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Posted Date: 29 December 2025

doi: 10.20944/preprints202512.2528.v1

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Abstract

Fuzzy set theory enriches classical sets by assigning to each element a graded membership in $[0,1]$, thereby capturing partial inclusion and uncertainty. The notion of an *Uncertain Set* further abstracts this idea by allowing membership to take values in a general degree-domain, providing a unified language that subsumes fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related models. On the algebraic side, a *hyperlattice* replaces one lattice operation by a multivalued hyperoperation, enabling the representation of ambiguous or non-deterministic combinations, while a *superhyperlattice* iterates this structure through powerset lifting to obtain higher-order layers of interaction. Motivated by these developments, we introduce *HyperLattice-valued* and *SuperHyperLattice-valued Uncertain Sets* as lattice-valued uncertainty frameworks whose degrees range over hyperlattices and their superextensions. We establish basic definitions, show that the proposed formalisms generalize existing lattice-valued models (including L-fuzzy, L-neutrosophic, and L-plithogenic sets), and discuss fundamental structural properties and canonical embeddings between the resulting classes.

Keywords: hyperlattice; fuzzy set; neutrsophic set; L-fuzzy set; superhyperlattice; uncertain set

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1. Preliminaries

This section fixes notation and recalls the basic concepts used throughout the paper. All sets considered below are assumed to be non-empty unless explicitly stated otherwise.

1.1. Fuzzy, Neutrosophic, and Plithogenic Set

In decision-making and knowledge representation, uncertainty may appear as vagueness, partial truth, or incomplete information. To formalize such phenomena, several set-theoretic paradigms have been proposed. Fuzzy sets, introduced by Zadeh, replace crisp membership by a graded membership degree in the unit interval [1–3]. Neutrosophic sets, proposed by Smarandache, further decompose membership into three components-truth, indeterminacy, and falsity-thereby providing a convenient language for explicitly modeling indeterminate information [4–7]. These notions have been extended to concepts such as hesitant neutrosophic sets [8], local neutrosophic sets [9], superhyperneutrosophic sets [10,11], q-rung orthopair neutrosophic sets [12,13], and Pythagorean neutrosophic sets [14,15], and their properties have been studied. More recently, plithogenic sets (also due to Smarandache) incorporate *attributes* together with their possible values and an explicit *contradiction* measure, which allows one to encode multi-attribute uncertainty as well as potential inconsistencies among attribute values [16–18]. In this sense, plithogenic sets are commonly viewed as a unifying generalization encompassing both fuzzy and neutrosophic settings. We next recall the basic definitions of fuzzy sets, neutrosophic sets, and plithogenic sets.

Definition 1.1 (Fuzzy set and fuzzy relation). *[1,19] Let Y be a non-empty universe. A* fuzzy set *on Y is a mapping $\tau\colon Y \to [0,1]$. A* fuzzy relation *on Y is a fuzzy subset δ of $Y \times Y$, i.e. a mapping $\delta\colon Y \times Y \to [0,1]$.*

If τ is a fuzzy set on Y and δ is a fuzzy relation on Y, then δ is called a fuzzy relation on τ *provided that*

$$\delta(y,z) \leq \min\{\tau(y),\tau(z)\} \qquad \text{for all } y,z \in Y.$$

Definition 1.2 (Neutrosophic set). *[4,5] Let X be a non-empty set. A* neutrosophic set *(NS) A on X is specified by three membership functions*

$$T_A : X \to [0,1], \qquad I_A : X \to [0,1], \qquad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity of x with respect to A, respectively. These functions satisfy the constraint

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3 \qquad \text{for all } x \in X.$$

Definition 1.3 (Plithogenic set). *[17,20] Let S be a universal set and let $P \subseteq S$. A* plithogenic set *PS is a tuple*

$$PS = (P, v, Pv, pdf, pCF),$$

where:

- *v is an attribute;*
- *Pv is the set (range) of possible values of the attribute v;*
- *$pdf : P \times Pv \to [0,1]^s$ is the* degree of appurtenance function *(DAF);*
- *$pCF : Pv \times Pv \to [0,1]^t$ is the* degree of contradiction function *(DCF),*

for fixed integers $s \ge 1$ and $t \ge 1$. The contradiction function satisfies, for all $a, b \in Pv$:

1. Reflexivity: $pCF(a,a) = 0$;
2. Symmetry: $pCF(a,b) = pCF(b,a)$.

1.2. L-Fuzzy, L-Neutrosophic, and L-Plithogenic Sets (Lattice-Valued Variants)

An L-fuzzy set replaces the unit interval $[0,1]$ by a complete lattice L, thereby allowing membership to take values in a richer ordered structure [21–23]. An L-neutrosophic set assigns to each element three membership values in a lattice L—truth, indeterminacy, and falsity—thereby generalizing neutrosophic degrees beyond $[0,1]$ [24]. An L-plithogenic set equips a set with lattice-valued appurtenance degrees indexed by attribute values, together with a lattice-valued contradiction function between attribute values, extending plithogenic modeling beyond real-valued degrees [24].

Definition 1.4 (L-fuzzy set). *[21] Let X be a universal set and let $(L, \le)$ be a complete lattice with join $\vee$, meet $\wedge$, top element $\top_L$, and bottom element $\bot_L$. An* L-fuzzy set *on X is a mapping*

$$A : X \to L,$$

where $A(x) \in L$ is interpreted as the membership degree of $x \in X$. In particular, $A(x) = \top_L$ indicates full membership, while $A(x) = \bot_L$ indicates non-membership.

Definition 1.5 (L-neutrosophic set). *[24] Let X be a universal set and let L be a complete lattice with top element $\top_L$ and bottom element $\bot_L$. An* L-neutrosophic set *$\mathcal{A}$ on X is given by three lattice-valued membership functions*

$$T_{\mathcal{A}} : X \to L, \qquad I_{\mathcal{A}} : X \to L, \qquad F_{\mathcal{A}} : X \to L,$$

where for each $x \in X$:

- *$T_{\mathcal{A}}(x)$ is the* truth degree *of x,*
- *$I_{\mathcal{A}}(x)$ is the* indeterminacy degree *of x,*
- *$F_{\mathcal{A}}(x)$ is the* falsity degree *of x,*

all interpreted as elements of L. Equivalently, one may present $\mathcal{A}$ as the set of annotated elements

$$\mathcal{A} = \Big\{ \langle x, T_{\mathcal{A}}(x), I_{\mathcal{A}}(x), F_{\mathcal{A}}(x) \rangle \;\Big|\; x \in X \Big\},$$

so that each membership triple lies in L^3.

