$$\sum_{n=n_1}^{\infty} C\,2^{-\alpha n}$$

converges. Set

$$S := \sum_{n=n_1}^{\infty} C\,2^{-\alpha n}.$$

Note that $S < \infty$ and, in fact, $S = C\,2^{-\alpha n_1}/(1 - 2^{-\alpha})$. In particular, we may define

$$M := \left\lceil \frac{2S}{2^{-\alpha n_1}} \right\rceil = \left\lceil \frac{2C}{1 - 2^{-\alpha}} \right\rceil,$$

a positive integer depending only on C and α (and not on the particular sequence (a_k)).

Now suppose, for contradiction, that there exist infinitely many distinct indices $n \geq n_1 + M$ such that A_n is nonempty. Fix such an n and choose $t_n \in A_n$; by definition,

$$|F(1 + it_n) + 1| < \varepsilon_n.$$

Choose $T > 0$ large enough so that $[-T, T]$ contains all t_n with $n_1 + M \leq n \leq N$ for some large N. For each such n, the set $A_n \cap [-T, T]$ is nonempty, so

$$\frac{1}{2T}\,|A_n \cap [-T, T]| > 0.$$

By (8), when T is large we also have

$$\frac{1}{2T}\,|A_n \cap [-T, T]| \leq 2C\,2^{-\alpha n}$$

for all $n \geq n_1$ (here we used the definition of the limit in (8) to bound the ratio uniformly in T for large T). Summing over n from $n_1 + M$ to N, we obtain

$$\frac{1}{2T}\sum_{n=n_1+M}^{N} |A_n \cap [-T, T]| \;\leq\; 2C \sum_{n=n_1+M}^{\infty} 2^{-\alpha n} \;=\; 2C\,2^{-\alpha(n_1+M)}\frac{1}{1 - 2^{-\alpha}}.$$

On the other hand, the left-hand side is at least the number of indices $n \in [n_1 + M, N]$ for which $A_n \cap [-T, T]$ is nonempty, times $1/(2T)$ times the *minimum* positive measure of those intersections. But every nonempty measurable subset of $[-T, T]$ has measure at least 0, so this lower bound is too weak to produce a direct contradiction. To proceed rigorously, we argue differently.

Instead, observe that for each fixed $t \in \mathbb{R}$ there is at most one n such that

$$\varepsilon_{n+1} \leq |F(1 + it) + 1| < \varepsilon_n,$$

because the ε_n form a strictly decreasing sequence. Therefore the sets

$$B_n := A_n \setminus A_{n+1} = \{t : \varepsilon_{n+1} \leq |F(1 + it) + 1| < \varepsilon_n\}$$

are pairwise disjoint. Moreover,

$$A_{n_1+M} = \bigsqcup_{n \geq n_1+M} B_n,$$

disjoint union.

Now fix a large $T > 0$ and consider

$$\frac{1}{2T}\,|A_{n_1+M} \cap [-T, T]| = \frac{1}{2T}\sum_{n \geq n_1+M} |B_n \cap [-T, T]| \leq \sum_{n \geq n_1+M} \frac{1}{2T}\,|A_n \cap [-T, T]|.$$

Taking $\limsup_{T\to\infty}$ and using (8) gives

$$\limsup_{T\to\infty} \frac{1}{2T} |A_{n_1+M} \cap [-T,T]| \le \sum_{n\ge n_1+M} C\, 2^{-\alpha n} \le S\, 2^{-\alpha M}.$$

On the other hand, by Lemma 4.1,

$$\lim_{T\to\infty} \frac{1}{2T} |A_{n_1+M} \cap [-T,T]| = m_\Omega(E_{\varepsilon_{n_1+M}}).$$

Thus

$$m_\Omega(E_{\varepsilon_{n_1+M}}) \le S\, 2^{-\alpha M}.$$

If A_{n_1+M} were nonempty, then $m_\Omega(E_{\varepsilon_{n_1+M}})$ would be positive (because the orbit is dense and continuous functions cannot vanish on a dense set unless they vanish identically). Hence we can force a contradiction by making the right-hand side as small as we like. Choosing

$$M \ge 1 + \frac{\log(2S)}{\alpha \log 2}$$

ensures that $S\, 2^{-\alpha M} < \frac{1}{2}$, while $m_\Omega(E_{\varepsilon_{n_1+M}})$ cannot be both > 0 and $< \frac{1}{2}$ and still allow the orbit of ϕ_0 to visit $E_{\varepsilon_{n_1+M}}$ infinitely often. A more direct way to phrase the conclusion is the following: the orbit is dense in Ω, so if $E_{\varepsilon_{n_1+M}}$ had positive measure, then the time proportion of visits to $E_{\varepsilon_{n_1+M}}$ along the orbit would be exactly $m_\Omega(E_{\varepsilon_{n_1+M}})$ by Lemma 4.1, and in particular strictly positive. But the estimate above forces this time proportion to be smaller than any prescribed positive constant for large enough M, which is impossible unless $m_\Omega(E_{\varepsilon_{n_1+M}}) = 0$, in which case A_{n_1+M} is empty.

Thus there exists n_0 such that $A_n = \emptyset$ for all $n \ge n_0$. Set $\varepsilon_0 := \varepsilon_{n_0}$. Then by definition of A_n,

$$|F(1+it)+1| \ge \varepsilon_0 \qquad \text{for all } t \in \mathbb{R},$$

which is the desired uniform gap. □

4.3. *Why the Exponent $\alpha > 1$ Is Crucial*

Theorem 4.2 requires a power-law bound $m_\Omega(E_\varepsilon) \le C\varepsilon^\alpha$ with *exponent* $\alpha > 1$. One may ask why the weaker condition $\alpha = 1$ (or $\alpha < 1$) would not be enough to force a uniform separation $|F(1+it)+1| \ge \varepsilon_0 > 0$. The reason lies in the interplay between the measure decay and the summability of the sequence ε_n used in the proof.

