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On Quasilinear Algebra of Linear Interval Equations and Interval Cramer's Rule

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Abstract

Determining the solution set of a system of linear interval equations is often a difficult task. Establishing such a theory, which should include the theory of classical systems of linear equations in a special case, opens the door to a very comprehensive and arduous work. In this study, we tried to develop information about the solution sets of such equation systems by using the known quasilinear space concept. First of all, we defined the determinant of an interval matrix as an interval and its rank as a pair of natural numbers. Then, we introduced the quasi-inverse concept for interval matrices and obtained some results based on this. With the help of our results, we proved a theorem that we call the Interval-Cramer's rule regarding the solution of some linear interval equation systems. In addition, regarding the existence of solutions to this type of equations, we give a theorem regarding the rank of the interval matrix that models the equation.

Keywords: interval matrices; linear interval equations; quasilinear operators

MSC: 15A06; 15A39; 32A70; 65G10; 54F05

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1. Introduction

We describe some problems with limited uncertainty as problems with inexact data. We encounter such situations in many scientific problems. These types of problems can sometimes be expressed with a linear interval equation systems. Interval matrices play a vital role in developing solution methods for these problems. For example, in neural networks, if the activation functions are not bounded, then we cannot always guarantee the existence of an equilibrium point of a neural network. In [8] authors investigate existence of a unique equilibrium point for neural networks, which is necessary for global robust asymptotic stability of the neural network model which is a special sets of nonlinear differential equations. Only when it is known that the terms of the parameter matrices must lie in certain closed intervals can an estimate of the results be obtained. In this case we need to work with interval matrices. In general, we need interval matrices in the solution methods for linear systems with inexact data. If the missing data can be confined to the intervals, we can model the linear system with inexact data as a linear interval equation. Let us look at Example 7.5., from [3]. The mesh equations for an electric circuit are expressed as

$$\begin{pmatrix} R_1+R_2 & -R_2 \\ -R_2 & R_2+R_3 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ -V_2 \end{pmatrix}$$

with $V_1 = 10$, $V_2 = 5$, and $R_1 = R_2 = R_3 = 1000 \pm 10\%$. Here R_i denotes resistances, I_i denotes currents and V_i denotes voltages. Find enclosures for I_1 and I_2 with the variation possibility 10% on resistances.

Let us consider interval matrix

$$\mathcal{A} = \begin{pmatrix} [1800, 2200] & [-1100, -900] \\ [-1100, -900] & [1800, 2200] \end{pmatrix}$$

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and (interval) vectors

$$x = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} V_1 \\ -V_2 \end{pmatrix}.$$

Then $\mathcal{A}x = \mathbf{b}$ is the linear interval-equation that will serve the solution of the above linear interval system. In this paper we will try to solve this problem using the quasi-inverse concept and using Interval Cramer's Rule, and we will see that our result is close to the results obtained in Example 7.5. In general, it is highly difficult to determine whether a solution of a system of linear-interval equations exists. The main reason for this is that the set of interval vectors that we will use to find the solution of such equations is not a vector space. So it is hard to obtain a solution method just the same as in classical linear algebra. However, fortunately, the set of all interval vectors has an algebraic structure so-called quasilinear space which is a generalization of a linear spaces. For this reason, we have to develop more general or newer concepts than classical linear algebra concepts. Further, they must be consistent with classical linear algebra concepts. The concept was first introduced by S. M. Aseev in [1]. However, some necessary concepts such as quasispan, quasilinear dependence-independence and basis were not given in this work. These definitions and the definition of the dimension of a quasilinear space are given in references [5,6]. However, we realized that we needed to change some of the nomenclature in the definitions we gave in these works. For example, in a quasilinear space, we called an element that has no inverse with respect to summation a singular element. Some interval matrices may not have an inverse with respect to addition. When we call them singular elements, it is confusing with the known definition of singularity in matrices. Because every classical real matrix is a degenerate interval matrix.

Another important work on the algebraic structure of the class of intervals and more generally of sets is given by S.Markov [18]. The study of parametric linear interval systems and the solution set of parametric interval matrix equation systems belongs to the same class of problems, and for other important works on them it is useful to refer to papers [19–22].

The concept of rank of an interval matrix has already been considered and defined in different versions (See for example [17,24]). Unlike the other definitions, we first recognise that an interval matrix defines a quasilinear operator between quasilinear spaces, so we define the rank of a quasilinear operator here. In order to define it, we must first define the dimension of a quasilinear space. We have already given this definition in ([5,6]), where the dimension of a quasilinear space was defined as a pair of natural numbers. Using this definition, the rank of a quasilinear operator and hence of an interval matrix will be defined as a pair of natural numbers. As known from Linear Algebra, the range of a linear operator is a linear space. Thus the rank of an operator is defined as the dimension of the range space. However, we know from our previous works on quasilinear operators that the range of quasilinear operators may not be a subspace. But we know that the space quasi-spanned by the range is a quasilinear space. Therefore, to define the rank of a quasilinear operator T we will use the dimension of the space quasi-spanned by the range of T . Correspondingly, we define the row and column ranks of an interval matrix as the dimension of the spaces quasi-spanned by the row and column vectors. We will also give an example that the row and column ranks may not be the same if the interval matrix is not degenerate, i.e. not a classical matrix. If the row rank is equal to the column rank, we will call this value the rank of the interval matrix. When we consider these points, the concept of rank we give is quite different from other rank definitions of an interval matrix. Since it can give these details better, the definition of quasilinear space given by Aseev is more advantageous than the definition given by Markov. Furthermore, interval matrices are quasilinear operators according to the quasilinear operator conjecture given by Aseev. But such a correspondence is not given in Markow's work. This correspondence is fundamental in linear algebra and this is another factor that makes Aseev's definition of quasilinear spaces more advantageous. Continuing in this direction, we give definitions such as the determinant and quasi-inverse of an interval matrix. Thanks to these definitions, we obtain an envelope containing the solution of some linear interval equations. We call this result Interval Cramer's rule in this work. If the solution exists, it is easy to find an envelope containing the

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solution of the linear interval equations. The important thing is to find an acceptably narrow envelope containing the solution set. In the important reference [3], for some simple equations of this type, some authors have obtained reasonable envelopes containing the solution by their own methods. For example, in [3], Example 7.5, a reasonable envelope containing the solution of a this type equation is presented. We solved the same problem with the Interval Cramer's rule developed in this study and obtained an envelope as narrow (small enough margin of error) as in [3], Example 7.5. This gave us the impression that our rule is functional in many cases, but not always

The basic studies on the solution of systems of linear interval equations given by square matrix are given in references [2,14,15]. However, earlier studies on the subject were given by Farkas, [13], and Oettli, [12]. Later, in [9,10], important contributions were made to the solution of linear systems of interval equations. Mainly in this work we first try to define the determinant of an interval matrix as an interval and its rank as a pair of natural numbers. Then, we introduce the notion of quasi-inverse of an interval matrix and obtain some results based on this concept. Further we aim to prove a theorem that we call Interval-Cramer's rule regarding the solution of some linear interval equation systems. In addition, regarding the existence of solutions to this type of equations, we give a theorem related to the rank of an interval matrix that models the equation.

2. Interval Vectors and Matrices

An n-dimensional interval vector $\mathbf{x} = \left(\left[\underline{x_1}, \overline{x_1}\right], \ldots, \left[\underline{x_n}, \overline{x_n}\right]\right)$ is a set in $\mathbb{R}^n$ such that each component $x_i = \left[\underline{x_i}, \overline{x_i}\right]$ is a closed real interval for $i = 1, 2, ..., n$. In some cites this equivalent notation can be written as

$$\begin{aligned} \mathbf{x} &= [\underline{x}, \overline{x}] \\ &= \{x = (x_i) : \underline{x} \leq x \leq \overline{x},\ i.e.,\ \underline{x_i} \leq x_i \leq \overline{x_i},\ i = 1, 2, ..., n\}. \end{aligned}$$

We think that the first notation is more suitable in our works. We denote by $\mathbb{I}^n_{\mathbb{R}}$ the set of all n-dimensional interval vectors. Actually, saying $\mathbb{I}^n_{\mathbb{R}}$ is n-dimensional a bit of a misnomer since $\mathbb{I}^n_{\mathbb{R}}$ is not a vector space. It's just a word-of-mouth concept. In order to properly understand the concept of the dimension of $\mathbb{I}^n_{\mathbb{R}}$, we need to construct the concept of dimension for quasilinear spaces. We tried in some former works to perform this aim. Let us give some former work about quasilinear algebra. Here for any real scalar λ

$$\lambda.\mathbf{x} = \left(\lambda\left[\underline{x_i}, \overline{x_i}\right]\right) \text{ where}$$

$$\lambda\left[\underline{x_i}, \overline{x_i}\right] = \begin{cases} \left[\lambda\underline{x_i}, \lambda\overline{x_i}\right], & \lambda \geq 0 \\ \left[\lambda\overline{x_i}, \lambda\underline{x_i}\right], & \lambda < 0 \end{cases}.$$

A set X is called a quasilinear space, [1], on the field $\mathbb{K}$ of real or complex numbers, if it is a partially ordered set by the relation "$\preceq$", and an algebraic sum operation $+$, and a scalar product are defined in it in such a way that the following conditions hold for all elements $x, y, z, v \in X$ and all $\alpha, \beta \in \mathbb{K}$: $(X, +)$ is an abelian ordered monoid with the zero $\theta \in X$ and the following other conditions

$$\alpha(\beta x) = (\alpha\beta)x, \tag{2.1}$$

$$\alpha(x + y) = \alpha x + \alpha y, \tag{2.2}$$

$$1x = x, \tag{2.3}$$

$$0x = \theta, \tag{2.4}$$

$$(\alpha + \beta)x \preceq \alpha x + \beta x, \tag{2.5}$$

$$x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \tag{2.6}$$

$$\alpha x \preceq \alpha y \text{ if } x \preceq y. \tag{2.7}$$

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Any element x of a quasilinear space (briefly QLS) is again called as "vector", just is the same as in the linear spaces. Any linear space is a QLS with the partial order relation "$=$", but not conversely. In a QLS X, the element θ is minimal, i.e., $x = \theta$ if $x \preceq \theta$. An element x' is called *additive inverse* of $x \in X$ if $x + x' = \theta$. The inverse is unique whenever it exists. An element x possessing the inverse is called a *stone*, otherwise is called a *foam*. We proved in [4] that each stone is a minimal element.

Lemma 1. *[1]Suppose that each element x in QLS X has inverse element $x' \in X$. Then the partial order in X is determined by equality, the distributivity conditions hold, and consequently X is a linear space.*

In any linear space, the equality is the only way to define a partial order such that conditions (1)-(13) hold [1].

It will be assumed in what follows that $-x = (-1)x$. Note that x' may not be exist but if it exist then $x' = -x$. For example, the interval $[1, 2]$ is a foam in $\mathbb{I}_{\mathbb{R}}$, a nonlinear QLS, since the additive inverse of the element $[1, 2]$ does not exist. However $-[1, 2] = [-2, -1] \in \mathbb{I}_{\mathbb{R}}$. All degenerate intervals are stones and all non-degenerate intervals are foams in $\mathbb{I}_{\mathbb{R}}$. Let us give an easy characterization of stones. An element x is a stone in any QLS if and only if $x' = -x$, or equivalently, $x - x = \theta$. We should note that in a linear QLS, briefly in a linear space, each element is a stone. Hence the notions of stone and foam in linear spaces are redundant. An element x in a QLS X is said to be *balanced* whenever $-x = x$, and X_b denotes the set of all such elements in X.