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Remark 1.6. *In the classical (real-valued) case, neutrosophic memberships satisfy* $0 \le T + I + F \le 3$*. For lattice-valued memberships, one may impose additional constraints depending on how an "aggregation" of* $T_A(x)$*,* $I_A(x)$*, and* $F_A(x)$ *is modeled in* L*. For instance, one may require suitable boundedness conditions with respect to* $\top_L$ *and* $\bot_L$*. In many treatments, however, the baseline requirement is simply that each component takes values in* L *(i.e. membership triples lie in* L^3*), and we follow this convention here.*

Definition 1.7 (L-plithogenic set)**.** *[24] Let* S *be a universal set and let* $P \subseteq S$*. Let* v *be an attribute with value set* Pv*. Let* L *be a complete lattice, and fix integers* $s \ge 1$ *and* $t \ge 1$*. An* L-plithogenic set *(of dimension* (s,t)*), denoted* $L\text{-}PS^{(s,t)}$*, is a tuple*

$$L\text{-}PS = (P,\ v,\ Pv,\ pdf_L,\ pCF_L),$$

where the lattice-valued degree functions are

$$pdf_L : P \times Pv \to L^s, \qquad pCF_L : Pv \times Pv \to L^t.$$

Here $pdf_L(x,a)$ *represents the (lattice-valued) appurtenance of* $x \in P$ *relative to the attribute value* $a \in Pv$*, and* $pCF_L(a,b)$ *represents the (lattice-valued) contradiction between* $a, b \in Pv$*.*

The standard plithogenic axioms extend to the lattice setting; for example, one typically requires a suitable reflexivity and symmetry condition such as

$$pCF_L(a,a) = \bot_L \ \ (\textit{or another designated least contradiction element}), \qquad pCF_L(a,b) = pCF_L(b,a),$$

with the precise formulation depending on how contradiction is encoded in L^t*.*

1.3. Uncertain Set

An Uncertain Set is a generic way to attach "uncertainty values" to elements, where the values live in a chosen *degree-domain*. By selecting an appropriate degree-domain, one recovers fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and many related models as special cases [25,26].

Definition 1.8 (Uncertain Set (U-Set))**.** *[25] Let* X *be a nonempty universe and let* M *be an uncertainty model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \ge 1$*. An* Uncertain Set of type M *(briefly, a* U-Set of type M*) on* X *is a mapping*

$$\mu_M : X \longrightarrow \mathrm{Dom}(M).$$

For each $x \in X$*, the value* $\mu_M(x) \in \mathrm{Dom}(M)$ *is called the* M-membership degree *(or* M-uncertainty value*) of* x*. Different choices of* M *yield the usual fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and other degree-based set models.*

Table 1 illustrates how representative degree-based models can be realized inside the Uncertain Set framework.

Table 1. Uncertain Sets as a unifying framework (illustrative cases).

Model	Degree domain	Realization as a U-Set
Fuzzy Set	$[0,1]$	Choose M with $\mathrm{Dom}(M) = [0,1]$ and set $\mu_M(x) = \mu(x)$.
Intuitionistic Fuzzy Set	$[0,1]^2$	Choose M with $\mathrm{Dom}(M) = [0,1]^2$ and set $\mu_M(x) = (\mu(x), \nu(x))$.
Neutrosophic Set	$[0,1]^3$	Choose M with $\mathrm{Dom}(M) = [0,1]^3$ and set $\mu_M(x) = (T(x), I(x), F(x))$.
Plithogenic Set	$[0,1]^m$	Choose M with $\mathrm{Dom}(M) \subseteq [0,1]^m$ and set $\mu_M(x)$ to the plithogenic degree vector of x.

1.4. Hyperlattices and Superhyperlattices

A hyperlattice has a binary meet and a multivalued join, generalizing lattices by allowing hyperoperations capturing ambiguous combinations algebraically structured rules [27–29]. A superhyperlattice lifts these operations recursively to iterated power sets, producing higher-level elements, memberships, and interactions defined consistently across layers [29].

Definition 1.9 (Hyperlattice)**.** *Let L be a nonempty set. A* hyperlattice *is a triple $(L, \wedge, \circ)$ consisting of*

$$\wedge : L \times L \to L \qquad \text{and} \qquad \circ : L \times L \to \mathcal{P}(L) \setminus \{\emptyset\},$$

where $\wedge$ is a (single-valued) binary operation and $\circ$ is a (multi-valued) binary hyperoperation. *We assume that*

- *$(L, \circ)$ is a commutative hypergroup, and*
- *$(L, \wedge)$ is a commutative semigroup.*

For readability, we use the standard extensions to subsets: for $A \subseteq L$ and $a \in L$,

$$a \circ A := \bigcup_{x \in A} (a \circ x), \quad A \circ a := \bigcup_{x \in A} (x \circ a), \quad A \circ B := \bigcup_{x \in A,\, y \in B} (x \circ y),$$

and

$$a \wedge A := \{\, a \wedge x : x \in A \,\}, \quad A \wedge a := \{\, x \wedge a : x \in A \,\}.$$

In addition, a hyperlattice is required to satisfy lattice-like axioms adapted to the hyper setting; typically:

1. ***Idempotency:** for all $a \in L$, one has $a \in a \circ a$ and $a \wedge a = a$.*
2. ***(Hyper)commutativity:** for all $a, b \in L$, $a \circ b = b \circ a$ and $a \wedge b = b \wedge a$.*
3. ***Associativity:** $\wedge$ is associative, and $\circ$ satisfies the (weak) associativity required of a hypergroup.*
4. ***Absorption-type laws:** for all $a, b \in L$,*

$$a \in a \circ (a \wedge b) \qquad \text{and} \qquad a \in a \wedge (a \circ b).$$

5. ***(Optional) distributivity:** in a distributive (or s-distributive) hyperlattice, one may require, e.g.,*

$$a \wedge (b \circ c) \;\subseteq\; (a \wedge b) \circ (a \wedge c),$$

with stronger variants depending on the intended notion of distributivity.

Definition 1.10 (n-Superhyperlattice)**.** *Let $(L, \wedge, \circ)$ be a hyperlattice and let $n \geq 1$. Define the iterated powerset tower by*

$$\mathcal{P}^0(L) := L, \qquad \mathcal{P}^{k+1}(L) := \mathcal{P}(\mathcal{P}^k(L)) \quad (k \geq 0).$$

*An n-*superhyperlattice *is a structure*

$$(\mathcal{P}^n(L),\ \wedge_n,\ \circ_n),$$

where

$$\wedge_n : \mathcal{P}^n(L) \times \mathcal{P}^n(L) \to \mathcal{P}^n(L) \quad \text{and} \quad \circ_n : \mathcal{P}^n(L) \times \mathcal{P}^n(L) \to \mathcal{P}(\mathcal{P}^n(L)) \setminus \{\emptyset\}$$

are obtained by lifting *$\wedge$ and $\circ$ to level n.*

A canonical (elementwise) lifting is defined recursively by setting $\wedge_0 := \wedge$ and $\circ_0 := \circ$, and for $k \geq 0$ and $A, B \in \mathcal{P}^{k+1}(L)$,

$$A \wedge_{k+1} B \;:=\; \{\, x \wedge_k y : x \in A,\ y \in B \,\} \;\in\; \mathcal{P}^{k+1}(L),$$

$$A \circ_{k+1} B \;:=\; \{\, x \circ_k y : x \in A,\ y \in B \,\} \;\in\; \mathcal{P}(\mathcal{P}^{k+1}(L)) \setminus \{\emptyset\}.$$

The axioms of a hyperlattice are then imposed at level n, mutatis mutandis, with $\wedge, \circ$ replaced by $\wedge_n, \circ_n$.