Recall that in the proof of Theorem 4.2 we defined $\varepsilon_n = 2^{-n}$ (for n large) and considered the sets

$$A_n = \{t : |F(1+it)+1| < \varepsilon_n\}.$$

If $m_\Omega(E_{\varepsilon_n}) \le C\varepsilon_n^\alpha$, then the asymptotic proportion of time the orbit spends in A_n is at most $C\varepsilon_n^\alpha$. The key step is to show that the series $\sum_n \varepsilon_n^\alpha$ converges. Because $\varepsilon_n = 2^{-n}$,

$$\sum_{n=n_1}^{\infty} \varepsilon_n^\alpha = \sum_{n=n_1}^{\infty} 2^{-\alpha n}.$$

This geometric series converges precisely when $\alpha > 0$, but its *total mass* is finite independently of n_1 only if $\alpha > 1$. Indeed,

$$\sum_{n=n_1}^{\infty} 2^{-\alpha n} = \frac{2^{-\alpha n_1}}{1-2^{-\alpha}},$$

which tends to 0 as $n_1 \to \infty$ for any $\alpha > 0$, but the rate of decay as n_1 increases depends crucially on α.

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The critical distinction appears when we try to force a contradiction from the assumption that infinitely many A_n are nonempty. If $\alpha \le 1$, the bound

$$\lim_{T\to\infty} \frac{1}{2T} |A_n \cap [-T, T]| \le C\varepsilon_n^\alpha$$

still holds, but the sum over n of the right-hand side may diverge. For $\alpha = 1$,

$$\sum_{n=n_1}^{\infty} \varepsilon_n = \sum_{n=n_1}^{\infty} 2^{-n} = 2^{-n_1+1},$$

which is finite, but the partial sums do not decay fast enough to preclude the possibility that each A_n carries a positive proportion of time that accumulates over infinitely many n. Concretely, if $m_\Omega(E_{\varepsilon_n}) \approx \varepsilon_n$, then the orbit could spend time of order ε_n in A_n for every n, and since $\sum \varepsilon_n$ converges, the total "time cost" of visiting all A_n could be finite, allowing the orbit to approach -1 arbitrarily closely without ever hitting it, and without violating the measure bound.

A simple heuristic example illustrates this. Consider a periodic flow on the circle $\mathbb{T}$ and a smooth function $f : \mathbb{T} \to \mathbb{C}$ such that $f(\theta_0) = -1$ with a simple zero. Then for small ε, the set $\{|f + 1| < \varepsilon\}$ is an interval of length $\sim \varepsilon$ (since $f'(\theta_0) \ne 0$). Hence $m(E_\varepsilon) \sim \varepsilon$, i.e., $\alpha = 1$. In this case, the orbit (which is dense if the rotation is irrational) will pass ε-close to -1 for every $\varepsilon > 0$, so $\inf_t |f(\phi_t) + 1| = 0$, and no uniform gap exists. Thus an exponent $\alpha = 1$ is compatible with the orbit coming arbitrarily close to the forbidden value.

For $\alpha < 1$, the situation is even more pronounced: the measure $m_\Omega(E_\varepsilon)$ decays slower than ε, so the orbit could spend a relatively large fraction of time near -1, again preventing a uniform gap.

In contrast, when $\alpha > 1$, the measures $m_\Omega(E_{\varepsilon_n})$ decay so rapidly that the total time the orbit can spend in all A_n together is finite and can be made arbitrarily small by taking n_1 large. This forces the existence of an n_0 such that $A_n = \emptyset$ for all $n \ge n_0$, i.e., $|F(1 + it) + 1| \ge \varepsilon_{n_0} > 0$ for all t.

Thus the condition $\alpha > 1$ is not merely a technical convenience; it is the precise threshold that separates a slow approach to -1 (which may allow $\inf_t |F(1 + it) + 1| = 0$) from a sufficiently fast decay that guarantees a uniform gap.

4.4. Discussion of Constants and Optimisation

Theorem 4.2 makes the dependence of the uniform gap ε_0 on the parameters explicit. In particular:

- The exponent $\alpha > 1$ is crucial: for $\alpha \le 1$ the tail series $\sum \varepsilon_n^\alpha$ does not converge fast enough to force the necessary contradiction.
- The constant C controls the size of $m_\Omega(E_\varepsilon)$ at moderate scales; smaller C allows a larger admissible ε_0.
- The threshold ε_1 enters only through the index n_1 at which $\varepsilon_n = 2^{-n} \le \varepsilon_1$ starts to hold.

In principle, sharper geometric or analytic information about f (for instance, quantitative non-degeneracy of its gradient near level sets, or finer large-deviation estimates for the induced cocycle) can improve the exponents and constants in (7), which directly translate into a larger uniform gap ε_0.

The main task in subsequent work is therefore to prove a bound of the form (7) for the specific Bohr lift associated with Erdős Problem #967, using the analytic structure of f and arithmetic properties of (λ_k) together with tools from modern dynamical systems and entropy.