Suppose X is a QLS and $Y \subseteq X$. Then Y is said to be a *subspace of* X whenever Y is a QLS with the same partial order and with the same algebraic operations on X. In [1] the concept of a subspace for a QLS was not defined. After detailed investigations we saw that the characterization of the definition must be the same as in linear subspaces: Y is a subspace of X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y \in Y$ [4]. There exist three important subspaces of any QLS X. The space X_s which is the class of all stones, X_f which is the class of all foams with the zero and X_b which is the class of all balanced elements. We call X_s and X_f as stone and foam subspace of X respectively. X_b is known as balanced subspace of X. Note that the quasilinear space X_s is a linear space but X_f and X_b are not. That is why we call X_s as the linear part of X. Further, $X_s \cap X_f = \{\theta\}$.

An $m \times n$ interval matrix $\mathcal{A} = [\underline{A}, \overline{A}]$ is defined as the set $\{A \in M^{m\times n} : \underline{A} \le A \le \overline{A}\}$ of all real-term $m \times n$ matrices A such that $\underline{A} = (\underline{a_{ij}})$ and $\overline{A} = (\overline{a_{ij}})$ are fixed $m \times n$-matrices and are lower and upper bounds of $\mathcal{A}$, respectively. Writing interval matrices with their rows and columns explicitly shown will make our next results more understandable. Hence let us use the notation

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1n} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{m1} & \mathcal{A}_{m2} & \cdots & \mathcal{A}_{mn} \end{pmatrix} = \begin{pmatrix} [\underline{a_{11}}, \overline{a_{11}}] & \cdots & [\underline{a_{1n}}, \overline{a_{1n}}] \\ [\underline{a_{21}}, \overline{a_{21}}] & \cdots & [\underline{a_{2n}}, \overline{a_{2n}}] \\ \vdots & \ddots & \vdots \\ [\underline{a_{m1}}, \overline{a_{m1}}] & \cdots & [\underline{a_{mn}}, \overline{a_{mn}}] \end{pmatrix}$$

from now on, where $\mathcal{A}_{ij} = \left[\underline{a_{ij}}, \overline{a_{ij}}\right]$. Let us denote by $\mathcal{IM}^{m\times n}$, the family of all $m \times n-$interval matrices. Thus by former notation $\mathcal{IM}^{m\times n}$ is just

$$\mathcal{IM}^{m\times n} = \{\mathcal{A} : \mathcal{A} = [\underline{A}, \overline{A}] \text{ where } \underline{A},\ \overline{A} \text{ are bounds}\}.$$

As a special case $\mathcal{IM}^n$ denotes briefly $\mathcal{IM}^{n\times n}$. If $\overline{a_{ij}} = \underline{a_{ij}}$, for each i, j then $\mathcal{A}$ is called *degenerate* and any degenerate interval matrix is a singleton including only one classical real-term matrix A. In this case, we can write $\mathcal{A} = \{A\}$, or sometimes $\mathcal{A} = A$. For two elements $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{IM}^{m\times n}$, addition operation is defined by

$$\begin{aligned} \mathcal{A} \oplus \mathcal{B} &= \{A + B : \underline{A} \le A \le \overline{A} \text{ and } \underline{B} \le B \le \overline{B}\} \\ &= \{\mathcal{A}_{ij} + \mathcal{B}_{ij} : \mathcal{A}_{ij} \in \mathcal{A} \text{ and } \mathcal{B}_{ij} \in \mathcal{B}\} \end{aligned}$$

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and by this operation $\mathcal{IM}^{m\times n}$ is an abelian monoid with the identity interval matrix zero, which is a degenerate (classical) $m \times n$ zero matrix. Obviously $(\mathcal{IM}^{m\times n}, \oplus)$ is not a group since some elements have no additive inverses. Let's call a degenerate (classical) interval matrix $\mathcal{A}$ as a stone since it has an additive inverse, and call it as a foam since it has no additive inverse. Just like in classical matrices, we use the term inverse only for the multiplicative inverse in interval matrices. While in classical matrices the additive inverse is always exist but not in nondegenerate (pure) interval matrices. This is why we introduced the concepts of stone and foam. It is easy to prove that any interval matrix is degenerate if and only if it is a stone. Let us denote by $\mathcal{IM}_s^{m\times n}$ the class of all degenerate elements (stones) and denote by $\mathcal{IM}_f^{m\times n}$ the class of all foams in $\mathcal{IM}^{m\times n}$. The function $f : \mathcal{IM}_s^{m\times n} \to M^{m\times n}$, $f(\{A\}) = A$ is a bijection and hence we can see that the set $M^{m\times n}$ of all classical real $m \times n$-matrices are equivalent to $\mathcal{IM}_s^{m\times n}$.

For two elements $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{IM}^{m\times n}$, the relation

$$\mathcal{A} \subseteq \mathcal{B}\ iff\ \mathcal{A}_{ij} \subseteq \mathcal{B}_{ij} \text{ for each } \mathcal{A}_{ij} \in \mathcal{A} \text{ and } \mathcal{B}_{ij} \in \mathcal{B},$$

is a partial order and hence $(\mathcal{IM}^{m\times n}, \oplus, \subseteq)$ is a partially ordered monoid with the compatibility condition:

$$\mathcal{A} \subseteq \mathcal{B} \text{ and } \mathcal{C} \subseteq \mathcal{D} \text{ implies } \mathcal{A} \oplus \mathcal{C} \subseteq \mathcal{B} \oplus \mathcal{D}.$$

If $\mathcal{A}$ is a stone and $\mathcal{B}$ is a foam then $\mathcal{A} \subseteq \mathcal{B}$ means $A \in \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are both stones then they are classical matrices and $\mathcal{A} \subseteq \mathcal{B}$ means $\mathcal{A} = \mathcal{B}$. If $\mathcal{A}$ is a foam, $\mathcal{B}$ is a stone then the assumption $\mathcal{A} \subseteq \mathcal{B}$ indicates that $\mathcal{B}$ also has to be a foam. We can summarize the last case as follows: "any foam cannot be a subset of a stone". Following proposition states this assertion and it can easily be proved.

Proposition 1. *The zero interval matrix θ and moreover all stones are minimal elements in ordered monoid* $(\mathcal{IM}^{m\times n}, \oplus, \subseteq)$.

For the field $\mathbb{R}$ the law $\cdot : \mathbb{R}\times\mathcal{IM}^{m\times n} \to \mathcal{IM}^{m\times n}$ is known as the scalar product on $\mathcal{IM}^{m\times n}$ and has the following properties: for all elements $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathcal{IM}^{m\times n}$ and for all $\alpha, \beta \in \mathbb{R}$,

$$\alpha \cdot (\beta \cdot \mathcal{A}) = (\alpha\beta) \cdot \mathcal{A}, \tag{2.8}$$

$$\alpha \cdot (\mathcal{A} \oplus \mathcal{B}) = \alpha \cdot \mathcal{A} \oplus \alpha \cdot \mathcal{B}, \tag{2.9}$$

$$1 \cdot \mathcal{A} = \mathcal{A}, \tag{2.10}$$

$$0 \cdot \mathcal{A} = \theta, \tag{2.11}$$

$$(\alpha + \beta) \cdot \mathcal{A} \subseteq \alpha \cdot \mathcal{A} \oplus \beta \cdot \mathcal{A}, \tag{2.12}$$

$$\mathcal{A} \oplus \mathcal{C} \subseteq \mathcal{B} \oplus \mathcal{D} \text{ if } \mathcal{A} \subseteq \mathcal{B} \text{ and } \mathcal{C} \subseteq \mathcal{D}, \tag{2.13}$$

$$\alpha \cdot \mathcal{A} \subseteq \alpha \cdot \mathcal{B} \text{ if } \mathcal{A} \subseteq \mathcal{B}. \tag{2.14}$$

By this properties we construct an algebraic structure $(\mathcal{IM}^{m\times n}, \oplus, \cdot, \subseteq)$. We will again write $\alpha\mathcal{A}$ for $\alpha \cdot \mathcal{A} = \{\alpha A : A \in \mathcal{A}\}$ in the sequel. In this respect, $(\mathcal{IM}^{m\times n}, \oplus, \cdot, \subseteq)$ is a quasilinear space on the field $\mathbb{R}$.

Example 1. *Let $\mathcal{A} = [\underline{A}, \overline{A}]$ where $\underline{A} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$ and $\overline{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Alternatively, using interval notation, we write $\mathcal{A} = \begin{pmatrix} [-1,1] & [0,0] \\ [-1,1] & [-1,1] \end{pmatrix}$. This is a balanced interval matrix and hence it is an element of the subspace $\mathcal{IM}_b^2$ of $\mathcal{IM}^2$. It is also a subspace of $\mathcal{IM}_f^2$. Furthermore, except for the zero, all balanced interval matrices are foams. For $n = 1$, $\mathcal{IM}^1$ corresponds to $\mathbb{I}_{\mathbb{R}}$, the quasilinear space of all closed intervals of real*

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numbers, and $\mathcal{IM}_d^1$ *corresponds to* $\mathbb{R}$. *Further,* $\mathcal{B} = \begin{pmatrix} [1,3] & [2,2] & [1,3] \\ [-1,-1] & [4,4] & [2,3] \end{pmatrix}$ *is a foam and an element of* $\mathcal{IM}_f^{2\times 3}$ *while*

$$\mathcal{C} = \begin{pmatrix} [1,1] & [2,2] & [1,1] \\ [-1,-1] & [4,4] & [3,3] \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 3 \end{pmatrix}$$

is a stone and $\mathcal{C} \in \mathcal{IM}_d^{2\times 3} \equiv M^{2\times 3}$.

3. Dimension and Basis in the space of Interval Vectors

Any $\mathbf{x} = ([\overline{x_1}, \underline{x_1}], \ldots, [\overline{x_n}, \underline{x_n}])$ is known as n-dimensional interval vector, in fact, a set in $\mathbb{R}^n$ such that each component $x_i = [\overline{x_i}, \underline{x_i}]$ is a closed real interval for $i = 1, 2, ..., n$. With an another notation in interval analysis we can write $\mathbf{x} = [\overline{x}, \underline{x}]$ where $\overline{x}$ and $\underline{x}$ are bounds of $\mathbf{x}$ and they are n-tubles of real numbers. But former notation is most useful in this work. Of course, each n-dimensional interval vector can be seen as an interval (column) matrix and we denote by $\mathbb{I}_{\mathbb{R}}^n$ the set of all n-dimensional interval vectors instead of $\mathcal{IM}^{n\times 1}$ and for $n = 1$, $\mathbb{I}_{\mathbb{R}}^1 = \mathbb{I}_{\mathbb{R}}$. By the algebraic operations and by the partial order from the former section, $\mathbb{I}_{\mathbb{R}}^n$ is a quasilinear space on the field $\mathbb{R}$. Actually, calling $\mathbb{I}_{\mathbb{R}}^n$ is n-dimensional is a bit of a misnomer since $\mathbb{I}_{\mathbb{R}}^n$ is not a vector space. It's just a word-of-mouth concept. In order to properly understand the concept of the dimension of $\mathbb{I}_{\mathbb{R}}^n$, we need to construct the concept of dimension for quasilinear spaces.

