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Article

# Is Weniger's Transformation Capable to Simulate Stieltjes Function Branch Cut?

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## Abstract

The resummation of Stieltjes series remains a key challenge in mathematical physics, especially when Padé approximants fail, as in the case of superfactorially divergent series. Weniger's  $\delta$ -transformation, which incorporates a priori structural information on Stieltjes series, namely the inverse factorial series representation of their converging factors, offers a superior framework with respect to Padé. Here, the problem of the pole distribution of the  $\delta$ -transformation is addressed. We show that the algebraic structure of the transformation, together with the intrinsic log-concavity of Stieltjes moments, satisfy the necessary conditions for having real poles. Moreover, by recasting the denominator of the  $\delta$ -transformation rational approximant as a high-order derivative of a log-concave polynomial and invoking the Gauss-Lucas theorem, a possible geometrical justification of the pole positioning along the negative real axis is proposed. While a fully rigorous proof remains an open challenge, our conjecture is substantiated by a comprehensive numerical investigation across an extensive catalog of Stieltjes series. In particular, our results provide systematic evidence that the mandatory branch cut conditions are respected even in the more delicate case of superfactorial growth, recently addressed from a converging factor perspective.

**Keywords:** mathematical physics; divergent series; stieltjes series; converging factors

**MSC:** 40A05; 65B10

## 1. Introduction

The resummation of Stieltjes series remains a pivotal challenge in mathematical physics, from strongly divergent expansions [1–3], to centenary problems in Celestial Mechanics [4,5]. Conventionally, this task is entrusted to Padé approximants, customarily implemented via Wynn's epsilon algorithm [6]. While a robust convergence theory exists for Padé approximants to Stieltjes series (see for instance Baker and Graves-Morris [7]), this supremacy is not absolute. In some important cases, such as the octic anharmonic oscillator, Padé-based methods fail to retrieve the correct energy spectrum. These limitations of Padé approximants stem from their inability to incorporate important *a priori* structural information regarding the character of the Stieltjes series. For example, Stieltjes functions truncation errors are governed by converging factors that must satisfy a linear first-order difference equation [8]. Since such equations are naturally solved by inverse factorial series, any resummation tool capable of integrating this structural feature possesses a decisive advantage. Weniger's  $\delta$ -transformation [9], was specifically engineered to exploit this property, by representing converging factors through inverse factorial expansions. Despite its computational superiority, a rigorous convergence theory for the  $\delta$ -transformation to Stieltjes series has not yet been achieved. For such a convergence theory to be constructed, a fundamental preliminary problem has to be addressed. The convergence of rational approximants of Stieltjes series is fundamentally dictated by the requirement that their poles accumulate along the corresponding Stieltjes function branch cuts [7]. Accordingly, also the  $\delta$ -transformation singularities must consistently reproduce the singularity pattern. This leads to the main question of the present paper:

is Weniger's transformation able to simulate the Stieltjes function branch cut?

We show that the algebraic structure of the  $\delta$ -transformation and the intrinsic log-concavity of Stieltjes moments (which is a direct consequence of Hankel's determinant positivity) guarantee that the necessary condition for the poles to be real is satisfied. Unfortunately, such a condition is not sufficient. Then, a geometrical interpretation of the denominator of the  $\delta$  rational approximant is here proposed, by recasting the latter as a high-order derivative of a simpler, still log-concave polynomial. This, in turn, allows us to invoke the Gauss-Lucas theorem to justify and visualize the geometric mapping of its (possibly complex) zeros onto the negative real axis. In order to substantiate our analytical approach, a comprehensive numerical investigation across an extensive catalog of Stieltjes series is offered to our readers. This catalog specifically targets also those Stieltjes series exhibiting a superfactorial growth recently explored in [10], for which traditional techniques based on Padé approximants are known to fail.

The paper is structured as follows: much of the material presented in Sec. 2 is drawn from my previous works, specifically the 2015 article that I have co-authored with Ernst Joachim Weniger [11] and those recently published between 2024 and 2025 about the converging factors of Stieltjes series [5, 8,10]. It has been summarized and rearranged uniquely to give enough self-consistency to the present paper. Readers which are more interested in the history of Levin-type sequence transformations are encouraged to go through the original publications that will be cited in the rest of the paper. Section 3 represents the core of the paper, where the main analytical strategy to address the pole distribution problem is outlined, while in Sec. 4 a catalogue of different classes of Stieltjes series is offered to our readers to check the validity of the main conjecture of the present work. Finally, a few conclusive words are given in Sec. 5. A couple of appendices containing the most tedious mathematical steps accompanies the paper.

## 2. Why Should Weniger's Transformation be Fit for Decoding Stieltjes Series?

Consider a nondecreasing, real-valued function  $\mu(t)$  defined for  $t \in [0, \infty]$ , possessing infinitely many points of increase. This ensures that the associated measure, say  $d\mu$ , is positive on  $[0, \infty)$ . It will also assumed that all moments,

$$\mu_m = \int_0^\infty t^m d\mu, \quad m \geq 0, \quad (1)$$

are finite and positive. Then, the formal power series

$$\sum_{m=0}^{\infty} \frac{(-)^m}{z^{m+1}} \mu_m, \quad (2)$$

is called a *Stieltjes series*. Such series turns out to be asymptotic, in the sense of Poincaré, for  $z \rightarrow \infty$ , to the function  $f(z)$  defined as

$$f(z) = \int_0^\infty \frac{d\mu}{z+t}, \quad |\arg(z)| < \pi, \quad (3)$$

which turns out to be analytic in the complex plane cut along the negative real axis (i.e.,  $\mathbb{C} \setminus (-\infty, 0]$ ), and is called *Stieltjes function*. Another way to express the above link is that  $f$  and  $\mu$  are linked by a *Stieltjes transform* [12].

The probably most known example of Stieltjes series is the Euler series [13], characterized by the moment sequence  $\{\mu_m = m!\}_{m=0}^\infty$ , and asymptotic to so-called Euler integral,

$$\int_0^\infty \frac{\exp(-t) dt}{z+t} = \exp(z) \Gamma(0, z), \quad |\arg(z)| < \pi, \quad (4)$$

which has the form given in Eq. (3) with  $d\mu = \exp(-t)dt$ . Decoding the asymptotic series in Eq. (2) to retrieve the correct value of  $f(z)$  is known as the Stieltjes moment problem. A sufficient criterion to guarantee unicity to the solution of the moment problem is the so-called Carleman condition, which requires that the following series:

$$\sum_{m=0}^{\infty} \mu_m^{-\frac{1}{2m}}, \quad (5)$$

be divergent. An important necessary condition for a given sequence  $\{\mu_n\}_{n=0}^{\infty}$  to represent the moment sequence of a Stieltjes series is the following: let  $\{u_n\}_{n=0}^{\infty}$  be a sequence. The Hankel determinants  $H_k(u_n)$  of this sequence are defined as follows (see for example [14, pp. 78 and 80]):

$$H_0(u_n) = 1, \quad H_1(u_n) = u_n, \quad n \in \mathbb{N}_0, \quad (6a)$$

$$H_k(u_n) = \begin{vmatrix} u_n & u_{n+1} & \cdots & u_{n+k-1} \\ u_{n+1} & u_{n+2} & \cdots & u_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n+k-1} & u_{n+k} & \cdots & u_{n+2k-2} \end{vmatrix}, \quad k \geq 2, \quad n \in \mathbb{N}_0, \quad (6b)$$

Hankel determinants play a very important role in the theory of Stieltjes series. A necessary condition, that a power series of the type of (2) is indeed a Stieltjes series, is that the Hankel determinants  $H_k(\mu_n)$  of the Stieltjes moments sequence are positive for all  $k, n \geq 0$  [7, Theorem 5.1.2].