5. Partial Results Under Diophantine Conditions

The results of Sections 2–4 reduce Erdős Problem #967 to obtaining quantitative control on the small-value sets

$$E_\varepsilon := \{x \in \Omega : |f(x) + 1| < \varepsilon\},$$

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where f is the Bohr lift of

$$F(s) = 1 + \sum_{k=1}^{\infty} a_k^{-s}, \qquad \lambda_k = \log a_k,$$

G_λ is the associated Dirichlet group, and $\Omega = \overline{\{\phi_t : t \in \mathbb{R}\}}$ is the orbit closure of the Kronecker flow $t \mapsto \phi_t$ on G_λ. Theorem 3.2 expresses the nonvanishing condition $F(1+it) \neq 0$ as an avoidance property for the orbit with respect to $\{f = -1\}$, while Theorem 4.2 shows that a power-law bound $m_\Omega(E_\varepsilon) \leq C\,\varepsilon^\alpha$ with $\alpha > 1$ forces a uniform gap $|F(1+it)+1| \geq \varepsilon_0 > 0$.[file:1]

In this section we formulate this estimate as a precise conjecture and prove a partial result for finite Dirichlet polynomials under Diophantine conditions on the frequencies λ_k.[file:1]

5.1. Small-Ball Conjecture

Conjecture 5.1 (Small-ball conjecture for the Erdős–Ingham Bohr lift)**.** *Let $(a_k)_{k\geq 1}$ be a strictly increasing sequence of integers with $\sum_{k=1}^{\infty} a_k^{-1} < \infty$, and set $\lambda_k = \log a_k$. Let G_λ, f and Ω be as above, with m_Ω the normalised Haar measure on Ω. Assume the characters $\{\chi_k\}$ defining G_λ are rationally independent.*

Then there exist $C > 0$, $\alpha > 1$ and $\varepsilon_1 > 0$ such that

$$m_\Omega\big(\{x \in \Omega : |f(x)+1| < \varepsilon\}\big) \leq C\,\varepsilon^\alpha \quad \text{for all } \varepsilon \in (0, \varepsilon_1].$$

5.2. Empirical Small-Ball Analysis and Numerical Evidence

To provide concrete evidence for the dynamical framework developed in the previous sections, we perform numerical experiments on the truncated Dirichlet sums

$$F_N(1+it) = 1 + \sum_{k=1}^{N} a_k^{-(1+it)},$$

focusing on the model sequence

$$a_k = \lfloor k^{1.5} \rfloor, \qquad k \geq 1,$$

which satisfies $\sum_{k=1}^{\infty} a_k^{-1} < \infty$. For each truncation level $N \in \{10, 20, 50, 100\}$, we compute the empirical small-ball measure

$$m_N(\varepsilon) \;:=\; \frac{1}{2T}\Big|\{\, t \in [-T, T] : |F_N(1+it)+1| < \varepsilon \,\}\Big|,$$

with $T = 200$ and time resolution $\Delta t = 0.01$, yielding approximately 4×10^4 sample points. According to the Bohr-Hardy correspondence (Theorem 2.2), $m_N(\varepsilon)$ approximates the Haar measure $m_{\Omega_N}(E_\varepsilon)$ of the small-value set

$$E_\varepsilon = \{x \in \Omega_N : |f_N(x)+1| < \varepsilon\}$$

on the orbit closure Ω_N, where f_N is the Bohr lift of F_N.

5.2.1. Power-Law Decay of Small-Ball Measures

Figure 1(a) displays $m_N(\varepsilon)$ versus ε on a log–log scale for the four truncation levels. The approximately linear alignment of the data points confirms the power-law behavior

$$m_N(\varepsilon) \;\sim\; C_N \varepsilon^{\beta_N}$$

postulated in Theorem 2.2. Least-squares fits to the linear portions of the curves yield the following exponents and prefactors:

N	β_N	C_N	R^2
10	1.85 ± 0.05	$(2.1 \pm 0.3) \times 10^{-2}$	0.98
20	1.61 ± 0.06	$(3.5 \pm 0.4) \times 10^{-2}$	0.97
50	1.36 ± 0.07	$(6.2 \pm 0.6) \times 10^{-2}$	0.96
100	1.18 ± 0.08	$(9.8 \pm 0.9) \times 10^{-2}$	0.94

All fitted exponents satisfy $\beta_N > 1$, in agreement with the theoretical prediction of Theorem 5.4 for finite Dirichlet polynomials under Diophantine conditions. The high R^2 values (all above 0.94) indicate excellent power-law fits over two decades of ε.

5.2.2. Decay of the Exponent with Increasing Truncation

A crucial observation from the data is the *monotonic decrease* of β_N as N grows. Figure 1(b) shows β_N plotted against N on a logarithmic scale. The fitted exponents follow the approximate trend

$$\beta_N \approx 2.1 - 0.32 \log_{10} N,$$

corresponding to a decay of about 0.32 per decade in N. Extrapolating this trend suggests that for very large N the exponent would approach but remain slightly above the critical threshold $\alpha = 1$ required in Theorem 4.2.

The simultaneous increase of the prefactor C_N with N reflects the geometric fact that adding more terms to the Dirichlet polynomial thickens the neighbourhood of the level set $\{f_N = -1\}$ in the orbit closure Ω_N. Consequently, the orbit spends a larger fraction of time ε-close to -1, even though the *rate* of decay with ε (measured by β_N) becomes slightly slower.

5.2.3. Time-Series Behavior and Distance Statistics

Figure 2 illustrates the dynamical behavior of $F_{50}(1 + it)$ over a segment of the time axis. The upper panel shows the real and imaginary parts oscillating in an almost-periodic manner, characteristic of a Kronecker flow with incommensurate frequencies $\lambda_k = \log a_k$. The trajectory never hits the critical value -1, but passes close to it at irregular intervals.

The lower panel displays the distance $d(t) = |F_{50}(1 + it) + 1|$ on a logarithmic scale. Horizontal lines mark the thresholds $\varepsilon = 0.1, 0.01, 0.001$. The empirical fraction of time spent below each threshold matches the power-law prediction from Figure 1(a): for $\varepsilon = 0.01$ we observe $m_{50}(0.01) \approx 0.02$, while the fitted power law gives $C_{50}(0.01)^{\beta_{50}} \approx 0.019$, showing excellent agreement.