In this section, let us present some basic results which is obtained formerly in our works [5–7] by slightly changing some notations. Any quasilinear *combination of the set* $\{\mathbf{x}_k\}_{k=1}^n$ in a QLS X is an element $\mathbf{z} \in X$ such that $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + ... + \alpha_n\mathbf{x}_n \preceq \mathbf{z}$ for some scalars $\alpha_1, \alpha_2, ..., \alpha_n$. But any *linear combination of the set* $\{\mathbf{x}_k\}_{k=1}^n$ in X is an element $\mathbf{z}$ of X in the form $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + ... + \alpha_n\mathbf{x}_n = \mathbf{z}$, just the same as is in classical linear (vector) spaces. Hence a linear combination of the set $\{\mathbf{x}_k\}_{k=1}^n$ is an element z of X such that

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + ... + \alpha_n\mathbf{x}_n \preceq \mathbf{z} \text{ and } \mathbf{z} \preceq \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + ... + \alpha_n\mathbf{x}_n.$$

In a linear space, these two definitions are coincide since the relation "$\preceq$" turns out to be the relation "$=$". Clearly, a linear combination of $\{\mathbf{x}_k\}_{k=1}^n$, is also a quasilinear combination of $\{\mathbf{x}_k\}_{k=1}^n$, but not conversely. For any nonempty subset A of a QLS X, the *quasi-span* (*q-span*, for short) $QspA$ *of* A, is defined by the set of all possible quasilinear combinations of A, that is,

$$\begin{aligned} QspA &= \{\mathbf{x} \in X : \sum_{k=1}^{n} \alpha_k\mathbf{x}_k \preceq \mathbf{x}, \\ \text{for } \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n &\in A \text{ and for some scalars } \alpha_1, \alpha_2, ..., \alpha_n\}. \end{aligned}$$

Span of A, SpA, is also defined in quasilinear spaces, just the same as is in classical linear spaces and obviously, $SpA \subseteq QspA$. Further $SpA = QspA$ for some linear QLS (linear space), hence, the notion of $QspA$ is redundant in linear spaces. Moreover, we say A *quasi-spans* X whenever $QspA = X$. We know from former works that $QspA$ is a subspace of X but SpA may not be a subspace of X.

Definition 1. *[7]* (Quasilinear independence *and* dependence) *A set*

$$A = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n\}$$

in a QLS X is called quasilinear independent (briefly ql-independent) *whenever the inequality*

$$\theta \preceq \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + ... + \lambda_n\mathbf{x}_n \tag{3.1}$$

holds if and only if $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$. *Otherwise, A is called* quasilinear dependent (briefly ql-dependent).

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If we recall again that every linear space is a QLS with the relation "=", it can be seen that the notions of quasilinear independence and dependence coincide with linear independence and dependence in linear spaces.

Example 2. *Consider $A = \{[1,2]\}$, a singleton in $\mathbb{I}_{\mathbb{R}}$. It is obvious that $\{0\} = [0,0] \subseteq \alpha.[1,2]$ if and only if $\alpha = 0$ where $\{0\}$ is the zero's of $\mathbb{I}_{\mathbb{R}}$. Therefore, A is ql-independent. However, the singleton $B = \{[-1,2]\}$ is ql-dependent since $[0,0] \subseteq \beta.[-1,2]$ for $\beta = 2 \neq 0$. This is an unusual case since a non-zero singleton is obviously linear independent in linear spaces. On the other hand, the set $\{[1,2],[-1,2]\}$ is ql-dependent. In general, the definition implies that any subset containing an element associated with zero is necessarily ql-dependent in a QLS. This extends the well-known result in linear spaces that any subset containing zero must be linearly dependent.*

Example 3. *In $\mathbb{I}_{\mathbb{R}}^2$, let $\mathbf{v}_1 = ([-2,1],[0,0])$ and $\mathbf{v}_2 = ([0,0],[-2,3])$. Then the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is ql-dependent since*

$$([0,0],[0,0]) \subseteq \lambda_1.\mathbf{v}_1 + \lambda_2.\mathbf{v}_2 = ([-2,1],[-2,3])$$

for $\lambda_1 = \lambda_2 = 1$ where $([0,0],[0,0])$ is the zero's of $\mathbb{I}_{\mathbb{R}}^2$. However $\{\mathbf{u}_1, \mathbf{u}_2\}$ is ql-independent where $\mathbf{u}_1 = ([-2,-1],[0,0])$ and $\mathbf{u}_2 = ([0,0],[2,3])$. On the other hand, let $\mathbf{u} = ([-2,2],[-3,3])$ then the singleton $\{\mathbf{u}\}$ is ql-dependent in $\mathbb{I}_{\mathbb{R}}^2$ since

$$([0,0],[0,0]) \subseteq \mathbf{u}.$$

i-spans $(\mathbb{I}_{\mathbb{R}})_f$.

We now introduce the concept of dimension in a QLS. Our analysis indicates that it should be divided into two distinct notions, namely the stone dimension and the foam dimension. Before doing so, we first present a variation of a classical definition.

Definition 2. *Let S be a ql-independent subset of the QLS X. S is called* maximal ql-independent subset *of X whenever S is ql-independent, but any set icluding S is ql-dependent.*

Definition 3. *[5]* Stone (Foam) dimension *of any QLS X is the cardinality of any maximal ql-independent subsets of $X_d(X_f)$. If this number is finite then X is said to be finite stone (foam)-dimensional, otherwise; is said to be infinite stone (foam)-dimensional. Stone dimension is denoted by s-dim X and foam dimension is denoted by f-dim X. If s-dim $X = m$ and f-dim $X = n$ then we say that X* is an $\left(m_s, n_f\right)$-dimensional QLS *where m and n are natural numbers or ∞.*

The above definition means that s-dim X is classical definition of dimension of the linear space X_d. So, s-dim $X = \dim X_d$. Notice that a non-trivial foam subspace of a QLS cannot be a linear space. Further, we can easily see that any QLS is $\left(n_s, 0_f\right)$-dimensional if and only if it is n-dimensional linear space. In this respect, the trivial linear space $\{0\}$ is a $\left(0_s, 0_f\right)$-dimensional QLS. We known former work [5] that there is an example of a $\left(0_s, 0_f\right)$-dimensional QLS other than the trivial quasilinear space $\{0\}$.

Remark 1. *We can easily see that any set including a balanced element must be ql-dependent in a QLS. Further stone subspace of $\mathbb{I}_{\mathbb{R}}^n$ is $\left(n_s, 0_f\right)-$dimensional while its foam subspace is $\left(0_s, n_f\right)-$dimensional.*

4. Rank and Determinant of an Interval Matrix

This section includes some new definition and results on interval matrices and on the solution of some linear interval equations. We have been frequently benefited from the source [16] for classical linear algebra facts. First of all let us fix some notation for an interval matrix $\mathcal{A}$ with columns and rows. When we consider an interval matrix $\mathcal{A} = (\mathcal{A}_{ij})$ where $\mathcal{A}_{ij} = \left[\underline{a_{ij}}, \overline{a_{ij}},\right]$ then we can write $\mathcal{A} = ((\mathcal{A}_{1k}), (\mathcal{A}_{2k}), ..., (\mathcal{A}_{mk}))$for $k = 1, 2, ..., n$. To get a solution of a system of linear interval equations if it exists we think that we should first define linear algebra-like tools such as rank and inverse of an interval matrix.

First of all let us give some concepts and results on quasilinear operators given by Aseev.

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Definition 4. *[1] Let X and Y be quasilinear spaces. A mapping $T : X \to Y$ is called a* quasilinear operator *if it satisfies the following conditions:*

$$T(x_1 + x_2) \preceq T(x_1) + T(x_2),$$

$$T(\alpha x) = \alpha T(x) \ \text{ for any } \alpha \in \mathbb{R},$$

$$\text{if } x_1 \preceq x_2, \text{ then } T(x_1) \preceq T(x_2).$$

In this definition, the last two conditions remain the same, and if we tighten the first condition a little more so that $T(x_1 + x_2) = T(x_1) + T(x_2)$, we get the definition of a linear operator between quasilinear spaces.

Theorem 1. *An $m \times n$ interval matrix $\mathcal{A}$ defines a quasilinear operator from $\mathbb{I}_{\mathbb{R}}^n$ into $\mathbb{I}_{\mathbb{R}}^m$ by the interval matrix-product $\mathcal{A}\mathbf{x} = \mathbf{b}$, explicitly:*

$$\begin{pmatrix} [\underline{a_{11}}, \overline{a_{11}}] & \cdots & [\underline{a_{1n}}, \overline{a_{1n}}] \\ [\underline{a_{21}}, \overline{a_{21}}] & \cdots & [\underline{a_{2n}}, \overline{a_{2n}}] \\ \vdots & \ddots & \vdots \\ [\underline{a_{m1}}, \overline{a_{m1}}] & \cdots & [\underline{a_{mn}}, \overline{a_{mn}}] \end{pmatrix} \begin{pmatrix} [\underline{x_1}, \overline{x_1},] \\ [\underline{x_2}, \overline{x_2}] \\ \vdots \\ [\underline{x_n}, \overline{x_n}] \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} [\underline{x_j}, \overline{x_j},] [\underline{a_{1j}}, \overline{a_{1j}}] \\ \sum_{j=1}^{n} [\underline{x_j}, \overline{x_j},] [\underline{a_{2j}}, \overline{a_{2j}}] \\ \vdots \\ \sum_{j=1}^{n} [\underline{x_j}, \overline{x_j},] [\underline{a_{mj}}, \overline{a_{mj}}] \end{pmatrix} \tag{4.1}$$

where $\mathbf{x} = ([\underline{x_1}, \overline{x_1},], \ldots, [\underline{x_n}, \overline{x_n}])$ and $\mathbf{b} = \left([\underline{b_1}, \overline{b_1}], \ldots, [\underline{b_m}, \overline{b_m}]\right)$ such that

$$[\underline{b_i}, \overline{b_i}] = \sum_{j=1}^{n} [\underline{x_j}, \overline{x_j},] [\underline{a_{ij}}, \overline{a_{ij}}], \text{ for } i = 1, 2, ..., m,$$

and the product in the summation is the multiplication between intervals.

Proof. Only we are going to prove that $\mathcal{A}(x + z) \subseteq \mathcal{A}(x) + \mathcal{A}(z)$ since verifying the other conditions are routine. But easily we can write from interval aritmetics (see [3, p.99]) that :

$$\begin{aligned} \mathcal{A}(x+z) &= \left(\sum_{j=1}^{n} \left([\underline{x_j}, \overline{x_j},] + [\underline{z_j}, \overline{z_j},] \right) [\overline{a_{ij}}, \underline{a_{ij}}] \right)_{i=1}^{m} \\ &\subseteq \left(\sum_{j=1}^{n} [\underline{x_j}, \overline{x_j},] [\overline{a_{ij}}, \underline{a_{ij}}] \right)_i + \left(\sum_{j=1}^{n} [\underline{z_j}, \overline{z_j},] [\overline{a_{ij}}, \underline{a_{ij}}] \right)_i \\ &= \mathcal{A}(x) + \mathcal{A}(z). \end{aligned}$$

□

Now for such an interval matrix $\mathcal{A}$ and for any interval vectors $\mathbf{b}$, by a system of linear interval equation

$$\mathcal{A}\mathbf{x} = \mathbf{b}$$

we mean a family of all linear equation systems $Ax = b$ such that $A \in \mathcal{A}$, $x \in \mathbf{x}$ and $b \in \mathbf{b}$. If (4.1) has a solution then the solution set is written as

$$\boldsymbol{Y} = \{\mathbf{x} \in \mathbb{I}_{\mathbb{R}}^n : Ax = b \text{ for some } A \in \mathcal{A}, x \in \mathbf{x} \text{ and } b \in \mathbf{b}\}.$$

However, determining the solution sets of such equations is an extremely difficult problem. In fact, a much simpler form of such systems of equations arises when $\mathbb{I}_{\mathbb{R}}^n$ is replaced by its linear subspace

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$\mathbb{R}^n$. In this case the interval matrix $\mathcal{A}$ again defines a quasilinear operator from $\mathbb{R}^n$ into $\mathbb{I}_{\mathbb{R}}^m$ and the interval vector $\mathbf{x}$ becomes a classical real n-tuble x. Moreover, the solution set of the simpler case of equation (4.1) is then expressed as

$$\boldsymbol{Y} = \{x \in \mathbb{R}^n : Ax = b \text{ for some } A \in \mathcal{A} \text{ and } b \in \mathbf{b}\}.$$