Any Stieltjes function  $f(z)$  can be expressed as the sum of the  $n$ th-order partial sum of the associated asymptotic series (2) and of a truncation error which has itself the form of a Stieltjes integral (see for example [9, Theorem 13-1]). More precisely, we have

$$f(z) = f_n(z) + r_n(z), \quad (7)$$

where  $f_n(z)$  denotes the  $n$ th-order partial sum,

$$f_n(z) = \sum_{m=0}^n \frac{(-1)^m}{z^{m+1}} \mu_m, \quad (8)$$

and the symbol  $r_n(z)$  denotes the  $n$ th-order remainder, which is formally defined by

$$r_n(z) = \left(-\frac{1}{z}\right)^{n+1} \int_0^{\infty} \frac{t^{n+1} d\mu}{z+t}, \quad |\arg(z)| < \pi. \quad (9)$$

It is worth recasting the truncation error as follows:

$$r_n(z) = \frac{(-)^{n+1}}{z^{n+1}} \mu_{n+1} \varphi_{n+1}(z), \quad (10)$$

where the quantity

$$\varphi_m(z) = \frac{1}{\mu_m} \int_0^{\infty} t^m \frac{d\mu}{t+z}, \quad m \in \mathbb{N}_0, \quad |\arg(z)| < \pi, \quad (11)$$

will be called the  $m$ th-order *converging factor* [15,16].<sup>1</sup> The search of techniques aimed at estimating convergence factors without resorting to the numerical evaluation of the integral in Eq. (11), played a role of pivotal importance in the development of Stieltjes asymptotic series decoding.

In particular, the modern era of sequence transformations started with two seminal articles by Shanks [17] and Wynn [6], respectively. Shanks introduced in 1955 a powerful sequence transforma-

<sup>1</sup> Actually, the definition of the converging factor  $\varphi_n$  used here differs by the classical definition by a factor  $z$ . This has been done for making the subsequent calculations easier.

tion aimed at computing Padé approximants. Wynn showed in 1956 that the Shanks transformation (thus also Padé approximants) can be computed effectively by means of a nonlinear recursive algorithm, the celebrated Wynn  $\varepsilon$ -algorithm [18, §3.9(iv) Shanks' Transformation]. To obtain approximations of the function  $f(z)$ , the  $\varepsilon$ -algorithm needs only the input of the numerical values of a finite substring of the partial sum sequence  $\{f_n(z)\}_{n=0}^{\infty}$ . In the case of Stieltjes series, however, important *a priori* additional informations on the index dependence of the truncation error are available, as shown for example by Eq. (9). Such *structural* information could then be employed to improve the efficiency of the transformation process. From Eq. (11), it appears that the value of the Stieltjes function  $f(z)$  could be retrieved, in principle, from the knowledge of only a *finite number* of single terms of the associated Stieltjes series, provided that the corresponding converging factor  $\varphi_{n+1}(z)$  could be estimated, in some way, starting from the knowledge of the *sole* moment sequence  $\{\mu_n\}_{n=0}^{\infty}$ . Levin-type transformation theory [19] is ultimately based on the research of suitable approximation models for the converging factor  $\varphi_n(z)$ .

In the following, we shall denote  $\varphi_n^{(k)}(z)$  the  $k$ th-order approximation (with  $k > 1$ ) of  $\varphi_n(z)$ , in such a way that, in some limiting sense, it could be possible to write

$$\lim_{k \rightarrow \infty} \varphi_n^{(k)}(z) = \varphi_n(z). \quad (12)$$

Converging factor approximants  $\varphi_n^{(k)}(z)$  are built up in order to contain only  $k$  unspecified parameters, occurring *linearly* within them. Accordingly, a systematic approach for the construction of Levin-type sequence transformations boils down to find suitable *linear* operators, say  $\hat{T}_k$ , which are able to *annihilate* the approximant itself,

$$\hat{T}_k \{ \varphi_n^{(k)} \} = 0, \quad (13)$$

for fixed  $k$  but for all  $n \in \mathbb{N}_0$ . In [9, Sections 7 - 9], E. J. Weniger showed how simple and powerful sequence transformations can systematically be obtained on using annihilation operators based upon the finite difference operator  $\Delta$ . More precisely, the operator  $\hat{T}_k \{ \cdot \}$  is written as follows:

$$\hat{T}_k \{ \cdot \} = \Delta^k \{ P_{k-1}(n) \cdot \}, \quad (14)$$

where the symbol  $P_{k-1}(n)$  denotes a polynomial of degree  $k - 1$  with respect the integer variable  $n$  [20, Section II], while the iterated difference operator  $\Delta^k$  can be explicitated through the help of [18, Eq. (25.1.1)], i.e.,

$$\Delta^k g(n) = (-1)^k \sum_{j=0}^k (-1)^j \binom{k}{j} g(n+j), \quad k \in \mathbb{N}. \quad (15)$$

As a consequence, it should be clear that the functional form of the convergence factor approximant  $\varphi_n^{(k)}(z)$  must be set in order for the product  $P_{k-1}(n)\varphi_n^{(k)}(z)$  to reduce itself to a  $n$ -polynomial having *degree less than  $k$* , and thus ready to be annihilated by  $\Delta^k$ .

