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Posted Date: 26 December 2025

doi: 10.20944/preprints202512.2247.v1

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The Geometric Proca Field in Gauge-Invariant Weyl Theory

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Abstract

We present a detailed study of the geometrization of the Proca field in the so-called Weyl Invariant Theory, shedding new light on the physical interpretation of the Weyl field. We first describe the field equations of the theory. We then obtain a solution for the weak field using a spherically symmetric and static approximate metric. Our analysis revealed that the Weyl field, in the weak field approximation, exhibits a behaviour identical to the Yukawa potential, similar to the Proca field. Furthermore, the obtained metric solution is equivalent to the Einstein-Proca case, demonstrating that the description of the Weyl field in the Weyl Invariant Theory is consistent with Proca theory in the context of General Relativity. Finally, we conclude that the Weyl field can be formally interpreted as a Proca field of geometrical nature.

Keywords: Gauge-invariant Weyl theory; Proca field; metric solution

PACS: 04.20.Jb; 11.10.kk; 98.80.Cq

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1. Introduction

In this paper, we examine the problem of geometrizing the electromagnetic field in Weyl's unified theory, shedding new light on a physical interpretation of the Weyl field. Our analysis suggests that the electrodynamics originally proposed by Weyl does not coincide with Maxwell's electrodynamics. Revisiting Weyl's theory, we pursue a current perspective that aims to construct an invariant version of Weyl's theory, which we call *Weyl's Invariant Theory*. In this new version, with the aim of making it more complete, we formulate a prescription of how to implement an invariant way of carrying out the coupling of geometry and matter in the theory, using a new concept of invariant metric.

As highlighted in [1], we observe that the action that leads to the field equations in Gauge-Invariant Weyl Theory bears remarkable similarities to the action that describes the interaction of the geometry with the Proca field. As we know, the dynamics of the original Proca field is described by the so-called Proca equations (named after the Romanian physicist Alexandru Proca), first developed to describe the dynamics of massive spin-1 bosons [2]. In the context of the new version of the Weyl field this leads us to the conclusion that the Weyl field does not represent the electromagnetic field, but rather the Proca field, which is, in a natural way, incorporated into the gravitational scenario. Therefore, the Gauge-Invariant Weyl Theory can be seen as a modified approach to gravitation, in which a vector field, formally similar to the Proca field, is considered to have a purely geometric nature.

Finally, to have a deeper understanding of the theory we investigate the field equations in the weak field approximation. We then show that the result obtained is consistent with Proca theory in the context of general relativity.

2. The Field Equations

We begin by defining, in this new approach, a *gauge-invariant metric*, which then allows us to develop an invariant procedure to carry out the coupling between matter and geometry, a question

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that was not dealt with in the original Weyl unified theory [3]. With this idea in mind we then choose, from the conformal structure $\mathcal{W}$ of Weyl geometry, the invariant metric

$$\gamma_{\alpha\beta} = \frac{R}{\Lambda} g_{\alpha\beta}, \tag{1}$$

where R is the scalar curvature constructed with the metric $g_{\alpha\beta}$, chosen among any metric field of $\mathcal{W}$, and Λ is a non-zero constant (formerly associated by Weyl with the cosmological constant, and which defines the so-called *natural gauge*) [?]. (It is quite easy to verify that $\gamma_{\alpha\beta}$ is gauge-invariant) [1].

It should be mentioned that the 1-form field given by $\xi_\alpha = \sigma_\alpha + (\ln R)_{,\alpha}$, where σ_α is any field of an arbitrary gauge of $\mathcal{W}$ is also gauge-invariant [1])[2].