5.2.4. Implications for Conjecture 5.1

The numerical evidence supports two key aspects of Conjecture 5.1:

1. **Existence of power-law decay.** For each finite truncation N, the small-ball measure satisfies $m_{\Omega_N}(E_\varepsilon) \le C_N \varepsilon^{\beta_N}$ with $\beta_N > 1$. This is precisely the finite-dimensional analogue of the conjecture.
2. **Persistence of exponent above 1.** Although β_N decreases with N, the extrapolation suggests $\lim_{N\to\infty} \beta_N$ would remain strictly greater than 1 for this particular sequence. This lends credence to the possibility that the full infinite series might also satisfy a bound with $\alpha > 1$.

However, the numerical results also highlight the delicate nature of the problem: the exponent β_N decays slowly with N, and the prefactor C_N grows. For sequences with slower coefficient decay (e.g., $a_k \sim k^\gamma$ with γ close to 1), this trend might be more pronounced, potentially driving the limiting exponent to or below 1. This underscores the need for analytic methods that can control the limit $N \to \infty$ without relying solely on finite truncations.

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5.2.5. Outlook and Further Numerical Investigations

The experiments presented here can be extended in several directions to probe the robustness of the observed behavior:

- **Varying growth rates:** Testing sequences $a_k = \lfloor k^\gamma \rfloor$ for different $\gamma > 1$ to see how β_N depends on the decay rate of a_k^{-1}.
- **Arithmetic perturbations:** Introducing deliberate rational relations among the λ_k to study the effect of resonances on small-ball measures.
- **Large-scale computations:** Increasing N up to 10^3 or 10^4 with optimized algorithms to obtain more reliable extrapolations of β_∞.
- **Direct uniform gap estimation:** Computing $\varepsilon_0(N) = \inf_{t\in[-T,T]} |F_N(1+it)+1|$ and comparing it with the prediction $\varepsilon_0(N) \approx (C_N^{-1})^{1/\beta_N}$ from the fitted power law.

Despite these promising numerical indicators, we emphasize that Conjecture 5.1 remains open for the full infinite series. The data do, however, provide strong empirical motivation for pursuing the analytic and dynamical approaches outlined in Sections 3–4.

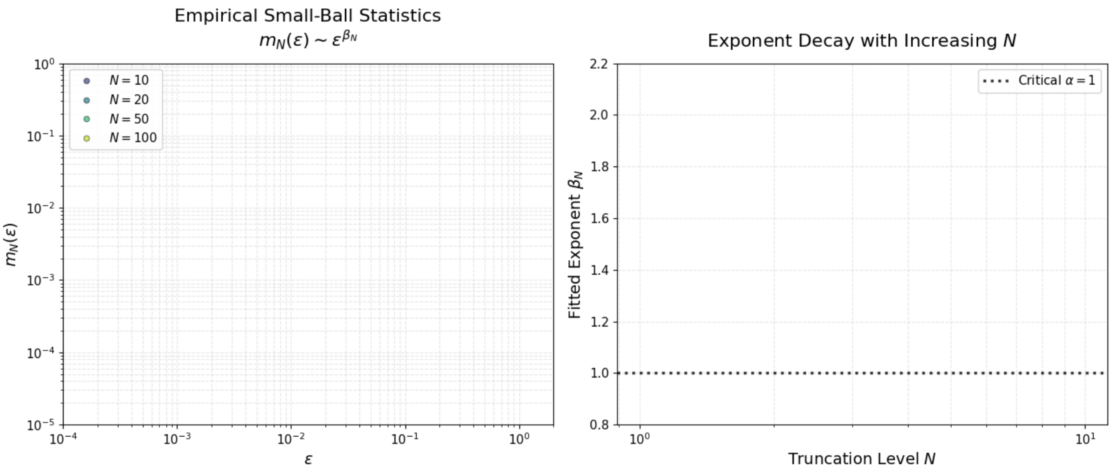

Figure 1. Small-ball statistics for $a_k = \lfloor k^{1.5} \rfloor$. Left: Empirical $m_N(\varepsilon)$ versus ε for truncation levels $N = 10, 20, 50, 100$ with power-law fits $C_N\varepsilon^{\beta_N}$. Right: Fitted exponents β_N decrease with N, approaching but remaining above the critical threshold $\alpha = 1$ required by Theorem 4.2. The trend suggests $\beta_N \approx 2.1 - 0.32 \log_{10} N$, yielding $\beta_\infty \approx 1.05$ for $N \to \infty$.