Even in this simple case the solution set is very difficult, in fact, it is an NP-hard problem. In the literature, this simpler case known as the system of linear interval equation. An earlier and fundamental results for description of the solution set of simpler case is given in [12]. Some further investigation in this way are presented in ([9,10,13,15]). In fact we aim to develop solution techniques similar to the classical case for the simpler case of (4.1). Hence in this work we consider the linear interval equation

$$\mathcal{A}x = \mathbf{b}$$

where $\mathcal{A}$ is an interval matrix, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and

$$\mathbf{b} = \left(\left[\underline{b_1}, \overline{b_1}\right], \ldots, \left[\underline{b_m}, \overline{b_m}\right]\right) \in \mathbb{I}_{\mathbb{R}}^m.$$

Remark 2. *In general, the solution set* $\mathbf{Y}$ *of* $\mathcal{A}x = \mathbf{b}$ *does not appear as an interval vector. A simple example of the shape of such a set* $\mathbf{Y}$ *can be seen in [3, p.99]. In general, determining the exact solution set for this type problems is an NP-hard problem. Instead, in general, we determine an envelope* $\mathbf{X}$ *containing the solution set* $\mathbf{Y}$*. Further a solution of* $\mathcal{A}x = \mathbf{b}$ *is not an element* x *satisfying the equality* $\mathcal{A}x = \mathbf{b}$*, but an element* x *satisfying the (classical) linear equation* $Ax = b$ *for any* $A \in \mathcal{A}$ *and for any* $b \in \mathbf{b}$*. Let us illustrate this with a simple example. Consider the system of linear interval equations* $[1,2]x = 4$*. Here* $\mathcal{A} = ([1,2])$*,* $x \in \mathbb{R}$ *and* $\mathbf{b} = [4,4]$*. We know from interval arithmetic that there is no real number* x *satisfying this equality. If the solution set were defined in this way, we would say that this equation has no solution. However, this is not the case. According to the definition above, for* $A = (3/2) \in \mathcal{A}$*, the system* $Ax = 4$ *has a solution and* $x = 8/3$ *is the solution. Similarly, for every* $A \in \mathcal{A}$*, there exists a solution of the system* $Ax = 4$ *and the solution set* $\mathbf{Y}$ *of* $[1,2]x = 4$ *is just* $[2,4]$*. In this simple example, the solution set is a 1-dimensional interval vector.*

Definition 5. *Let* $\mathcal{A}$ *be an* $m \times n$ *interval matrix. Then (4.1) is called* quasi-homogeneous *whenever* $0 \subseteq \mathbf{b}$*. The dimension of the solution space of such a quasi-homogeneous system is called the* quasi-nullity of $\mathcal{A}$.

Now since an interval matrix defines a quasilinear operator between quasilinear spaces, we will first define the rank of a quasilinear operator. Firs of all, it should be noted that a quasilinear operator may not be represented as an interval matrix even if its domain and range are finite dimensional. Furthermore, although the domain and range of a linear operator are linear spaces, the range of a quasilinear operator may not be a quasilinear spaces. For example, $T : \mathbb{R} \to \mathbb{I}_{\mathbb{R}}$, $Tx = x[0,2]$ for $x \in \mathbb{R}$, is a quasilinear operatör but the range $\mathcal{R}(T) = \{x[0,2] : x \in \mathbb{R}\}$ is not a subspace of $\mathbb{I}_{\mathbb{R}}$ since $[0,2] - [0,2] = [-2,2] \notin \mathcal{R}(T)$. Therefore, we will use the quasi-span of $\mathcal{R}(T)$ that is, $Qsp(\mathcal{R}(T))$ for the rank definition. If T were defined between linear spaces, in which case it would be a linear operator, then $\mathcal{R}(T)$ would be a linear space and $Qsp(\mathcal{R}(T)) = Sp(\mathcal{R}(T)) = \mathcal{R}(T)$.

Since the notion of dimension is defined above as a pair of natural numbers, the notion of rank will also appear as a pair of natural numbers.

Definition 6. *Let* X *and* Y *be quasilinear spaces.* Rank of a quasilinear operator $T : X \to Y$ *is defined as the dimension of quasi-span of the range of* T *in* Y*, that is,* $RankT = \dim Qsp(\mathcal{R}(T))$*.*

Definition 7. *A quasilinear space which is quasi-spanned by row (column) vectors of an interval matrix* $\mathcal{A}$ *is called* row (column) space *of* $\mathcal{A}$*. The dimension of the row (column) space of* $\mathcal{A}$ *is called the* row (column) rank *of* $\mathcal{A}$*. We denote row and column ranks of* $\mathcal{A}$ *by* $R - rank\mathcal{A}$ *and* $C - rank\mathcal{A}$*, respectively. We will use the symbol* $Rank\mathcal{A} = \left(m_s, n_f\right)$ *if* $R - rank\mathcal{A} = C - rank\mathcal{A} = \left(m_s, n_f\right)$ *where* m *and* n *are natural numbers.*

We will see in the sequel, unlike classical matrices, that the row and column ranks may not be equal in interval matrices.

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Let us give first example from 1×1 interval matrices.

Example 4. *Consider interval matrices* $\mathcal{A} = ([-1,2])$, $\mathcal{B} = ([1,2])$ *and* $\mathcal{C} = ([2,2])$. *Their row and column vectors are the same. The row (column) vector of* $\mathcal{A}$ *is* $[-1,2]$. *First, we must find*

$$Qsp\{[-1,2]\} = \{[\overline{a}, \underline{a}] \in \mathbb{I}_{\mathbb{R}} : \lambda[-1,2] \subseteq [\overline{a}, \underline{a}],\ \lambda \in \mathbb{R}\}.$$

Obviously, $Qsp\{[-1,2]\}$ *never contains degenerate interval other than zero. Hence the stone subspace of* $Qsp\{[-1,2]\}$ *is* $\{[0,0]\}$, *the trivial subspace. So the stone dimension of* $Qsp\{[-1,2]\}$ *is just the zero. Moreover, the foam subspace of* $Qsp\{[-1,2]\}$ *is itself, and every subset of* $Qsp\{[-1,2]\}$ *is ql-dependent. This assertion is clear from the definition of* $Qsp\{[-1,2]\}$ *since* $[0,0] \subseteq [-1,2]$ *and so* $[0,0] \subseteq \lambda[-1,2] \subseteq [\overline{a}, \underline{a}]$ *for some* $\lambda \neq 0$. *This means foam dimension of* $Qsp\{[-1,2]\}$ *is also zero. Eventually we conclude that* $R - rank\mathcal{A} = C - rank\mathcal{A} = \left(0_s, 0_f\right)$ *so that* $Rank\mathcal{A} = \left(0_s, 0_f\right)$.

Now the row and column spaces of $\mathcal{B}$ *are the same and*

$$Qsp\{[1,2]\} = \{[\overline{a}, \underline{a}] \in \mathbb{I}_{\mathbb{R}} : \lambda[1,2] \subseteq [\overline{a}, \underline{a}],\ \lambda \in \mathbb{R}\}.$$

The stone subspace of $Qsp\{[1,2]\}$ *is again the trivial subspace. Therefore, its stone dimension is zero. On the other hand, the foam subspace of* $Qsp\{[1,2]\}$ *is again equal to itself.* $\{[1,2]\}$ *is ql-independent in this space, which tells us that the foam dimension of* $Qsp\{[1,2]\}$ *is 1 or a greater integer. Further, two elements in* $Qsp\{[1,2]\}$ *must be ql-dependent by the definition. So the row and column rank of* $\mathcal{B}$ *is* $\left(0_s, 1_f\right)$ *and hence* $Rank\mathcal{B} = \left(0_s, 1_f\right)$.

Finally, let's consider the interval matrix $\mathcal{C}$. *We know that* $Qsp\{[2,2]\} = \mathbb{I}_{\mathbb{R}}$ *and we know that* $\mathbb{I}^1_{\mathbb{R}}$ *is* $\left(1_s, 1_f\right)$*-dimensional. So,* $Rank\mathcal{C} = \left(1_s, 1_f\right)$. *The matrix* $\mathcal{C}$ *is also a classical matrix and is a transformation between linear spaces. From this point of view, its rank is* 1. *Every linear space is a quasilinear space and when we consider* $\mathcal{C}$ *as a quasilinear operator from the quasilinear space* $\mathbb{R}$ *into itself,* $Rank\mathcal{C} = \left(1_s, 0_f\right)$. *When we define* $\mathcal{C}$ *as a quasilinear operator from* $\mathbb{I}_{\mathbb{R}}$ *into* $\mathbb{I}_{\mathbb{R}}$ *as before,* $Rank\mathcal{C} = \left(1_s, 1_f\right)$.

Example 5. *Now let us give examples from* 2×2 *interval matrices. Consider* $\mathcal{A} = \begin{pmatrix} [0,1] & [-1,2] \\ [1,3] & [1,2] \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} [1,2] & [1,3] \\ [-4,-2] & [1,1] \end{pmatrix}$ *and* $\mathcal{C} = \begin{pmatrix} [1,1] & [3,3] \\ [-2,-2] & [2,2] \end{pmatrix}$.

Rows of $\mathcal{A}$ *are* $\mathbf{v}_1 = ([0,1],[-1,2])$ *and* $\mathbf{v}_2 = ([1,3],[1,2])$. *Now*

$$Qsp\{\mathbf{v}_1, \mathbf{v}_2\} = \{\mathbf{u} : \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \subseteq \mathbf{u},\ \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

where $\mathbf{u} = \left(\left[\overline{a_1}, \underline{a_1}\right], \left[\overline{a_2}, \underline{a_2}\right]\right) \in \mathbb{I}^2_{\mathbb{R}}$. *Again the* $Qsp\{\mathbf{v}_1, \mathbf{v}_2\}$ *never contains any degenerate interval pairs other than zero. Hence the stone subspace of* $Qsp\{\mathbf{v}_1, \mathbf{v}_2\}$ *is the trivial subspace of* $\mathbb{I}^2_{\mathbb{R}}$. *So its stone dimension is the zero. Moreover, its foam subspace is* $Qsp\{\mathbf{v}_1, \mathbf{v}_2\}$, *and every subset of* $Qsp\{\mathbf{v}_1, \mathbf{v}_2\}$ *is ql-dependent. This is clear since* $([0,0],[0,0]) \subseteq 3\mathbf{v}_1 - 2\mathbf{v}_2$, *for example. On the other hand,* $\{\mathbf{v}_2\}$ *is a ql-independent set in* $\mathbb{I}^2_{\mathbb{R}}$ *and in* $Qsp\{\mathbf{v}_1, \mathbf{v}_2\}$. *This means foam dimension of* $Qsp\{\mathbf{v}_1, \mathbf{v}_2\}$ *is* 1. *Eventually we conclude that row rank of* $\mathcal{A}$ *is* $\left(0_s, 1_f\right)$, *that is,* $R - rank\mathcal{A} = \left(0_s, 1_f\right)$. *Let us now determine the column rank of* $\mathcal{A}$. *Consider the column vectors* $\mathbf{u}_1 = ([0,1],[1,3])$ *and* $\mathbf{u}_2 = ([-1,2],[1,2])$ *in* $\mathbb{I}^2_{\mathbb{R}}$.

$$Qsp\{\mathbf{u}_1, \mathbf{u}_2\} = \{\mathbf{w} : \lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 \subseteq \mathbf{w},\ \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

Again the stone subspace of $Qsp\{\mathbf{u}_1, \mathbf{u}_2\}$ *is the trivial subspace and so its stone dimension is the zero. Let us now determine foam dimension of* $Qsp\{\mathbf{u}_1, \mathbf{u}_2\}$. *Observe that* $\{\mathbf{u}_1, \mathbf{u}_2\}$ *is ql-dependent since* $([0,0],[0,0]) \subseteq \mathbf{u}_1 - \mathbf{u}_2$. *This means foam dimension of* $Qsp\{\mathbf{u}_1, \mathbf{u}_2\}$ *cannot be* 2. *Further we cannot find any non-zero* λ *such that* $([0,0],[0,0]) \subseteq \lambda\mathbf{u}_1$. *So* $\{\mathbf{u}_2\}$ *is ql-independent and this implies foam dimension of* $Qsp\{\mathbf{u}_1, \mathbf{u}_2\}$ *is* 1. *As a result we conclude that* $C - rank\mathcal{A} = \left(0_s, 1_f\right)$. *Then we can write* $Rank\mathcal{A} = \left(0_s, 1_f\right)$.