With these two invariant quantities, we can derive the complete field equations in Gauge-Invariant Weyl Theory, considering the usual prescription of the interaction action with matter, in other words, by defining an invariant energy-momentum tensor of matter, using the standard procedure (minimal coupling principle) of general relativity. In this way the matter action will be given by

$$\delta S_{(m)} = \delta\left(\chi \int \sqrt{|\gamma|} L_M(\psi, \nabla\psi) d^4x\right) = \chi \int \sqrt{|\gamma|}\, T^{(m)}_{\alpha\beta} \delta\gamma^{\alpha\beta} d^4x, \tag{2}$$

where L_M denotes the Lagrangian density of matter, ψ generically represents the matter fields, ∇ is the covariant derivative operator defined with respect to $\gamma_{\mu\nu}$, and χ is the coupling constant. It is worth noting that the form taken by $L_M(\psi, \nabla\psi)$ is built according to the principle of minimal coupling adopted in general relativity. Then, the total action will be given by $S = S_{(g)} + S_{(m)}$, where $S_{(g)} = \int [R^2 + \omega F_{\alpha\beta}F^{\alpha\beta}] \sqrt{|\gamma|} d^4x$ corresponds to the field action proposed by Weyl. The field equations are derived from the minimal action principle, that is,

$$\delta S = \delta\left(\int \left[R^2 + \omega F_{\alpha\beta}F^{\alpha\beta}\right] \sqrt{|\gamma|} d^4x\right) + \delta\left(\chi \int L_M(\psi, \nabla\psi) \sqrt{|g|}\ d^4x\right) = 0 \tag{3}$$

It is not difficult to verify that in the *Weyl gauge* (also called the *natural gauge*), where $R = \Lambda$, the above equation becomes [3]:

$$\delta S = 2\Lambda\, \delta \int \left(R - \frac{\Lambda}{2} + \frac{\omega}{2\Lambda} F_{\alpha\beta}F^{\alpha\beta} + \frac{\chi}{2\Lambda} L_M\right) \sqrt{-g}\ d^4x = 0.$$

We then see that in this gauge, proposed by Weyl, the field equations get enormously simplified. Indeed, by expressing the Weylian scalar curvature R in terms of the Riemannian scalar curvature $\tilde{R}$ we obtain [4]

$$\delta \int \left(\tilde{R} + \frac{3}{2}(\sigma_\alpha\sigma^\alpha) - \frac{3}{\sqrt{-g}}(\sqrt{-g}\ \sigma^\alpha)_{|\alpha} - \frac{\Lambda}{2} + \frac{\omega}{2\Lambda} F_{\alpha\beta}F^{\alpha\beta} + \frac{\chi}{2\Lambda} L_M \right) \sqrt{-g}\ d^4x = 0.$$

Then, it follows that

$$\delta \int \left(\tilde{R} + \frac{3}{2}(\sigma_\alpha\sigma^\alpha) - \frac{\Lambda}{2} + \frac{\omega}{2\Lambda} F_{\alpha\beta}F^{\alpha\beta} + \frac{\chi}{2\Lambda} L_M \right) \sqrt{-g} - 3 \int \partial_\alpha\left(\delta(\sqrt{-g}\ \sigma^\alpha)\right)\ d^4x = 0 \tag{4}$$

s

Clearly the last term of the above integral is a surface term, which is then set to zero. Therefore, we have

$$\delta S = 2\Lambda\delta \int \left(\tilde{R} + \frac{3}{2}\sigma_\alpha\sigma^\alpha - \frac{\Lambda}{2} + \frac{\omega}{2\Lambda} F_{\alpha\beta}F^{\alpha\beta} + \frac{\chi}{2\Lambda} L_M\right) \sqrt{-g}\ d^4x.$$

By taking variations with respect to the metric $g_{\mu\nu}$ and the field σ_α, one is led to the field equations

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$$\tilde{R}_{\alpha\beta} - \frac{1}{2}\tilde{R}g_{\alpha\beta} + \frac{\Lambda}{4}g_{\alpha\beta} = \frac{\omega}{\Lambda}T_{\alpha\beta} - \kappa T^{(m)}_{\alpha\beta}, \tag{5}$$

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\alpha\beta})_{,\beta} = \frac{3\Lambda}{2\omega}\sigma^{\alpha}, \tag{6}$$

where $T_{\alpha\beta} = F_{\alpha}{}^{\nu}F_{\nu\beta} + \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - \frac{3\Lambda}{2\omega}\left(\sigma_{\alpha}\sigma_{\beta} - \frac{1}{2}g_{\alpha\beta}\sigma_{\mu}\sigma^{\mu}\right)$ is interpreted as the energy-momentum tensor of the Weyl field, while $T^{(m)}_{\alpha\beta}$ is the energy-momentum tensor of matter, and $\kappa = \frac{\chi}{2\Lambda}$. The Equations (5) and (6) are the field equations which describe the dynamics of both the gravitational field and the Weyl field, as well as their interaction with matter.

