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Discrete-to-Continuum Limits of Graph-Regularized Energy Functionals on Irregular Domains

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Abstract

Graph-based energy functionals constitute a fundamental tool in the mathematical modeling of networked systems, numerical schemes, and data-driven variational methods. In many applications, such energies are defined on discrete structures that serve as approximations of an underlying continuum domain. While the discrete-to-continuum behavior of such functionals has been extensively studied for regular graph sequences (e.g., uniform lattices, quasi-uniform point clouds), substantially less is known in the presence of graph irregularity. In this work, we analyze a class of graph-regularized energy functionals defined on highly irregular discrete domains. We introduce structural assumptions that allow for non-uniform connectivity, heterogeneous weights, and varying vertex densities, and we study their impact on compactness and Γ-convergence. Our main results establish sufficient conditions for convergence toward a continuum variational problem involving a p-Laplacian type energy with an L^q fidelity term. We demonstrate that irregular graph geometry may both preserve and destroy convergence depending on specific scaling regimes and connectivity patterns. Several illustrative examples and counterexamples are provided to show the sharpness of our assumptions. The analysis reveals a delicate interplay between local connectivity properties and global geometric constraints in determining the limiting behavior.

Keywords: discrete-to-continuum limits; Γ-convergence; graph-based energies; irregular domains; variational methods; non-uniform graphs

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1. Introduction

1.1. Background and Motivation

Variational methods on discrete structures have become a central topic in contemporary mathematical analysis. They arise naturally in a wide variety of contexts, including the modeling of physical systems on networks (e.g., elastic networks, electrical grids), image and signal processing (graph-based denoising, inpainting), machine learning (graph neural networks, semi-supervised learning), and the numerical approximation of partial differential equations via finite difference or finite element schemes. In many of these applications, graphs provide a flexible framework for representing discrete interactions, while variational principles encode regularity, stability, or optimality properties of interest.

A fundamental problem in this setting concerns the relationship between discrete variational models and their continuum counterparts. When a graph is viewed as a discretization of a continuous domain $\Omega \subset \mathbb{R}^d$, it is natural to ask whether the solutions of the discrete problem converge, in an appropriate sense, to solutions of a continuum variational problem as the graph becomes increasingly fine (i.e., as the number of vertices increases and the typical edge length decreases). This question is of both theoretical and practical importance, as it underlies the consistency and reliability of graph-based models: if the discrete problem approximates a well-posed continuum problem, one can expect robustness with respect to discretization details and confidence in predictions derived from the discrete model.

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For regular graph sequences, such as uniform lattice discretizations or graphs generated by quasi-uniform sampling of a smooth domain, this question has been studied extensively. Classical results based on Γ-convergence [1,2] provide a rigorous framework for establishing convergence of energies and, consequently, of minimizers. In these settings, geometric regularity (uniform vertex distribution, bounded degree, isotropic connectivity) ensures compactness and allows for a precise identification of the limiting functional, which typically takes the form of a Dirichlet-type energy $\int_\Omega |\nabla u|^p dx$ plus possibly additional terms.

However, many graphs arising in real-world applications are far from regular. Examples include transportation networks (road networks, flight routes), social networks, biological interaction networks (protein-protein interactions, neural connectivity), and data graphs constructed from non-uniformly sampled point clouds (e.g., manifold learning scenarios where data points concentrate near lower-dimensional structures). Such graphs often exhibit strong irregularities, including heterogeneous degree distributions (power-law degree sequences), long-range edges (small-world phenomena), highly variable edge weights (reflecting varying interaction strengths), and non-uniform vertex densities (clustering). These features violate the assumptions underlying classical convergence results and may lead to qualitatively different limiting behavior: the limiting energy might become anisotropic, degenerate, or even trivial (infinite for all non-constant functions).

1.2. Main Contributions

The objective of this paper is to investigate discrete-to-continuum limits for graph-regularized energy functionals defined on irregular graph sequences. Rather than imposing strong geometric constraints (like uniform degree bounds or isotropic neighborhood structure), we aim to identify minimal structural conditions that ensure compactness and convergence. Our analysis emphasizes the role of:

- *Scaling regimes:* The relationship between the number of vertices, typical edge length, and edge weights.
- *Connectivity:* Local and global connectivity properties that prevent pathological behavior.
- *Measure consistency:* Weak convergence of empirical vertex measures to a reference measure on Ω.
- *Coercivity:* Parameters that prevent minimizers from escaping to infinity.

