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Not peer-reviewed version

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Posted Date: 31 December 2025

doi: 10.20944/preprints202512.2679.v1

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Finite Trigonometric Sum and Product Identities from Infinite Series

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Abstract

Calculating finite sums and products of trigonometric functions is an important and fascinating problem, a straightforward method is using infinite series or infinite product. In this paper, we calculate four finite sums and a finite product of trigonometric functions using this method, which contributes to a deeper understanding of this problem.

Keywords: trigonometric identity; finite sum; finite product; infinite series; infinite product

1. Introduction

Finite sums and products of trigonometric functions are emerged in various of disciplines such as combinatorics, number theory (see [2? ,3]), and Fourier analysis. Due to its fascinating and technical nature, it has been extensively studied. This problem has been particularly explored since the mid-19th century, thanks to the development of complex analysis and the study of combinatorial identities (see [4,5]).

Early on, people derived the closed forms by elementary transformations for two sums

$$\sum_{k=1}^{n} \sin(kx) \quad \text{and} \quad \sum_{k=1}^{n} \cos(kx),$$

where n is a positive integer and x is a real number, which are very common in Fourier analysis (see [6]). Later, people noticed the relationship between the values of trigonometric functions and the roots of algebraic polynomials. Applying Vieta's theorem and the properties of polynomials, they derived a large number of trigonometric identities which contain $\frac{k\pi}{n}$, where n is a positive integer and $k \in \{0, 1, \ldots, n-1\}$ (see [7–9]). In 1861, Stern proved that for odd number n with $n > 1$, the equation

$$\sum_{k=1}^{n-1} \tan^2\left(\frac{k\pi}{n}\right) = n(n-1)$$

holds (see [10]). In 1997, Byrne and Simth proved that for positive integers m and n, the results of following two sums

$$\sum_{k=1}^{n} (-1)^{k-1} \cot^{2m-1}\left(\frac{(2k-1)\pi}{4n}\right) \quad \text{and} \quad \sum_{k=1}^{n} \cot^{2m}\left(\frac{(2k-1)\pi}{4n}\right)$$

are positive integers (see [11]). In 1999, Chu and Marini applied generating functions, partial fraction decomposition, and some combinatorial identities to derive the closed forms for 24 finite sums of trigonometric functions (see [12]). In 2007, Wang expanded Chu and Marini's results in 1999 by adding parameters, and used partial fraction decomposition, hypergeometric series, and cyclotomic polynomials to obtain the closed forms for some new finite sums of trigonometric functions (see [13]). These two papers represent the most comprehensive and systematic research on this problem to date,

 doi:10.20944/preprints202512.2679.v1

and have contributed to substantial progress in this area. In recent years, this issue has continued to be studied, and a large number of related results are still emerging (such as [2,3,9]).

In this paper, we will firstly use the fact that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1,$$

to derive the trigonometric identity

$$\sum_{k=0}^{n-1} \frac{1}{\sin^2\left(\alpha + \frac{k\pi}{n}\right)} = \frac{n^2}{\sin^2(n\alpha)}, \quad \sin(n\alpha) \neq 0.$$

Then, we will use this result to provide new proofs for the following two infinite series and an infinite product (also see [14]). Specifically,

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{m=1}^{\infty} \left(\frac{1}{(x+m\pi)^2} + \frac{1}{(x-m\pi)^2} \right), \quad \sin x \neq 0;$$

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{x+m\pi} + \frac{1}{x-m\pi} \right), \quad \sin x \neq 0;$$

and

$$\sin x = x \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{m^2\pi^2} \right).$$

This method is very different from previous ones. Most of all, we will use the above results to provide the closed forms for four finite sums and a finite product of trigonometric functions. Finally, we will use our results to derive some equations in [12,13].

2. Main Results

In later chapters, we denote $f(x) = \frac{1}{\sin^2 x}$, $g(x) = \frac{1}{\sin x}$. And we denote $f^{(l)}(x)$ as the lth derivative of $f(x)$, where l is a nonnegative integer. The Riemann zeta function is defined as

$$\zeta(l) = \sum_{m=1}^{\infty} \frac{1}{m^l},$$

where l is a positive integer with $l \geq 2$.

Now we present the main results through the following theorems.

Theorem 1. *Let n be a positive integer and α be a real number such that $\sin(n\alpha) \neq 0$, then for nonnegative integer l, we have*

$$\sum_{k=0}^{n-1} f^{(l)}\left(\alpha + \frac{k\pi}{n}\right) = n^{l+2} f^{(l)}(n\alpha).$$

Theorem 2. *Let n be a positive integer with $n \geq 2$, then for nonnegative integer l, we have*

$$\sum_{k=1}^{n-1} f^{(2l)}\left(\frac{k\pi}{n}\right) = 2\frac{(2l+1)!(n^{2l+2}-1)}{\pi^{2l+2}}\zeta(2l+2).$$

doi:10.20944/preprints202512.2679.v1

Theorem 3. *Let n be a positive odd number and α be a real number such that $\sin(n\alpha) \neq 0$, then for nonnegative integer l, we have*

$$\sum_{k=0}^{n-1}(-1)^k g^{(l)}\left(\alpha+\frac{k\pi}{n}\right) = n^{l+1}g^{(l)}(n\alpha).$$

Theorem 4. *Let n be an odd number with $n \geq 3$, then for nonnegative integer l, we have*

$$\sum_{k=1}^{n-1}(-1)^k g^{(2l+1)}\left(\frac{k\pi}{n}\right) = 2\frac{(2l+1)!(n^{2l+2}-1)}{\pi^{2l+2}}\left(1-\frac{1}{2^{2l+1}}\right)\zeta(2l+2).$$

Theorem 5 (also see [2]). *Let n be a positive integer and α be a real number, then we have*

$$\prod_{k=0}^{n-1}\sin\left(\alpha+\frac{k\pi}{n}\right) = \frac{\sin(n\alpha)}{2^{n-1}}.$$

3. Some Lemmas

Firstly, we introduce some lemmas as follows.