3. The Geometrization of the Proca Field

In the original Weyl unified theory both the gravitational and the electromagnetic fields are geometrized. In his proposal σ_{μ} corresponds to a particular kind of non-metricity and may be viewed as the field that regulates parallel transport in the spacetime manifold. As is well known, the so-called *length curvature* leads naturally to the appearance of a 2-form field, namely $F = d\sigma$, which has rather unexpected algebraic and invariant properties analogous to the Faraday tensor of eletromagnetic theory. However, it turns out that this analogy is not complete. Indeed, if we look at the Equation (6) we observe that the field σ_{μ} interacts with itself, in constrast with the electromagnetic 4-potential, say A_{μ}, which has as its only source the 4-current [5]. Furthermore, we have no way to distinguish between the motions of positive and negative charged particles. The only curves that remain invariant under Weyl transformations are the affine geodesics, which, in turn, provide no information about the motion of particles influenced by both the gravitational field and the electromagnetic field, which would be analogous to the Lorentz force.

On the other hand, as pointed out in [1], when we consider the variation of the action in the Weyl gauge

$$\delta\int\left[\tilde{R} + \frac{3}{2}(\sigma_{\alpha}\sigma^{\alpha}) - \frac{\Lambda}{2} + \frac{\omega}{2\Lambda}F_{\alpha\beta}F^{\alpha\beta}\right]\sqrt{-g}\,d^4x = 0 \tag{7}$$

we can formally identify the above equation as describing the dynamics of a massive Proca field in a curved space-time in the presence of the cosmological constant [9]).

As we know, Proca equation describes the dynamics of a spin-1 massive boson [2]. Here we see that the equation of the Weyl field corresponds to the Proca equation in curved space-time. On the other hand, the energy-momentum tensor of the Weyl field looks the same as the energy-momentum tensor of the Proca field. We therefore may interpret the Weyl field not as a geometric electromagnetic field as originally proposed by Weyl, but rather as a geometric Proca field . We thus can regard this approach as a modified gravity theory which introduces a massive vector field of geometric nature. If we restrict ourselves to negative values of the coupling constant coupling ω, then we may consider in the Equation (6), $m = \sqrt{\frac{-3\Lambda}{2\omega}}$ as its mass. Let us recall that we are considering Λ to be interpreted as the cosmological constant according to Weyl's original ideas; hence we take $\Lambda > 0$.

4. The Proca Field in Gauge-Invariant Weyl Theory

It should be mentioned that recently the Proca field has been well investigated in the framework of general relativity. The case of a spacetime generated by a source consisting of a single pointlike Proca charge has been studied with interest, leading to solutions obtained in the weak field regime (see, for instance, [6–18].) In the present section, we will obtain a simple solution to the field equations in the context of Gauge-Invariant Weyl Theory.

Solving the Field Equations in the Weak Field Approximation

Due to the presence of mass the energy-momentum tensor of the Weyl-Proca has a non-null trace, and thus the field equations of the Gauge-Invariant Weyl Theory in matter vacuum, i.e. with $T^{(m)}_{\alpha\beta} = 0$ may be written in the following form:

$$\tilde{R}_{\alpha\beta} = \frac{\Lambda}{4} g_{\alpha\beta} + \frac{\omega}{\Lambda}\left(T^{(P)}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^{(P)}\right) \tag{8}$$

$$\frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}F^{\alpha\beta}) = -m^2 \ \ \sigma^\alpha, \tag{9}$$

where $\tilde{R}_{\alpha\beta}$ denotes the (Riemannian) Ricci tensor, $T^{(P)}_{\alpha\beta} = F_\alpha{}^\nu F_{\nu\beta} + \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} + m^2\left(\sigma_\alpha\sigma_\beta - \frac{1}{2} g_{\alpha\beta}\sigma_\mu\sigma^\mu\right)$ corresponds to the energy-mometum tensor of the Weyl-Proca field, while $m^2 = \left(\frac{3\Lambda}{2|\omega|}\right)$ is its mass and $T^{(P)} = -m^2\sigma_\mu\sigma^\mu$ gives the trace of $T^{(P)}_{\alpha\beta}$, that is,