Our main contributions are:

1. We introduce a general framework for studying Γ-convergence of graph-based energies on irregular domains, formalized through a set of structural assumptions (Section 3).
2. We establish compactness results for sequences with bounded energy under these assumptions (Section 5).
3. We prove Γ-convergence to a continuum p-Dirichlet energy with an L^q fidelity term, identifying the precise scaling conditions required (Section 6).
4. We provide explicit examples of irregular graph sequences that satisfy our assumptions and converge to the expected limit, as well as counterexamples showing what can go wrong when assumptions are violated (Section 7).

The results demonstrate that irregularity per se does not preclude convergence, but it imposes precise constraints on how the graph sequence must be scaled. In particular, we identify a critical scaling window in which the discrete energy approximates a continuum Sobolev norm.

1.3. Related Work

Discrete-to-continuum limits of graph-based energies have been studied in various contexts. Early work focused primarily on lattice-based discretizations and finite-difference schemes [5]. More recently, attention has shifted toward graph-based models arising in data analysis and machine learning, where irregularity is intrinsic. The use of Γ-convergence in this setting has proven particularly effective, as it provides a robust notion of convergence for variational problems [4].

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Notable contributions include the work of García Trillos and Slepčev [4] on continuum limits of graph total variation, which assumes uniform sampling and bounded degree; and the analysis of Bourgain et al. [5] on nonlocal variational problems, which considers more general kernels but still imposes symmetry and decay conditions. Our work extends these studies by allowing for significant heterogeneity in vertex distribution and connectivity patterns. We also relate to the literature on graph-based learning [3], where understanding the continuum limit helps explain the performance of algorithms on large datasets.

Despite this progress, most existing results rely on strong regularity assumptions, such as uniform sampling or bounded degree. The present work contributes to this literature by relaxing these assumptions and by explicitly addressing the effects of irregular graph geometry.

1.4. Organization of the Paper

The remainder of this paper is organized as follows. Section 2 introduces notation, basic definitions, and recalls key concepts from Γ-convergence. Section 3 presents our structural assumptions on graph sequences. Section 4 defines the class of energy functionals we study. Section 5 establishes compactness results. Section 6 contains our main Γ-convergence theorem and its proof. Section 7 provides examples and counterexamples. Section 8 discusses implications, open problems, and directions for future research.

2. Preliminaries

2.1. Graphs as Discrete Metric Measure Spaces

Let $G = (V, E, w)$ be a finite, simple, weighted, undirected graph, where:

- $V = \{1, \dots, n\}$ is the vertex set.
- $E \subset V \times V$ is the edge set (excluding self-loops).
- $w : E \to (0, \infty)$ assigns positive weights to edges.

We extend w to all pairs by setting $w(i, j) = 0$ if $(i, j) \notin E$. The degree of vertex i is $d_i = \sum_{j \in V} w(i, j)$.

We assume each vertex $i \in V$ is associated with a point x_i in a bounded domain $\Omega \subset \mathbb{R}^d$ (with Lipschitz boundary). This embedding allows us to compare discrete functions on V with continuum functions on Ω. The vertex set V inherits the Euclidean metric from Ω, and we define the graph metric d_G as the shortest-path distance weighted by w^{-1} (or by hop-count if all weights are 1).

The empirical measure associated with V is

$$\mu_n = \frac{1}{|V|} \sum_{i \in V} \delta_{x_i}.$$

We denote by $\mathcal{P}(\Omega)$ the space of Borel probability measures on Ω.

2.2. Function Spaces and Convergence

For a graph $G = (V, E, w)$, we consider functions $u : V \to \mathbb{R}$. The space $L^p(V, \mu_n)$ $(1 \le p < \infty)$ is endowed with the norm

$$\|u\|_{L^p(V,\mu_n)} = \left(\frac{1}{|V|} \sum_{i \in V} |u(i)|^p \right)^{1/p}.$$

For $p = \infty$, $\|u\|_{L^\infty(V)} = \max_{i \in V} |u(i)|$.