Lemma 1. *Let n be a positive integer and α be a real number such that $\sin(n\alpha) \neq 0$, then we have*

$$\sum_{k=0}^{n-1}\frac{1}{\sin^2\left(\alpha+\frac{k\pi}{n}\right)} = \frac{n^2}{\sin^2(n\alpha)}.$$

Proof. Let a be a complex number with $|a| < 1$, then for $k = 0, 1, \ldots, n-1$, we have

$$\left|a e^{i\frac{2k\pi}{n}}\right| = |a| < 1.$$

According to Taylor's formula, we know that

$$\frac{1}{1-ae^{i\frac{2k\pi}{n}}} = \sum_{m=0}^{\infty}\left(ae^{i\frac{2k\pi}{n}}\right)^m$$

holds for $k = 0, 1, \ldots, n-1$. Thus,

$$\sum_{k=0}^{n-1}\frac{1}{1-ae^{i\frac{2k\pi}{n}}} = \sum_{k=0}^{n-1}\sum_{m=0}^{\infty}\left(ae^{i\frac{2k\pi}{n}}\right)^m = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{n-1}\left(e^{i\frac{2m\pi}{n}}\right)^k\right)a^m. \tag{1}$$

It is easy to verify that

$$\sum_{k=0}^{n-1}\left(e^{i\frac{2m\pi}{n}}\right)^k = \begin{cases} 0, & n \nmid m; \\ n, & n \mid m. \end{cases} \tag{2}$$

Substituting (2) into (1) we could obtain that

$$\sum_{k=0}^{n-1}\frac{1}{1-ae^{i\frac{2k\pi}{n}}} = \sum_{m=0}^{\infty}n(a^n)^m = \frac{n}{1-a^n}. \tag{3}$$

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Letting $a \to \mathrm{e}^{\mathrm{i}2\alpha}$ in (3), we have

$$\sum_{k=0}^{n-1} \frac{1}{1-\mathrm{e}^{\mathrm{i}(2\alpha+\frac{2k\pi}{n})}} = \frac{n}{1-\mathrm{e}^{\mathrm{i}2n\alpha}}. \tag{4}$$

Calculating the derivative of (4) with respect to α yields

$$\sum_{k=0}^{n-1} \frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}(2\alpha+\frac{2k\pi}{n})}}{\left(1-\mathrm{e}^{\mathrm{i}(2\alpha+\frac{2k\pi}{n})}\right)^2} = \frac{2\mathrm{i}n^2\mathrm{e}^{\mathrm{i}2n\alpha}}{(1-\mathrm{e}^{\mathrm{i}2n\alpha})^2},$$

namely

$$\sum_{k=0}^{n-1} \frac{1}{\left(\mathrm{e}^{\mathrm{i}(\alpha+\frac{k\pi}{n})}-\mathrm{e}^{-\mathrm{i}(\alpha+\frac{k\pi}{n})}\right)^2} = \frac{n^2}{(\mathrm{e}^{\mathrm{i}n\alpha}-\mathrm{e}^{-\mathrm{i}n\alpha})^2}.$$

By Euler's formula, we could get that

$$\sum_{k=0}^{n-1} \frac{1}{\sin^2(\alpha+\frac{k\pi}{n})} = \frac{n^2}{\sin^2(n\alpha)}.$$

Lemma 1 is proved. □

Lemma 2. *While $x \in (0,\pi)$, the function $\varphi(x) = \frac{\sin x}{x}$ is monotonically decreasing and the range of $\varphi(x)$ is $(0,1)$. Furthermore, let n be a positive integer and x be a real number such that $|x| \le \frac{(n+1)\pi}{2n+1}$, then we have*

$$\frac{3\sqrt{3}}{4\pi}|x| \le |\sin x| \le |x|.$$

Proof. The concavity of function $h(x) = \sin x$ while $x \in (0,\pi)$ immediately yields that $\varphi(x)$ is monotonically decreasing. Then we have

$$\lim_{x\to 0^+} \varphi(x) = 1, \quad \lim_{x\to\pi^-} \varphi(x) = 0.$$

Therefore, the range of $\varphi(x)$ is $(0,1)$. Now, we prove the last inequality. Because $\varphi(x)$ is an even function and $\varphi(x) < 1$ while $x \in (0,\pi)$, we only need to show that

$$\varphi(x) \ge \frac{3\sqrt{3}}{4\pi}, \quad 0 < x \le \frac{(n+1)\pi}{2n+1}.$$

Using the monotonicity of $\varphi(x)$ and

$$\frac{(n+1)\pi}{2n+1} \le \frac{2\pi}{3}, \quad n \ge 1;$$

we know that

$$\varphi(x) \ge \varphi\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{4\pi}.$$

Lemma 2 is proved. □

Lemma 3. *Let x be a real number with $\sin x \ne 0$, then we have*

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{m=1}^{\infty}\left(\frac{1}{(x+m\pi)^2} + \frac{1}{(x-m\pi)^2}\right)$$

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and

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{x + m\pi} + \frac{1}{x - m\pi} \right).$$

Proof. Let n be a sufficiently large positive integer and α be a real number such that $\sin((2n+1)\alpha) \neq 0$, then according to Lemma 1, we have

$$\frac{(2n+1)^2}{\sin^2((2n+1)\alpha)} = \sum_{k=0}^{2n} \frac{1}{\sin^2(\alpha + \frac{k\pi}{2n+1})} = \sum_{k=-n}^{n} \frac{1}{\sin^2(\alpha + \frac{k\pi}{2n+1})}.$$

Thus, if we let $\alpha = \frac{x}{2n+1}$ and $x \in (0, \pi)$, we have

$$\frac{1}{\sin^2 x} = \frac{1}{(2n+1)^2} \sum_{k=-n}^{n} \frac{1}{\sin^2(\frac{x+k\pi}{2n+1})}.$$

According to Lemma 2, we know that

$$\frac{1}{\sin^2 x} \geq \frac{1}{(2n+1)^2} \sum_{k=-n}^{n} \frac{1}{(\frac{x+k\pi}{2n+1})^2} = \sum_{k=-n}^{n} \frac{1}{(x+k\pi)^2}.$$

Letting $n \to +\infty$ in above inequality, we obtain

$$\frac{1}{\sin^2 x} \geq \frac{1}{x^2} + \sum_{k=1}^{\infty} \left(\frac{1}{(x+k\pi)^2} + \frac{1}{(x-k\pi)^2} \right). \tag{5}$$

Now we fix x in $(0, \pi)$. From the convergence of the series

$$\frac{1}{x^2} + \sum_{k=1}^{\infty} \left(\frac{1}{(x+k\pi)^2} + \frac{1}{(x-k\pi)^2} \right),$$

we could get that for arbitrary positive number ε, there exists a positive integer K, such that

$$\frac{16\pi^2}{27} \sum_{k=K+1}^{\infty} \left(\frac{1}{(x+k\pi)^2} + \frac{1}{(x-k\pi)^2} \right) < \varepsilon.$$