Although it is not possible to establish general strategies aimed at guessing the mathematical structure of the converging factors, for the class of Stieltjes series it has been proved [8] that the converging factor in Eq. (11) can always be represented as an *inverse factorial series*. The proof was ultimately based on the fact that: (i) the converging factor defined in Eq. (11) must necessarily satisfy the following first-order difference equation [8]:

$$\varphi_{n+1} = \frac{\mu_n}{\mu_{n+1}} (1 - z \varphi_n), \quad n \geq 0, \quad (16)$$

and that (ii) *inverse factorial* series constitute a natural tool for solving difference equations, similarly as *inverse power* series are customarily used to solve differential equations. For reader's convenience, it is worth reminding that the basic definitions and properties of factorial series can be found, for

instance, in recent extensive reviews, like for instance [8,11,21]. In particular, in [8] it was shown that the solution of Eq. (16) can *always* be set in the following form:

$$\varphi_n(z) = \sum_{j=0}^{\infty} \frac{a_j}{(n+\beta)_j}, \quad n \in \mathbb{N}_0, \quad (17)$$

where  $\beta > 0$  and  $\{a_k\}_{k=0}^{\infty}$  denotes a sequence which is *independent* of  $n$ . The series in the right side of Eq. (17) is the mathematical object known as *inverse factorial* expansion, with the symbol  $(n+\beta)_j$  denoting Pochhammer symbol. From Eqs. (12) and (17), it would then be natural to conclude that a very suitable model for the  $k$ th-order approximant of the converging factor  $\varphi_n(z)$  of a typical Stieltjes series should be written as follow:

$$\varphi_n^{(k)}(z) = \sum_{j=0}^{k-1} \frac{a_j}{(n+\beta)_j}, \quad k > 1, \quad (18)$$

for any  $n \in \mathbb{N}$ . As a consequence, the annihilation operator  $\hat{T}_k$  in Eq. (14) would immediately follow on taking into account the fact that the following quantity:

$$(n+\beta)_{k-1} \sum_{j=0}^{k-1} \frac{a_j}{(n+\beta)_j} = \sum_{j=0}^{k-1} \frac{(n+\beta)_{k-1}}{(n+\beta)_j} a_j, \quad (19)$$

represents itself a polynomial of degree  $k-1$  in  $n$ , and thus can be annihilated simply by  $\Delta^k$ . Then, it is sufficient to let  $P_{k-1}(n) = (n+\beta)_{k-1}$  in Eq. (14) to have

$$\hat{T}_k\{\cdot\} = \Delta^k\{(n+\beta)_{k-1}\cdot\}. \quad (20)$$

Then, from Eqs. (7) - (10) we have

$$f(z) - f_n(z) = z \frac{(-)^{n+1}}{z^{n+2}} \mu_{n+1} \varphi_{n+1}(z) = z \Delta f_n(z) \varphi_{n+1}(z), \quad (21)$$

where in the last step use has been made of Eq. (8), i.e.,

$$\Delta f_n(z) = f_{n+1}(z) - f_n(z) = \frac{(-1)^{n+1}}{z^{n+2}} \mu_{n+1}. \quad (22)$$

On replacing the quantity  $\varphi_{n+1}(z)$  with its  $k$ th-order approximant  $\varphi_{n+1}^{(k)}(z)$ , it follows at once:

$$z \varphi_{n+1}^{(k)} \simeq \frac{f}{\Delta f_n} - \frac{f_n}{\Delta f_n}, \quad (23)$$

and on applying the annihilation operator  $\hat{T}_k$  to both sides of Eq. (23), we finally arrive to

$$0 = \hat{T}_k\{z \varphi_{n+1}^{(k)}\} \simeq \hat{T}_k\left\{\frac{1}{\Delta f_n}\right\} f - \hat{T}_k\left\{\frac{f_n}{\Delta f_n}\right\}, \quad (24)$$

or, equivalently,

$$f(z) \simeq \frac{\hat{T}_k\left\{\frac{f_n(z)}{\Delta f_n(z)}\right\}}{\hat{T}_k\left\{\frac{1}{\Delta f_n(z)}\right\}}. \quad (25)$$

More rigorously, the right side of Eq. (25) should be interpreted, once  $n$  and  $\beta$  have been given, as the  $k$ th-order term of a *new sequence*, say  $\{\delta_k^{(n)}(\beta)\}_{k=0}^\infty$ , defined by

$$\delta_k^{(n)}(\beta) = \frac{\Delta^k \left\{ (n + \beta)_{k-1} \frac{f_n(z)}{\Delta f_n(z)} \right\}}{\Delta^k \left\{ (n + \beta)_{k-1} \frac{1}{\Delta f_n(z)} \right\}}, \quad \beta > 0, n \in \mathbb{N}_0, \quad (26)$$

where use has been made of Eq. (20). Equation (26) defines the so-called *Weniger*, or *delta* transformation, of the original Stieltjes series partial sum sequence  $\{f_n(z)\}_{n=0}^\infty$  [18, Chapter 3.9(v) Levin's and Weniger's Transformations]. E. J. Weniger first employed his  $\delta$ -transformation for the evaluation of auxiliary functions in molecular electronic structure calculations [22]. Later, it was successfully used for the evaluation of special functions [9,23–32], the summation of divergent perturbation expansions [20,26–28,33–46], as well as for the prediction of unknown perturbation series coefficients [1,38,39,44]. In the last fifteen years,  $\delta$ -transformation has also been employed in optics in the study of nonparaxial free-space propagation of optical wavefields [47–51], as well as in the numerical evaluation of several types of stable and unstable diffraction catastrophes [52–60].

The relevance of Weniger's transformation in the decoding process of divergent Stieltjes asymptotic series has been put into evidence about ten years ago in [11], where it was rigorously proved that the  $\delta$  sequence in Eq. (26) evaluated at  $n = 0$  and  $\beta = 1$ , once applied to the partial sum sequence of the Euler series, *does converge* to the Euler integral (4), i.e.,

$$\lim_{k \rightarrow \infty} \delta_k^{(0)}(1) = f(z), \quad (27)$$

and that, in accomplishing such task, it turns out to be “exponentially faster” than Padé approximants.

E. J. Weniger and I thought that trying to conceive a general convergence theory for the resummation of Stieltjes series through Levin-type transformations, similar to that already existing for Padé approximants [7], could have been an ambitious scientific project, worthy of being pursued. On August 10th, 2022, sadly too soon, E. J. Weniger passed away. Since then, I am trying to carry on our joint project with particular care to his  $\delta$  transformation which, differently from other Levin-type transformations, continues to reveal a precious source of interesting analytical as well as numerical results. Weniger's transformation proved to be able to decode divergent Stieltjes series which do not satisfy Carleman's condition in Eq. (5), as those arising from perturbative treatments of high-order anharmonic quantum oscillators, where Padé unavoidably failed. In [10], a deep analysis on the converging factors of an important class of superfactorially divergent Stieltjes series related to such problem, has been carried out on the basis of the theoretical results established in [8] about the solution of the difference equation recalled in Eq. (16). These results corroborate our feeling that  $\delta$ -transformation in Eq. (26) could be able to potentially substitute Padé approximants as the principal Stieltjes series decoding tool. To this end, however, a preliminary fundamental approximation problem has to be addressed and solved. The next section (Sec. 3) which represents the core of the present paper, is devoted to present the problem and to delineate a possible strategy for its resolution. In Sec. 4, an important catalogue of Stieltjes series will be offered to check and validate the main conclusions drawn in Sec. 3.