A sequence of discrete functions $\{u_n\}$ (with $u_n : V_n \to \mathbb{R}$) is said to converge to a continuum function $u \in L^p(\Omega)$ if the piecewise-constant interpolants $\bar{u}_n : \Omega \to \mathbb{R}$, defined by $\bar{u}_n(x) = u_n(i)$ for x in a Voronoi cell of x_i, converge to u in $L^p(\Omega)$. More precisely, we define the extension operator $P_n : L^p(V_n, \mu_n) \to L^p(\Omega)$ by

$$(P_n u)(x) = \sum_{i \in V_n} u(i) \chi_{C_i}(x),$$

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where $\{C_i\}$ is a partition of Ω such that $x_i \in C_i$ and $\operatorname{diam}(C_i) \to 0$ as $n \to \infty$. We say $u_n \to u$ in L^p if $P_n u_n \to u$ in $L^p(\Omega)$.

2.3. Γ-Convergence

We recall the definition and key properties of Γ-convergence [1].

Definition 1 (Γ-convergence)**.** *Let (X, d) be a metric space and $F_n : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ a sequence of functionals. We say F_n Γ-converges to $F : X \to \overline{\mathbb{R}}$ (written $F_n \xrightarrow{\Gamma} F$) if for every $x \in X$:*

1. (Liminf inequality) *For every sequence $\{x_n\} \subset X$ with $x_n \to x$,*
$$F(x) \leq \liminf_{n\to\infty} F_n(x_n).$$
2. (Limsup inequality) *There exists a sequence $\{x_n\} \subset X$ with $x_n \to x$ (called a recovery sequence) such that*
$$F(x) \geq \limsup_{n\to\infty} F_n(x_n).$$

The fundamental consequence of Γ-convergence is the convergence of minimizers:

Proposition 1 (Convergence of minimizers)**.** *If $F_n \xrightarrow{\Gamma} F$ and $\{x_n\}$ is a sequence of minimizers of F_n that converges to some $x \in X$, then x minimizes F. Moreover, if each F_n has a minimizer and the sequence $\{F_n\}$ is equi-coercive, then minimizers of F_n exist and any cluster point of such minimizers minimizes F.*

In our context, $X = L^p(\Omega)$ and F_n will be the graph-based energy (possibly extended to $+\infty$ for non-discrete functions).

3. Structural Assumptions

We consider a sequence of graphs $\{G_n\}_{n\in\mathbb{N}}$ with $G_n = (V_n, E_n, w_n)$, where $|V_n| \to \infty$. Each vertex $i \in V_n$ corresponds to a point $x_i^n \in \Omega \subset \mathbb{R}^d$, Ω bounded with Lipschitz boundary. We impose the following assumptions:

Assumption 1 (Measure convergence)**.** *The empirical measures $\mu_n = \frac{1}{|V_n|}\sum_{i\in V_n} \delta_{x_i^n}$ converge weakly to the Lebesgue measure $\mathcal{L}^d$ restricted to Ω (normalized to be a probability measure). That is, for every $f \in C(\overline{\Omega})$,*
$$\frac{1}{|V_n|}\sum_{i\in V_n} f(x_i^n) \to \frac{1}{\mathcal{L}^d(\Omega)}\int_\Omega f(x)\,dx.$$
Moreover, there exists $C > 0$ such that for all n and all $i \in V_n$,
$$\frac{1}{|V_n|} \leq \mathcal{L}^d(C_i^n) \leq \frac{C}{|V_n|},$$
where $\{C_i^n\}$ is the Voronoi partition associated with $\{x_i^n\}$.

This assumption ensures vertices are distributed somewhat uniformly over Ω, without severe clustering or holes. The Voronoi condition controls the local density.

Assumption 2 (Connectivity and length scale)**.** *Each G_n is connected. Define the graph length scale*
$$\varepsilon_n = \sup_{x\in\Omega}\inf_{i\in V_n}|x - x_i^n| + \max_{(i,j)\in E_n}|x_i^n - x_j^n|.$$
We assume $\varepsilon_n \to 0$ as $n \to \infty$. Moreover, there exists $\delta > 0$ such that for all n and all $i, j \in V_n$ with $|x_i^n - x_j^n| \leq \delta\varepsilon_n$, there is a path in G_n from i to j of at most K edges, where K is independent of n, i, j.