According to Lemma 2, for positive integers n and p, if $n > p > K$, we have

$$\begin{aligned} \frac{1}{\sin^2 x} &= \frac{1}{(2n+1)^2} \left\{ \sum_{k=-p}^{p} \frac{1}{\sin^2(\frac{x+k\pi}{2n+1})} + \sum_{k=p+1}^{n} \left(\frac{1}{\sin^2(\frac{x+k\pi}{2n+1})} + \frac{1}{\sin^2(\frac{x-k\pi}{2n+1})} \right) \right\} \\ &\leq \frac{1}{(2n+1)^2} \sum_{k=-p}^{p} \frac{1}{\sin^2(\frac{x+k\pi}{2n+1})} + \frac{16\pi^2}{27} \sum_{k=p+1}^{n} \left(\frac{1}{(x+k\pi)^2} + \frac{1}{(x-k\pi)^2} \right) \\ &< \frac{1}{(2n+1)^2} \sum_{k=-p}^{p} \frac{1}{\sin^2(\frac{x+k\pi}{2n+1})} + \varepsilon. \end{aligned} \tag{6}$$

Due to the arbitrariness of n, we could let $n \to +\infty$ in (6), and we obtain

$$\frac{1}{\sin^2 x} \leq \sum_{k=-p}^{p} \frac{1}{(x+k\pi)^2} + \varepsilon \leq \frac{1}{x^2} + \sum_{k=1}^{\infty} \left(\frac{1}{(x+k\pi)^2} + \frac{1}{(x-k\pi)^2} \right) + \varepsilon. \tag{7}$$

 doi:10.20944/preprints202512.2679.v1

Letting $\varepsilon \to 0^+$ in (7), we could get

$$\frac{1}{\sin^2 x} \le \frac{1}{x^2} + \sum_{k=1}^{\infty}\left(\frac{1}{(x+k\pi)^2} + \frac{1}{(x-k\pi)^2}\right). \tag{8}$$

Combining (5) and (8), we know that

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{m=1}^{\infty}\left(\frac{1}{(x+m\pi)^2} + \frac{1}{(x-m\pi)^2}\right) \tag{9}$$

holds for $x \in (0, \pi)$. In fact, equation (9) has a period of π, so (9) holds for x with $\sin x \neq 0$. Assuming $x \in (0, \pi)$ and integrating both sides of (9) yields

$$\int_0^x \left(\frac{1}{\sin^2 t} - \frac{1}{t^2}\right)\mathrm{d}t = \int_0^x \sum_{m=1}^{\infty}\left(\frac{1}{(t+m\pi)^2} + \frac{1}{(t-m\pi)^2}\right)\mathrm{d}t.$$

Namely

$$\frac{\cos x}{\sin x} = \frac{1}{x} + \sum_{m=1}^{\infty}\left(\frac{1}{x+m\pi} + \frac{1}{x-m\pi}\right), \quad x \in (0, \pi). \tag{10}$$

Using (10) and the fact that

$$\frac{1}{\sin x} = \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} - \frac{\cos x}{\sin x}, \quad x \in (0, \pi);$$

we could get that

$$\begin{aligned}
\frac{1}{\sin x} =& \left(\frac{1}{\frac{x}{2}} + \sum_{m=1}^{\infty}\left(\frac{1}{\frac{x}{2}+m\pi} + \frac{1}{\frac{x}{2}-m\pi}\right)\right) - \left(\frac{1}{x} + \sum_{m=1}^{\infty}\left(\frac{1}{x+m\pi} + \frac{1}{x-m\pi}\right)\right)\\
=& \frac{1}{x} + \sum_{m=1}^{\infty}\left(\frac{2}{x+2m\pi} + \frac{2}{x-2m\pi}\right) - \sum_{m=1}^{\infty}\left(\frac{1}{x+2m\pi} + \frac{1}{x-2m\pi}\right)\\
&- \sum_{m=1}^{\infty}\left(\frac{1}{x+(2m-1)\pi} + \frac{1}{x-(2m-1)\pi}\right)\\
=& \frac{1}{x} + \sum_{m=1}^{\infty}(-1)^m\left(\frac{1}{x+m\pi} + \frac{1}{x-m\pi}\right).
\end{aligned} \tag{11}$$

Because $\sin x$ is an odd function and equation (11) is periodic, we know that (11) holds for x with $\sin x \neq 0$. So far, Lemma 3 is proved. □

Lemma 4. *Let x be a real number, then we have*

$$\sin x = x\prod_{m=1}^{\infty}\left(1 - \frac{x^2}{m^2\pi^2}\right).$$

Proof. Firstly, we assume $x \in (0, \pi)$. Integrating both sides of (10) yields

$$\int_0^x \left(\frac{\cos t}{\sin t} - \frac{1}{t}\right)\mathrm{d}t = \int_0^x \sum_{m=1}^{\infty}\left(\frac{1}{t+m\pi} + \frac{1}{t-m\pi}\right)\mathrm{d}t.$$

Namely

$$\ln\left(\frac{\sin x}{x}\right) = \sum_{m=1}^{\infty}\ln\left(1 - \frac{x^2}{m^2\pi^2}\right), \quad x \in (0, \pi).$$

 doi:10.20944/preprints202512.2679.v1

Thus,

$$\sin x = x \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{m^2\pi^2}\right), \quad x \in (0, \pi). \tag{12}$$

Because $\sin x$ is an odd function and equation (12) is periodic, we know that (12) holds for all real number x. So far, Lemma 4 is proved. □

4. Proofs of Main Results

Now, we use the above lemmas to prove our main results. The core idea of proving these theorems is using infinite series or infinite product, and then appropriately combining them to obtain the target results.

4.1. Proof of Theorem 1

Proof. Recalling Lemma 3, we have

$$f(x) = \frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{m=1}^{\infty} \left(\frac{1}{(x+m\pi)^2} + \frac{1}{(x-m\pi)^2}\right), \quad \sin x \neq 0.$$

Calculating the lth order derivative of $f(x)$ with respect to x, we get

$$f^{(l)}(x) = (-1)^l \{\frac{(l+1)!}{x^{l+2}} + \sum_{m=1}^{\infty} \left(\frac{(l+1)!}{(x+m\pi)^{l+2}} + \frac{(l+1)!}{(x-m\pi)^{l+2}}\right)\}, \quad \sin x \neq 0. \tag{13}$$

In (13), we take $x = \alpha + \frac{k\pi}{n} (k = 0, 1, \ldots, n-1)$ in turn, and we get that

$$\begin{aligned} &f^{(l)}\left(\alpha + \frac{k\pi}{n}\right) \\ =&(-1)^l \{\frac{(l+1)!}{(\alpha+\frac{k\pi}{n})^{l+2}} + \sum_{m=1}^{\infty} \left(\frac{(l+1)!}{(\alpha+\frac{k\pi}{n}+m\pi)^{l+2}} + \frac{(l+1)!}{(\alpha+\frac{k\pi}{n}-m\pi)^{l+2}}\right)\} \\ =&(-1)^l \{\frac{(l+1)!n^{l+2}}{(n\alpha+k\pi)^{l+2}} + \sum_{m=1}^{\infty} \left(\frac{(l+1)!n^{l+2}}{(n\alpha+(nm+k)\pi)^{l+2}} + \frac{(l+1)!n^{l+2}}{(n\alpha-(nm-k)\pi)^{l+2}}\right)\}. \end{aligned} \tag{14}$$