### 3. Is Weniger's Transformation Capable to Simulate the Stieltjes Function Branch Cut?

#### 3.1. Preliminaries

Stieltjes functions (or transforms) in Eq. (3) are defined throughout the whole complex plane but a cut along the *negative* real axis  $z < 0$ . This, in turn, does imply that any rational approximant of a Stieltjes function, like Padé or  $\delta$ , must necessarily behave coherently with such a prescription. For Padé approximants acting on Stieltjes series, a well established theory is available since several

decades. It is worth reminding the following important quotation from the classical book by Baker and Graves-Morris [7]

*Whether or not the formal asymptotic series of  $f(z)$  has a zero radius of convergence, the Padé approximants of the series are vital for its analysis and are useful for its numerical evaluation [...]. We can prove convergence of the Padé approximants largely because we can prove that the poles of the Padé approximants lie on the cuts of the Stieltjes function.*

Accordingly, it is beyond dispute the fact that a general convergence theory of Weniger's transformation on Stieltjes series would necessarily stem on a similar ground. In other words,

*is it true that all zeros of the denominator of Eq. (26) be confined to the sole negative real axis?*

In the present section, a possible strategy for addressing such problem will be proposed, while in the next section several numerical, as well as analytical evidences of the validity of this conjecture will be illustrated.

To start our analysis, we substitute from Eq. (15) into the denominator of Eq. (26), which gives

$$\Delta^k \left\{ \frac{(n+\beta)_{k-1}}{\Delta f_n} \right\} = (-)^{n+k+1} z^{n+2} Q_k(z), \quad (28)$$

where the  $k$ th-degree polynomial  $Q_k(z)$  is defined as

$$Q_k(z) = \sum_{j=0}^k \binom{k}{j} \frac{(n+\beta+j)_{k-1}}{\mu_{n+1+j}} z^j, \quad (29)$$

and will be the main character of the rest of the history.

### 3.2. A Necessary Condition to Be Satisfied for Simulating the Branch Cut

It is worth introducing, at this point, the following auxiliary  $k$ th-degree polynomial:

$$\mathcal{P}_k(z) = \sum_{j=0}^k \binom{k}{j} \frac{1}{\mu_{n+1+j}} z^j. \quad (30)$$

First of all, we shall prove that, when the sequence  $\{\mu_n\}_{n=0}^{\infty}$  contains the moments of a Stieltjes series, both polynomials  $\mathcal{P}_k(z)$  and  $Q_k(z)$  satisfy Newton's *necessary* condition for all zeros to be real. To this end, it is sufficient to note that the sequence  $\{\mu_n\}_{n=0}^{\infty}$  necessarily satisfies the following condition:

$$\frac{\mu_{n+1+j}^2}{\mu_{n+j} \mu_{n+j+2}} \leq 1, \quad n \geq 0, k \geq 1, \quad j = 1, 2, \dots, k-1, \quad (31)$$

which follows from the positivity of the Hankel determinant  $H_2(\mu_{n+j})$  defined in Eq. (6). In other words, for a typical Stieltjes series, the sequence  $\{1/\mu_n\}_{n=0}^{\infty}$  turns out to be log-concave, which is a necessary condition for the polynomial  $\mathcal{P}_k(z)$  to have all real zeros. Similar considerations can be done for  $Q_k(z)$ , for which the necessary condition becomes

$$\frac{\mu_{n+1+j}^2}{\mu_{n+j} \mu_{n+j+2}} \frac{(n+\beta+j-1)_{k-1} (n+\beta+j+1)_{k-1}}{(n+\beta+j)_{k-1}^2} \leq 1, \quad j = 1, 2, \dots, k-1, \quad (32)$$

which is automatically satisfied on taking Eq. (31) into account and the fact that

$$\frac{(n+\beta+j-1)_{k-1} (n+\beta+j+1)_{k-1}}{(n+\beta+j)_{k-1}^2} = \frac{1 - \frac{1}{n+\beta+j}}{1 - \frac{1}{n+\beta+j+k-1}}, \quad (33)$$

as it is trivial to prove. The condition in Eq. (32) is satisfied by any Stieltjes series, and guarantees that the *necessary* condition for all poles of the rational approximant into Eq. (26) to be real is fulfilled. Unfortunately, such condition is not sufficient.

### 3.3. An Alternative Expression of $\mathcal{Q}_k$ Polynomials

It should be noted how the factor  $(n + \beta + j)_{k-1} z^j$  appearing into the definition of  $\mathcal{Q}_k(z)$  can be recast as follows:

$$(n + \beta + j)_{k-1} z^j = z^{1-n-\beta} \frac{d^{k-1}}{dz^{k-1}} z^{j+n+\beta-1+k-1}, \quad (34)$$

which, once substituted in Eq. (29), after simple algebra leads to

$$\mathcal{Q}_k(z) = z^{1-n-\beta} \frac{d^{k-1}}{dz^{k-1}} \left\{ z^{n+\beta-1+k-1} \mathcal{P}_k(z) \right\}. \quad (35)$$

Henceforth, only for the sake of simplicity, it will be assumed  $n = 0$  and  $\beta = 1$ , which represents the choice customarily employed in most practical applications of Weniger's transformation. In this way, we have

$$\mathcal{Q}_k(z) = \frac{d^{k-1}}{dz^{k-1}} \left\{ z^{k-1} \mathcal{P}_k(z) \right\}, \quad (36)$$

which represents a key relation for the scope of the present paper.

In Ref. [11] we proved that, for the Euler series, the polynomial  $\mathcal{Q}_k(z)$  turns out to be proportional to an hypergeometric polynomial, precisely

$$\mathcal{Q}_k(z) \propto {}_2F_2 \left( \begin{matrix} -k, k \\ 1, 2 \end{matrix}; -z \right), \quad (37)$$

for which it was possible to prove the reality of all its zeros [11, Theorem 5.1].

Thanks to the connection provided by Eq. (36), the exploration of the zero location of a typical polynomial  $\mathcal{Q}_k(z)$  could be done, in principle, on invoking the Gauss-Lucas theorem [61, Ch. 2]. First of all, we note that the polynomial  $z^{k-1} \mathcal{P}_k(z)$  has one zero of multiplicity  $k - 1$  at  $z = 0$  and  $k$  additional zeros coincident with those of  $\mathcal{P}_k(z)$  which, due to the positivity of its coefficients, are located within the open complex half-space  $\text{Re}\{z\} < 0$ . Also, we know that  $\mathcal{P}_k(z)$  satisfies the log-concavity condition given in Eq. (31) which, as previously recalled, is only necessary. In this situation, an even number of zeros of  $\mathcal{P}_k(z)$ , and thus of  $z^{k-1} \mathcal{P}_k(z)$ , could be in principle complex conjugated. This, in turn, implies that the convex hull of the zeros of  $\mathcal{P}_k(z)$  must be symmetric with respect to the real axis, and is contained in the left half-space  $\text{Re}\{z\} < 0$ , with the inclusion of  $z = 0$ . By the Gauss-Lucas theorem, the zeros of each successive derivative in Eq. (36) must be located *inside* the convex hull of the zeros of the previous derivative of  $z^{k-1} \mathcal{P}_k(z)$ . Since all zeros of  $\mathcal{P}_k(z)$  lie in the open left-plane  $\text{Re}\{z\} < 0$ , the same holds for  $\mathcal{Q}_k(z)$ . Furthermore, the multiplicity  $k - 1$  of the zeros at  $z = 0$  decreases by one at each differentiation step and, since by definition we have

$$\mathcal{Q}_k(0) = \frac{(k-1)!}{\mu_1} > 0, \quad (38)$$

$\mathcal{Q}_k(z)$  cannot have zeros at the origin and all its zeros will be contained in  $\text{Re}\{z\} < 0$  as well. Accordingly, only two scenarios are possible: (i) all zeros or (ii) not all zeros of  $\mathcal{P}_k(z)$  are real. In the first scenario, it is immediate to convince that all zeros of  $\mathcal{Q}_k(z)$  will be real and negative. As far as the second scenario is concerned, the problem of proving the reality of zeros of  $\mathcal{Q}_k(z)$  remains open. In the following section, the prescriptions of the Gauss-Lucas theorem will be analyzed on a large catalogue of Stieltjes series.