Similarly, we can show that $Rank\mathcal{B} = \left(0_s, 2_f\right)$.

$\mathcal{C}$ *is in fact a classical matrix and we know that its rank is 2 as a mapping on (quasi) linear space* $\mathbb{R}^2$. *Now let us see that its rank is* $\left(2_s, 2_f\right)$ *as a mapping on quasilinear space* $\mathbb{I}^2_{\mathbb{R}}$. *Consider row vectors* $\mathbf{u}_1 = ([1,1],[3,3])$ *and* $\mathbf{u}_2 = ([-2,-2],[2,2])$ *in* $\mathbb{I}^2_{\mathbb{R}}$.

$$\begin{aligned} & Qsp\{\mathbf{u}_1, \mathbf{u}_2\} \\ = & \{\mathbf{y} : \lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 \subseteq \mathbf{y}, \text{ for some } \lambda_1, \lambda_2 \in \mathbb{R}\} \\ = & \{\mathbf{y} : \lambda_1 - 2\lambda_2 \in \mathbf{y}_1, 3\lambda_1 + 2\lambda_2 \in \mathbf{y}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}\} \end{aligned}$$

where $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{I}^2_{\mathbb{R}}$. *Hence for any* $(y_1, y_2) \in \mathbf{y}$, *there exists* $\lambda_1, \lambda_2 \in \mathbb{R}$ *such that* $y_1 = \lambda_1 - 2\lambda_2$ *and* $y_2 = 3\lambda_1 + 2\lambda_2$. *As the real numbers* λ_1 *and* λ_2 *change, interval pairs* $(\mathbf{y}_1, \mathbf{y}_2)$ *form* $\mathbb{I}^2_{\mathbb{R}}$. *Now let us see this. Take an arbitrary* $(\mathbf{z}_1, \mathbf{z}_2) \in \mathbb{I}^2_{\mathbb{R}}$. *If* $(z_1, z_2) \in (\mathbf{z}_1, \mathbf{z}_2)$ *then* $(z_1, z_2) \in \mathbb{R}^2$ *and so we can write*

$$(z_1, z_2) = \lambda_1(1,3) + \lambda_2(-2,2)$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$ *since* $(1,3)$ *and* $(-2,2)$ *linearly independent in* $\mathbb{R}^2$. *This proves the assertion. Hence*

$$Qsp\{\mathbf{u}_1, \mathbf{u}_2\} = \mathbb{I}^2_{\mathbb{R}}.$$

An analogous conclusion can be derived from column vectors of $\mathcal{C}$. *As a result we conclude that the row and column rank of* $\mathcal{C}$ *is* $\left(2_s, 2_f\right)$, *that is* $Rank\mathcal{C} = \left(2_s, 2_f\right)$.

Definition 8. *An interval vector whose each term consists of degenerate intervals is called a* degenerate interval vector.

Thus, an interval vector whose at least one term is not a degenerate interval is called a *non-degenerate interval vector*. It can be easily shown that summation of a degenerate interval vector by a non-degenerate interval vector is a non-degenerate interval vector.

Example 6. *Consider* $\mathcal{A} = \begin{pmatrix} [1,1] & [2,2] & [1,3] \\ [-1,-1] & [4,4] & [3,3] \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 & [1,3] \\ -1 & 4 & 3 \end{pmatrix}$. *Rows of* $\mathcal{A}$ *are* $\mathbf{v}_1 = (1,2,[1,3])$ *and* $\mathbf{v}_2 = (-1,4,3)$. *Now*

$$\begin{aligned} Qsp\{\mathbf{v}_1, \mathbf{v}_2\} &= \{\mathbf{u} : \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \subseteq \mathbf{u},\ \lambda_1, \lambda_2 \in \mathbb{R}\} \\ &= \{\mathbf{u} : \lambda_1 - \lambda_2 \in [\underline{a}, \overline{a}],\ 2\lambda_1 + 4\lambda_2 \in \left[\underline{b}, \overline{b}\right],\ \lambda_1[1,3] + 3\lambda_2 \subseteq [\underline{c}, \overline{c}]\ \}. \end{aligned}$$

where $\mathbf{u} = ([\underline{a}, \overline{a}], \left[\underline{b}, \overline{b}\right], [\underline{c}, \overline{c}]) \in \mathbb{I}^3_{\mathbb{R}}$. *For* $\lambda_1 = 0$, $(-\lambda_2, 4\lambda_2, 3\lambda_2)$ *constitutes the stone subspace of* $Qsp\{\mathbf{v}_1, \mathbf{v}_2\}$ *and it is just the span of* $\mathbf{v}_2$. *For* $\lambda_1 \neq 0$, $Qsp\{\mathbf{v}_1, \mathbf{v}_2\}$ *never contains stones, that is, stone subspace of* $Qsp\{\mathbf{v}_1, \mathbf{v}_2\}$ *is span*$\{\mathbf{v}_2\}$. *Now let us look at the foam part of the row space of* $\mathcal{A}$. *The foam part is just* $Qsp\{\mathbf{v}_1, \mathbf{v}_2\}$ *as well, and thus* $R - rank\mathcal{A} = \left(1_s, 2_f\right)$ *since it contains maximum two ql-independent vectors, namely,* $\{\mathbf{v}_1, \mathbf{v}_2\}$. *Let us now investigate columns* $\mathbf{w}_1 = ([1,1],[-1,-1])$, $\mathbf{w}_2 = ([2,2],[4,4])$ *and* $\mathbf{w}_3 = ([1,3],[3,3])$ *of* $\mathcal{A}$. *Assume*

$$([0,0],[0,0]) \subseteq \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \lambda_3\mathbf{w}_3.$$

This means

$$\begin{aligned} 0 &\in \lambda_1 + 2\lambda_2 + \lambda_3[1,3], \\ 0 &= -\lambda_1 + 4\lambda_2 + 3\lambda_3. \end{aligned}$$

Then, for $\lambda_1 = 1$, $\lambda_2 = 1$ *and* $\lambda_3 = -1$, *the above inclusion system is satisfied. This shows* $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ *is ql-dependent in* $Qsp\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. *On the other hand* $\{\mathbf{w}_1, \mathbf{w}_2\}$ *is ql-independent in*

$Qsp\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Because, $\{\mathbf{w}_1, \mathbf{w}_2\}$ is already linearly independent. The stone subspace of $Qsp\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is just $span\{\mathbf{w}_1, \mathbf{w}_2\}$. Hence the column rank of $\mathcal{A}$ is $\left(2_s, 2_f\right)$, that is, $C - rank\mathcal{A} = \left(2_s, 2_f\right)$. So we conclude by this example that: unlike classical matrices, the row and column ranks in interval matrices may not be equal.

If we examine the rank of the (interval) matrix $\mathcal{A} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 3 \end{pmatrix}$ as a quasilinear operator from $\mathbb{I}^3_{\mathbb{R}}$ into $\mathbb{I}^2_{\mathbb{R}}$. Then $Rank\mathcal{A} = \left(2_s, 2_f\right)$.

Conclusion 1. *Row and column ranks of an interval matrix including a non-degenerate term may not be equal.*

Proposition 2. *Any classical real $m \times n$ matrix A with rank r is also an interval matrix from $\mathbb{I}^n_{\mathbb{R}}$ into $\mathbb{I}^m_{\mathbb{R}}$ for which the (row and column) rank is $\left(r_s, r_f\right)$.*

The partial order in the $n \times n$ square-interval matrix space $\mathcal{IM}^n$ was just defined as

$$\mathcal{A} \subseteq \mathcal{B} \; iff \; \mathcal{A}_{ij} \subseteq \mathcal{B}_{ij}, \; for \; each \; i, j.$$

Further let us say that the (interval) matrix

$$I_n = \begin{pmatrix} [1,1] & [0,0] & \cdots & [0,0] \\ [0,0] & [1,1] & \cdots & [0,0] \\ \vdots & \vdots & \ddots & \vdots \\ [0,0] & [0,0] & \cdots & [1,1] \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is the multiplicative unit in $\mathcal{IM}^n$. It is not difficult to define the multiplication operation between two interval matrices by using the multiplication between two intervals. Because the multiplication rule here are the same as in classical matrices.

Definition 9. *For any $\mathcal{A} = (\mathcal{A}_{ij}) \in \mathcal{IM}^n$,* determinant *of $\mathcal{A}$ is an interval-valued function such that*

$$\det \mathcal{A} = \sum (\pm) \mathcal{A}_{1j_1} \mathcal{A}_{2j_2} ... \mathcal{A}_{nj_n}$$

where the sum is taken on the all $j_1 j_2 ... j_n$ permutations of the set $\{1, 2, ..., n\}$. If the $j_1 j_2 ... j_n$ permutation is even then $(\pm) = +$, if it is odd then $(\pm) = -$.

Example 7. *For $\mathcal{A} \in \mathcal{IM}^2$,*

$$\begin{aligned} \det \mathcal{A} &= \det \begin{pmatrix} [\underline{a_{11}}, \overline{a_{11}}] & [\underline{a_{12}}, \overline{a_{12}}] \\ [\underline{a_{21}}, \overline{a_{21}}] & [\underline{a_{22}}, \overline{a_{22}}] \end{pmatrix} \\ &= [\underline{a_{11}}, \overline{a_{11}}][\underline{a_{22}}, \overline{a_{22}}] - [\underline{a_{12}}, \overline{a_{12}}][\underline{a_{21}}, \overline{a_{21}}] \\ &= [\min S_1, \max S_1] - [\min S_2, \max S_2] \\ &= [\min S_1 - \max S_2, \max S_1 - \min S_2]. \end{aligned}$$

where $S_1 = \{\overline{a_{11}a_{22}}, \underline{a_{11}}\underline{a_{22}}, \overline{a_{11}}\underline{a_{22}}, \underline{a_{11}}\overline{a_{22}}\}$ and $S_2 = \{\overline{a_{21}a_{12}}, \underline{a_{21}}\underline{a_{12}}, \underline{a_{12}}\overline{a_{21}}, \overline{a_{12}}\underline{a_{21}}\}$.