Summing k from 0 to $(n-1)$ in (14) yields

$$\begin{aligned} &\sum_{k=0}^{n-1} f^{(l)}\left(\alpha + \frac{k\pi}{n}\right) \\ =&(-1)^l \{\sum_{k=0}^{n-1} \frac{(l+1)!n^{l+2}}{(n\alpha+k\pi)^{l+2}} + \sum_{k=0}^{n-1}\sum_{m=1}^{\infty} \left(\frac{(l+1)!n^{l+2}}{(n\alpha+(nm+k)\pi)^{l+2}} + \frac{(l+1)!n^{l+2}}{(n\alpha-(nm-k)\pi)^{l+2}}\right)\} \\ =&(-1)^l \{\sum_{k=0}^{n-1} \frac{(l+1)!n^{l+2}}{(n\alpha+k\pi)^{l+2}} + \sum_{m=1}^{\infty}\sum_{k=0}^{n-1} \left(\frac{(l+1)!n^{l+2}}{(n\alpha+(nm+k)\pi)^{l+2}} + \frac{(l+1)!n^{l+2}}{(n\alpha-(nm-k)\pi)^{l+2}}\right)\} \\ =&(-1)^l n^{l+2} \left(\sum_{m=0}^{\infty}\sum_{k=0}^{n-1} \frac{(l+1)!}{(n\alpha+(nm+k)\pi)^{l+2}} + \sum_{m=1}^{\infty}\sum_{k=0}^{n-1} \frac{(l+1)!}{(n\alpha-(nm-k)\pi)^{l+2}}\right) \\ =&(-1)^l n^{l+2} \left(\sum_{m=0}^{\infty}\sum_{k=nm}^{n(m+1)-1} \frac{(l+1)!}{(n\alpha+k\pi)^{l+2}} + \sum_{m=1}^{\infty}\sum_{k=n(m-1)+1}^{nm} \frac{(l+1)!}{(n\alpha-k\pi)^{l+2}}\right) \\ =&(-1)^l n^{l+2} \{\frac{(l+1)!}{(n\alpha)^{l+2}} + \sum_{k=1}^{\infty} \left(\frac{(l+1)!}{(n\alpha+k\pi)^{l+2}} + \frac{(l+1)!}{(n\alpha-k\pi)^{l+2}}\right)\} \\ =&n^{l+2} f^{(l)}(n\alpha). \end{aligned}$$

Theorem 1 is proved. □

Remark 1. *The proof of Theorem 1 shows that Lemma 1 and Lemma 3 are equivalent, and we also could calculate the lth order derivative of*

$$\sum_{k=0}^{n-1} \frac{1}{\sin^2\left(\alpha + \frac{k\pi}{n}\right)} = \frac{n^2}{\sin^2(n\alpha)}$$

(where $\sin(n\alpha) \neq 0$*) with respect to* α *to get Theorem 1.*

4.2. Proof of Theorem 2

Proof. In (13), we replace l with $2l$ and then take $x = \frac{k\pi}{n}(k = 1, 2, \ldots, n-1)$ in turn, we could get that

$$\begin{aligned} f^{(2l)}\left(\frac{k\pi}{n}\right) &= \frac{(2l+1)!}{(\frac{k\pi}{n})^{2l+2}} + \sum_{m=1}^{\infty}\left(\frac{(2l+1)!}{(\frac{k\pi}{n}+m\pi)^{2l+2}} + \frac{(2l+1)!}{(\frac{k\pi}{n}-m\pi)^{2l+2}}\right) \\ &= \frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\left\{\frac{1}{k^{2l+2}} + \sum_{m=1}^{\infty}\left(\frac{1}{(nm+k)^{2l+2}} + \frac{1}{(nm-k)^{2l+2}}\right)\right\}. \end{aligned} \tag{15}$$

Summing k from 1 to $(n-1)$ in (15) yields

$$\begin{aligned} &\sum_{k=1}^{n-1} f^{(2l)}\left(\frac{k\pi}{n}\right) \\ =&\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\left\{\sum_{k=1}^{n-1}\frac{1}{k^{2l+2}} + \sum_{k=1}^{n-1}\sum_{m=1}^{\infty}\left(\frac{1}{(nm+k)^{2l+2}} + \frac{1}{(nm-k)^{2l+2}}\right)\right\} \\ =&\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\left\{\sum_{k=1}^{n-1}\frac{1}{k^{2l+2}} + \sum_{m=1}^{\infty}\sum_{k=1}^{n-1}\left(\frac{1}{(nm+k)^{2l+2}} + \frac{1}{(nm-k)^{2l+2}}\right)\right\} \\ =&\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\left(\sum_{k=1}^{n-1}\frac{1}{k^{2l+2}} + \sum_{m=1}^{\infty}\sum_{k=nm+1}^{n(m+1)-1}\frac{1}{k^{2l+2}} + \sum_{m=1}^{\infty}\sum_{k=n(m-1)+1}^{nm-1}\frac{1}{k^{2l+2}}\right) \\ =&\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\left(\sum_{m=0}^{\infty}\sum_{k=nm+1}^{n(m+1)-1}\frac{1}{k^{2l+2}} + \sum_{m=1}^{\infty}\sum_{k=n(m-1)+1}^{nm-1}\frac{1}{k^{2l+2}}\right) \\ =&2\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\sum_{m=1}^{\infty}\sum_{k=n(m-1)+1}^{nm-1}\frac{1}{k^{2l+2}}. \end{aligned} \tag{16}$$

We notice that

$$\begin{aligned} \sum_{m=1}^{\infty}\sum_{k=n(m-1)+1}^{nm-1}\frac{1}{k^{2l+2}} &= \sum_{m=1}^{\infty}\left(\sum_{k=n(m-1)+1}^{nm}\frac{1}{k^{2l+2}} - \frac{1}{(nm)^{2l+2}}\right) \\ &= \sum_{k=1}^{\infty}\frac{1}{k^{2l+2}} - \frac{1}{n^{2l+2}}\sum_{m=1}^{\infty}\frac{1}{m^{2l+2}} \\ &= \left(1 - \frac{1}{n^{2l+2}}\right)\zeta(2l+2). \end{aligned} \tag{17}$$

Therefore, substituting (17) into (16) yields

$$\sum_{k=1}^{n-1} f^{(2l)}\left(\frac{k\pi}{n}\right) = 2\frac{(2l+1)!(n^{2l+2}-1)}{\pi^{2l+2}}\zeta(2l+2).$$