## 4. Madamina, il catalogo è questo

### 4.1. Preliminaries

In the present section, we present a catalogue of Stieltjes series for which the polynomial  $\mathcal{P}_k(z)$  can often be expressed in closed form, allowing for a rigorous analysis of the distribution of its zeros. In most cases, we show that  $\mathcal{P}_k(z)$  possessed only real zeros and, by virtue of the results of the previous section, that also the polynomial  $\mathcal{Q}_k(z)$  is stable. Moreover, a few numerical examples illustrating situations in which  $\mathcal{P}_k(z)$ , and thus  $z^{k-1}\mathcal{P}_k(z)$ , admits some complex zeros, will also be presented, in which the repeated action of the differentiation operator progressively reduces their number until they are eventually eliminated altogether. The catalogue has mainly been taken from [12, Ch. 12], with the addition of two important examples of Stieltjes series recently addressed [5,10].

### 4.2. A Class of Superfactorially Divergent Stieltjes Asymptotic Series

In Ref. [10], an important class of superfactorially divergent Stieltjes series has been studied as far as their converging factors were concerned. The elements of this class constitute natural generalizations of the Euler series. They are characterized by the following moment sequence  $\{\mu_n\}_{n=0}^{\infty}$ :

$$\mu_n = \Gamma(\nu n + q + 1), \quad n \in \mathbb{N}_0, \quad (39)$$

where  $\nu \in \mathbb{N}$  and  $q \in (-1, 1)$ . The Euler series corresponds to the pair  $(\nu, q) = (1, 0)$ . It should be noted that Carleman's condition is not satisfied for  $\nu > 2$ , and also that Padé are not capable to decode the Stieltjes series to the corresponding Stieltjes function. The numerical analysis carried out in [10], has shown how the inverse factorial expansion of the converging factor in Eq. (17) provided excellent estimates also for values of  $\nu > 3$ . Now, on substituting from Eq. (39) in Eq. (30) we have

$$\mathcal{P}_k(z) = \sum_{j=0}^k \binom{k}{j} \frac{1}{\Gamma[\nu(j+1) + q + 1]} z^j, \quad (40)$$

and in Appendix A it is proved that

$$\mathcal{P}_k(z) = \frac{1}{\Gamma(\nu + q + 1)} {}_1F_{\nu} \left( \begin{matrix} -k \\ 1 + \boldsymbol{\rho} \end{matrix}; \xi \right), \quad (41)$$

where, for simplicity, we set  $\xi = -\frac{z}{\nu}$  and the  $\nu$ -dimensional vector  $\boldsymbol{\rho}$  is defined by

$$\rho_j = \frac{q + j}{\nu}, \quad j = 1, 2, \dots, \nu. \quad (42)$$

Proving that the hypergeometric polynomial in Eq. (41) has only real zeros can be done by using similar arguments to those employed in [11, Theorem 5.1], in particular the connection with the so-called *Pólya frequency functions* [62,63]. In particular, since in [62, Theorem 4.1] it was shown that the generalized hypergeometric series

$${}_pF_q \left( \begin{matrix} \alpha_1 + m_1, \dots, \alpha_p + m_p \\ \alpha_1, \dots, \alpha_q \end{matrix}; x \right) = \sum_{j=0}^{\infty} \frac{(\alpha_1 + m_1)_j \cdots (\alpha_p + m_p)_j}{(\alpha_1)_j \cdots (\alpha_q)_j} \frac{x^j}{j!}, \quad (43)$$

with  $p \leq q$ ,  $\alpha_1, \dots, \alpha_q > 0$ , and  $m_1, \dots, m_p \in \mathbb{N}$  is a Pólya frequency function, according to [64, Lemma 5], it follows at once that the associated terminating generalized hypergeometric series

$$\sum_{j=0}^k \frac{(-k)_j (\alpha_1 + m_1)_j \cdots (\alpha_p + m_p)_j}{(\alpha_1)_j \cdots (\alpha_q)_j} \frac{(-x)^j}{j!} = {}_{p+1}F_q \left( \begin{matrix} -k, \alpha_1 + m_1, \dots, \alpha_p + m_p \\ \alpha_1, \dots, \alpha_q \end{matrix}; -x \right), \quad (44)$$

with  $k \in \mathbb{N}$ , has only real zeros. This, in turn, implies that also the polynomial in Eq. (41) has only real zeros.

#### 4.3. Laguerre Distribution

Another generalization of the Euler series is the Stieltjes function whose moment sequence is given by

$$\mu_n = (\alpha)_n, \quad n = 0, 1, 2, \dots, \quad (45)$$

where  $\alpha$  denotes a positive real number [12, Ch. 12]. The Euler series corresponds to the choice  $\alpha = 1$ . Henrici called this distribution *Laguerre distribution*, with the measure  $d\mu$  being given by [12, Eq. (12.12-12)]

$$d\mu = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \exp(-t) dt, \quad t \geq 0. \quad (46)$$

Again, on substituting from Eq. (45) in Eq. (30) it is found that

$$\mathcal{P}(z) = \frac{1}{\alpha} {}_1F_1\left(\begin{matrix} -k \\ 1 + \alpha \end{matrix}; -z\right), \quad (47)$$

which is a particular case of what we have found in Sec. 4.2. We thus conclude that also for the Laguerre distribution, Weniger's transformation is able to correctly simulate the cut.