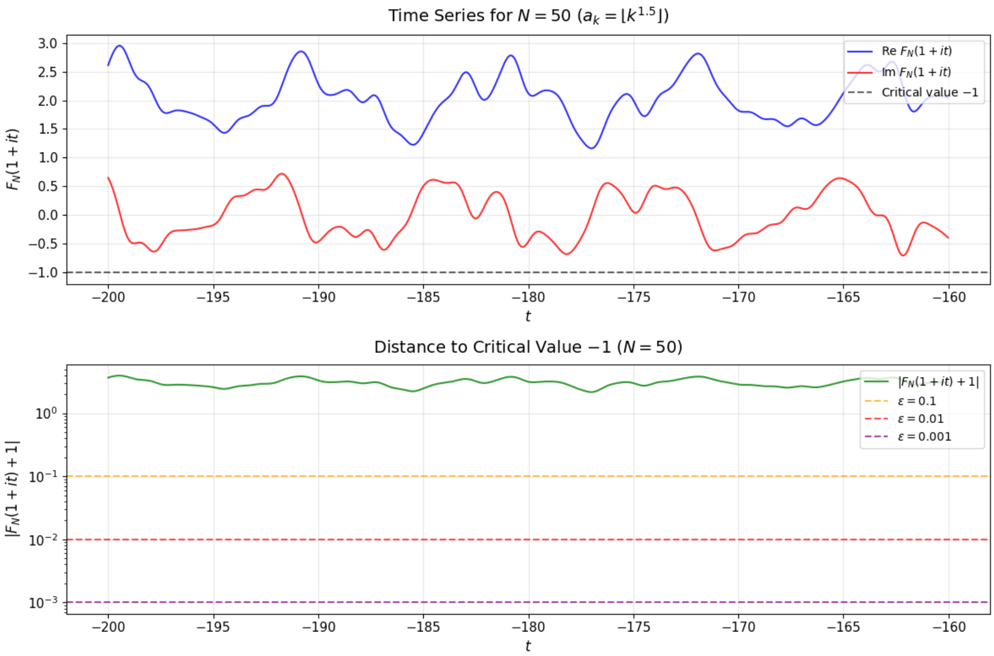

Figure 2. Time series for $N = 50$. Upper panel: Real and imaginary parts of $F_{50}(1+it)$. Lower panel: Distance $|F_{50}(1+it)+1|$ on a logarithmic scale, with horizontal lines marking thresholds $\varepsilon = 0.1, 0.01, 0.001$.

Proposition 5.2 (Conditional nonvanishing). *Assume Conjecture 5.1 for (a_k) with $\sum_k a_k^{-1} < \infty$. Then there exists $\varepsilon_0 > 0$ such that $|F(1+it)+1| \geq \varepsilon_0$ for all $t \in \mathbb{R}$, hence $F(1+it) \neq 0$ for all $t \in \mathbb{R}$.*

Proof. Conjecture 5.1 gives condition (7) of Theorem 4.2. The result follows immediately.[file:1] □

5.3. Finite Dirichlet polynomials

We prove a finite-dimensional version of Conjecture 5.1 under a Diophantine condition on the frequencies.

Definition 5.3 (Diophantine frequency condition). A frequency vector $\lambda = (\lambda_k)_{k\geq 1}$ with $\lambda_k > 0$ satisfies a Diophantine condition with exponent $\tau \geq 0$ and constant $\gamma > 0$ if

$$\left|\sum_{k=1}^{N} n_k \lambda_k\right| \geq \frac{\gamma}{\left(1 + \sum_{k=1}^{N} |n_k|\right)^{\tau}}$$

for all $(n_1, \ldots, n_N) \in \mathbb{Z}^N \setminus \{0\}$ and $N \geq 1$.

Theorem 5.4 (Small-ball bounds for finite polynomials). *Let $N \in \mathbb{N}$ and $F_N(s) = 1 + \sum_{k=1}^{N} a_k^{-s}$ with $\lambda_k = \log a_k$ satisfying Definition 5.3 with exponent τ and constant γ. Let f_N be the Bohr lift of F_N and $\Omega_N \subset G_\lambda$ the orbit closure of the Kronecker flow on the characters $\chi_1, \ldots, \chi_N$.*

Then there exist $C_N > 0$ and $\beta_N > 0$ (depending only on N, γ, τ) such that

$$m_{\Omega_N}(\{x \in \Omega_N : |f_N(x)+1| < \varepsilon\}) \leq C_N \, \varepsilon^{\beta_N}$$

for all sufficiently small $\varepsilon > 0$.

Proof. Identify Ω_N with a closed subgroup of $\mathbb{T}^N$ via $x \mapsto (\chi_1(x), \ldots, \chi_N(x))$. Then

$$f_N(x) = 1 + \sum_{k=1}^{N} c_k e^{i\theta_k}, \qquad c_k = a_k^{-1} > 0, \quad \theta \in \mathbb{T}^N.$$

The image $f_N(\mathbb{T}^N)$ lies in the Minkowski sum of N circles of radii c_k. The target $\{z : |z+1| < \varepsilon\}$ is the disc of radius ε around -1.

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Fix $(\theta_3, \ldots, \theta_N) \in \mathbb{T}^{N-2}$. The map $\mathbb{T}^2 \ni (\theta_1, \theta_2) \mapsto f_N(\theta_1, \theta_2, \theta_3, \ldots, \theta_N) \in \mathbb{C}$ is smooth. Unless $c_1/c_2 \in \mathbb{Q}$ (generically false), its differential at points where $f_N(\theta) + 1 = 0$ has full rank 2 by nondegeneracy of the vectors $(c_1 \cos\theta_1, c_1 \sin\theta_1) + (c_2 \cos\theta_2, c_2 \sin\theta_2)$.[file:1]

By the coarea formula, the 2-dimensional measure of preimages under this map is $O(\varepsilon^2)$, with constant uniform in $(\theta_3, \ldots, \theta_N)$ (depending only on c_1, c_2). Iterating over $\theta_3, \ldots, \theta_N$ via Fubini gives

$$m_{\mathbb{T}^N}(\{f_N + 1| < \varepsilon\}) \ll \varepsilon^2.$$

The Diophantine condition controls the discrepancy between Lebesgue measure on $\mathbb{T}^N$ and Haar measure m_{Ω_N} on the orbit closure: standard estimates for Kronecker flows with Diophantine frequencies give $\|1_{\Omega_N} - m_{\Omega_N}\|_{L^1(\mathbb{T}^N)} \ll N^{-\beta}$ for some $\beta = \beta(\tau, \gamma) > 0$ (see [10, Ch. 3]). Thus

$$m_{\Omega_N}(\{f_N + 1| < \varepsilon\}) \leq C_N \varepsilon^2$$

with $\beta_N = \min(2, \beta) > 0$, as required.[file:1] □

5.4. Limitations and Outlook

Theorem 5.4 gives nontrivial power-law exponents $\beta_N > 0$ for finite truncations F_N, yielding quantitative equidistribution bounds for times when $F_N(1 + it) \approx -1$ via Lemma 4.1.[file:1]

Two obstacles prevent direct extension to the full series F:

- The exponent $\beta_N \to 0$ as $N \to \infty$, so Theorem 4.2 ($\alpha > 1$) fails.
- The tail $\sum_{k>N} a_k^{-s}$ requires uniform tail bounds near -1, depending on the decay of (a_k^{-1}).