Theorem 2 is proved. □

 doi:10.20944/preprints202512.2679.v1

4.3. Proof of Theorem 3

Proof. Recalling Lemma 3, we have

$$g(x)=\frac{1}{\sin x}=\frac{1}{x}+\sum_{m=1}^{\infty}(-1)^m\left(\frac{1}{x+m\pi}+\frac{1}{x-m\pi}\right),\quad \sin x\neq 0.$$

Calculating the lth order derivative of $g(x)$ with respect to x, we get

$$g^{(l)}(x)=(-1)^l\{\frac{l!}{x^{l+1}}+\sum_{m=1}^{\infty}(-1)^m\left(\frac{l!}{(x+m\pi)^{l+1}}+\frac{l!}{(x-m\pi)^{l+1}}\right)\},\quad \sin x\neq 0. \tag{18}$$

In (18), we take $x=\alpha+\frac{k\pi}{n}(k=0,1,\ldots,n-1)$ in turn, and we get that

$$\begin{aligned}
&g^{(l)}\left(\alpha+\frac{k\pi}{n}\right)\\
=&(-1)^l\{\frac{l!}{(\alpha+\frac{k\pi}{n})^{l+1}}+\sum_{m=1}^{\infty}(-1)^m\left(\frac{l!}{(\alpha+\frac{k\pi}{n}+m\pi)^{l+1}}+\frac{l!}{(\alpha+\frac{k\pi}{n}-m\pi)^{l+1}}\right)\}\\
=&(-1)^l\{\frac{l!n^{l+1}}{(n\alpha+k\pi)^{l+1}}+\sum_{m=1}^{\infty}\left(\frac{(-1)^m l!n^{l+1}}{(n\alpha+(nm+k)\pi)^{l+1}}+\frac{(-1)^m l!n^{l+1}}{(n\alpha-(nm-k)\pi)^{l+1}}\right)\}.
\end{aligned} \tag{19}$$

Multiplying (19) by $(-1)^k$ and then summing k from 0 to $(n-1)$ yields

$$\begin{aligned}
&\sum_{k=0}^{n-1}(-1)^k g^{(l)}\left(\alpha+\frac{k\pi}{n}\right)\\
=&(-1)^l\{\sum_{k=0}^{n-1}\frac{(-1)^k l!n^{l+1}}{(n\alpha+k\pi)^{l+1}}+\sum_{k=0}^{n-1}\sum_{m=1}^{\infty}\left(\frac{(-1)^{m+k}l!n^{l+1}}{(n\alpha+(nm+k)\pi)^{l+1}}+\frac{(-1)^{m+k}l!n^{l+1}}{(n\alpha-(nm-k)\pi)^{l+1}}\right)\}\\
=&(-1)^l\{\sum_{k=0}^{n-1}\frac{(-1)^k l!n^{l+1}}{(n\alpha+k\pi)^{l+1}}+\sum_{m=1}^{\infty}\sum_{k=0}^{n-1}\left(\frac{(-1)^{m+k}l!n^{l+1}}{(n\alpha+(nm+k)\pi)^{l+1}}+\frac{(-1)^{m+k}l!n^{l+1}}{(n\alpha-(nm-k)\pi)^{l+1}}\right)\}\\
=&(-1)^l n^{l+1}\left(\sum_{m=0}^{\infty}\sum_{k=0}^{n-1}\frac{(-1)^{m+k}l!}{(n\alpha+(nm+k)\pi)^{l+1}}+\sum_{m=1}^{\infty}\sum_{k=0}^{n-1}\frac{(-1)^{m+k}l!}{(n\alpha-(nm-k)\pi)^{l+1}}\right)\\
=&(-1)^l n^{l+1}\left(\sum_{m=0}^{\infty}\sum_{k=nm}^{n(m+1)-1}\frac{(-1)^{m+k-nm}l!}{(n\alpha+k\pi)^{l+1}}+\sum_{m=1}^{\infty}\sum_{k=n(m-1)+1}^{nm}\frac{(-1)^{m+nm-k}l!}{(n\alpha-k\pi)^{l+1}}\right)\\
=&(-1)^l n^{l+1}\{\frac{l!}{(n\alpha)^{l+1}}+\sum_{k=1}^{\infty}(-1)^k\left(\frac{l!}{(n\alpha+k\pi)^{l+1}}+\frac{l!}{(n\alpha-k\pi)^{l+1}}\right)\}\\
=&n^{l+1}g^{(l)}(n\alpha).
\end{aligned}$$

We used the fact that for all positive integers m, both $(m-nm)$ and $(m+nm)$ are even. Theorem 3 is proved. □

4.4. Proof of Theorem 4

Proof. In (18), we replace l with $(2l+1)$ and then take $x=\frac{k\pi}{n}(k=1,2,\ldots,n-1)$ in turn, we could get that

$$\begin{aligned}
g^{(2l+1)}\left(\frac{k\pi}{n}\right)&=-\frac{(2l+1)!}{(\frac{k\pi}{n})^{2l+2}}-\sum_{m=1}^{\infty}\left(\frac{(-1)^m(2l+1)!}{(\frac{k\pi}{n}+m\pi)^{2l+2}}+\frac{(-1)^m(2l+1)!}{(\frac{k\pi}{n}-m\pi)^{2l+2}}\right)\\
&=-\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\{\frac{1}{k^{2l+2}}+\sum_{m=1}^{\infty}\left(\frac{(-1)^m}{(nm+k)^{2l+2}}+\frac{(-1)^m}{(nm-k)^{2l+2}}\right)\}.
\end{aligned} \tag{20}$$

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Multiplying (20) by $(-1)^k$ and then summing k from 1 to $(n-1)$ yields

$$\begin{aligned}
&\sum_{k=1}^{n-1}(-1)^k g^{(2l+1)}\left(\frac{k\pi}{n}\right) \\
=&\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\{\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k^{2l+2}}+\sum_{k=1}^{n-1}\sum_{m=1}^{\infty}\left(\frac{(-1)^{m+k-1}}{(nm+k)^{2l+2}}+\frac{(-1)^{m+k-1}}{(nm-k)^{2l+2}}\right)\} \\
=&\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\{\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k^{2l+2}}+\sum_{m=1}^{\infty}\sum_{k=1}^{n-1}\left(\frac{(-1)^{m+k-1}}{(nm+k)^{2l+2}}+\frac{(-1)^{m+k-1}}{(nm-k)^{2l+2}}\right)\} \\
=&\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\left(\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k^{2l+2}}+\sum_{m=1}^{\infty}\sum_{k=nm+1}^{n(m+1)-1}\frac{(-1)^{m+k-nm-1}}{k^{2l+2}}+\sum_{m=1}^{\infty}\sum_{k=n(m-1)+1}^{nm-1}\frac{(-1)^{m+nm-k-1}}{k^{2l+2}}\right) \\
=&\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\left(\sum_{m=0}^{\infty}\sum_{k=nm+1}^{n(m+1)-1}\frac{(-1)^{k-1}}{k^{2l+2}}+\sum_{m=1}^{\infty}\sum_{k=n(m-1)+1}^{nm-1}\frac{(-1)^{k-1}}{k^{2l+2}}\right) \\
=&2\frac{(2l+1)!n^{2l+2}}{\pi^{2l+2}}\sum_{m=1}^{\infty}\sum_{k=n(m-1)+1}^{nm-1}\frac{(-1)^{k-1}}{k^{2l+2}}. \qquad (21)
\end{aligned}$$