#### 4.4. The Modified Bessel Function of the Second Kind $K_\nu(z)$

In an important 1990 paper, E. J. Weniger and J. Čížek proved that the modified Bessel function of the second kind,  $K_\nu(z)$ , is a Stieltjes function [32]. More precisely, on limiting only to the interval  $0 \leq \nu < 1/2$  [32], the associated moment sequence turns out to be [32]

$$\mu_n = \frac{\left(\frac{1}{2} + \nu\right)_n \left(\frac{1}{2} - \nu\right)_n}{2^n n!}, \quad n = 0, 1, 2, \dots \quad (48)$$

with the associated measure [32]

$$d\mu = \frac{(2/\pi)^{1/2}}{\Gamma(1/2 + \nu) \Gamma(1/2 - \nu)} t^{-1/2} \exp(-t) K_\nu(t) dt, \quad t \geq 0. \quad (49)$$

In Appendix B it is proved that also in the present case the polynomial  $\mathcal{P}_k(z)$  can be expressed in closed form terms via hypergeometric polynomials, precisely

$$\mathcal{P}_k(z) = \frac{2}{\frac{1}{4} - \nu^2} {}_2F_2\left(\begin{matrix} -k, 2 \\ \frac{3}{2} + \nu, \frac{3}{2} - \nu \end{matrix}; -2z\right). \quad (50)$$

Differently from the previous sections, we were not able to prove the reality of its zeros, and we did not find any proof about it in the past literature. However, several numerical simulations, carried out with the help of Wolfram Mathematica 14.3 for different values of  $\nu$  and for different orders  $k$ , have confirmed our conjecture that the polynomial  $\mathcal{P}_k(z)$  defined in Eq. (50) is stable and, accordingly, also  $\mathcal{Q}_k(z)$ . The rigorous proof remains an open question.

#### 4.5. The Gamma Function

In the above quoted book of Henrici [12, Ch. 12], the Binet formula for the natural logarithm of the Gamma function,  $\log \Gamma(z)$ , is recalled, together with its connection to the Stieltjes series whose moment sequence is [12, Eq. (12.12-18)]

$$\mu_n = \frac{(-1)^n B_{2n+2}}{(2n+1)(2n+2)}, \quad n = 0, 1, 2, \dots \quad (51)$$

where the symbol  $B_m$  denotes the  $m$ th-order Bernoulli number, whose asymptotics gives at once

$$\mu_n = \frac{2}{(2\pi)^{n+1}} \frac{(2n+2)!}{(2n+2)(2n+1)} \zeta(2n+2) \sim \frac{2}{(2\pi)^{n+1}} (2n)!, \quad n \rightarrow \infty, \quad (52)$$

where  $\zeta(z)$  denotes the Riemann zeta function. Accordingly, the moments asymptotically tend to grow with the superfactorially law studied in Sec. 4.2. Several numerical simulations carried out for different values of  $k$  have verified the stability of the corresponding polynomial  $\mathcal{P}_k(z)$  and, consequently, also of  $\mathcal{Q}_k(z)$ .

#### 4.6. Jacobi Distribution

An interesting example of Stieltjes function is offered by the so-called Jacobi distribution, characterized by the following moment sequence [12, Ch. 12]:

$$\mu_n = \frac{(\alpha)_n}{(\gamma)_n}, \quad n \in \mathbb{N}_0, \quad (53)$$

where  $0 < \alpha < \gamma$ . In this case, both polynomials  $\mathcal{P}_k(z)$  and  $\mathcal{Q}_k(z)$  can be evaluated in closed form by using Wolfram Mathematica 14.3, which gives at once

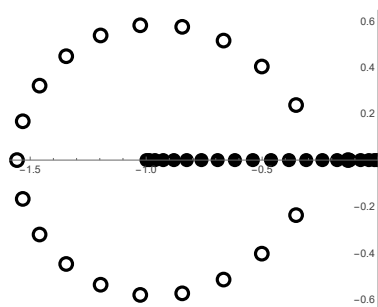
$$\mathcal{P}_k(z) = \sum_{j=0}^k \binom{k}{j} \frac{(\gamma)_{j+1}}{(\alpha)_{j+1}} z^j = \frac{\gamma}{\alpha} {}_2F_1\left(\begin{matrix} -k, \gamma+1 \\ \alpha+1 \end{matrix}; -z\right). \quad (54)$$

and

$$\mathcal{Q}_k(z) = \sum_{j=0}^k \binom{k}{j} (j+1)_{k-1} \frac{(\gamma)_{j+1}}{(\alpha)_{j+1}} z^j = \frac{\gamma}{\alpha} \Gamma(k) {}_3F_2\left(\begin{matrix} -k, k, \gamma+1 \\ 1, \alpha+1 \end{matrix}; -z\right), \quad (55)$$

respectively. Now, if  $\alpha$  and  $\gamma$  differ by an integer number, the theorem quoted at the end of Sec. 4.2 guarantees that all zeros of  $\mathcal{P}_k(z)$  are real and, consequently, also those of  $\mathcal{Q}_k(z)$ .

If not, it is worth giving a few numerical simulations aimed at confirming our conjecture about the action of the differential operator in Eq. (36). To this end, in Figure 1 the spatial distribution, across the complex plane, of the zeros of the polynomial  $\mathcal{P}_k(z)$  are shown (open circles) for  $k = 20$  and for the pair  $(\alpha, \gamma) = \left(1, \frac{3}{2}\right)$ . In the same figure, also the locations of  $\mathcal{Q}_k(z)$ 's zeros are reported (dots), all of them being real. It is worth monitoring, at least from a visual/numerical perspective, the action of the

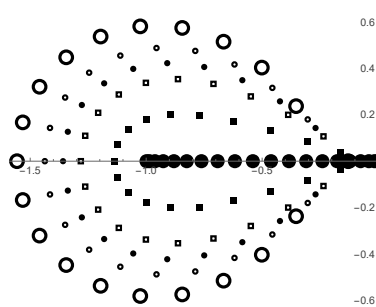


**Figure 1.** Locations, on the complex plane, of the zeros of the polynomial  $\mathcal{P}_k(z)$  (open circles) and of the polynomial  $\mathcal{Q}_k(z)$  (dots), for  $k = 20$  and for the pair  $(\alpha, \gamma) = \left(1, \frac{3}{2}\right)$ .

derivative operator in Eq. (36) in relation with the Gauss-Lucas theorem. To this end, consider the polynomial, say  $\mathcal{R}_{k,m}(z)$ , defined as

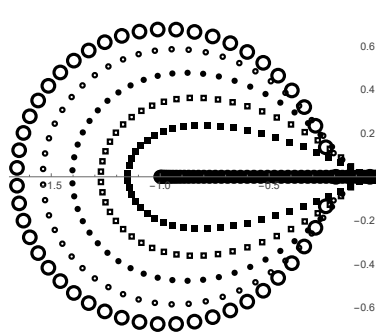
$$\mathcal{R}_{k,m}(z) = \frac{d^m}{dz^m} \left\{ z^{k-1} \mathcal{P}_k(z) \right\}, \quad m = 1, 2, \dots, k-1, \quad (56)$$

in such a way  $Q_k(z) = \mathcal{R}_{k,k-1}(z)$ . In Figure 2, the zero distributions of Figure 1 are plotted together with the distributions of the complex zeros of the polynomial  $\mathcal{R}_{k,m}(z)$  defined in Eq. (56), obtained for  $m = 3$  (small open circles),  $m = 5$  (small black circles),  $m = 7$  (small open squares), and  $m = 11$  (small black squares).



**Figure 2.** The same as in Figure 1, together with the distributions of the complex zeros of the polynomial  $\mathcal{R}_{k,m}(z)$  defined in Eq. (56), for  $m = 3$  (small open circles),  $m = 5$  (small black circles),  $m = 7$  (small open squares), and  $m = 11$  (small black squares).