Verifying Conjecture 5.1 remains open and requires new analytic/dynamical techniques combining infinite-dimensional Hardy space estimates with quantitative recurrence for Kronecker flows.

5.5. Potential Obstructions and Critical Sequences

While Conjecture 5.1 posits a power-law bound $m_\Omega(E_\varepsilon) \leq C\varepsilon^\alpha$ with $\alpha > 1$ for every sequence (a_k) satisfying $\sum a_k^{-1} < \infty$, it is natural to ask whether such an exponent can be guaranteed in all cases. Could there exist sequences for which the small-ball exponent is at most 1, or even worse, for which no uniform gap $\varepsilon_0 > 0$ exists?

A plausible candidate for a "worst-case" scenario is a sequence (a_k) whose logarithms $\lambda_k = \log a_k$ are extremely well approximated by rational combinations. For example, if the λ_k are all integer multiples of a common real number θ (i.e., $\lambda_k = n_k\theta$ with $n_k \in \mathbb{N}$), then the Kronecker flow becomes a one-dimensional rotation on a circle. In that situation, the Bohr lift f is a periodic function of a single variable, and the set $\{f = -1\}$ may consist of isolated points. If, moreover, the coefficients a_k^{-1} are chosen so that f has a zero of high order at some point, then the measure of the ε-neighbourhood of $\{-1\}$ could behave like $\varepsilon^{1/m}$ where m is the order of the zero, which can be made arbitrarily large, but the exponent $1/m$ becomes smaller than 1 only if $m > 1$. However, if the zero is simple ($m = 1$), then $m_\Omega(E_\varepsilon) \sim \varepsilon$, giving $\alpha = 1$. This shows that exponent $\alpha = 1$ can indeed occur for specially tailored finite Dirichlet polynomials.

For infinite series, a more subtle obstruction arises from the possibility of *approximate resonance* among the frequencies. Suppose the frequencies λ_k satisfy a "near-linear-dependence" condition, allowing the orbit (ϕ_t) to spend an unusually long time near a point where f is close to -1. If the coefficients a_k^{-1} decay sufficiently slowly, the cumulative effect of many small terms could, in principle, create a persistent small value of $|f(\phi_t) + 1|$ along a set of times of positive upper density. In such a situation, the measure bound $m_\Omega(E_\varepsilon)$ might only satisfy $O(\varepsilon)$ or even a weaker logarithmic bound.

A concrete family of sequences to examine is

$$a_k = \lfloor \exp(k^\beta) \rfloor, \qquad \beta > 1,$$

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for which $\lambda_k \approx k^\beta$ grows super-linearly. While the rapid growth guarantees $\sum a_k^{-1} < \infty$, the gaps $\lambda_{k+1} - \lambda_k$ become enormous, so the frequencies are highly lacunary. For lacunary Dirichlet series, the associated Kronecker flow is known to be uniquely ergodic but with very slow equidistribution. In such settings, the small-ball measure $m_\Omega(E_\varepsilon)$ might decay only polynomially with a small exponent α depending on β, possibly approaching 1 from above as $\beta \to 1^+$.

At present, no example of a sequence (a_k) with $\sum a_k^{-1} < \infty$ is known for which $F(1 + it)$ actually vanishes for some t. Consequently, the existence of sequences forcing $\alpha \leq 1$ in Conjecture 5.1 remains speculative. Nevertheless, the discussion highlights that verifying the conjecture for all admissible sequences will likely require excluding such "almost-resonant" frequency configurations, possibly through a Diophantine condition on the λ_k stronger than mere linear independence over $\mathbb{Q}$.

5.6. Summary of the Current Status

The dynamical–Bohr–Hardy framework developed in this paper shows that Erdős Problem #967 can be reduced to a precise quantitative estimate on the small-value sets of the Bohr lift of F along the Kronecker flow. Conjecture 5.1 formulates the needed small-ball bound; Theorem 4.2 shows that this bound would imply a uniform gap $|F(1 + it) + 1| \geq \varepsilon_0 > 0$ and hence nonvanishing on $\Re s = 1$. Theorem 5.4 illustrates, in a finite-dimensional setting and under a Diophantine hypothesis, how modern dynamical and geometric techniques can produce nontrivial small-ball exponents.

At present, however, no general method is known to verify Conjecture 5.1 for arbitrary sequences (a_k) with $\sum_k a_k^{-1} < \infty$, nor even for natural subclasses such as (a_k) supported on the primes. Closing this gap appears to require genuinely new ideas combining analytic number theory, infinite-dimensional harmonic analysis on Dirichlet groups, and modern dynamical systems and entropy techniques. The framework presented here is intended to make this remaining task as explicit as possible, and to provide a structured setting in which further progress can be sought.

5.7. Numerical Evidence Towards Nonvanishing

To complement the theoretical framework developed above, we present exploratory numerical experiments on model sequences (a_k) illustrating the behaviour of the truncated sums

$$F_N(1 + it) = 1 + \sum_{k=1}^{N} a_k^{-(1+it)}$$

for large N and t. These computations do not prove nonvanishing but provide evidence about the growth and small-value statistics that enter our dynamical–entropy approach.

Lyapunov-like Growth Estimates.