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The parameter ε_n represents the "resolution" of the graph: the first term controls how well vertices cover Ω, the second term controls the maximum edge length. The path condition ensures local connectivity at the scale ε_n.

Assumption 3 (Weight scaling)**.** *The edge weights scale as*

$$w_n(i,j) = \frac{\eta\left(\frac{|x_i^n - x_j^n|}{\varepsilon_n}\right)}{\varepsilon_n^d \cdot (\text{vertex degree normalization})},$$

where $\eta : [0,\infty) \to [0,\infty)$ is a non-increasing function with compact support (or exponential decay), $\eta(0) > 0$, and $\int_{\mathbb{R}^d} \eta(|z|)\,dz = 1$. The vertex degree normalization is either 1 (symmetric weights) or d_i (degree-normalized weights). More generally, we require that for some $\alpha, \beta > 0$,

$$\alpha \frac{\eta\left(\frac{|x_i^n - x_j^n|}{\varepsilon_n}\right)}{\varepsilon_n^d} \leq w_n(i,j) \leq \beta \frac{\eta\left(\frac{|x_i^n - x_j^n|}{\varepsilon_n}\right)}{\varepsilon_n^d} \quad \text{for all } (i,j) \in E_n.$$

This assumption links the discrete weights to a continuum kernel η. The scaling ε_n^{-d} ensures the discrete sum approximates an integral.

Assumption 4 (Coercivity and scaling)**.** *The regularization parameter λ_n satisfies $\lambda_n \to \lambda \geq 0$. The exponents p, q satisfy $1 < p < \infty$, $1 \leq q < \infty$. Additionally, we assume the scaling relation*

$$\frac{\varepsilon_n^p}{(\text{vertex density factor})} \to \sigma \in (0,\infty),$$

where the vertex density factor is typically $|V_n|^{-p/d}$ (ensuring the correct continuum scaling).

This assumption ensures the discrete gradient term and fidelity term balance appropriately in the limit.

Remark 1. *Assumption 3 allows for significant irregularity: vertices can have highly variable degrees, edges can be anisotropic (though η is radially symmetric), and the graph can contain both short and long edges (but long edges are penalized by η). The key is that the* average *connectivity resembles the kernel η.*

4. Graph-Regularized Energy Functionals

For each n, we define the discrete energy $E_n : L^p(V_n, \mu_n) \to [0,\infty]$ by

$$E_n(u) = \underbrace{\frac{1}{|V_n|} \sum_{i,j \in V_n} w_n(i,j)|u(i) - u(j)|^p}_{\text{graph gradient term } G_n(u)} + \lambda_n \underbrace{\frac{1}{|V_n|} \sum_{i \in V_n} |u(i)|^q}_{\text{fidelity term } F_n(u)}.$$

The prefactor $1/|V_n|$ turns the sums into discrete integrals with respect to μ_n. For convenience, we extend E_n to $L^p(\Omega)$ by setting $E_n(u) = +\infty$ if u is not of the form $P_n^{-1}(v)$ for some $v : V_n \to \mathbb{R}$.

The term $G_n(u)$ penalizes differences between values at connected vertices, encouraging smoothness along graph edges. The fidelity term $F_n(u)$ ties the function to data or prevents trivial minimizers. In semi-supervised learning, λ_n would be small and the fidelity term would only be applied to labeled vertices; here we consider the fully penalized version for simplicity.

We also define the continuum limit functional $E : L^p(\Omega) \to [0,\infty]$ by

$$E(u) = \begin{cases} \sigma C_{\eta,p} \int_\Omega |\nabla u(x)|^p\,dx + \lambda \int_\Omega |u(x)|^q\,dx & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

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where $C_{\eta,p} = \frac{1}{p}\int_{\mathbb{R}^d} \eta(|z|)|z_1|^p\,dz$ (assuming η is radially symmetric) and σ is the scaling constant from Assumption 4. For non-smooth u, $E(u) = +\infty$.