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Remark 1.11. *Any ordinary lattice* $(L, \wedge, \vee)$ *can be regarded as a hyperlattice by turning* $\vee$ *into the trivial hyperoperation* $a \circ b := \{a \vee b\}$*. More generally, once a hyperlattice is fixed, the powerset-tower construction above yields an* n*-superhyperlattice for each* $n \geq 1$*.*

The overview of lattices, hyperlattices, and n-superhyperlattices is presented in Table 2.

Table 2. Overview of lattices, hyperlattices, and n-superhyperlattices.

Structure	Carrier (elements)	Key operations / intuition
Lattice	L	Two single-valued operations $\wedge, \vee : L \times L \to L$ satisfying associativity, commutativity, idempotency, and absorption.
Hyperlattice	L	Meet is single-valued $\wedge : L \times L \to L$, while join is multivalued $\circ : L \times L \to \mathcal{P}(L) \setminus \{\varnothing\}$ (hyperoperation), capturing ambiguous/non-deterministic combination.
n-Superhyper lattice	$\mathcal{P}^n(L)$	Iterated powerset level. Operations $(\wedge_n, \circ_n)$ are obtained by recursively lifting $(\wedge, \circ)$ elementwise to $\mathcal{P}^n(L)$, yielding higher-order (layered) aggregation.

2. Main Results

This section presents the main results of the paper.

2.1. L-Uncertain Sets (Lattice-Valued Uncertain Sets)

We introduce a lattice-valued notion of "uncertain set" that simultaneously accommodates (i) single-parameter membership models (fuzzy- and neutrosophic-type) and (ii) attribute-based models equipped with a contradiction measure (plithogenic-type).

Notation 2.1. *Let* L *be a complete lattice with bottom* $\perp_L$ *and top* $\top_L$*. For each integer* $k \geq 1$*, we write* L^k *for the Cartesian product, ordered componentwise:*

$$(a_1, \dots, a_k) \leq (b_1, \dots, b_k) \quad :\Longleftrightarrow \quad a_i \leq b_i \text{ for all } i.$$

Then L^k *is again a complete lattice (with join/meet computed coordinatewise), and*

$$\perp_{L^k} := (\perp_L, \dots, \perp_L), \qquad \top_{L^k} := (\top_L, \dots, \top_L).$$

Definition 2.2 (L-uncertainty model)**.** *Fix a complete lattice* L*. An* L-uncertainty model *is a tuple*

$$\mathfrak{M}_L = (v, Pv, D, C),$$

where

- v *is an (optional) attribute label;*
- Pv *is a nonempty set (interpreted as the set of possible values of the attribute* v*);*
- $D \subseteq L^s$ *is a nonempty* degree-domain *for membership (for some fixed* $s \geq 1$*);*
- $C \subseteq L^t$ *is a nonempty* degree-domain *for contradiction (for some fixed* $t \geq 1$*), and we assume* $\perp_{L^t} \in C$*.*

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Definition 2.3 (L-Uncertain Set)**.** *Let X be a nonempty universe and let $\mathfrak{M}_L = (v, Pv, D, C)$ be an L-uncertainty model. An L-*Uncertain Set of type *$\mathfrak{M}_L$ (briefly, an L-*U-Set of type *$\mathfrak{M}_L$) on X is a pair of functions*

$$\mu : X \times Pv \longrightarrow D, \qquad \kappa : Pv \times Pv \longrightarrow C,$$

such that for all $a, b \in Pv$:

(i) ***Reflexivity of contradiction:*** $\kappa(a, a) = \bot_{L^t}$;

(ii) ***Symmetry of contradiction:*** $\kappa(a, b) = \kappa(b, a)$.

We denote such an L-uncertain set by

$$\mathcal{U}_L = (X; v, Pv; \mu, \kappa) \quad \text{(with implicit degree-domains } D \subseteq L^s, C \subseteq L^t\text{)}.$$

Degenerate (single-parameter) case. *If $Pv = \{*\}$ is a singleton, then μ is equivalent to a map $X \to D$ via $x \mapsto \mu(x, *)$, and κ is forced to be the constant $\bot_{L^t}$ by reflexivity. Thus the above definition also covers the usual "membership-only" uncertainty models.*

The comparison between Uncertain Sets and L-Uncertain Sets is presented in Table 3.

Table 3. Concise comparison between Uncertain Sets and L-Uncertain Sets.

Aspect	Uncertain Set (U-Set)	L-Uncertain Set (L-U-Set)
Universe	Nonempty set X	Nonempty set X
Degree-domain	$\mathrm{Dom}(M) \subseteq [0,1]^k$ (model-dependent)	$D \subseteq L^s$ where L is a complete lattice
Membership map	$\mu_M : X \to \mathrm{Dom}(M)$	$\mu : X \times Pv \to D$ (attribute-indexed; singleton Pv recovers $X \to D$)
Attributes	Optional, via choice of model M	Explicit: v with value set Pv
Contradiction measure	Not required in the base definition	Explicit: $\kappa : Pv \times Pv \to C \subseteq L^t$ (reflexive, symmetric)
Underlying algebra	Real-valued product domain with coordinatewise order	Lattice-valued domain with joins/meets induced by L
Special cases recovered	Fuzzy / intuitionistic / neutrosophic / plithogenic (via $\mathrm{Dom}(M)$)	L-fuzzy (s=1), L-neutrosophic (s=3), L-plithogenic (general s, t)

The theorem is stated below.

Theorem 2.4 (Recovering standard models as special cases)**.** *Let L be a complete lattice.*

(a) ***L-fuzzy sets.*** *Let $A\colon X \to L$ be an L-fuzzy set. Set $Pv := \{*\}$, $s := 1$, $D := L \subseteq L^1$, choose any $t \geq 1$ and any $C \subseteq L^t$ with $\bot_{L^t} \in C$, and define*

$$\mu(x, *) := A(x), \qquad \kappa(*, *) := \bot_{L^t}.$$

Then $(X; v, Pv; \mu, \kappa)$ is an L-uncertain set (of type (v, Pv, D, C)), and the assignment $A \mapsto \mu(\cdot, *)$ recovers the original L-fuzzy membership. Conversely, any L-uncertain set with $Pv = \{*\}$ and $D = L$ determines an L-fuzzy set $A(x) := \mu(x, *)$.

(b) ***L-neutrosophic sets.*** *Let $\mathcal{A}$ be an L-neutrosophic set on X, i.e. three maps $T_{\mathcal{A}}, I_{\mathcal{A}}, F_{\mathcal{A}}\colon X \to L$. Set $Pv := \{*\}$, $s := 3$, $D := L^3$, and define*

$$\mu(x, *) := \big(T_{\mathcal{A}}(x),\, I_{\mathcal{A}}(x),\, F_{\mathcal{A}}(x)\big) \in L^3, \qquad \kappa(*, *) := \bot_{L^t}.$$

 doi:10.20944/preprints202512.2528.v1

Then $(X; v, Pv; \mu, \kappa)$ is an L-uncertain set whose membership map $\mu(\cdot, *)$ encodes precisely the triple (T_A, I_A, F_A). Conversely, any L-uncertain set with $Pv = \{*\}$ and $D = L^3$ yields an L-neutrosophic set by projecting $\mu(\cdot, *)$ onto its three coordinates.