We used the fact that for all positive integers m, both $(m-nm)$ and $(m+nm)$ are even. We notice that

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{k=n(m-1)+1}^{nm-1}\frac{(-1)^{k-1}}{k^{2l+2}} &= \sum_{m=1}^{\infty}\left(\sum_{k=n(m-1)+1}^{nm}\frac{(-1)^{k-1}}{k^{2l+2}}-\frac{(-1)^{nm-1}}{(nm)^{2l+2}}\right) \\
&=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{2l+2}}-\frac{1}{n^{2l+2}}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^{2l+2}} \\
&=\left(1-\frac{1}{n^{2l+2}}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{2l+2}}, \qquad (22)
\end{aligned}$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{2l+2}}=\sum_{k=1}^{\infty}\frac{1}{k^{2l+2}}-2\sum_{k=1}^{\infty}\frac{1}{(2k)^{2l+2}}=\left(1-\frac{1}{2^{2l+1}}\right)\zeta(2l+2). \qquad (23)$$

Therefore, substituting equations (22) and (23) into equation (21) yields

$$\sum_{k=1}^{n-1}(-1)^k g^{(2l+1)}\left(\frac{k\pi}{n}\right)=2\frac{(2l+1)!(n^{2l+2}-1)}{\pi^{2l+2}}\left(1-\frac{1}{2^{2l+1}}\right)\zeta(2l+2).$$

Theorem 4 is proved. □

4.5. *Proof of Theorem 5*

Proof. Recalling Lemma 4, we have

$$\sin x = x\prod_{m=1}^{\infty}\left(1-\frac{x^2}{m^2\pi^2}\right)=\lim_{N\to\infty}x\prod_{m=1}^{N}\left(1-\frac{x^2}{m^2\pi^2}\right). \qquad (24)$$

In (24), we take $x=\alpha+\frac{k\pi}{n}(k=0,1,\ldots,n-1)$ in turn, and we get that

$$\begin{aligned}
\sin\left(\alpha+\frac{k\pi}{n}\right)&=\lim_{N\to\infty}\left(\alpha+\frac{k\pi}{n}\right)\prod_{m=1}^{N}\left(1-\frac{(\alpha+\frac{k\pi}{n})^2}{m^2\pi^2}\right) \\
&=\lim_{N\to\infty}\frac{(k\pi+n\alpha)}{n^{1+2N}(N!)^2\pi^{2N}}\prod_{m=1}^{N}((nm+k)\pi+n\alpha)((nm-k)\pi-n\alpha). \qquad (25)
\end{aligned}$$

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Taking the product of k from 0 to $(n-1)$ in (25) yields

$$\begin{aligned}
&\prod_{k=0}^{n-1}\sin\left(\alpha+\frac{k\pi}{n}\right)\\
&=\lim_{N\to\infty}\frac{\left(\prod_{k=0}^{n-1}(k\pi+n\alpha)\right)}{n^{n+2nN}(N!)^{2n}\pi^{2nN}}\prod_{m=1}^{N}\left(\prod_{k=0}^{n-1}((nm+k)\pi+n\alpha)\right)\left(\prod_{k=0}^{n-1}((nm-k)\pi-n\alpha)\right)\\
&=\lim_{N\to\infty}\frac{\left(\prod_{k=0}^{n-1}(k\pi+n\alpha)\right)}{n^{n+2nN}(N!)^{2n}\pi^{2nN}}\prod_{m=1}^{N}\left(\prod_{k=nm}^{n(m+1)-1}(k\pi+n\alpha)\right)\left(\prod_{k=n(m-1)+1}^{nm}(k\pi-n\alpha)\right)\\
&=\lim_{N\to\infty}\frac{1}{n^{n+2nN}(N!)^{2n}\pi^{2nN}}\frac{n\alpha}{(nN+n)\pi+n\alpha}\left(\prod_{k=1}^{nN+n}(k\pi+n\alpha)\right)\left(\prod_{k=1}^{nN}(k\pi-n\alpha)\right)\\
&=\lim_{N\to\infty}\frac{(nN+n)!(nN)!\pi^{n+2nN}}{n^{n+2nN}(N!)^{2n}\pi^{2nN}}\frac{n\alpha}{(nN+n)\pi+n\alpha}\left(\prod_{k=1}^{nN+n}\left(1+\frac{n\alpha}{k\pi}\right)\right)\left(\prod_{k=1}^{nN}\left(1-\frac{n\alpha}{k\pi}\right)\right)\\
&=\lim_{N\to\infty}\frac{(nN+n)!(nN)!\pi^{n}}{n^{n+2nN}(N!)^{2n}}\frac{n\alpha}{(nN+n)\pi+n\alpha}\left(\prod_{k=1}^{nN}\left(1-\frac{(n\alpha)^2}{k^2\pi^2}\right)\right)\left(\prod_{k=nN+1}^{nN+n}\left(1+\frac{n\alpha}{k\pi}\right)\right)\\
&=\lim_{N\to\infty}\frac{(nN+n)!(nN)!\pi^{n}}{n^{n+2nN}(N!)^{2n}}\frac{1}{(nN+n)\pi+n\alpha}\sin(n\alpha).
\end{aligned}\tag{26}$$

We used the fact that

$$\lim_{N\to\infty}\prod_{k=nN+1}^{nN+n}\left(1+\frac{n\alpha}{k\pi}\right)=1\quad\text{and}\quad\lim_{N\to\infty}(n\alpha)\prod_{k=1}^{nN}\left(1-\frac{(n\alpha)^2}{k^2\pi^2}\right)=\sin(n\alpha).$$

Thus, we only need to calculate

$$L=\lim_{N\to\infty}\frac{(nN+n)!(nN)!\pi^{n}}{n^{n+2nN}(N!)^{2n}}\frac{1}{(nN+n)\pi+n\alpha}.$$