As a further example, Figure 3 shows the same as in Figure 2 but for the couple  $(\alpha, \gamma) = (1, 1 + \frac{\pi}{3})$  and  $k = 50$ , and similarly for  $m = 7$  (small open circles),  $m = 15$  (small black circles),  $m = 23$  (small open squares), and  $m = 31$  (small black squares). Several other numerical simulations have been carried out, with the help of Wolfram Mathematica 14.3, for different values of  $k$  and  $(\alpha, \gamma)$ , all of them confirming our conjecture that all zeros of  $Q_k(z)$  in Eq. (55) are real.



**Figure 3.** The same as in Figure 2, but for the couple  $(\alpha, \gamma) = (1, 1 + \frac{\pi}{3})$  and  $k = 50$ , and for  $m = 7$  (small open circles),  $m = 15$  (small black circles),  $m = 23$  (small open squares), and  $m = 31$  (small black squares).

#### 4.7. The Bessel Solution of Kepler's Equation

The last example we are going to deal with is purely numerical. In a recent paper [5], a very classical and old problem has been tackled from a new original perspective. The problem consists in solving the following transcendental equation with respect the unknown  $\psi$ :

$$M = \psi - \epsilon \sin \psi, \tag{57}$$

where  $M \in [0, \pi]$  and  $\epsilon \in [0, 1]$  are real positive parameters. Equation (57) is the celebrated elliptic Kepler Equation, which plays a central role in Celestial Mechanics [65]. Among hundreds of different methods that have been conceived along more than four centuries to solve Kepler's equation [4], in Ref. [5] the following Fourier series solution, originally proposed by Friedrich Wilhelm Bessel, namely

$$\psi = M + S(\epsilon; M), \tag{58}$$

where [4] (Ch. 3)

$$S(\epsilon; M) = \sum_{n=1}^{\infty} \frac{2J_n(n\epsilon)}{n} \sin nM, \tag{59}$$

has been addressed under a new perspective. After introducing the complex function  $\mathbb{S}(\epsilon; M)$ , defined through the Kapteyn series

$$\mathbb{S}(\epsilon; M) = \sum_{n=1}^{\infty} \frac{2J_n(n\epsilon)}{n} \exp(inM), \quad (60)$$

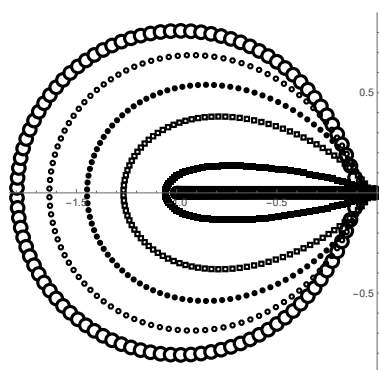
in such a way that  $S = \text{Im}\{\mathbb{S}\}$ , it was shown that  $\mathbb{S}(\epsilon, M)$  is a Stieltjes series [5]. In particular, on introducing the parameter  $\lambda < 0$ , defined as

$$\lambda = \chi + \frac{1}{2} \log \frac{1-\chi}{1+\chi}, \quad (61)$$

where  $\chi = \sqrt{1-\epsilon^2}$ , it was proved that the function  $\mathbb{S}(\epsilon, M)$  defined in Eq. (60) is a Stieltjes series characterized by the following moment sequence [5]:

$$\mu_n = \frac{2J_{n+1}[(n+1)\epsilon]}{(n+1)} \exp[-\lambda(n+1)], \quad n \in \mathbb{N}_0. \quad (62)$$

Differently from what happened for Jacobi's distribution, in the present case it is not possible to express neither  $\mathcal{P}_k(z)$  nor  $\mathcal{Q}_k(z)$  in closed form. Again, all subsequent experiments will be carried out with the help Wolfram Mathematica 14.3. In particular, only two parameters are involved, namely  $k$  and  $\epsilon$ . To illustrate a single example, in Figure 2 the location of the zeros of  $\mathcal{P}_k(z)$  (large open circles) and of  $\mathcal{Q}_k(z)$  (large dots) are reported for  $\epsilon = 99/100$  and for  $k = 100$ . Moreover, in order to show the action of the derivative operator in agreement with the Gauss-Lucas theorem, the same as in Figure 2 is also shown, for  $m = 17$  (small open circles),  $m = 37$  (small black circles),  $m = 57$  (small open squares), and  $m = 83$  (small black squares). Several other simulations, not shown here, have confirmed that, also for the Stieltjes series  $\mathbb{S}(\epsilon, M)$  defined in Eq. (60), all poles of the Weniger rational approximants are real and negative.



**Figure 4.** The same as in Figure 2, but for the Stieltjes series  $\mathbb{S}(\epsilon, M)$  defined in Eq. (60), evaluated at  $\epsilon = 99/100$ , for  $k = 100$ , and for  $m = 17$  (small open circles),  $m = 37$  (small black circles),  $m = 57$  (small open squares), and  $m = 83$  (small black squares).

## 5. Conclusions

Nonlinear sequence transformations in general, and Weniger's  $\delta$ -transformation in particular, are nowadays considered computational tools of great importance for the resummation of several divergent series occurring in applied mathematics and theoretical physics problems. Padé approximants remain a cornerstone for the resummation of Stieltjes series, but their effectiveness drops dramatically within specific contexts, such as for instance the perturbative treatment of the octic quantum anharmonic oscillator, where Carleman's condition is violated. Converging factors of Stieltjes series are represented by inverse factorial series, a fact which poses Weniger's  $\delta$ -transformation as a superior alternative to Padé, which are not designed to incorporate such important *a priori* structural information.

In the present paper, we have addressed a fundamental problem to be solved for building a convergence theory of Weniger's transformation to Stieltjes series. For any rational approximant of a Stieltjes function to be meaningful, its poles must necessarily be located along the branch cut of the function itself, on the real negative half-axis. This mandatory requirement must be satisfied by the  $k$ th-order  $\delta$ -transformation denominator in Eq. (26). To this end, we have shown that the denominator can be recast in the form of the  $(k - 1)$ th-order derivative of the polynomial  $z^{k-1}\mathcal{P}_k(z)$ , where  $\mathcal{P}_k(z)$  is the binomial sum of the inverse of the first  $k$  Stieltjes series moments. Due to the positivity of the Hankel determinants,  $\mathcal{P}_k(z)$  is proven to be log-concave, as well as  $\mathcal{Q}_k(z)$ , which is a necessary (but not sufficient) condition for their zeros to be real. The Gauss-Lucas theorem has then used to visualize and justify the mapping of the (possibly complex) zeros of  $\mathcal{P}_k(z)$  into the (real) zeros of  $\mathcal{Q}_k(z)$ . Although a formal proof that the  $\delta$ -transformation simulates the branch cut of a general Stieltjes series remains an open problem, the empirical evidence provided in the present paper is compelling. All numerical/analytical results found in our extensive catalog, which includes the class of series with superfactorial growth studied in [10], as well as the Stieltjes series recently discovered in [5], confirm that all poles of the  $\delta$ -transformation are restricted to the real axis, thus substantiate the robustness of the method proposed and validate our conjecture about the capability of  $\delta$ -transformation of mimicking the branch cut of the corresponding Stieltjes functions.