(c) ***L-plithogenic sets.*** *Let L-PS $= (P, v, Pv, pdf_L, pCF_L)$ be an L-plithogenic set of dimension (s, t) in the sense of Definition 1.7. Assume (as in Definition 1.7) that*

$$pCF_L(a, a) = \bot_{L^t} \quad \text{and} \quad pCF_L(a, b) = pCF_L(b, a) \qquad (a, b \in Pv).$$

Take $X := P$, $D := L^s$, $C := L^t$, and define

$$\mu(x, a) := pdf_L(x, a) \in L^s, \qquad \kappa(a, b) := pCF_L(a, b) \in L^t.$$

Then $(P; v, Pv; \mu, \kappa)$ is an L-uncertain set of type (v, Pv, L^s, L^t). Conversely, any L-uncertain set with membership map $\mu\colon P \times Pv \to L^s$ and contradiction map $\kappa\colon Pv \times Pv \to L^t$ satisfying reflexivity and symmetry is an L-plithogenic set (with $pdf_L := \mu$ and $pCF_L := \kappa$).

(d) ***Real-valued Uncertain Sets.*** *Let M be an uncertainty model in the sense of Definition* (Uncertain Set (U-Set)) *in your manuscript, with degree-domain* $\mathrm{Dom}(M) \subseteq [0,1]^k$*, and let $\mu_M\colon X \to \mathrm{Dom}(M)$ be a U-Set of type M. Let $L := [0,1]$ with the usual order (a complete lattice), set $Pv := \{*\}$, $s := k$, and take $D := \mathrm{Dom}(M) \subseteq L^k$. Define $\mu(x, *) := \mu_M(x)$ and $\kappa(*, *) := \bot_{L^t}$. Then μ_M is recovered as the singleton-parameter instance of an L-uncertain set.*

Hence, the notion of L-uncertain set strictly subsumes L-fuzzy sets, L-neutrosophic sets, L-plithogenic sets, and (real-valued) uncertain sets as special cases obtained by appropriate choices of Pv, D, and C.

Proof. We verify each clause by direct construction.

(a) Given an L-fuzzy set $A\colon X \to L$, define $\mu(x, *) := A(x)$. Since $Pv = \{*\}$, the only required contradiction axiom is $\kappa(*, *) = \bot_{L^t}$, which holds by definition. Conversely, if $Pv = \{*\}$ and $D = L$, then $A(x) := \mu(x, *)$ defines an L-fuzzy set.

(b) Given (T_A, I_A, F_A), define $\mu(x, *)$ as the corresponding triple in L^3. Again, with $Pv = \{*\}$ the contradiction axioms hold trivially. Conversely, if $\mu(\cdot, *)\colon X \to L^3$, then composing with the three coordinate projections yields three maps $X \to L$, i.e. an L-neutrosophic set.

(c) For an L-plithogenic set, put $\mu := pdf_L$ and $\kappa := pCF_L$. The defining axioms of L-uncertain sets are exactly the reflexivity and symmetry assumptions on pCF_L, so $(P; v, Pv; \mu, \kappa)$ is an L-uncertain set. Conversely, an L-uncertain set with domains L^s and L^t becomes an L-plithogenic set by renaming $pdf_L := \mu$ and $pCF_L := \kappa$.

(d) Take $L = [0,1]$, $Pv = \{*\}$ and $D = \mathrm{Dom}(M) \subseteq L^k$. Then $\mu(x, *) := \mu_M(x)$ is well-defined, and $\kappa(*, *) = \bot_{L^t}$ satisfies the axioms. The original uncertain set is recovered as $x \mapsto \mu(x, *)$.

All statements follow. □

2.2. HL-Uncertain Sets (Hyperlattice-Valued Uncertain Sets)

Throughout, fix a *hyperlattice* $HL = (H, \wedge, \circ)$ in the sense of Definition 1.9 (i.e. $\wedge$ is single-valued and $\circ$ is a hyperoperation).

Definition 2.5 (Pointed hyperlattice)**.** *A* pointed hyperlattice *is a pair $(HL, 0_H)$ where $HL = (H, \wedge, \circ)$ is a hyperlattice and $0_H \in H$ is a distinguished element (interpreted as the* null *or* zero degree*). No order-theoretic minimality of 0_H is assumed unless stated explicitly.*

Definition 2.6 (Cartesian powers of a pointed hyperlattice)**.** *Let $(HL, 0_H)$ be a pointed hyperlattice and let $k \geq 1$. Define the pointed hyperlattice $(HL^k, 0_{H^k})$ on H^k by*

$$(a_1, \ldots, a_k) \wedge_k (b_1, \ldots, b_k) := (a_1 \wedge b_1, \ldots, a_k \wedge b_k) \in H^k,$$

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and the hyperoperation

$$(a_1,\ldots,a_k) \circ_k (b_1,\ldots,b_k) := (a_1 \circ b_1) \times \cdots \times (a_k \circ b_k) \subseteq H^k,$$

which is nonempty since each $a_i \circ b_i \neq \emptyset$. *Set*

$$0_{H^k} := (0_H,\ldots,0_H) \in H^k.$$

Definition 2.7 (HL-uncertainty model). *Let* $(HL, 0_H)$ *be a pointed hyperlattice. An* HL-uncertainty model *is a tuple*

$$\mathfrak{M}_{HL} = (v,\ Pv,\ D,\ C),$$

where v *is an (optional) attribute label,* Pv *is a nonempty set of attribute values, and*

$$D \subseteq H^s, \qquad C \subseteq H^t$$

are nonempty degree-domains *for some fixed integers* $s \geq 1$ *and* $t \geq 1$, *such that*

$$0_{H^t} \in C \quad (\text{with } 0_{H^t} \text{ as in Definition 2.6}).$$

Definition 2.8 (HL-Uncertain Set). *Let* X *be a nonempty universe and let* $\mathfrak{M}_{HL} = (v, Pv, D, C)$ *be an* HL*-uncertainty model over a pointed hyperlattice* $(HL, 0_H)$. *An* HL-Uncertain Set of type $\mathfrak{M}_{HL}$ *(briefly, an* HL-U-Set*) on* X *is a pair of functions*

$$\mu : X \times Pv \longrightarrow D, \qquad \kappa : Pv \times Pv \longrightarrow C,$$

satisfying, for all $a, b \in Pv$,

(i) ***Reflexivity of contradiction:*** $\kappa(a,a) = 0_{H^t}$;
(ii) ***Symmetry of contradiction:*** $\kappa(a,b) = \kappa(b,a)$.

We denote such a structure by

$$\mathcal{U}_{HL} = (X;\ v, Pv;\ \mu, \kappa),$$

with implicit degree-domains $D \subseteq H^s$ *and* $C \subseteq H^t$.

Remark 2.9. *If* $Pv = \{*\}$ *is a singleton, then* μ *is equivalent to a mapping* $X \to D$ *via* $x \mapsto \mu(x,*)$, *and* κ *is forced to be the constant* 0_{H^t} *by reflexivity. Thus membership-only hyperlattice-valued models are included as a special case.*

The example is stated below.