4.1. Interpretation of Scaling

The scaling of G_n is crucial. Write $G_n(u)$ as:

$$G_n(u) = \frac{1}{|V_n|^2}\sum_{i,j}\frac{w_n(i,j)|V_n|}{1}|u(i)-u(j)|^p.$$

Under Assumption 3, $w_n(i,j) \sim \varepsilon_n^{-d}\eta(|x_i - x_j|/\varepsilon_n)$. The factor $|V_n|$ is typically $\sim \varepsilon_n^{-d}$ (since vertices cover Ω), so $w_n(i,j)|V_n| \sim \varepsilon_n^{-2d}\eta(\cdots)$. The difference $|u(i)-u(j)|^p$ scales like ε_n^p if u is Lipschitz. Hence,

$$\text{term} \sim \varepsilon_n^{-2d}\cdot\varepsilon_n^p = \varepsilon_n^{p-2d}.$$

To get a finite limit, we need to multiply by an appropriate factor. The prefactor $1/|V_n|$ in $G_n(u)$ gives an extra ε_n^d, so overall scaling is ε_n^{p-d}. This matches the scaling of the continuum Dirichlet energy when $\varepsilon_n \sim |V_n|^{-1/d}$.

5. Compactness Results

We now establish compactness for sequences with uniformly bounded energy.

Lemma 1 (Discrete Poincaré inequality)**.** *Under Assumptions 1 and 2, there exists a constant $C_P > 0$ (independent of n) such that for all $u : V_n \to \mathbb{R}$,*

$$\frac{1}{|V_n|}\sum_{i\in V_n}|u(i)-\bar{u}_n|^p \le C_P\cdot\frac{1}{|V_n|}\sum_{i,j\in V_n} w_n(i,j)|u(i)-u(j)|^p,$$

where $\bar{u}_n = \frac{1}{|V_n|}\sum_{i\in V_n} u(i)$.

Proof. (Sketch) The proof uses the connectivity of G_n and the fact that weights are bounded below on edges of length $O(\varepsilon_n)$. One shows that for any edge (i,j) with $|x_i - x_j| \le \varepsilon_n$, the weight $w_n(i,j)$ is at least $c\varepsilon_n^{-d}$. A discrete version of the classical Poincaré inequality on graphs [4] then yields the result. The constant C_P depends on Ω, p, and the constants in the assumptions. □

Theorem 1 (Compactness)**.** *Let $\{u_n\}$ be a sequence with $u_n : V_n \to \mathbb{R}$ and $\sup_n E_n(u_n) < \infty$. Under Assumptions 1–4, there exists a subsequence (still denoted $\{u_n\}$) and a function $u \in W^{1,p}(\Omega)$ such that $P_n u_n \to u$ strongly in $L^p(\Omega)$ and weakly in $W^{1,p}(\Omega)$.*

Proof. The proof proceeds in several steps:

Step 1: L^p bound. From $E_n(u_n) < M$, we have $F_n(u_n) < M/\lambda_n \le C$. Hence $\|u_n\|_{L^q(V_n,\mu_n)} \le C$. Since Ω is bounded, $L^q \subset L^p$ (possibly with a constant), so $\|u_n\|_{L^p(V_n,\mu_n)} \le C'$.

Step 2: Bounded variation. Define the discrete gradient magnitude

$$|\nabla_n u_n|(i) = \left(\frac{1}{|V_n|}\sum_{j\in V_n} w_n(i,j)|u_n(i)-u_n(j)|^p\right)^{1/p}.$$

Then $G_n(u_n) = \frac{1}{|V_n|}\sum_i |\nabla_n u_n|(i)^p$. Since $G_n(u_n) < M$, we have $\||\nabla_n u_n|\|_{L^p(V_n,\mu_n)} \le M^{1/p}$.

Step 3: Control of oscillations. For any two vertices i,j connected by an edge, $|u_n(i)-u_n(j)|^p \le w_n(i,j)^{-1}\cdot(|V_n|\cdot\text{contribution to } G_n(u_n))$. Under Assumption 3, $w_n(i,j)^{-1} \le C\varepsilon_n^d$. Since $|V_n| \sim \varepsilon_n^{-d}$, the product is $O(1)$. Thus $|u_n(i)-u_n(j)| \le C\varepsilon_n$ for adjacent vertices (up to p-powers). By connectivity, this controls differences between any two vertices within distance $O(\varepsilon_n)$.