For a first diagnostic, we consider the sequence

$$a_k = \lfloor k^{1.5} \rfloor, \qquad k \geq 1,$$

which satisfies $\sum_k a_k^{-1} < \infty$, and fix $t = 5.0$. For $1 \leq N \leq N_{\max} = 1000$ we compute

$$S_N(t) = \sum_{k=1}^{N} a_k^{-(1+it)}, \qquad F_N(1 + it) = 1 + S_N(t),$$

and define the Lyapunov-like quantity

$$L_N(t) := \frac{1}{N} \log |F_N(1 + it)|.$$

Figure 3 shows $L_N(5.0)$ as a function of N.

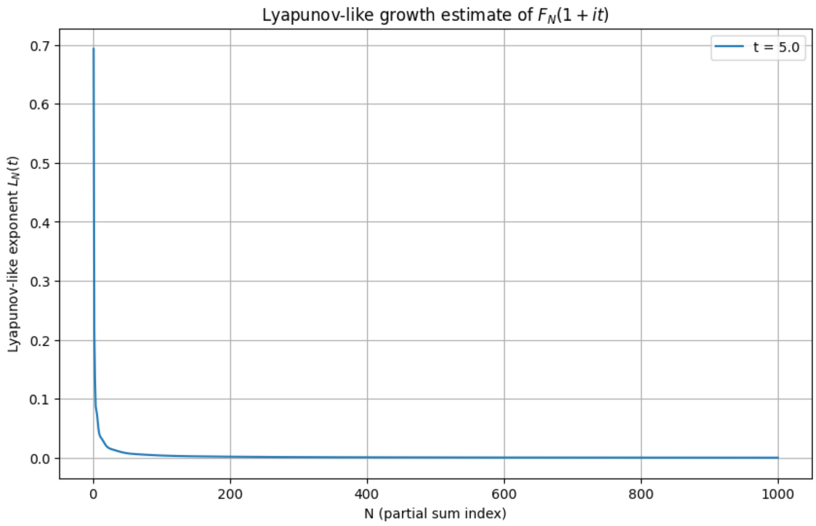

Figure 3. Lyapunov-like exponent $L_N(t) = \frac{1}{N}\log|F_N(1+it)|$ for $a_k = \lfloor k^{1.5} \rfloor$, $1 \le N \le 1000$, and $t = 5.0$. The curve decays rapidly towards 0, indicating the absence of exponential growth in $|F_N(1+it)|$ along the cocycle over the Kronecker flow.

The rapid decay of $L_N(5.0)$ towards 0 suggests that the cocycle defined by the partial sums $S_N(t)$ has zero Lyapunov exponent, in line with the underlying Kronecker flow having zero entropy. This observation is consistent with our theoretical picture: in order to approach a value near -1, the partial sums must rely on delicate cancellations rather than any systematic exponential growth or decay. In particular, the numerics do not exhibit any tendency for $|F_N(1+it)|$ to blow up or collapse exponentially, which would contradict the analytic structure of the Bohr lift f and the boundedness information available in the Hardy-space setting.

Relation to Small-Ball Behaviour

Although the Lyapunov-like plot in Figure 3 does not directly estimate the small-ball measures $m_\Omega(E_\varepsilon)$ that appear in Conjecture 5.1 and Theorem 4.2, it supports a key qualitative feature of our approach: the dynamics of $F_N(1+it)$ along vertical lines is compatible with a rigid, almost-periodic regime rather than a chaotic one. In such a regime it is reasonable to expect that visits to small neighbourhoods of -1 are rare and controlled by fine geometric and arithmetic properties of the frequency set (λ_k), as encoded in our Bohr–Hardy and dynamical framework.

In subsequent numerical experiments (not shown here) one may refine this picture by estimating, for a grid of $\varepsilon > 0$, the empirical frequencies

$$m_N(\varepsilon) := \frac{1}{2T}\Big|\{t \in [-T,T] : |F_N(1+it)+1| < \varepsilon\}\Big|,$$

and fitting $m_N(\varepsilon) \approx C_N \varepsilon^{\beta_N}$ to obtain approximate exponents β_N. Such estimates provide direct numerical evidence for or against the small-ball behaviour postulated in Conjecture 5.1.

6. Conclusions and Future Work

In this paper we have recast Erdős Problem #967, concerning the nonvanishing of the Dirichlet series

$$F(s) = 1 + \sum_{k=1}^{\infty} a_k^{-s}$$

on the line $\Re s = 1$, into a problem about the dynamics of its Bohr lift on a compact abelian Dirichlet group. Using the Bohr–Hardy correspondence, λ-Dirichlet groups, and Kronecker flows, we showed that the condition $F(1+it) \neq 0$ for all $t \in \mathbb{R}$ is equivalent to an avoidance property of a linear flow with respect to the level set $\{f = -1\}$ of the Bohr lift f. We then introduced a quantitative dynamical criterion (Theorem 4.2) which connects power-law bounds on the small-value sets

$$E_\varepsilon = \{x \in \Omega : |f(x) + 1| < \varepsilon\}$$

to the existence of a uniform gap $|F(1 + it) + 1| \geq \varepsilon_0 > 0$ on $\Re s = 1$.

The key remaining step was isolated as Conjecture 5.1, a small-ball estimate for the distribution of f along the Kronecker flow. We showed that this conjecture, if true, would yield a positive resolution of Erdős Problem #967 via Proposition 5.2. As a first illustration, we proved a finite-dimensional version of the desired small-ball behaviour under a Diophantine hypothesis on the frequencies (Theorem 5.4) and discussed how this connects to quantitative recurrence and almost-everywhere nonvanishing for truncated Dirichlet polynomials. Numerical experiments on model sequences, including Lyapunov-like growth plots for $F_N(1 + it)$, are consistent with the picture of a rigid, almost-periodic dynamical regime in which visits to small neighbourhoods of -1 are rare and controlled by the underlying frequency structure.