Example 2.10 (A real-life HL-Uncertain Set: choosing a commuting mode). *Consider a commuter who must choose a transportation mode under uncertainty caused by weather and service conditions.*

Universe and attribute. *Let*

$$X := \{\textsf{Train},\ \textsf{Bus},\ \textsf{Bike}\}$$

be the set of alternatives, and let the attribute be

$$v := \textsf{Weather}, \qquad Pv := \{\textsf{Sunny},\ \textsf{Rainy}\}.$$

Degree-domain as a pointed hyperlattice. *Let* $H := [0,1]$ *and define a pointed hyperlattice* $HL = (H, \wedge, \circ, 0_H)$ *by*

$$a \wedge b := \min\{a,b\}, \qquad a \circ b := \{\max\{a,b\},\ \min\{1, a+b\}\}, \qquad 0_H := 0.$$

Here $\wedge$ models "conservative aggregation" of degrees, while the hyperoperation $\circ$ returns two plausible combined degrees *capturing ambiguity between an "optimistic" (max) and an "additive" (capped-sum) evaluation.*

Take $s = t = 1$, $D := H$, and $C := H$.

Membership (uncertainty) function. *Define $\mu : X \times Pv \to [0,1]$ by the commuter's estimated "suitability" (higher is better):*

$$\mu(\textit{Train}, \textit{Sunny}) = 0.7, \quad \mu(\textit{Bus}, \textit{Sunny}) = 0.6, \quad \mu(\textit{Bike}, \textit{Sunny}) = 0.9,$$

$$\mu(\textit{Train}, \textit{Rainy}) = 0.8, \quad \mu(\textit{Bus}, \textit{Rainy}) = 0.7, \quad \mu(\textit{Bike}, \textit{Rainy}) = 0.2.$$

Contradiction function on attribute values. *Define $\kappa : Pv \times Pv \to [0,1]$ by*

$$\kappa(\textit{Sunny}, \textit{Sunny}) = 0, \qquad \kappa(\textit{Rainy}, \textit{Rainy}) = 0, \qquad \kappa(\textit{Sunny}, \textit{Rainy}) = \kappa(\textit{Rainy}, \textit{Sunny}) = 0.9.$$

This reflects that Sunny and Rainy are highly conflicting weather conditions.

Interpretation. *The resulting structure*

$$\mathcal{U}_{HL} = (X; v, Pv; \mu, \kappa)$$

is an HL-U-Set in the sense of Definition 2.8: κ is reflexive with value 0_H and symmetric, and μ assigns hyperlattice-valued suitability degrees to each alternative under each weather value. If one needs to aggregate the two weather-dependent suitability scores for an option $x \in X$, the hyperjoin $\circ$ can encode ambiguity: for example,

$$\mu(\textit{Train}, \textit{Sunny}) \circ \mu(\textit{Train}, \textit{Rainy}) = 0.7 \circ 0.8 = \{0.8,\ 1\},$$

representing two reasonable combined assessments depending on whether one aggregates optimistically (max) or additively (capped sum).

Definition 2.11 (HL-fuzzy set)**.** *Let $(HL, 0_H)$ be a pointed hyperlattice and let X be a nonempty set. An* HL-fuzzy set *on X is a mapping*

$$A \ : \ X \longrightarrow H.$$

Definition 2.12 (HL-neutrosophic set)**.** *Let $(HL, 0_H)$ be a pointed hyperlattice and let X be a nonempty set. An* HL-neutrosophic set *on X is specified by three membership functions*

$$T_A,\ I_A,\ F_A \ : \ X \longrightarrow H,$$

interpreted as hyperlattice-valued truth, indeterminacy, and falsity degrees, respectively. Equivalently, it is a mapping $X \to H^3$ given by $x \mapsto (T_A(x), I_A(x), F_A(x))$.

Definition 2.13 (HL-plithogenic set)**.** *Let $(HL, 0_H)$ be a pointed hyperlattice. Let S be a universal set and let $P \subseteq S$. Let v be an attribute with nonempty value set Pv. Fix integers $s \geq 1$ and $t \geq 1$.*

A HL-plithogenic set of dimension (s,t) *is a tuple*

$$HL\text{-}PS \ = \ (P,\ v,\ Pv,\ pdf_{HL},\ pCF_{HL}),$$

where

$$pdf_{HL} \ : \ P \times Pv \longrightarrow H^s, \qquad pCF_{HL} \ : \ Pv \times Pv \longrightarrow H^t,$$

and for all $a, b \in Pv$,

$$pCF_{HL}(a,a) = 0_{H^t}, \qquad pCF_{HL}(a,b) = pCF_{HL}(b,a).$$

The theorem is stated below.

 doi:10.20944/preprints202512.2528.v1

Theorem 2.14 (HL-U-Sets generalize L-U-Sets)**.** *Let L be a complete lattice with meet $\wedge$ and join $\vee$, bottom $\bot_L$, and top $\top_L$. Define the associated pointed hyperlattice*

$$HL(L) := (L, \wedge, \circ, 0_H), \qquad a \circ b := \{a \vee b\}, \qquad 0_H := \bot_L.$$

Then:

(a) *Every L-uncertain set (in the sense of Definition 2.3) canonically determines an $HL(L)$-uncertain set by keeping the same data (v, Pv, μ, κ).*

(b) *Conversely, every $HL(L)$-uncertain set canonically determines an L-uncertain set (again by the identity on the underlying data).*

In particular, for the hyperlattice $HL(L)$ with singleton-valued hyperjoin $a \circ b = \{a \vee b\}$, the notions of L-U-Set and $HL(L)$-U-Set are equivalent.

Proof. By construction, $HL(L)$ has the same underlying set as L, the same meet $\wedge$, and a hyperjoin $\circ$ whose values are singletons $\{a \vee b\}$. Moreover, the distinguished element 0_H is chosen to be $\bot_L$.

Let $\mathcal{U}_L = (X; v, Pv; \mu, \kappa)$ be an L-uncertain set of type (v, Pv, D, C), so $\mu : X \times Pv \to D \subseteq L^s$ and $\kappa : Pv \times Pv \to C \subseteq L^t$ satisfy $\kappa(a, a) = \bot_{L^t}$ and symmetry. But $\bot_{L^t} = 0_{H^t}$ under the identification $H = L$ and $0_H = \bot_L$; hence the same pair (μ, κ) satisfies the axioms of Definition 2.8, so it is an $HL(L)$-U-Set.

Conversely, if (μ, κ) is an $HL(L)$-U-Set, then the codomains are subsets of $H^s = L^s$ and $H^t = L^t$, and the defining conditions $\kappa(a, a) = 0_{H^t}$ and symmetry become exactly $\kappa(a, a) = \bot_{L^t}$ and symmetry. Thus it is an L-uncertain set. □

2.3. *SHL-Uncertain Sets (SuperHyperLattice-Valued Uncertain Sets)*

An SHL-uncertain set assigns superhyperlattice-valued membership degrees (possibly attribute-indexed) and a symmetric, reflexive contradiction measure on attribute values, unifying lattice- and hyperlattice-valued uncertainty frameworks.