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Step 4: Compact embedding. The previous steps show that the piecewise-constant extensions $\bar{u}_n = P_n u_n$ have uniformly bounded L^p norm and are asymptotically equicontinuous in the sense that their oscillations on scale ε_n are controlled. Using the compact embedding of BV into L^p and the fact that $\varepsilon_n \to 0$, we extract a subsequence converging strongly in $L^p(\Omega)$ to some u.

Step 5: Sobolev regularity. The bound on $|\nabla_n u_n|$ implies (after technical arguments) that the limit u belongs to $W^{1,p}(\Omega)$ and that $\nabla_n u_n$ converges weakly to ∇u in an appropriate sense. This uses the non-degeneracy of weights (Assumption 3) and the measure convergence (Assumption 1). □

Remark 2. *The connectivity assumption is crucial. If G_n consisted of two large connected components with only a single edge between them, a function could jump arbitrarily across that edge without increasing $G_n(u_n)$ much. The limit might then be a function that is discontinuous across an interface, i.e., in BV but not in $W^{1,p}$.*

6. Discrete-to-Continuum Limit via Γ-Convergence

We now state and prove our main convergence result.

6.1. Liminf Inequality

Proposition 2 (Liminf inequality). *Let $u_n \to u$ in $L^p(\Omega)$ with $\sup_n E_n(u_n) < \infty$. Then*

$$E(u) \leq \liminf_{n\to\infty} E_n(u_n).$$

Proof. By Theorem 1, $u \in W^{1,p}(\Omega)$. We need to show

$$\sigma C_{\eta,p} \int_\Omega |\nabla u|^p dx + \lambda \int_\Omega |u|^q dx \leq \liminf_{n\to\infty} (G_n(u_n) + \lambda_n F_n(u_n)).$$

For the fidelity term, since $u_n \to u$ in L^p and $\lambda_n \to \lambda$, by lower semicontinuity of the L^q norm we have

$$\lambda \int_\Omega |u|^q dx \leq \liminf_{n\to\infty} \lambda_n \int_\Omega |P_n u_n|^q dx = \liminf_{n\to\infty} \lambda_n F_n(u_n).$$

For the gradient term, the argument is more involved. We sketch the idea: For smooth u, one can approximate the gradient by finite differences:

$$|u(x_i) - u(x_j)|^p \approx |\nabla u(x_i) \cdot (x_i - x_j)|^p \leq |\nabla u(x_i)|^p |x_i - x_j|^p.$$

Then

$$\begin{aligned} G_n(u_n) &\approx \frac{1}{|V_n|} \sum_{i,j} w_n(i,j) |\nabla u(x_i)|^p |x_i - x_j|^p \\ &= \frac{1}{|V_n|} \sum_i |\nabla u(x_i)|^p \sum_j w_n(i,j) |x_i - x_j|^p. \end{aligned}$$

Under Assumption 3,

$$\sum_j w_n(i,j)|x_i - x_j|^p \approx \varepsilon_n^{-d} \sum_j \eta\left(\frac{|x_i - x_j|}{\varepsilon_n}\right) |x_i - x_j|^p \approx \varepsilon_n^p \int_{\mathbb{R}^d} \eta(|z|)|z|^p dz.$$

Since $|V_n| \sim \varepsilon_n^{-d}$, the factor $\frac{1}{|V_n|}\varepsilon_n^p$ tends to σ. Collecting constants yields $C_{\eta,p} \int |\nabla u|^p dx$.

For non-smooth u, we use a density argument: approximate u by smooth functions and use the lower semicontinuity of the lim inf. □

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6.2. Limsup Inequality

Proposition 3 (Limsup inequality). *For every $u \in W^{1,p}(\Omega)$, there exists a sequence $\{u_n\}$ with $u_n : V_n \to \mathbb{R}$ such that $P_n u_n \to u$ in $L^p(\Omega)$ and*

$$E(u) \geq \limsup_{n\to\infty} E_n(u_n).$$

Proof. We first assume $u \in C^\infty(\overline{\Omega})$. Define $u_n(i) = u(x_i^n)$. Then clearly $P_n u_n \to u$ uniformly. Compute:

$$\begin{aligned} G_n(u_n) &= \frac{1}{|V_n|}\sum_{i,j} w_n(i,j)|u(x_i) - u(x_j)|^p \\ &= \frac{1}{|V_n|}\sum_{i,j} w_n(i,j)|\nabla u(x_i)\cdot(x_i - x_j) + O(|x_i - x_j|^2)|^p. \end{aligned}$$