## Appendix A. Proof of Eq. (41)

The polynomial in Eq. (40) can be expressed in terms of a terminating generalized hypergeometric series by recasting  $\Gamma[\nu(j + 1) + q + 1]$  in terms of gamma functions depending not on  $\nu\xi$  but on  $\xi$  on using Gauss' multiplication formula, i.e.,

$$\Gamma(\nu\xi) = (2\pi)^{(1-\nu)/2} \nu^{\nu\xi-1/2} \prod_{\lambda=0}^{\nu-1} \Gamma(\xi + \lambda/\nu), \quad \xi \in \mathbb{C}, \quad \nu \in \mathbb{N}. \quad (\text{A1})$$

This, in particular, implies that

$$\begin{aligned} \Gamma[\nu(j + 1) + q + 1] &= \Gamma\left[\nu\left(j + 1 + \frac{q + 1}{\nu}\right)\right] = \\ &= (2\pi)^{(1-\nu)/2} \nu^{\nu(j+1)+q+1/2} \prod_{\lambda=0}^{\nu-1} \Gamma\left(j + 1 + \frac{q + 1 + \lambda}{\nu}\right), \end{aligned} \quad (\text{A2})$$

which, on letting  $j = 0$ , gives at once

$$\Gamma(\nu + q + 1) = \Gamma\left[\nu\left(1 + \frac{q + 1}{\nu}\right)\right] = (2\pi)^{(1-\nu)/2} \nu^{\nu+q+1/2} \prod_{\lambda=0}^{\nu-1} \Gamma\left(1 + \frac{q + 1 + \lambda}{\nu}\right). \quad (\text{A3})$$

Then, from Eq. (A2) we have

$$\begin{aligned} \frac{z^j}{\Gamma[\nu(j + 1) + q + 1]} &= \frac{(2\pi)^{(1-\nu)/2}}{\nu^{\nu(j+1)+q+1/2}} \prod_{\lambda=0}^{\nu-1} \frac{z^j}{\Gamma\left(j + 1 + \frac{q + 1 + \lambda}{\nu}\right)} = \\ &= \frac{(2\pi)^{(1-\nu)/2}}{\nu^{\nu+q+1/2}} \left(\frac{z}{\nu}\right)^j \prod_{\lambda=0}^{\nu-1} \frac{1}{\Gamma\left(1 + \frac{q + 1 + \lambda}{\nu}\right)} \frac{1}{\left(1 + \frac{q + 1 + \lambda}{\nu}\right)_j} = \\ &= \frac{(z/\nu)^j}{\Gamma(\nu + q + 1)} \prod_{\lambda=0}^{\nu-1} \frac{1}{\left(1 + \frac{q + 1 + \lambda}{\nu}\right)_j}, \end{aligned} \quad (\text{A4})$$

where in the last passage use has been made of Eq. (A3). Finally, on substituting from Eq. (A4) in Eq. (40) and on letting  $-z/v^\nu \rightarrow \xi$ , we obtain

$$\mathcal{P}_k(z) = \frac{1}{\Gamma(\nu + q + 1)} \sum_{j=0}^k \binom{k}{j} \frac{(-\xi)^j}{\prod_{\lambda=0}^{\nu-1} \left(1 + \frac{q+1+\lambda}{\nu}\right)_j}, \quad (\text{A5})$$

which, on recalling that

$$\frac{(-k)_j}{j!} = \begin{cases} (-1)^j \binom{k}{j} & 0 \leq j \leq k, \\ 0 & j > k, \end{cases} \quad (\text{A6})$$

can be recast as follows:

$$\begin{aligned} \mathcal{P}_k(z) &= \frac{1}{\Gamma(\nu + q + 1)} \sum_{j=0}^k \frac{\xi^j}{j!} \frac{(-k)_j}{\prod_{\lambda=0}^{\nu-1} \left(1 + \frac{q+1+\lambda}{\nu}\right)_j} = \\ &= \frac{1}{\Gamma(\nu + q + 1)} {}_1F_\nu \left( \begin{matrix} -k \\ 1 + \frac{q+1}{\nu}, \dots, 1 + \frac{q+\nu}{\nu} \end{matrix}; \xi \right). \end{aligned} \quad (\text{A7})$$

## Appendix B. Proof of Eq. (50)

On substituting from Eq. (48) in Eq. (40) we have

$$\mathcal{P}_k(z) = 2 \sum_{j=0}^k \binom{k}{j} \frac{(j+1)!}{\left(\frac{1}{2} + \nu\right)_{j+1} \left(\frac{1}{2} - \nu\right)_{j+1}} (2z)^j. \quad (\text{A8})$$

Now, since  $(j+1)! = (2)_j$  and

$$\begin{aligned} \left(\frac{1}{2} \pm \nu\right)_{j+1} &= \frac{\Gamma\left(\frac{1}{2} \pm \nu + j + 1\right)}{\Gamma\left(\frac{1}{2} \pm \nu\right)} = \frac{\Gamma\left(\frac{3}{2} \pm \nu + j\right)}{\Gamma\left(\frac{1}{2} \pm \nu\right)} = \\ &= \frac{\Gamma\left(\frac{3}{2} \pm \nu\right) \Gamma\left(\frac{3}{2} \pm \nu + j\right)}{\Gamma\left(\frac{1}{2} \pm \nu\right) \Gamma\left(\frac{3}{2} \pm \nu\right)} = \left(\frac{1}{2} \pm \nu\right) \left(\frac{3}{2} \pm \nu\right)_j, \end{aligned} \quad (\text{A9})$$

after rearranging and simplifying we finally obtain

$$\mathcal{P}_k(z) = \frac{2}{\frac{1}{4} - \nu^2} \sum_{j=0}^k \binom{k}{j} \frac{(2)_j}{\left(\frac{3}{2} + \nu\right)_j \left(\frac{3}{2} - \nu\right)_j} (2z)^j, \quad (\text{A10})$$

or, on using again Eq. (A6),

$$\begin{aligned} \mathcal{P}_k(z) &= \frac{2}{\frac{1}{4} - \nu^2} \sum_{j=0}^k \frac{(-k)_j (2)_j}{\left(\frac{3}{2} + \nu\right)_j \left(\frac{3}{2} - \nu\right)_j} \frac{(-2z)^j}{j!} = \\ &= \frac{2}{\frac{1}{4} - \nu^2} {}_2F_2 \left( \begin{matrix} 3 \\ \frac{3}{2} + \nu, \frac{3}{2} - \nu \end{matrix}; -2z \right) \quad \square \end{aligned} \quad (\text{A11})$$

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