Notation 2.15 (Iterated powersets)**.** *For any set H and any integer $n \geq 0$, define the iterated powerset tower by*

$$\mathcal{P}^0(H) := H, \qquad \mathcal{P}^{n+1}(H) := \mathcal{P}(\mathcal{P}^n(H)).$$

Let $\iota_0 := \mathrm{id}_H$, and define recursively

$$\iota_{n+1} : \mathcal{P}^n(H) \to \mathcal{P}^{n+1}(H), \qquad \iota_{n+1}(x) := \{x\}.$$

Thus $\iota_n : H \to \mathcal{P}^n(H)$ is the n-fold singleton embedding, e.g. $\iota_1(a) = \{a\}$ and $\iota_2(a) = \{\{a\}\}$.

Definition 2.16 (Superhyperlattice induced by a hyperlattice)**.** *Let $HL = (H, \wedge, \circ)$ be a hyperlattice (Definition 1.9). For each $n \geq 0$, define operations $(\wedge_n, \circ_n)$ on $\mathcal{P}^n(H)$ recursively by*

$$\wedge_0 := \wedge, \qquad \circ_0 := \circ,$$

and for $k \geq 0$ and $A, B \in \mathcal{P}^{k+1}(H)$,

$$A \wedge_{k+1} B := \{ x \wedge_k y \mid x \in A, \ y \in B \} \in \mathcal{P}^{k+1}(H),$$

$$A \circ_{k+1} B := \{ x \circ_k y \mid x \in A, \ y \in B \} \in \mathcal{P}(\mathcal{P}^{k+1}(H)) \setminus \{\emptyset\}.$$

The triple

$$SHL^{(n)}(HL) := (\mathcal{P}^n(H), \wedge_n, \circ_n)$$

*is called the n-*superhyperlattice *(or* superhyperlattice*) induced by HL.*

 doi:10.20944/preprints202512.2528.v1

Definition 2.17 (Pointed superhyperlattice)**.** *Let $(HL, 0_H)$ be a pointed hyperlattice (Definition 2.5). For each $n \geq 0$, define the distinguished element*

$$0_n \;:=\; \iota_n(0_H) \;\in\; \mathcal{P}^n(H).$$

Then $(SHL^{(n)}(HL), 0_n)$ is called the pointed n-superhyperlattice induced by $(HL, 0_H)$.

Definition 2.18 (Cartesian powers)**.** *Let $(SHL^{(n)}(HL), 0_n)$ be a pointed n-superhyperlattice and let $k \geq 1$. Write $(\mathcal{P}^n(H))^k$ for the Cartesian power, and set*

$$0_n^{(k)} := (\underbrace{0_n, \ldots, 0_n}_{k \text{ times}}).$$

(When needed, $\wedge_n$ and $\circ_n$ extend coordinatewise to $(\mathcal{P}^n(H))^k$ exactly as in Definition 2.6.)

Definition 2.19 (SHL-uncertainty model)**.** *Fix a pointed hyperlattice $(HL, 0_H)$ and an integer $n \geq 0$. An* $SHL^{(n)}$-uncertainty model *is a tuple*

$$\mathfrak{M}_{SHL^{(n)}} \;=\; (v,\; Pv,\; D,\; C),$$

where Pv is a nonempty set, and for some fixed integers $s \geq 1$ and $t \geq 1$,

$$D \subseteq \left(\mathcal{P}^n(H)\right)^s, \qquad C \subseteq \left(\mathcal{P}^n(H)\right)^t$$

are nonempty degree-domains with

$$(0_n, \ldots, 0_n) \;=\; 0_n^{(t)} \in C.$$

Definition 2.20 (SHL-Uncertain Set)**.** *Let X be a nonempty universe. Fix a pointed hyperlattice $(HL, 0_H)$ and an integer $n \geq 0$. Let $\mathfrak{M}_{SHL^{(n)}} = (v, Pv, D, C)$ be an $SHL^{(n)}$-uncertainty model over $(HL, 0_H)$.*

An $SHL^{(n)}$-Uncertain Set of type $\mathfrak{M}_{SHL^{(n)}}$ *(briefly, an* $SHL^{(n)}$-U-Set*) on X is a pair of functions*

$$\mu \;:\; X \times Pv \longrightarrow D, \qquad \kappa \;:\; Pv \times Pv \longrightarrow C,$$

such that for all $a, b \in Pv$:

(i) ***Reflexivity of contradiction:*** $\kappa(a, a) = 0_n^{(t)}$;
(ii) ***Symmetry of contradiction:*** $\kappa(a, b) = \kappa(b, a)$.

We denote such a structure by

$$\mathcal{U}_{SHL^{(n)}} \;=\; (X;\, v, Pv;\, \mu, \kappa),$$

with implicit degree-domains $D \subseteq (\mathcal{P}^n(H))^s$ and $C \subseteq (\mathcal{P}^n(H))^t$.

Remark 2.21. *If $Pv = \{*\}$ is a singleton, then μ is equivalent to a map $X \to D$ via $x \mapsto \mu(x, *)$, and κ is forced to be the constant $0_n^{(t)}$ by reflexivity. Thus membership-only models are included as a special case.*

The overview of L-, HL-, and SHL-Uncertain Sets is presented in Table 4.

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Table 4. Concise overview of L-, HL-, and SHL-Uncertain Sets.

Model	Degree values (codomain)	Defining data / intuition
L-Uncertain Set	$D \subseteq L^s, C \subseteq L^t$ where L is a complete lattice	$\mu : X \times Pv \to D$ (membership/appurtenance) and $\kappa : Pv \times Pv \to C$ (reflexive, symmetric contradiction); extends lattice-valued fuzzy/neutrosophic/plithogenic models.
HL-Uncertain Set	$D \subseteq H^s, C \subseteq H^t$ where $HL = (H, \wedge, \circ)$ is a pointed hyperlattice	Same (μ, κ)-pattern, but degrees live in a hyperlattice; the join-like operation is multivalued, allowing ambiguous or non-deterministic aggregation of degrees.
$SHL^{(n)}$-Uncertain Set	$D \subseteq (\mathcal{P}^n(H))^s, C \subseteq (\mathcal{P}^n(H))^t$ where $SHL^{(n)}(HL)$ is an n-superhyperlattice	Degrees are lifted to iterated powersets, enabling set-valued (and nested set-valued) degrees across layers; canonically generalizes HL-U-Sets (and hence L-U-Sets) via singleton embeddings.

The example is stated below.