Using the Taylor expansion and properties of η, one shows

$$\lim_{n\to\infty} G_n(u_n) = \sigma C_{\eta,p}\int_\Omega |\nabla u|^p dx.$$

Similarly,

$$\lim_{n\to\infty} \lambda_n F_n(u_n) = \lambda \int_\Omega |u|^q dx.$$

Thus $E_n(u_n) \to E(u)$.

For general $u \in W^{1,p}(\Omega)$, approximate by smooth functions $u_k \to u$ in $W^{1,p}(\Omega)$. For each k, choose n_k large enough so that $|E_{n_k}(u_k) - E(u_k)| < 1/k$ and $\|P_{n_k}u_k - u\|_{L^p} < 1/k$. Then diagonalize. □

6.3. Main theorem

Combining the previous results yields our main theorem:

Theorem 2 (Γ-convergence). *Under Assumptions 1–4, the sequence of functionals $\{E_n\}$ Γ-converges to E in the $L^p(\Omega)$ topology. Moreover, if $\{u_n^*\}$ is a sequence of minimizers of E_n (with $E_n(u_n^*) \leq \inf E_n + \delta_n$, $\delta_n \to 0$), then every cluster point of $\{P_n u_n^*\}$ in $L^p(\Omega)$ is a minimizer of E.*

Proof. The Γ-convergence follows from Propositions 2 and 3. The convergence of minimizers follows from the general theory of Γ-convergence combined with the compactness result (Theorem 1), which provides the required equi-coercivity. □

Remark 3 (Role of irregularity). *The theorem shows that irregularity (non-uniform vertex distribution, variable degrees) is allowed as long as it is "averaged out" in the limit. The assumptions ensure that locally, the graph looks like a rescaling of the kernel η. Global irregularities (e.g., a few vertices with extremely high degree) are permitted if they occupy negligible measure.*

7. Examples and Counterexamples

7.1. Convergent Irregular Graphs

Example 1 (Random geometric graph with density ρ). *Let $\rho : \Omega \to (0,\infty)$ be a bounded, strictly positive density. Sample points $\{x_i^n\}$ i.i.d. from distribution with density proportional to ρ. Connect two points x_i, x_j if $|x_i - x_j| < \varepsilon_n$, with weight $w_n(i,j) = \varepsilon_n^{-d}/\rho(x_i)$ (degree normalization). Then Assumptions 1–3 hold almost surely as $n \to \infty$ if $\varepsilon_n \to 0$ and $n\varepsilon_n^d/\log n \to \infty$ (connectivity threshold). The limiting energy becomes*

$$E(u) = \sigma C_{\eta,p}\int_\Omega \frac{1}{\rho(x)}|\nabla u(x)|^p \rho(x)dx + \lambda\int_\Omega |u|^q\rho(x)dx = \sigma C_{\eta,p}\int_\Omega |\nabla u|^p dx + \lambda \int_\Omega |u|^q \rho(x)dx.$$

The vertex density affects only the fidelity term, not the gradient term.

Example 2 (Graph with hubs). *Let most vertices be distributed uniformly, but add m_n "hub" vertices at locations $y_1, \ldots, y_{m_n}$, each connected to many other vertices. Suppose $m_n/|V_n| \to 0$ and each hub has degree $\sim d_n$ with $d_n/|V_n| \to 0$. Then the hubs occupy negligible measure and do not affect the limit. The graph remains connected, and the Γ-limit is the same as without hubs.*

7.2. Counterexamples: When Convergence Fails

Example 3 (Disconnected clusters). *Suppose $\Omega = \Omega_1 \cup \Omega_2$ separated by distance $\delta > 0$. Place $n/2$ vertices in each component, connect all pairs within distance $\varepsilon_n \ll \delta$ within each component, but place no edges between components. Then G_n is disconnected (two components). For any sequence u_n with $u_n \equiv 1$ on Ω_1, $u_n \equiv 0$ on Ω_2, we have $G_n(u_n) = 0$ but the limit u is discontinuous. Compactness fails in $W^{1,p}$ (the limit is in BV). The Γ-limit would be an energy that allows jumps across the interface.*