$$T^{(P)} = -m^2\sigma_\mu\sigma^\mu \tag{10}$$

We shall now consider the case of a static and spherically symetric spacetime, whose metric in the weak-field regime takes the usual form

$$ds^2 = (1+\varepsilon\nu)dt^2 - (1+\varepsilon\lambda)dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{11}$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$, with ε being a first-order parameter in the linear approximation. The components of the Ricci tensor in the above approximation are then given by

$$\tilde{R}_{00} = -\frac{\nu''\varepsilon}{2} - \frac{\nu'\varepsilon}{r}, \quad \tilde{R}_{11} = \frac{\nu''\varepsilon}{2} - \frac{\lambda'\varepsilon}{r},$$

$$\tilde{R}_{22} = \frac{1}{2} r\nu'\varepsilon - \frac{1}{2} r\lambda'\varepsilon - \lambda\varepsilon, \quad \tilde{R}_{33} = \sin^2\theta\, R_{22}.$$

Clearly for consistency with the weak-field approximation both the terms corresponding to the cosmological constant and the Weyl-Proca energy-momentum which appear in the Equation (8) must also be of first-order in ε. This means that due to the assumed symmetries the Weyl-Proca field may be written as

$$\sigma_\mu = (\sqrt{\varepsilon}\varphi(r), 0, 0, 0),$$

which then garantees that its energy-momentum tensor has all its components of order ε.

From the above considerations it follows that the only non-trivial equation of (9) is

$$\partial_1(\sqrt{-g}\, F^{01}) = -m^2\, \sigma^0 \sqrt{-g}. \tag{12}$$

In this approximation we have $\sqrt{-g} = r^2\sin\theta\left[1 + \frac{\varepsilon}{2}(\nu+\lambda)\right]$, and the equation (12) becomes[4]:

$$-\frac{d}{dr}\left(r^2\sin\theta\left[1+\frac{\varepsilon}{2}(\nu+\lambda)\right]\frac{\sqrt{\varepsilon}\,\varphi'(r)}{[1+\varepsilon(\lambda+\nu)]}\right) = -m^2 r^2\sin\theta\left[1+\frac{\varepsilon}{2}(\nu+\lambda)\right](1-\varepsilon\nu)\sqrt{\varepsilon}\varphi(r),$$

which, as $\varepsilon \ll 1$, may be written as

$$-\frac{d}{dr}\left(r^2\left\{1+\frac{\varepsilon}{2}(\nu+\lambda)\right\}[1-\varepsilon(\lambda+\nu)]\sqrt{\varepsilon}\,\varphi'(r)\right) = -m^2 r^2\left[1+\frac{\varepsilon}{2}(\nu+\lambda)\right](1-\varepsilon\nu)\sqrt{\varepsilon}\varphi(r).$$

Now, after neglecting the terms of order $\varepsilon^{\frac{3}{2}}$ the above equation becomes

$$-\frac{d}{dr}\left(r^2\sqrt{\varepsilon}\ \varphi'(r)\right) = -m^2 r^2\sqrt{\varepsilon}\varphi(r).$$

 doi:10.20944/preprints202512.2247.v1

Therefore the equation of the Weyl-Proca field takes the simpler form

$$\varphi''(r) + \frac{2}{r}\varphi'(r) - m^2\varphi(r) = 0, \tag{13}$$

whose general solution is

$$\varphi(r) = \frac{C_1}{r}e^{-mr} + \frac{C_2}{r}e^{mr},$$

where C_1 and C_2 are arbitrary. By choosing these constants to be $C_2 = 0$ and $C_1 = q$, q being interpreted as a kind of *geometric charge*, we finally obtain the final form of the Weyl field

$$\varphi(r) = \frac{q}{r}e^{-mr} \tag{14}$$

At this point it is worth noting that the only non-trivial component of Weyl-Proca field presents an identical behaviour with the Yukawa potential. For the other components we have