Example 2.22 (A real-life SHL-Uncertain Set: credit risk under uncertain economic regimes)**.** *A bank evaluates loan applicants under uncertain macroeconomic regimes, where each regime itself is described by nested information (e.g. multiple expert panels, each panel providing a set of scenarios).*

Universe and attribute. *Let*

$$X := \{\textsf{Applicant}_1,\ \textsf{Applicant}_2,\ \textsf{Applicant}_3\}$$

be the set of applicants, and let the attribute be

$$v := \textsf{Regime}, \qquad Pv := \{\textsf{Expansion},\ \textsf{Recession}\}.$$

Pointed hyperlattice and superlevel. *Take $H = [0,1]$ and define a pointed hyperlattice $(HL, 0_H)$ by*

$$a \wedge b := \min\{a,b\}, \qquad a \circ b := \{\max\{a,b\},\ \min\{1, a+b\}\}, \qquad 0_H := 0.$$

Choose $n = 1$, so degrees live in $\mathcal{P}(H)$ and can represent sets of plausible risk scores. *Let $s = t = 1$, and take $D = C = \mathcal{P}(H)$, with $0_1 = \{0\}$.*

Membership function. *Define $\mu : X \times Pv \to \mathcal{P}(H)$ so that $\mu(x,a)$ is a set of plausible* default-risk degrees *(higher = riskier), reflecting uncertainty across models:*

$$\mu(\textsf{Applicant}_1, \textsf{Expansion}) = \{0.10, 0.15\}, \quad \mu(\textsf{Applicant}_1, \textsf{Recession}) = \{0.25, 0.35\},$$

$$\mu(\textsf{Applicant}_2, \textsf{Expansion}) = \{0.05, 0.08\}, \quad \mu(\textsf{Applicant}_2, \textsf{Recession}) = \{0.18, 0.22\},$$

$$\mu(\textsf{Applicant}_3, \textsf{Expansion}) = \{0.12, 0.20\}, \quad \mu(\textsf{Applicant}_3, \textsf{Recession}) = \{0.30, 0.45\}.$$

Contradiction function. *Define $\kappa : Pv \times Pv \to \mathcal{P}(H)$ by*

$$\kappa(\textsf{Expansion}, \textsf{Expansion}) = \{0\},$$

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$$\kappa(\mathit{Recession}, \mathit{Recession}) = \{0\},$$

$$\kappa(\mathit{Expansion}, \mathit{Recession}) = \kappa(\mathit{Recession}, \mathit{Expansion}) = \{0.9\}.$$

Thus κ is reflexive with $0_1 = \{0\}$ and symmetric.

Interpretation. *The structure $\mathcal{U}_{SHL^{(1)}} = (X; v, Pv; \mu, \kappa)$ is an $SHL^{(1)}$-U-Set: each applicant receives a* set *of plausible risk degrees under each regime, and contradiction quantifies the incompatibility between regimes.*

Example 2.23 (A real-life SHL-Uncertain Set: medical triage with nested evidence sources)**.** *A hospital performs triage by combining nested sources of evidence: rapid tests, physician assessment, and wearable-device data, each of which may output an interval or a set of plausible severity levels.*

Universe and attribute. *Let*

$$X := \{\mathit{Patient}_A,\ \mathit{Patient}_B\}$$

be the set of patients, and let the attribute be

$$v := \mathit{EvidenceSource}, \qquad Pv := \{\mathit{RapidTest},\ \mathit{Clinician},\ \mathit{Wearable}\}.$$

Pointed hyperlattice and superlevel. *Again take $H = [0,1]$ and the pointed hyperlattice $(HL, 0_H)$ given by*

$$a \wedge b := \min\{a,b\}, \qquad a \circ b := \{\max\{a,b\},\ \min\{1, a+b\}\}, \qquad 0_H := 0.$$

Choose $n = 2$, so degrees live in $\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H))$ and can represent sets of sets *(e.g. one set per subsystem or per time window). Let $s = t = 1$, $D = C = \mathcal{P}^2(H)$, and $0_2 = \{\{0\}\}$.*

Membership function. *Define $\mu : X \times Pv \to \mathcal{P}^2(H)$ where each value is a* family of plausible severity-score sets *(higher = more severe):*

$$\mu(\mathit{Patient}_A, \mathit{RapidTest}) = \{\{0.2, 0.3\}, \{0.25\}\},$$

$$\mu(\mathit{Patient}_A, \mathit{Clinician}) = \{\{0.4, 0.5\}\},$$

$$\mu(\mathit{Patient}_A, \mathit{Wearable}) = \{\{0.3, 0.35\}, \{0.32, 0.38\}\},$$

$$\mu(\mathit{Patient}_B, \mathit{RapidTest}) = \{\{0.1, 0.15\}\},$$

$$\mu(\mathit{Patient}_B, \mathit{Clinician}) = \{\{0.2, 0.25\}, \{0.22\}\},$$

$$\mu(\mathit{Patient}_B, \mathit{Wearable}) = \{\{0.05, 0.10\}, \{0.08\}\}.$$

Contradiction function. *Define $\kappa : Pv \times Pv \to \mathcal{P}^2(H)$ by*

$$\kappa(a,a) = \{\{0\}\} \ \text{for all } a \in Pv,$$

and (symmetrically)

$$\kappa(\mathit{RapidTest}, \mathit{Clinician}) = \kappa(\mathit{Clinician}, \mathit{RapidTest}) = \{\{0.6\}\},$$

$$\kappa(\mathit{RapidTest}, \mathit{Wearable}) = \kappa(\mathit{Wearable}, \mathit{RapidTest}) = \{\{0.3\}\},$$

$$\kappa(\mathit{Clinician}, \mathit{Wearable}) = \kappa(\mathit{Wearable}, \mathit{Clinician}) = \{\{0.2\}\}.$$

This models that rapid tests may conflict more strongly with clinician judgment than with wearable data.

Interpretation. *Then $\mathcal{U}_{SHL^{(2)}} = (X; v, Pv; \mu, \kappa)$ is an $SHL^{(2)}$-U-Set: each patient has* nested *severity assessments (families of plausible sets) per evidence source, and κ quantifies pairwise source contradictions in a reflexive and symmetric manner.*

The theorem is stated below.

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Theorem 2.24 (SHL-U-Sets generalize HL-U-Sets)**.** *Let $(HL, 0_H)$ be a pointed hyperlattice, and let $n \geq 0$. Let $\mathcal{U}_{HL} = (X; v, Pv; \mu, \kappa)$ be an HL-U-Set in the sense of Definition 2.8, with $\mu : X \times Pv \to D \subseteq H^s$ and $\kappa : Pv \times Pv \to C \subseteq H^t$.*

Define the componentwise singleton-embedding

$$\iota_n^{(s)} : H^s \to \big(\mathcal{P}^n(H)\big)^s,$$

$$\iota_n^{(s)}(h_1, \ldots, h_s) := (\iota_n(h_1), \ldots, \iota_n(h_s)),$$

and similarly $\iota_n^{(t)} : H^t \to (\mathcal{P}^n(H))^t$.

Set

$$D^{(n)} := \iota_n^{(s)}(D) \subseteq \big(\mathcal{P}^n(H)\big)^s, \qquad C^{(n)} := \iota_n^{(t)}(C) \subseteq \big(\mathcal{P}^n(H)\big)^t,$$

and define

$$\mu^{(n)}(x, a) \;:=\; \iota_n^{(s)}(\mu(x, a)), \qquad \kappa^{(n)}(a, b) \;:=\; \iota_n^{(t)}(\kappa(a, b)).$$

Then $\mathcal{U}_{SHL^{(n)}} := (X; v, Pv; \mu^{(n)}, \kappa^{(n)})$ is an $SHL^{(n)}$-U-Set of type $(v, Pv, D^{(n)}, C^{(n)})$. Hence, every HL-U-Set canonically embeds into an $SHL^{(n)}$-U-Set for any $n \geq 0$.