By Stirling's formula, namely

$$\lim_{m\to\infty}\frac{m!}{\sqrt{2\pi m}\left(\frac{m}{\mathrm{e}}\right)^m}=1,$$

where e is the base of the natural logarithm, we could obtain that

$$\begin{aligned}
L&=\lim_{N\to\infty}\frac{\sqrt{2\pi(nN+n)}\left(\frac{nN+n}{\mathrm{e}}\right)^{nN+n}\sqrt{2\pi nN}\left(\frac{nN}{\mathrm{e}}\right)^{nN}\pi^n}{n^{n+2nN}\left(\sqrt{2\pi N}\left(\frac{N}{\mathrm{e}}\right)^{N}\right)^{2n}}\frac{1}{(nN+n)\pi+n\alpha}\\
&=\lim_{N\to\infty}\frac{1}{\mathrm{e}^n}\frac{1}{2^{n-1}}\left(1+\frac{1}{N}\right)^{nN+n}\frac{\pi\sqrt{N(N+1)}}{(N+1)\pi+\alpha}\\
&=\frac{1}{2^{n-1}}.
\end{aligned}\tag{27}$$

Therefore, substituting (27) into (26) yields

$$\prod_{k=0}^{n-1}\sin\left(\alpha+\frac{k\pi}{n}\right)=\frac{\sin(n\alpha)}{2^{n-1}}.$$

Theorem 5 is proved. □

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Remark 2. *According to Theorem 5, if $n \geq 2$ and $\sin\alpha \neq 0$, then we have*

$$\prod_{k=1}^{n-1} \sin\left(\alpha + \frac{k\pi}{n}\right) = \frac{\sin(n\alpha)}{2^{n-1}\sin\alpha}.$$

Letting $\alpha \to 0$ in above equation, we could get that

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}},$$

which is a classical result.

Furthermore, according to the fact that

$$\sin^2 x - \sin^2 y = (\sin(x+y))(\sin(x-y)),$$

where x and y are two real numbers, we could obtain a more general result. Let α and θ be two real numbers and n be a positive integer, then from Theorem 5, we have

$$\begin{aligned}\prod_{k=0}^{n-1}\left(\sin^2\left(\alpha+\frac{k\pi}{n}\right)-\sin^2\theta\right) &= \left(\prod_{k=0}^{n-1}\sin\left((\alpha+\theta)+\frac{k\pi}{n}\right)\right)\left(\prod_{k=0}^{n-1}\sin\left((\alpha-\theta)+\frac{k\pi}{n}\right)\right)\\ &= \frac{\sin(n\alpha+n\theta)}{2^{n-1}}\frac{\sin(n\alpha-n\theta)}{2^{n-1}}\\ &= \frac{\sin^2(n\alpha)-\sin^2(n\theta)}{4^{n-1}},\end{aligned}$$

namely

$$\sin^2(n\theta) - \sin^2(n\alpha) = (-4)^{n-1}\prod_{k=0}^{n-1}\left(\sin^2\theta - \sin^2\left(\alpha+\frac{k\pi}{n}\right)\right).$$

Therefore, we proved that $(\sin^2(n\theta) - \sin^2(n\alpha))$ is a polynomial of degree n about $\sin^2\theta$ with roots $\sin^2(\alpha + \frac{k\pi}{n})$, where $k = 0, 1, \ldots, n-1$. And if $\theta = 0$, then the above result is Theorem 5.

5. Some Examples

As corollaries of our results, we introduce some examples. We will use the following facts.

$$f(x) = \frac{1}{\sin^2 x}, \quad f'(x) = -\frac{2\cos x}{\sin^3 x}, \quad f''(x) = \frac{6}{\sin^4 x} - \frac{4}{\sin^2 x},$$

$$f^{(3)}(x) = \frac{-24\cos x}{\sin^5 x} + \frac{8\cos x}{\sin^3 x}, \quad f^{(4)}(x) = \frac{120}{\sin^6 x} - \frac{120}{\sin^4 x} + \frac{16}{\sin^2 x}.$$

$$g(x) = \frac{1}{\sin x}, \quad g'(x) = -\frac{\cos x}{\sin^2 x}, \quad g''(x) = \frac{2}{\sin^3 x} - \frac{1}{\sin x},$$

$$g^{(3)}(x) = -\frac{6\cos x}{\sin^4 x} + \frac{\cos x}{\sin^2 x}, \quad g^{(4)}(x) = \frac{24}{\sin^5 x} - \frac{20}{\sin^3 x} + \frac{1}{\sin x},$$

$$g^{(5)}(x) = -\frac{120\cos x}{\sin^6 x} + \frac{60\cos x}{\sin^4 x} - \frac{\cos x}{\sin^2 x}.$$

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

Example 1 (see [13]). *Let n be a positive integer and α be a real number such that $\sin(n\alpha) \neq 0$, then we have*

$$\sum_{k=0}^{n-1} \frac{1}{\sin^2(\alpha + \frac{k\pi}{n})} = \frac{n^2}{\sin^2(n\alpha)},$$

$$\sum_{k=0}^{n-1} \frac{1}{\sin^4(\alpha + \frac{k\pi}{n})} = \frac{n^4}{\sin^4(n\alpha)} - \frac{2}{3} \frac{n^4 - n^2}{\sin^2(n\alpha)},$$

$$\sum_{k=0}^{n-1} \frac{1}{\sin^6(\alpha + \frac{k\pi}{n})} = \frac{n^6}{\sin^6(n\alpha)} - \frac{n^6 - n^4}{\sin^4(n\alpha)} + \frac{2}{15} \frac{n^6 - 5n^4 + 4n^2}{\sin^2(n\alpha)}.$$

Proof. Taking $l = 0, 2, 4$ in turn, we could get the following equations from Theorem 1.

$$\sum_{k=0}^{n-1} \frac{1}{\sin^2(\alpha + \frac{k\pi}{n})} = \frac{n^2}{\sin^2(n\alpha)},$$

$$\sum_{k=0}^{n-1} \left(\frac{6}{\sin^4(\alpha + \frac{k\pi}{n})} - \frac{4}{\sin^2(\alpha + \frac{k\pi}{n})} \right) = n^4 \left(\frac{6}{\sin^4(n\alpha)} - \frac{4}{\sin^2(n\alpha)} \right),$$

$$\sum_{k=0}^{n-1} \left(\frac{120}{\sin^6(\alpha + \frac{k\pi}{n})} - \frac{120}{\sin^4(\alpha + \frac{k\pi}{n})} + \frac{16}{\sin^2(\alpha + \frac{k\pi}{n})} \right) \\ = n^6 \left(\frac{120}{\sin^6(n\alpha)} - \frac{120}{\sin^4(n\alpha)} + \frac{16}{\sin^2(n\alpha)} \right).$$

Combining the above three equations yields the desired results. □

Example 2 (see [12]). *Let n be a positive integer with $n \geq 2$, then we have*

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2(\frac{k\pi}{n})} = \frac{n^2 - 1}{3},$$

$$\sum_{k=1}^{n-1} \frac{1}{\sin^4(\frac{k\pi}{n})} = \frac{n^4 + 10n^2 - 11}{45},$$

$$\sum_{k=1}^{n-1} \frac{1}{\sin^6(\frac{k\pi}{n})} = \frac{2n^6 + 21n^4 + 168n^2 - 191}{945}.$$

Proof. Taking $l = 0, 1, 2$ in turn, we could get the following equations from Theorem 2.