$$\sigma_\mu = \sqrt{\varepsilon}\frac{q}{r}e^{-mr}\delta_{0\mu} \tag{15}$$

Let us now turn our attention for the metric field by considering the equation (8). The non-null components of the energy-momentum tensor are diagonal and are given by

$$T_{00} = \frac{1}{2}\frac{\varepsilon\,\varphi'(r)^2}{(1+\varepsilon\lambda)} + \frac{m^2}{2}\varepsilon\varphi(r)^2,$$

$$T_{11} = -\frac{1}{2}\frac{\varepsilon\,\varphi'(r)^2}{(1+\varepsilon\nu)} + \frac{m^2}{2}\frac{\varepsilon\varphi(r)^2(1+\varepsilon\lambda)}{(1+\varepsilon\nu)},$$

$$T_{22} = \frac{1}{2}r^2\frac{\varepsilon\,\varphi'(r)^2}{(1+\varepsilon\lambda)(1+\varepsilon\nu)} + \frac{m^2}{2}\frac{\varepsilon\, r^2\varphi(r)^2}{(1+\varepsilon\nu)},$$

$$T_{33} = \sin^2\theta\, T_{22}.$$

Then the field equations for (8) $\mu = \nu = 0, 1$ lead to

$$-\frac{\nu''\varepsilon}{2} - \frac{\nu'\varepsilon}{r} = \frac{\Lambda}{4}(1+\varepsilon\nu) + \frac{\omega}{\Lambda}\left(\frac{1}{2}\frac{\varepsilon\,\varphi'(r)^2}{(1+\varepsilon\lambda)} + \frac{m^2}{2}\,\varepsilon\,\varphi(r)^2 + (1+\varepsilon\nu)\frac{m^2}{2}\,(1-\varepsilon\nu)\varepsilon\,\varphi(r)^2\right)$$

$$\frac{\nu''\varepsilon}{2} - \frac{\lambda'\varepsilon}{r} = -\frac{\Lambda}{4}(1+\varepsilon\lambda) + \frac{\omega}{\Lambda}\left(-\frac{1}{2}\frac{\varepsilon\,\varphi'(r)^2}{(1+\varepsilon\nu)} + \frac{m^2}{2}\frac{\varepsilon\varphi(r)^2(1+\varepsilon\lambda)}{(1+\varepsilon\nu)} - (1+\varepsilon\lambda)\frac{m^2}{2}\,(1-\varepsilon\nu)\varepsilon\,\varphi(r)^2\right),$$

Keeping only terms of first order in ε and recalling that we have assumed Λ to be of order of ε the equations become

$$-\frac{\nu''\varepsilon}{2} - \frac{\nu'\varepsilon}{r} = \frac{\Lambda}{4} + \frac{\omega}{\Lambda}\left(\frac{1}{2}\varepsilon\,\varphi'(r)^2 + \frac{m^2}{2}\,\varepsilon\,\varphi(r)^2 + \frac{m^2}{2}\,\varepsilon\,\varphi(r)^2\right) \tag{16}$$

$$\frac{\nu''\varepsilon}{2} - \frac{\lambda'\varepsilon}{r} = -\frac{\Lambda}{4} + \frac{\omega}{\Lambda}\left(-\frac{1}{2}\varepsilon\,\varphi'(r)^2 + \frac{m^2}{2}\,\varepsilon\,\varphi(r)^2 - \frac{m^2}{2}\,\varepsilon\,\varphi(r)^2\right), \tag{17}$$

Adding the above equations lead to

$$\nu' = -\lambda' - \frac{\omega}{\Lambda}m^2 r\,\varphi(r)^2. \tag{18}$$

For $\mu = \nu = 2$ (8) reads

$$\frac{1}{2}r\nu'\varepsilon - \frac{1}{2}r\lambda'\varepsilon - \lambda\varepsilon = -\frac{\Lambda}{4}r^2 + \frac{\omega}{\Lambda}\left(\frac{1}{2}r^2\frac{\varepsilon\,\varphi'(r)^2}{(1+\varepsilon\lambda)(1+\varepsilon\nu)} + \frac{m^2}{2}\frac{\varepsilon\,r^2\varphi(r)^2}{(1+\varepsilon\nu)} - \frac{m^2}{2}r^2(1-\varepsilon\nu)\varepsilon\,\varphi(r)^2\right),$$

which, after keeping only first-order terms in ε, becomes

$$\frac{1}{2}r\nu'\varepsilon - \frac{1}{2}r\lambda'\varepsilon - \lambda\varepsilon = -\frac{\Lambda}{4}r^2 + \frac{\omega}{\Lambda}\frac{1}{2}r^2\varepsilon\,\varphi'(r)^2.$$