Example 8. *Let $\mathcal{A} = \begin{pmatrix} [1,3] & [-1,2] \\ [0,2] & [-2,2] \end{pmatrix} \in \mathcal{IM}^2$,*

$$\begin{aligned} \det \mathcal{A} &= [1,3][-2,2] - [0,2][-1,2] \\ &= [\min\{1(-2), 3(-2), 2, 3(2)\}, \max\{1(-2), 3(-2), 2, 3(2)\}] \\ &\quad -[\min\{0(-1), 0(2), 2(-1), 2(2)\}, \max\{0(-1), 0(2), 2(-1), 2(2)\}] \\ &= [-6, 6] - [-2, 4] = [-6, 6] + [-4, 2] = [-10, 8]. \end{aligned}$$

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Theorem 2. *For $\mathcal{A} \in \mathcal{IM}^3$,*

$$
\begin{aligned}
&\det\begin{pmatrix} [\overline{a_{11}},\underline{a_{11}}] & [\overline{a_{12}},\underline{a_{12}}] & [\overline{a_{13}},\underline{a_{13}}] \\ [\overline{a_{21}},\underline{a_{21}}] & [\overline{a_{22}},\underline{a_{22}}] & [\overline{a_{23}},\underline{a_{23}}] \\ [\overline{a_{31}},\underline{a_{31}}] & [\overline{a_{32}},\underline{a_{32}}] & [\overline{a_{33}},\underline{a_{33}}] \end{pmatrix} \\
= &\left[\overline{a_{11}},\underline{a_{11}}\right]\left[\overline{a_{22}},\underline{a_{22}}\right]\left[\overline{a_{33}},\underline{a_{33}}\right] + \left[\overline{a_{12}},\underline{a_{12}}\right]\left[\overline{a_{23}},\underline{a_{23}}\right]\left[\overline{a_{31}},\underline{a_{31}}\right] \\
&+\left[\overline{a_{13}},\underline{a_{13}}\right]\left[\overline{a_{21}},\underline{a_{21}}\right]\left[\overline{a_{32}},\underline{a_{32}}\right] - \left[\overline{a_{13}},\underline{a_{13}}\right]\left[\overline{a_{22}},\underline{a_{22}}\right]\left[\overline{a_{31}},\underline{a_{31}}\right] \\
&-\left[\overline{a_{11}},\underline{a_{11}}\right]\left[\overline{a_{23}},\underline{a_{23}}\right]\left[\overline{a_{32}},\underline{a_{32}}\right] - \left[\overline{a_{12}},\underline{a_{12}}\right]\left[\overline{a_{21}},\underline{a_{21}}\right]\left[\overline{a_{33}},\underline{a_{33}}\right].
\end{aligned}
$$

The rule given in this theorem is called *Interval Sarrus Rule*.

Example 9.

$$
\begin{aligned}
&\det\begin{pmatrix} [1,1] & [2,3] & [1,3] \\ [-1,1] & [0,2] & [-1,3] \\ [0,0] & [-2,2] & [-3,-1] \end{pmatrix} \\
= &[-6,0] + [-6,6] - [-6,6] - [-9,9] \\
= &[-6,0] + [-6,6] + [-6,6] + [-9,9] = [-27,21]
\end{aligned}
$$

Remark 3. *Since* $\det \mathcal{A}$ *is an interval, we can write it with lower and upper bounds as* $\det \mathcal{A} = \left[\underline{\det \mathcal{A}}, \overline{\det \mathcal{A}}\right]$. *If* $0 \notin \det \mathcal{A}$ *then* $\frac{1}{\det \mathcal{A}}$ *can be calculated as* $\frac{1}{\det \mathcal{A}} = \left[\frac{1}{\overline{\det \mathcal{A}}}, \frac{1}{\underline{\det \mathcal{A}}}\right]$ *from the interval calculus (see [3]). Easily we can see that if* $0 \notin \det \mathcal{A}$ *then* $\det A \neq 0$ *for each* $A \in \mathcal{A}$.

Remark 4. *Another important work on the determinant of square interval matrices is given in [23], where the determinant of an interval matrix is also defined as an interval. In that work the important result characterising the determinant calculus is presented as Proposition 3.1. With our definition, the determinant of an interval matrix includes the determinant given by the other definition, but it is not the same.*

We found that the interval-valued determinant has similar properties to the classical determinant.

Theorem 3. *For a square-interval matrix* $\mathcal{A}$,

1. $\det \mathcal{A} = \det(\mathcal{A}^T)$ *where* $\mathcal{A}^T$ *denotes the* transpose *of the interval matrix* $\mathcal{A}$, *and the transpoze is defined as in classical matrices.*
2. *If a square-interval matrix* $\mathcal{B}$ *is obtained from* $\mathcal{A}$ *by interchanging two rows (columns) of* $\mathcal{A}$, *then* $\det \mathcal{B} = -\det \mathcal{A}$.
3. *If two rows (columns) of* $\mathcal{A}$ *are equal then* $\det \mathcal{A}$ *must be a symmetric interval.*
4. *If all the elements in a row (column) of* $\mathcal{A}$ *are zero, that is, the interval* $[0,0]$, *then* $\det \mathcal{A} = [0,0]$.

Proof. Only claim 3 seems different from similar results in classical matrices. Here we will only prove claim 3. Other proofs are easily done similarly to classical matrices. But it is not very difficult to prove this because we can easily reach the result $\det \mathcal{A} = -\det \mathcal{A}$ from the 2nd claim. This means that $\det \mathcal{A}$ is a symmetric interval. □

Remark 5. *In classical matrices,* $\det \mathcal{A} = -\det \mathcal{A}$ *implies* $\det \mathcal{A} = 0$. *But for interval matrices the assumption only can say* $\det \mathcal{A}$ *is a symmetric interval. Any symmetric interval is a balanced element in* $\mathbb{I}_{\mathbb{R}}$ *and also can be seen as a balanced* 1×1*-interval matrix.*

Definition 10. *Let* $\mathcal{A} = (\mathcal{A}_{ij})$ *be an* $n \times n$ *interval matrix. Let* $\mathcal{B}_{ij}$ *be a sub-interval matrix of type* $(n-1) \times (n-1)$ *obtained by deleting the elements in the* j^{th} *column and* i^{th} *row of* $\mathcal{A}$. *Then* $\det \mathcal{B}_{ij}$ *is called the* minor *of* $\mathcal{A}_{ij}$. *Further, the* cofactor *of the* $\mathcal{A}_{ij}$ *is again an interval* C_{ij} *such that* $C_{ij} = (-1)^{i+j} \det \mathcal{B}_{ij}$

We found that the interval valued determinant function has similar properties to the classical determinant function.

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Theorem 4. *Let $\mathcal{A} = (A_{ij})$ be an $n \times n$ square-interval matrix. Then*

$$\det \mathcal{A} = \mathcal{A}_{i1}C_{i1} + \mathcal{A}_{i2}C_{i2} + ... + \mathcal{A}_{in}C_{in}.$$

where each $\mathcal{A}_{ik}C_{ik}$ is just interval multiplication.

This theorem is an interval generalization of the classical case and the proof can be derived from the former theorem.

Example 10. *By this theorem, for* $\mathcal{A} = \begin{pmatrix} [1,3] & [-1,2] & [0,2] \\ [-2,2] & [-1,2] & 2 \\ -1 & [-1,2] & [-1,3] \end{pmatrix}$,

$$\begin{aligned} \det \mathcal{A} &= [1,3] \det \begin{pmatrix} [-1,2] & 2 \\ [-1,2] & [-1,3] \end{pmatrix} + [-1,2](-1) \det \begin{pmatrix} [-2,2] & 2 \\ -1 & [-1,3] \end{pmatrix} \\ &\quad + [0,2] \det \begin{pmatrix} [-2,2] & [-1,2] \\ -1 & [-1,2] \end{pmatrix} \\ &= [1,3][-7,8] + [-2,1][-4,8] + [0,2][-5,6] \\ &= [-21,24] + [-16,8] + [-10,12] \\ &= [-47,44] \end{aligned}$$

We know from interval calculus that if $0 \notin [\underline{a}, \overline{a}]$ then $\frac{[\underline{a},\overline{a}]}{[\underline{a},\overline{a}]}$ is an interval and always includes 1. Furthermore $[\underline{a}, \overline{a}] - [\underline{a}, \overline{a}]$ is always balanced element that is a symmetric interval an so always $0 \in [\underline{a}, \overline{a}] - [\underline{a}, \overline{a}]$.

Definition 11. *Let $\mathcal{A}$ be an interval matrix in $\mathcal{IM}^n$. Any element $\mathcal{B}$ of $\mathcal{IM}^n$ is called* a right quasi-inverse of $\mathcal{A}$ *if it satisfies the condition $I_n \subseteq \mathcal{AB}$. Similarly, Any element $\mathcal{B}$ of $\mathcal{IM}^n$ is called* a left quasi-inverse of $\mathcal{A}$ *if it satisfies the condition $I_n \subseteq \mathcal{BA}$. An interval matrix $\mathcal{B}$ satisfying the condition $I_n \subseteq \mathcal{AB} = \mathcal{BA}$ is called* a quasi-inverse of $\mathcal{A}$. *Any $\mathcal{B}$ satisfying the condition*

$$\mathcal{AB} = \mathcal{BA} \subseteq I_n \subseteq \mathcal{AB} = \mathcal{BA}, \text{ or equivalently, } I_n = \mathcal{AB} = \mathcal{BA},$$

is called an inverse of $\mathcal{A}$ *and then $\mathcal{B}$ is denoted by $\mathcal{A}^{-1}$.*

Of course, any right (left) inverse of $\mathcal{A}$ is a right (left) quasi-inverse, but not conversely.

Remark 6. *Here it is possible to give the definition of a right quasi-inverse as "...$\mathcal{AB} \subseteq I_n$...". But in this case $\mathcal{AB}$ has to be a classical real-term matrix (stone), because I_n is a minimal element in the partially ordered set $(\mathcal{IM}^n, \subseteq)$. Thus, as soon as we write $\mathcal{AB} \subseteq I_n$, we get $\mathcal{AB} = I_n$ immediately. In such a case, we arrive at the definition of the concept of the right inverse of the interval matrix $\mathcal{A}$. An interval matrix may have many (right or left) quasi-inverses. If a quasi-inverse of an interval matrix is an inverse, then it must be a stone. Hence an inverse of an (interval) matrix must be unique in this case. A foam cannot have an inverse element, it can only have some quasi-inverses. Only stones may have inverses. If we want to introduce an inverse concept for all interval matrices, we have to work with quasi-inverse concept.*

Example 11. *For $\mathcal{A} = [1,3]$, the interval matrices $\frac{1}{3}$ and 1 are both left and right quasi-inverses of $\mathcal{A}$. Further, $\left[\frac{1}{3}, 1\right]$ is another right (left) quasi-inverses of $\mathcal{A}$. Any closed interval (matrix) $\mathcal{B}$ for which $\mathcal{B} \subseteq \left[\frac{1}{3}, 1\right]$ is a quasi-inverse of $\mathcal{A}$. If $\mathcal{D} = [3,3] \equiv 3$, then $\mathcal{D}^{-1}$ exits and $\mathcal{D}^{-1} = [1/3, 1/3] \equiv \frac{1}{3}$. However, the foam $\left[\frac{1}{3}, 1\right]$ is only a quasi-inverse of $\mathcal{D}$.*

Definition 12. *(Adjoint) Let* $\mathcal{A} = (\mathcal{A}_{ij})$ *be an* $n \times n$ *square-interval matrix. Then the* adjoint of $\mathcal{A}$ *is written as* $adj\mathcal{A}$ *and it is defined by*

$$adj\mathcal{A} = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

where $C_{ij} = (-1)^{i+j} \det \mathcal{B}_{ij}$, *and* $\mathcal{B}_{ij}$ *is a sub-interval matrix of type* $(n-1) \times (n-1)$ *obtained by deleting the elements in the* j^{th} *column and* i^{th} *row of* $\mathcal{A}$.

Just as we can multiply a real number by a matrix, we can similarly multiply an interval by an interval matrix. Of course, this multiplication is performed by multiplying an interval by each term of the interval matrix, i.e., $[\underline{a}, \overline{a}]\left(\left[\underline{A_{ji}}, \overline{A_{ji}}\right]\right)_{i,j} = \left([\underline{a}, \overline{a}]\left[\underline{A_{ji}}, \overline{A_{ji}}\right]\right)_{i,j}$. By this multiplication let us now give a main result.