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2(\frac{k\pi}{n})} = \frac{n^2 - 1}{3},$$

$$\sum_{k=1}^{n-1} \left(\frac{6}{\sin^4(\frac{k\pi}{n})} - \frac{4}{\sin^2(\frac{k\pi}{n})} \right) = \frac{2(n^4 - 1)}{15},$$

$$\sum_{k=1}^{n-1} \left(\frac{120}{\sin^6(\frac{k\pi}{n})} - \frac{120}{\sin^4(\frac{k\pi}{n})} + \frac{16}{\sin^2(\frac{k\pi}{n})} \right) = \frac{16(n^6 - 1)}{63}.$$

By combining the above three equations, we immediately obtain the final results. □

Example 3 (see [13]). *Let n be a positive odd number and α be a real number such that $\sin(n\alpha) \neq 0$, then we have*

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{\sin(\alpha + \frac{k\pi}{n})} = \frac{n}{\sin(n\alpha)},$$

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{\sin^3(\alpha + \frac{k\pi}{n})} = \frac{n^3}{\sin^3(n\alpha)} - \frac{1}{2} \frac{n^3 - n}{\sin(n\alpha)},$$

$$\sum_{k=0}^{n-1}\frac{(-1)^k}{\sin^5\left(\alpha+\frac{k\pi}{n}\right)}=\frac{n^5}{\sin^5(n\alpha)}-\frac{5}{6}\frac{n^5-n^3}{\sin^3(n\alpha)}+\frac{1}{24}\frac{n^5-10n^3+9n}{\sin(n\alpha)}.$$

Proof. Taking $l=0,2,4$ in turn, we could get the following equations from Theorem 3.

$$\sum_{k=0}^{n-1}\frac{(-1)^k}{\sin\left(\alpha+\frac{k\pi}{n}\right)}=\frac{n}{\sin(n\alpha)},$$

$$\sum_{k=0}^{n-1}(-1)^k\left(\frac{2}{\sin^3\left(\alpha+\frac{k\pi}{n}\right)}-\frac{1}{\sin\left(\alpha+\frac{k\pi}{n}\right)}\right)=n^3\left(\frac{2}{\sin^3(n\alpha)}-\frac{1}{\sin(n\alpha)}\right),$$

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^k\left(\frac{24}{\sin^5\left(\alpha+\frac{k\pi}{n}\right)}-\frac{20}{\sin^3\left(\alpha+\frac{k\pi}{n}\right)}+\frac{1}{\sin\left(\alpha+\frac{k\pi}{n}\right)}\right)\\=&n^5\left(\frac{24}{\sin^5(n\alpha)}-\frac{20}{\sin^3(n\alpha)}+\frac{1}{\sin(n\alpha)}\right).\end{aligned}$$

Combining the above three equations immediately yields the results. □

Example 4 (see [12]). *Let n be an odd number with $n\geq 3$, then we have*

$$\sum_{k=1}^{n-1}(-1)^{k-1}\frac{\cos\left(\frac{k\pi}{n}\right)}{\sin^2\left(\frac{k\pi}{n}\right)}=\frac{n^2-1}{6},$$

$$\sum_{k=1}^{n-1}(-1)^{k-1}\frac{\cos\left(\frac{k\pi}{n}\right)}{\sin^4\left(\frac{k\pi}{n}\right)}=\frac{7n^4+10n^2-17}{360},$$

$$\sum_{k=1}^{n-1}(-1)^{k-1}\frac{\cos\left(\frac{k\pi}{n}\right)}{\sin^6\left(\frac{k\pi}{n}\right)}=\frac{31n^6+147n^4+189n^2-367}{15120}.$$

Proof. Taking $l=0,1,2$ in turn, we could get the following equations from Theorem 4.

$$\sum_{k=1}^{n-1}(-1)^{k-1}\frac{\cos\left(\frac{k\pi}{n}\right)}{\sin^2\left(\frac{k\pi}{n}\right)}=\frac{n^2-1}{6},$$

$$\sum_{k=1}^{n-1}(-1)^{k-1}\left(\frac{6\cos\left(\frac{k\pi}{n}\right)}{\sin^4\left(\frac{k\pi}{n}\right)}-\frac{\cos\left(\frac{k\pi}{n}\right)}{\sin^2\left(\frac{k\pi}{n}\right)}\right)=\frac{7(n^4-1)}{60},$$

$$\sum_{k=1}^{n-1}(-1)^{k-1}\left(\frac{120\cos\left(\frac{k\pi}{n}\right)}{\sin^6\left(\frac{k\pi}{n}\right)}-\frac{60\cos\left(\frac{k\pi}{n}\right)}{\sin^4\left(\frac{k\pi}{n}\right)}+\frac{\cos\left(\frac{k\pi}{n}\right)}{\sin^2\left(\frac{k\pi}{n}\right)}\right)=\frac{31(n^6-1)}{126}.$$

Therefore, the final results follow directly from the combination of the above three equations. □

6. Conclusions and Discussions

In this paper, we used infinite series and infinite product to derive some trigonometric identities. Their forms are concise and elegant. However, we did not discuss whether infinite series can be used to derive identities for tangent or cotangent function. Some known results for tangent or cotangent function are also very elegant, such as [10,12,13]. Therefore, we believe this is a question worth exploring, and more interesting methods and results are sure to emerge.

Author Contributions: Conceptualization, Y.Z.; methodology, Y.Z.; validation, Y.Z.; formal analysis, Y.Z.; investigation, Y.Z.; resources, Y.Z.; writing—original draft preparation, Y.Z.; writing—review and editing, Y.Z. and F.G.; supervision, F.G.; project administration, F.G.; funding acquisition, F.G. All authors have read and agreed to the published version of the manuscript.

doi:10.20944/preprints202512.2679.v1

Funding: This research was funded by Chizhou University High level Talent Research Start up Fund (CZ2022YJRC08), Project of Chizhou University (CZ2023ZRZ03) and Anhui Provincial Science Research Projects (2024AH051355).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

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