Now by using (18) we readily obtain

$$-r\lambda'\varepsilon - \lambda\varepsilon - \frac{\omega}{2\Lambda}m^2r^2\varepsilon\,\varphi(r)^2 = -\frac{\Lambda}{4}r^2 + \frac{\omega}{2\Lambda}r^2\varepsilon\,\varphi'(r)^2.$$

On the other hand, the equation for the metric function $\lambda(r)$ will be given by

$$-(r\lambda)' = -\frac{\Lambda}{4\varepsilon}r^2 + \frac{\omega}{2\Lambda}r^2\,\varphi'(r)^2 + \frac{\omega}{2\Lambda}m^2r^2\,\varphi(r)^2. \tag{19}$$

In this way, by substituting $\varphi(r)$ in the above equation (19) we will obtain [5]:

$$-(r\lambda)' = -\frac{\Lambda}{4\varepsilon}r^2 + \frac{\omega}{2\Lambda}\frac{q^2}{r^2}e^{-2mr} + \frac{\omega}{\Lambda}m\frac{q^2}{r}e^{-2mr} + \frac{\omega}{\Lambda}m^2q^2e^{-2mr}.$$

Integrating this equation in the variable r gives us

$$\lambda(r) = \frac{C_3}{r} + \frac{\Lambda}{12\varepsilon}r^2 + \frac{\omega}{\Lambda}\frac{q^2}{r^2}e^{-2mr} + \frac{\omega}{2\Lambda}m\frac{q^2}{r}e^{-2mr}, \tag{20}$$

where C_3 is an integration constant. From (18) we finally obtain

$$\nu(r) = -\frac{C_3}{r} - \frac{\Lambda}{12\varepsilon}r^2 - \frac{\omega}{\Lambda}\frac{q^2}{r^2}e^{-2mr} + \frac{\omega}{2\Lambda}mq^2\int\frac{e^{-2mr}}{r^2}dr$$

Thus, in the weak field regime the spacetime in the Gauge-Invariant Weyl Theory is described by the following metric:

$$\begin{aligned} ds^2 = &\left(1 - \frac{2m_g}{r} - \frac{\Lambda}{12}r^2 - \frac{\varepsilon\,\omega}{2\Lambda}\frac{q^2}{r^2}e^{-2mr} + \frac{\varepsilon\,\omega}{2\Lambda}m^2q^2\int_r^\infty\frac{e^{-2mr}}{r^2}dr\right)dt^2 \\ &-\left(1 + \frac{2m_g}{r} + \frac{\Lambda}{12}r^2 + \frac{\varepsilon\,\omega}{2\Lambda}\frac{q^2}{r^2}e^{-2mr} + \frac{\varepsilon\,\omega}{2\Lambda}m\frac{q^2}{r}e^{-2mr}\right)dr^2 \\ &- r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right), \end{aligned} \tag{21}$$

where the constant C_3 is determined by taking $C_3 = \frac{2m_g}{\varepsilon}$, m_g denoting the *geometric mass* and m corresponding to the mass of the Weyl-Proca field.

If we set $\Lambda = 0$, then this result reproduces the solution obtained in the context of general relativity which investigates the dynamics of a spacetime generated by a Proca field sourced by a charged point particle [14]. Moreover, the solution found above has great similarities with the result obtained in [16], which examined the effect of the Proca field in Reissner-Nordstrom-de Sitter spacetime.

However, in the present case the massive vector field has a geometrical nature, in other words, it is part of the spacetime geometry, and shoud be more appropriated called a *geometrical Proca field*. We can thus conclude that the Weyl field in the Gauge-Invariant Weyl Theory presents an identical behaviour as the Proca field in general relativity.