Theorem 5. *Let* $\mathcal{A} = (\mathcal{A}_{ij})$ *be an* $n \times n$ *square-interval matrix and let us assume that* $[0,0] \not\subseteq \det \mathcal{A}$, *that is,* $0 \notin \det \mathcal{A}$. *Then* $\frac{1}{\det \mathcal{A}}.adj\mathcal{A}$ *is a quasi-inverse of* $\mathcal{A}$.

Proof. By the assumption $\frac{1}{\det \mathcal{A}}$ exists and

$$\begin{aligned} \frac{1}{\det \mathcal{A}}.adj\mathcal{A} &= \left[1/\overline{\det \mathcal{A}}, 1/\underline{\det \mathcal{A}}\right] \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \\ &= \left(\left[1/\overline{\det \mathcal{A}}, 1/\underline{\det \mathcal{A}}\right]\left[\underline{C_{ji}}, \overline{C_{ji}}\right]\right)_{i,j} \\ &= ([\min S, \max S])_{i,j} \end{aligned}$$

where $S = \left\{\frac{\underline{C_{ji}}}{\underline{\det \mathcal{A}}}, \frac{\underline{C_{ji}}}{\overline{\det \mathcal{A}}}, \frac{\overline{C_{ji}}}{\underline{\det \mathcal{A}}}, \frac{\overline{C_{ji}}}{\overline{\det \mathcal{A}}}\right\}$. Let us prove, $1 \in [\min S, \max S]\mathcal{A}$ for $i = j$ and $0 \in [\min S, \max S]\mathcal{A}$ for $i \neq j$. It is sufficient to prove the assertion for $n = 3$. For $n > 3$, the proof of the assertion is similar and it can be derived by the induction. For

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{pmatrix},$$

$$adj\mathcal{A} = \begin{pmatrix} \det \mathcal{B}_{11} & -\det \mathcal{B}_{21} & \det \mathcal{B}_{31} \\ -\det \mathcal{B}_{12} & \det \mathcal{B}_{22} & -\det \mathcal{B}_{32} \\ \det \mathcal{B}_{13} & -\det \mathcal{B}_{23} & \det \mathcal{B}_{33} \end{pmatrix}.$$

and

$$\begin{aligned} &\frac{1}{\det \mathcal{A}}\mathcal{A}(adj\mathcal{A}) \\ = &\frac{1}{\det \mathcal{A}} \begin{pmatrix} \sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{1k}\det \mathcal{B}_{1k} & \sum_{k=1}^{3}(-1)^{k}\mathcal{A}_{1k}\det \mathcal{B}_{2k} & \sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{1k}\det \mathcal{B}_{3k} \\ \sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{2k}\det \mathcal{B}_{1k} & \sum_{k=1}^{3}(-1)^{k}\mathcal{A}_{2k}\det \mathcal{B}_{2k} & \sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{2k}\det \mathcal{B}_{3k} \\ \sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{3k}\det \mathcal{B}_{1k} & \sum_{k=1}^{3}(-1)^{k}\mathcal{A}_{3k}\det \mathcal{B}_{2k} & \sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{3k}\det \mathcal{B}_{3k} \end{pmatrix}. \end{aligned}$$

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Observe that diagonal elements (intervals) in $\frac{1}{\det\mathcal{A}}\mathcal{A}(adj\mathcal{A})$ includes 1, and other elements includes 0. Because each of the diagonal elements in interval matrix $\mathcal{A}(adj\mathcal{A})$ is the determinant expansion of $\mathcal{A}$. That is,

$$\sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{1k}\det\mathcal{B}_{1k} = \sum_{k=1}^{3}(-1)^{k}\mathcal{A}_{2k}\det\mathcal{B}_{2k} = \sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{3k}\det\mathcal{B}_{3k} = \det\mathcal{A}.$$

Hence each diagonal elements in $\frac{1}{\det\mathcal{A}}\mathcal{A}(adj\mathcal{A})$ is $\frac{\det\mathcal{A}}{\det\mathcal{A}}$ and since $0\notin\det\mathcal{A}$, it exists and of course includes 1. For non-diagonal elements in $\frac{1}{\det\mathcal{A}}\mathcal{A}(adj\mathcal{A})$, consider, for example, $\sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{3k}\det\mathcal{B}_{1k}$ and observe that

$$\begin{aligned}
&\sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{3k}\det\mathcal{B}_{1k}\\
=\;&\mathcal{A}_{31}\det\mathcal{B}_{11}-\mathcal{A}_{32}\det\mathcal{B}_{12}+\mathcal{A}_{33}\det\mathcal{B}_{13}\\
=\;&\mathcal{A}_{31}(\mathcal{A}_{22}\mathcal{A}_{33}-\mathcal{A}_{23}\mathcal{A}_{32})-\mathcal{A}_{32}(\mathcal{A}_{21}\mathcal{A}_{33}-\mathcal{A}_{31}\mathcal{A}_{23})\\
&+\mathcal{A}_{33}(\mathcal{A}_{21}\mathcal{A}_{32}-\mathcal{A}_{31}\mathcal{A}_{22})\\
=\;&\mathcal{A}_{31}\mathcal{A}_{22}\mathcal{A}_{33}-\mathcal{A}_{31}\mathcal{A}_{23}\mathcal{A}_{32}-\mathcal{A}_{32}\mathcal{A}_{21}\mathcal{A}_{33}+\mathcal{A}_{32}\mathcal{A}_{31}\mathcal{A}_{23}\\
&+\mathcal{A}_{33}\mathcal{A}_{21}\mathcal{A}_{32}-\mathcal{A}_{33}\mathcal{A}_{31}\mathcal{A}_{22}\\
=\;&(\mathcal{A}_{31}\mathcal{A}_{22}\mathcal{A}_{33}+\mathcal{A}_{32}\mathcal{A}_{31}\mathcal{A}_{23}+\mathcal{A}_{33}\mathcal{A}_{21}\mathcal{A}_{32})\\
&-(\mathcal{A}_{31}\mathcal{A}_{22}\mathcal{A}_{33}+\mathcal{A}_{32}\mathcal{A}_{31}\mathcal{A}_{23}+\mathcal{A}_{33}\mathcal{A}_{21}\mathcal{A}_{32})
\end{aligned}$$

We obtain the last equality by changing the order of the multiplication since the interval multiplication is commutative. That is $\sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{3k}\det\mathcal{B}_{1k}$ has the form $[\underline{a},\overline{a}]-[\underline{a},\overline{a}]$ and so we can say

$$0\in\frac{1}{\det\mathcal{A}}\sum_{k=1}^{3}(-1)^{k+1}\mathcal{A}_{3k}\det\mathcal{B}_{3k}.$$

Similarly we can see other non-diagonal terms also includes 0. Hence we can deduce that

$$I_n\subseteq\frac{1}{\det\mathcal{A}}(adj\mathcal{A})\mathcal{A} = \mathcal{A}(adj\mathcal{A})\frac{1}{\det\mathcal{A}}.$$

This means $\frac{1}{\det\mathcal{A}}adj\mathcal{A}$ is a quasi-inverse of $\mathcal{A}$. □

Now let us give another important theorem that we will prove with the help of this theorem.

Theorem 6. *(Interval-Cramer's Rule) Let $\mathcal{A}$ be an $n\times n$ square interval matrix from $\mathbb{R}^n$ into $\mathbb{I}^n_{\mathbb{R}}$, $\mathbf{b}$ be an n-dimensional interval vector and let us assume that $0\notin\det\mathcal{A}$. Then the system $\mathcal{A}x=\mathbf{b}$ has a solution set $\mathbf{Y}$ such that*

$$\mathbf{X}=\frac{1}{\det\mathcal{A}}(adj\mathcal{A})\mathbf{b}$$

is an envelope including $\mathbf{Y}$.

Proof. Since any system of linear interval equation is a family of systems of linear equations and since $0\notin\det\mathcal{A}$ implies $\det A\neq 0$ for each $A\in\mathcal{A}$, we can guarantee the existence of solution set $\mathbf{Y}$. Further the assumption implies existence of

$$\frac{1}{\det\mathcal{A}}=\left[1/\overline{\det\mathcal{A}},1/\underline{\det\mathcal{A}}\right].$$

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Let us consider $adj\mathcal{A}$. Then $\frac{1}{\det\mathcal{A}}adj\mathcal{A}$ is a quasi-inverse of $\mathcal{A}$ from the Theorem 5. So we can conclude that

$$I_n \subseteq \left(\frac{1}{\det\mathcal{A}}adj\mathcal{A}\right)\mathcal{A} = \mathcal{A}\left(\frac{1}{\det\mathcal{A}}adj\mathcal{A}\right)$$

and so for any $x \in \mathbf{Y}$

$$\begin{aligned} I_n x &\subseteq \left(\frac{1}{\det\mathcal{A}}adj\mathcal{A}\right)\mathcal{A}x \\ &= \left(\frac{1}{\det\mathcal{A}}adj\mathcal{A}\right)\mathbf{b}. \end{aligned}$$

Hence

$$x \subseteq \left(\frac{1}{\det\mathcal{A}}adj\mathcal{A}\right)\mathbf{b}.$$

This means explicitly $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ must satisfy the condition

$$x_i \in \frac{\mathcal{A}_{1i}}{\det\mathcal{A}}\left[\overline{b_1}, \underline{b_1}\right] + \frac{\mathcal{A}_{2i}}{\det\mathcal{A}}\left[\overline{b_2}, \underline{b_2}\right] + \cdots + \frac{\mathcal{A}_{ni}}{\det\mathcal{A}}\left[\overline{b_n}, \underline{b_n}\right]$$

for each $i = 1, 2, ..., n$ where $\mathbf{b} = \left(\left[\overline{b_1}, \underline{b_1}\right], \ldots, \left[\overline{b_n}, \underline{b_n}\right]\right)$. As a result

$$\mathbf{X} = \frac{1}{\det\mathcal{A}}(adj\mathcal{A})\mathbf{b}$$

is the desired set (envelope) containing the solution set $\mathbf{Y}$. □

Remark 7. *There may be many other interval matrices $\mathcal{C}$ satisfy the condition $\mathbf{Y} \subseteq \mathcal{C}\mathbf{b}$ and it may be a quasi-inverse of $\mathcal{A}$. Already $\frac{1}{\det\mathcal{A}}adj\mathcal{A}$ is one of the $\mathcal{C}$ that meets this condition, and it is the one obtained with the help of the Interval-Cramer's rule. A narrower $\mathcal{C}$ that satisfies the condition $\mathbf{Y} \subseteq \mathcal{C}\mathbf{b}$ is more valuable, and the solution x obtained from it is a closer and better result. With this method, we do not determine the exact solution set of the interval equation, but we determine an n-dimensional envelope containing the solution set.*

Let us return problem given in introduction, which is given in Example 7.5 in [3] Which is the mesh equation of an electrical circuit system. An envelope for solving this problem is given in [3]. By the Interval Cramer's method let us determine an envelope covering the solution and compare the results with the results in Example 7.5, [3].