Example 4 (Vanishing weights). *Suppose weights decay too fast: $w_n(i,j) = \exp(-|x_i - x_j|/\varepsilon_n^2)/\varepsilon_n^d$. Then for fixed adjacent vertices, $w_n(i,j) \to 0$ exponentially fast. The discrete gradient term becomes too weak to control oscillations. Consequently, any bounded function has $G_n(u_n) \to 0$, so the Γ-limit is just the fidelity term (if $\lambda > 0$) or zero (if $\lambda = 0$). The gradient structure is lost.*

Example 5 (Incorrect scaling). *If ε_n decays faster than $|V_n|^{-1/d}$ (e.g., $\varepsilon_n = |V_n|^{-1}$ in $d = 2$), then $\varepsilon_n^p/|V_n|^{-p/d} \to 0$. The discrete gradient term vanishes relative to the continuum scaling. The limit becomes $E(u) = \lambda \int |u|^q dx$ (trivial gradient). Conversely, if ε_n decays too slowly, the discrete gradient term blows up, and the Γ-limit is $+\infty$ for non-constant functions.*

8. Discussion and Open Problems

8.1. Interpretation and Implications

Our results demonstrate that graph-based energies can converge to continuum variational problems even under substantial irregularity, provided the graph sequence satisfies certain averaging conditions. The key requirements are:

1. *Local connectivity:* At scale ε_n, the graph should be connected in a uniform way.
2. *Weight consistency:* Edge weights should approximate a rescaled kernel η.
3. *Measure convergence:* Vertices should distribute according to a density ρ.
4. *Balanced scaling:* ε_n should scale as $|V_n|^{-1/d}$ to match continuum.

These conditions are natural in many applications. For example, in machine learning with graph-based semi-supervised learning, one often constructs a k-nearest neighbor graph or an ε-graph from data points. If data are sampled from a density ρ, our results suggest that the graph Laplacian regularizer will approximate a weighted Dirichlet energy $\int |\nabla u|^2 \rho(x) dx$ in the limit of large data. This provides theoretical justification for the widely observed empirical success of such methods.

8.2. Open Problems and Future Directions

1. *Random graph models:* Our assumptions are deterministic. It would be valuable to formulate probabilistic versions (e.g., random geometric graphs, k-NN graphs, kernel graphs) and prove that they satisfy the assumptions almost surely under appropriate conditions.
2. *Anisotropic kernels:* We assumed η is radial. Anisotropic kernels $\eta(z)$ (with $\int zz^T \eta(z) dz$ positive definite) would lead to anisotropic limiting energies $\int \langle A\nabla u, \nabla u\rangle^{p/2} dx$. This could model directed or structured interactions.
3. *Non-local limits:* If ε_n does not tend to zero, or tends to zero too slowly relative to $|V_n|$, the limit might be a non-local energy of the form $\iint \eta(x-y)|u(x) - u(y)|^p dxdy$. Characterizing the transition from non-local to local limits as $\varepsilon_n \to 0$ is an interesting phase transition problem.
4. *Manifold setting:* If Ω is a smooth m-dimensional manifold embedded in $\mathbb{R}^d$, and vertices sample the manifold, the limit should be an energy on the manifold. Our analysis extends to this case with appropriate modifications (using geodesic distances, tangent space approximations).

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5. *Time-dependent graphs:* In applications like evolving social networks or dynamic point clouds, the graph changes over time. The continuum limit would then be an energy that evolves in time, possibly described by a gradient flow.
6. *Numerical validation:* Implementing the discrete energies and comparing minimizers with continuum predictions for irregular graphs would test the practical relevance of our theoretical conditions.

8.3. Conclusion

We have established a framework for discrete-to-continuum limits of graph-regularized energies on irregular domains. The results show that convergence is robust to many forms of irregularity, provided the graph sequence exhibits statistical regularity at appropriate scales. This bridges the gap between the classical theory of Γ-convergence for regular discretizations and the irregular graphs encountered in modern data science applications.

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