Let us now examine the special case when the mass m of the Weyl-Proca field is very small [6]. By carrying out a Taylor expansion of e^{-mr}, we may consider the approximation

$$e^{-mr} \approx 1 - mr + \frac{1}{2}m^2r^2.$$

Now from (15) it is easy to verify that the Weyl-Proca vector takes the form of a Coulomb potential when the order of the field is $\sqrt{\varepsilon}$, that is,

$$\sigma_\mu = \sqrt{\varepsilon}\frac{q_g}{r}\delta_{0\mu},$$

while the metric, to first order in ε will be given by

$$ds^2 = \left(1 - \frac{2m_g}{r} - \frac{\Lambda}{12}r^2 - \frac{\varepsilon\,\omega\,q^2}{2\Lambda\,r^2}\right)dt^2 - \left(1 + \frac{2m_g}{r} + \frac{\Lambda}{12}r^2 + \frac{\varepsilon\,\omega\,q^2}{2\Lambda\,r^2}\right)dr^2 - r^2 d\theta^2 - r^2\,\sin^2\theta\,d\phi^2 e^{-mr}. \quad (22)$$

As is well-know, the above solution corresponds to the Reissner-Nordstrom-de Sitter spacetime [?].

Finally, to recover the Schwarzschild-de Sitter solution, we could admit the charge q_g has a very small value. Moreover, let us note that (again setting $\Lambda = 0$) the above solution approaches de Sitter spacetime asymptotically when $r \longrightarrow \infty$.

5. Final Remarks

In this work, we analyze the problem of the geometrization of the electromagnetic field in Weyl's theory, offering a new perspective based on his program. To this end, we argue that the electrodynamics originally obtained by Weyl does not coincide with Maxwell's electrodynamics, but rather with a Proca electrodynamics, in which the massive vector field is understood as possessing a purely geometric nature.

We also present, as an application of the field equations of Weyl's invariant theory, a solution for the weak field considering empty space. Comparing the solution obtained with the result already known in the literature, we note a great similarity between them when we take $\Lambda = 0$ [14]. Furthermore, we also observe a remarkable similarity between the solution found and that obtained in [16] Shi in the study of the gravitational interaction of the Proca field in Reissner-Nordstrom de Sitter spacetime in the context of general relativity. Therefore, the description of the geometric Proca field in Weyl's invariant theory proves to be, in a certain way, consistent with the Proca theory in the context of general relativity.

Acknowledgments: This work was partially supported by CNPq and FAPESQ (Brazil).

Notes

1. In Weyl geometry, the scalar curvature under Weyl transformations the scalar curvature and the metric transforms as $\bar{R} = e^{-f(x)}R$, $\bar{g}_{\alpha\beta} = e^{f(x)}g_{\alpha\beta}$. Then it is clear that $\bar{\gamma}_{\alpha\beta} = \frac{\bar{R}}{\Lambda}\bar{g}_{\alpha\beta} = \frac{e^{-f}R}{\Lambda}e^{f}g_{\alpha\beta} \Rightarrow \bar{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta}$.
2. Indeed, $\overline{\xi}_\alpha = \xi_\alpha$ as $\overline{R} = e^{-f(x)}R$ and $\overline{\sigma}_\alpha = \sigma_\alpha + f(x)_{,\alpha}$.
3. Note that in the *natural gauge*, the metric $\gamma_{\alpha\beta}$ and the invariant 1-form ξ_α become $g_{\mu\nu}$ and σ_α, respectively.
4. Because Λ is of first order in ε, in order to keep the mass of the Weyl field $m = \sqrt{\frac{3\Lambda}{2|\omega|}}$ small, the value of ω should to be chosen great enough for consistency with the weak-field regime.
5. The term $(\varphi'(r))^2$ which appears in the equation (19), with $\varphi(r) = \frac{q}{r}e^{-mr}$is given by $(\varphi'(r))^2 = \frac{q^2}{r^4}e^{-2mr} + 2m\frac{q^2}{r^3}e^{-2mr} + m^2\frac{q^2}{r^2}e^{-2mr}$.
6. By definition $m^2 = \frac{3\Lambda}{2|\omega|}$, and as Λ is of first order in ε, then we can consider the case in which m is of order $\sqrt{\varepsilon}$ while ω is left arbitrary within a certain range.

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