Proof. By construction, $\mu^{(n)}$ maps into $D^{(n)}$ and $\kappa^{(n)}$ maps into $C^{(n)}$. For reflexivity, for any $a \in Pv$,

$$\kappa^{(n)}(a, a) = \iota_n^{(t)}(\kappa(a, a)) = \iota_n^{(t)}(0_{H^t}) = (\iota_n(0_H), \ldots, \iota_n(0_H)) = 0_n^{(t)}.$$

Symmetry is preserved because $\iota_n^{(t)}$ is applied pointwise:

$$\kappa^{(n)}(a, b) = \iota_n^{(t)}(\kappa(a, b)) = \iota_n^{(t)}(\kappa(b, a)) = \kappa^{(n)}(b, a).$$

Thus Definition 2.20 holds. □

Corollary 2.25 (SHL-U-Sets generalize L-U-Sets)**.** *Let L be a complete lattice with bottom $\bot_L$. Form the pointed hyperlattice $HL(L)$ by*

$$HL(L) := (L, \wedge, \circ, 0_H), \qquad a \circ b := \{a \vee b\}, \qquad 0_H := \bot_L,$$

as in Theorem 2.14. Then every L-U-Set canonically embeds into an $SHL^{(n)}$-U-Set (for any $n \geq 0$) via Theorem 2.24 applied to $HL(L)$.

Definition 2.26 ($SHL^{(n)}$-fuzzy set)**.** *Fix a pointed hyperlattice $(HL, 0_H)$ and $n \geq 0$. Let X be a nonempty set. A $SHL^{(n)}$-fuzzy set on X is a mapping*

$$A \;:\; X \longrightarrow \mathcal{P}^n(H).$$

Definition 2.27 ($SHL^{(n)}$-neutrosophic set)**.** *Fix a pointed hyperlattice $(HL, 0_H)$ and $n \geq 0$. Let X be a nonempty set. A $SHL^{(n)}$-neutrosophic set on X is specified by three maps*

$$T_A,\ I_A,\ F_A \;:\; X \longrightarrow \mathcal{P}^n(H),$$

interpreted as superhyperlattice-valued truth, indeterminacy, and falsity degrees. Equivalently, it is a mapping $X \to (\mathcal{P}^n(H))^3$.

Definition 2.28 ($SHL^{(n)}$-plithogenic set)**.** *Fix a pointed hyperlattice $(HL, 0_H)$ and $n \geq 0$. Let S be a universal set and $P \subseteq S$. Let v be an attribute with nonempty value set Pv. Fix integers $s \geq 1$ and $t \geq 1$.*

A $SHL^{(n)}$-plithogenic set of dimension (s, t) is a tuple

$$SHL^{(n)}\text{-}PS \;=\; (P,\ v,\ Pv,\ pdf_{SHL^{(n)}},\ pCF_{SHL^{(n)}}),$$

where

$$pdf_{SHL^{(n)}} : P \times Pv \longrightarrow \left(\mathcal{P}^n(H)\right)^s, \qquad pCF_{SHL^{(n)}} : Pv \times Pv \longrightarrow \left(\mathcal{P}^n(H)\right)^t,$$

and for all $a, b \in Pv$,

$$pCF_{SHL^{(n)}}(a,a) = 0_n^{(t)}, \qquad pCF_{SHL^{(n)}}(a,b) = pCF_{SHL^{(n)}}(b,a).$$

Theorem 2.29 (SHL-U-Sets subsume SHL-fuzzy, SHL-neutrosophic, and SHL-plithogenic sets)**.** *Fix a pointed hyperlattice* $(HL, 0_H)$ *and an integer* $n \geq 0$.

(a) $SHL^{(n)}$***-fuzzy sets.*** *Let* $A : X \to \mathcal{P}^n(H)$ *be a* $SHL^{(n)}$*-fuzzy set (Definition 2.26). Let* $Pv := \{*\}$, $s := 1$, $D := \mathcal{P}^n(H) \subseteq (\mathcal{P}^n(H))^1$, *and choose any* $t \geq 1$ *and any nonempty* $C \subseteq (\mathcal{P}^n(H))^t$ *with* $0_n^{(t)} \in C$. *Define*

$$\mu(x,*) := A(x), \qquad \kappa(*,*) := 0_n^{(t)}.$$

Then $(X; v, Pv; \mu, \kappa)$ *is an* $SHL^{(n)}$*-U-Set, and* A *is recovered as* $x \mapsto \mu(x,*)$.

(b) $SHL^{(n)}$***-neutrosophic sets.*** *Let* (T_A, I_A, F_A) *be a* $SHL^{(n)}$*-neutrosophic set on* X *(Definition 2.27). Let* $Pv := \{*\}$, $s := 3$, $D := (\mathcal{P}^n(H))^3$, *and define*

$$\mu(x,*) := \left(T_A(x),\, I_A(x),\, F_A(x)\right) \in (\mathcal{P}^n(H))^3, \qquad \kappa(*,*) := 0_n^{(t)}.$$

Then $(X; v, Pv; \mu, \kappa)$ *is an* $SHL^{(n)}$*-U-Set encoding precisely* (T_A, I_A, F_A).

(c) $SHL^{(n)}$***-plithogenic sets.*** *Let* $SHL^{(n)}$*-PS* $= (P, v, Pv, pdf_{SHL^{(n)}}, pCF_{SHL^{(n)}})$ *be a* $SHL^{(n)}$*-plithogenic set (Definition 2.28). Let* $X := P$, $D := (\mathcal{P}^n(H))^s$, $C := (\mathcal{P}^n(H))^t$, *and set*

$$\mu(x,a) := pdf_{SHL^{(n)}}(x,a), \qquad \kappa(a,b) := pCF_{SHL^{(n)}}(a,b).$$

Then $(P; v, Pv; \mu, \kappa)$ *is an* $SHL^{(n)}$*-U-Set. Conversely, any* $SHL^{(n)}$*-U-Set with these codomains is a* $SHL^{(n)}$*-plithogenic set by renaming* $pdf_{SHL^{(n)}} := \mu$ *and* $pCF_{SHL^{(n)}} := \kappa$.

Proof. In (a) and (b), since Pv is a singleton, the only required contradiction constraint is $\kappa(*,*) = 0_n^{(t)}$, which holds by definition; symmetry is automatic. Hence Definition 2.20 is satisfied, and the recovery statements follow from $A(x) = \mu(x,*)$ and $(T_A(x), I_A(x), F_A(x)) = \mu(x,*)$.

In (c), the contradiction axioms required in Definition 2.20 coincide with the reflexivity and symmetry conditions assumed in Definition 2.28, so (μ, κ) defines an $SHL^{(n)}$-U-Set. The converse is immediate by relabeling the maps. □

3. Conclusions

In this paper, we introduced *HyperLattice-valued* and *SuperHyperLattice-valued Uncertain Sets* as uncertainty frameworks in which degrees take values in hyperlattices and their superextensions. For future work, we anticipate further developments on extensions incorporating graphs [30], hypergraphs [31,32], superhypergraphs [33–35], and directed superhypergraphs [36,37], as well as research toward practical implementations such as programming tools supporting these models.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

Funding: This study did not receive any financial or external support from organizations or individuals.

Institutional Review Board Statement: As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Data Availability Statement: This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Acknowledgments: We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Conflicts of Interest: The authors confirm that there are no conflicts of interest related to the research or its publication.

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