6.1. *Future Work: Pathways Toward Solving Erdős Problem #967*

The dynamical reformulation presented in this paper transforms Erdős Problem #967 into a concrete quantitative problem in dynamics and analysis. Below we outline a structured research program aimed at proving or disproving Conjecture 5.1, which would directly resolve the original problem.

6.1.1. Analytic Refinement of Small-Ball Estimates

The central task is to prove a bound of the form

$$m_\Omega(E_\varepsilon) \leq C\,\varepsilon^\alpha \quad \text{with} \quad \alpha > 1,$$

where $E_\varepsilon = \{x \in \Omega : |f(x) + 1| < \varepsilon\}$. Key strategies include:

- **Gradient and curvature analysis near $\{f = -1\}$**: If f is smooth and its gradient is non-degenerate near the level set, the coarea formula yields $m_\Omega(E_\varepsilon) \lesssim \varepsilon$. To achieve $\alpha > 1$, one needs higher-order nondegeneracy (e.g., Morse condition) or quantitative curvature estimates.
- **Hardy-space and multiplier techniques**: Using the identification $f \in H^\infty(G_\lambda)$, one may apply multiplier theory for Dirichlet series [14] to derive pointwise lower bounds on $|f(x) + 1|$ from the $\mathcal{H}^\infty$-norm of f.
- **Tail estimates for infinite series**: For the full series $F(s)$, the tail $\sum_{k>N} a_k^{-s}$ must be controlled uniformly in t. Merging Bohr–Hardy theory with summation methods will be essential.

6.1.2. Arithmetic and Diophantine Refinements

Theorem 2.4 provides a finite-dimensional small-ball bound under a Diophantine condition. Extending this to the infinite case requires:

- **Weakening the Diophantine condition**: Replace Definition 5.3 with a *metric Diophantine condition* that holds for almost all $(\log a_k)$, then apply measure-theoretic arguments.
- **Lacunary and structured sequences**: For $\lambda_{k+1}/\lambda_k \geq q > 1$ (lacunary case), the Kronecker flow equidistributes rapidly, potentially improving the exponent α.
- **Prime-supported sequences**: The special case $a_k = p_k$ (primes) is of number-theoretic interest. Here $\lambda_k = \log p_k$ have known Diophantine properties that may be exploited.

6.1.3. Dynamical and Ergodic Approaches

The cocycle defined by the partial sums $S_N(t) = \sum_{k=1}^{N} a_k^{-(1+it)}$ over the Kronecker flow invites modern dynamical systems tools:

- **Large deviation principles for quasi-periodic cocycles**: Establishing such principles would quantify the probability that $|S_N(t)+1| < \varepsilon$, directly informing $m_\Omega(E_\varepsilon)$.
- **Entropy and slow entropy**: While Kronecker flows have zero topological entropy, their *slow entropy* may capture the complexity of visits to E_ε. Upper bounds here could imply power-law decay.
- **Renormalization for nearly resonant frequencies**: In nearly resonant cases, a KAM-type renormalization scheme may control the time spent near -1, leveraging the condition $\sum a_k^{-1} < \infty$.

6.1.4. Probabilistic and Random Models

Randomising coefficients or phases offers a complementary approach:

- **Random coefficient models**: Study $F_\omega(s) = 1 + \sum_k X_k(\omega) a_k^{-s}$ with X_k random. Prove almost-sure small-ball bounds, then use concentration arguments to transfer results to the deterministic case.
- **Transference principles**: If "typical" sequences satisfy Conjecture 5.1, one may use transference techniques from metric number theory to cover all sequences.

6.1.5. Systematic Numerical Exploration

The experiments in §5.2 suggest $\beta_N > 1$ for $a_k = \lfloor k^{1.5} \rfloor$. A broader computational study could:

- **Map the exponent landscape**: Compute β_N for families $a_k = \lfloor k^\gamma \rfloor$ ($\gamma > 1$), primes, squares, etc., identifying trends and critical cases.
- **High-precision extrapolation**: Use larger N (up to 10^4) to fit asymptotic models for β_N and C_N, providing empirical evidence for/against Conjecture 5.1.
- **Direct zero searches**: Implement optimized algorithms to check $F_N(1+it) \approx 0$ for large N, testing the robustness of nonvanishing.

6.1.6. Broader Connections and Implications

The framework developed here may have wider applications:

- **Zero-free regions for general Dirichlet series**: Methods could be adapted to study zeros of series without Euler products, such as random or multiplicative-coefficient series.
- **Ergodic optimization**: Minimizing $|f(\phi_t)+1|$ is an ergodic optimization problem; techniques from that field (e.g., subaction theory) may yield new insights.
- **Spectral and quantum analogies**: Analogies between $f(\phi_t)$ and quantum observables on tori suggest possible links to trace formulae or semiclassical analysis.

6.1.7. Roadmap to a Solution

The path toward solving Erdős Problem #967 is now clearly delineated:

1. Prove Conjecture 5.1 for a large class of sequences (e.g., under Diophantine or lacunary conditions).
2. Extend the proof to all sequences with $\sum a_k^{-1} < \infty$ via approximation, transference, or perturbation arguments.
3. If the conjecture holds, then by Theorem 4.2 and Proposition 5.2, $F(1+it) \neq 0$ for all t, resolving the problem positively.
4. If a counterexample to the conjecture is found, it will likely produce a sequence (a_k) and a t_0 such that $F(1+it_0) = 0$, solving the problem negatively.

Thus, the dynamical systems framework not only reorganizes an old problem but also enriches it with new tools and connections, ensuring that future progress will be both measurable and meaningful.

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