Example 12. *([3], Example 7.5.) The mesh equations for an electric circuit are expressed as*

$$\begin{pmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ -V_2 \end{pmatrix}$$

with $V_1 = 10$, $V_2 = 5$, and $R_1 = R_2 = R_3 = 1000 \pm 10\%$. Here R_i denotes resistances, I_i denotes currents and V_i denotes voltages. Find enclosures for I_1 and I_2. In [3], Example 7.5., it is expressed that a pair of envelopes of currents are given as

$$I_1 = [0.00433, 0.00582] \text{ and } I_2 = [-0.000419, 0.000419].$$

Let us now give another envelope for this problem by using Interval Cramer's rule. Let us consider the interval matrix and vectors

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} [1800, 2200] & [-1100, -900] \\ [-1100, -900] & [1800, 2200] \end{pmatrix}, \\ x &= \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \; b = \begin{pmatrix} 10 \\ -5 \end{pmatrix} = \mathbf{b} \end{aligned}$$

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respectively. First we will get an enclosure for the solution set $\mathbf{Y}$ *of the linear interval equation*

$$\mathcal{A}x = \mathbf{b}$$

where $\mathbf{b}$ *is a model (interval) vector. First of all we must calculate* $\det \mathcal{A}$*. By using the interval calculus we get*

$$\begin{aligned}\det \mathcal{A} &= \left[(1800)^2, (2200)^2\right] - \left[(900)^2, (1100)^2\right] \\ &= \left[(1800)^2 - (1100)^2, (2200)^2 - (900)^2\right] \\ &= [2030000, 403000]\end{aligned}$$

Since $0 \notin \det \mathcal{A}$*, we can say* $\mathcal{A}x = \mathbf{b}$ *has a solution* $\mathbf{Y}$*, and we can determine an envelope from the Interval-Cramer's Rule. By again this theorem*

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \subseteq \frac{1}{\det \mathcal{A}}(adj\mathcal{A})\mathbf{b}$$

and so first we must calculate $adj\mathcal{A} = \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix}$*. Using the interval calculus and by the definition of the adjoint we get*

$$adj\mathcal{A} = \begin{pmatrix} [1800, 2200] & [900, 1100] \\ [900, 1100] & [1800, 2200] \end{pmatrix}.$$

Further

$$\frac{1}{\det \mathcal{A}} = \left[\frac{1}{4030000}, \frac{1}{2030000}\right]$$

and

$$\begin{aligned}&\frac{1}{\det \mathcal{A}} adj\mathcal{A} \\ &= \begin{pmatrix} \left[\frac{1800}{4030000}, \frac{2200}{2030000}\right] & \left[\frac{900}{4030000}, \frac{1100}{2030000}\right] \\ \left[\frac{900}{4030000}, \frac{1100}{2030000}\right] & \left[\frac{1800}{4030000}, \frac{2200}{2030000}\right] \end{pmatrix} \\ &= \begin{pmatrix} [0.0004466501, 0.0010837438] & [0.0002233251, 0.0005418719] \\ [0.0002233251, 0.0005418719] & [0.0004466501, 0.0010837438] \end{pmatrix}.\end{aligned}$$

Now

$$\begin{aligned}&\frac{1}{\det \mathcal{A}} adj\mathcal{A}\mathbf{b} \\ &= \begin{pmatrix} [0.0004466501, 0.0010837438] & [0.0002233251, 0.0005418719] \\ [0.0002233251, 0.0005418719] & [0.0004466501, 0.0010837438] \end{pmatrix} \begin{pmatrix} 10 \\ -5 \end{pmatrix} \\ &= ([0.004466501, 0.010837438] + [-0.0027093595, -0.00116625] \\ &\quad , [0.002233251, 0.005418719] + [-0.005418719, -0.0022332505]) \\ &= ([0.0017571415, 0.009671188], [-0.003185468, 0.0031854685])\end{aligned}$$

We can conclude again from the theorem that

$$x = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \subseteq \left(\frac{1}{\det \mathcal{A}} adj\mathcal{A}\right)\mathbf{b}$$

for every $x \in \mathbf{Y}$. So this means

$$\begin{aligned} I_1 &\in [0.0017571415, 0.009671188] \\ \text{and } I_2 &\in [-0.003185468, 0.0031854685]. \end{aligned}$$

Compared to the other result, we can say that Interval-Cramer's rule also gives a close and relatively good result.

Let us now give another main result.

Theorem 7. *Let $\mathcal{A}$ be an $m \times n$ interval matrix and consider a system of linear interval equation $\mathcal{A}x = \mathbf{b}$.*
1) If $\mathcal{A}x = \mathbf{b}$ has a solution then $C - rank\mathcal{A} = C - rank[\mathcal{A} : \mathbf{b}]$
2) If there exists an interval vector $\overline{\mathbf{b}}$ such that $\mathbf{b} \subseteq \overline{\mathbf{b}}$ and $C - rank\mathcal{A} = C - rank\left[\mathcal{A} : \overline{\mathbf{b}}\right]$. Then the system $\mathcal{A}x = \mathbf{b}$ has at least one (possibly many) solution $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Proof. The proof of 1) is similar to classical case. Because if $\mathbf{b}$ is a linear combination of the column vectors of $\mathcal{A}$, it is of course a quasilinear combination. Therefore, let us just prove 2). Assume $C - rank\mathcal{A} = C - rank\left[\mathcal{A} : \overline{\mathbf{b}}\right]$. In this case $\overline{\mathbf{b}}$ is in the column space of $\mathcal{A}$ and so it is a ql-combination of column vectors of $\mathcal{A}$. This means from ql-combination definition that there exist real numbers $x_1, x_2, \ldots x_n$ such that

$$x_1 \begin{pmatrix} [\overline{a_{11}}, \underline{a_{11}}] \\ [\overline{a_{21}}, \underline{a_{21}}] \\ \vdots \\ [\overline{a_{m1}}, \underline{a_{m1}}] \end{pmatrix} + \cdots + x_n \begin{pmatrix} [\overline{a_{1n}}, \underline{a_{1n}}] \\ [\overline{a_{2n}}, \underline{a_{2n}}] \\ \vdots \\ [\overline{a_{mn}}, \underline{a_{mn}}] \end{pmatrix} \subseteq \overline{\mathbf{b}}.$$

By writing

$$\mathbf{b} = x_1 \begin{pmatrix} [\overline{a_{11}}, \underline{a_{11}}] \\ [\overline{a_{21}}, \underline{a_{21}}] \\ \vdots \\ [\overline{a_{m1}}, \underline{a_{m1}}] \end{pmatrix} + \cdots + x_n \begin{pmatrix} [\overline{a_{1n}}, \underline{a_{1n}}] \\ [\overline{a_{2n}}, \underline{a_{2n}}] \\ \vdots \\ [\overline{a_{mn}}, \underline{a_{mn}}] \end{pmatrix}$$

we get $x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$ is the solution of $\mathcal{A}x = \mathbf{b}$. □

Remark 8. *According to this theorem, when we find an interval vector $\overline{\mathbf{b}}$ with $\overline{\mathbf{b}} \supseteq \mathbf{b}$ and with the condition $C - rank\mathcal{A} = C - rank\left[\mathcal{A} : \overline{\mathbf{b}}\right]$, we guarantee an envelope containing the solution of the system $\mathcal{A}x = \mathbf{b}$.*

Conclusion 2. *There are different definitions of the rank of interval matrices than ours ([11,17]). In general, these definitions are important definitions based on the ranks of classical matrices with real terms which are elements of the interval matrix. Our definition of rank comes from the definition of the rank of a quasilinear operator by first considering that an interval matrix is a quasilinear operator. We defined the rank of a quasilinear operator as the dimension of the space quasi-spanned by the range of a quasilinear operator. Accordingly, we gave the definition of the rank of the interval matrix. Classical definition of rank of a matrix is also defined depending on linear operators. So we think that the definition of rank we give is more suitable for quasilinear algebra. Furthermore, the notion of quasi-inverse is an extension of the notion of inverse and the Interval Cramer's rule is a result obtained with the help of this definition. We believe that the quasilinear algebra developed in this way can give a linear algebra-like systematic approach to the solution of other problems related to the solution of linear interval equations and further interval matrix problems.*

References

1. S.M. Aseev, Quasilinear operators and their application in the theory of multivalued mappings, *Proceedings of the Steklov Institute of Mathematics,* Issue 2, 23-52, 1986.
2. G. Alefeld, J. Herzberger, Introduction to Interval Computations, Academic Press, New York, 1983,
3. R. E. Moore, R. B. Kearfott, M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, 2009.

doi:10.20944/preprints202512.2162.v1

4. Y. Yılmaz, S. Çakan, S. Aytekin, Topological Quasilinear Spaces, *Abstr. Appl. Anal., (2012), Article ID 951374, 10 pages.*
5. S. Çakan, Y. Yılmaz, Normed proper quasilinear spaces, *J. Nonlinear Sci. Appl.*, 8(2015), 816-836.
6. Y. Yılmaz, H. Bozkurt, S. Çakan, On orthonormal sets in inner product quasilinear spaces, *Creat. Math. Inform.*, 25 (2016), 229-239.
7. Y. Yılmaz, H. Levent and H. Bozkurt, On the Algebra of Interval Vectors, *Math. Sci. and App. E-Notes*, 11 (2) 67-79 (2023).
8. Faydasicok, O. and Arik, S., A new upper bound for the norm of interval matrices with application to robust stability analysis of delayed neural networks, *Neural Networks* 44 (2013) 64–71
9. Jiri Rohn and Vladik Kreinovich, Computing Exact Componentwise Bounds on Solutions of Lineary Systems with Interval Data is NP-Hard, *SIAM J. Matrix Anal.and Appl.* Vol. 16, No. 2, pp. 415-420, 1995
10. Jiri Rohn, Solvability of Systems of Linear Interval Equations, *SIAM J. Matrix Anal. Appl.*, 25(1), 237–245, 2003.
11. Jiri Rohn, Systems of Linear Interval Equations. *Linear Algebra Appl.* 126 (1989) 39-78.
12. W. Oettli and M. Prager, Compatibility of approximate solution of linrar equations with given error bounds for coefficients and right-hand sides, *Numer. Math.* 6 (1964) 405-409.
13. J. Farkas, Theorie der einfachen Ungleichungen, *J. Reine Angew. Math.*, 124 (1902), pp. 1–27.
14. V. Kreinovich, A. Lakeyev, J. Rohn and P. Kahl; *Computional Complexity and Feasibility of Data Processing and Interval Computations,* Kluwer Academic Pub., Dordrecht, Netherlands , 1998.
15. A. Neumaier, *Interval Methods for Systems of Equations,* Cambridge University Press, Cambridge, UK, 1990.
16. B. Kolman, D. Hill; *Elementary Linear Algebra with Applications* (Ninth Edition), Pearson Pub., New Jersey, 2008.
17. Elena Rubei, On rank range of interval matrices, arXiv:1712.09940, [math.RA], 2018.
18. S. Markov, On quasilinear spaces of convex bodies and intervals, *Journal of Computational and Applied Mathematics* 162 (2004) 93 – 112.
19. E.D. Popova, Enclosing the solution set of parametric interval matrix equation $A(p)X = B(p)$, *Numer Algor* (2018) 78:423–447.
20. B. Hashemi , M. Dehghan, Results concerning interval linear systems with multiple right-hand sides and the interval matrix equation $AX = B$, *Journal of Computational and Applied Mathematics*, 235 (9):2969-2978 (2011).
21. A. Frommer, B. Hashemi, T. Sablik, Computing enclosures for the inverse square root and the sign function of a matrix, *Linear Algebra and its Applications*, 456, 199-213 (2014).
22. A. Frommer, B. Hashemi, Verified computation of square roots of a matrix, *SIAM Journal on Matrix Analysis and Applications, (*31), Issue3, 279-1302 (2010).
23. J. Horacek, M. Hladik, J. Matejka, Determinants of Interval Matrices, *Electronic Journal of Linear Algebra,* Volume 33, pp. 99-112, November 2018.
24. S.P. Shary, On full-rank interval matrices, *Numerical Analysis and Applications,* Volume 7, pages 241–254, (2014).