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Article

# On the Johnson–Tzitzeica Theorems and the Yang–Baxter Equations

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## Abstract

The current paper deals with the sets of Johnson circles, and to the famous Yang–Baxter equations. The Johnson–Tzitzeica theorem could be considered a result related to symmetry. we will also present an asymmetric construction related to the Johnson–Tzitzeica theorem. Then, there is a special section discussing on the Yang–Baxter equations. The concluding section attempts to connect the two above topics.

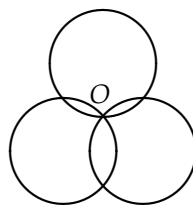
**Keywords:** Johnson–Tzitzeica theorems; Euler formula; Yang–Baxter equations; algebras; coalgebras

**MSC:** 00A05; 00A27; 16B50; 16T25; 30D05

## 1. Introduction

The Yang–Baxter equations are important tools in mathematics, physics and computer science (see [1–4]). The Yang–Baxter equations create a unifying framework for the duality of finite dimensional algebras and finite dimensional coalgebras. This idea will be explored in the realm of geometry in the current paper.

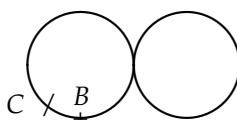
In geometry, a set of Johnson circles comprises three circles of equal radius  $r$  sharing one common point of intersection  $O$  (see the Figure 1).



**Figure 1.** The symmetry of a set of Johnson circles.

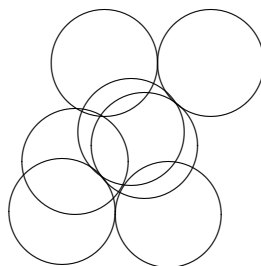
In a Johnson configuration the circles there are usually four intersections points ([5]). In the Johnson–Tzitzeica theorem “the figure is so simple (especially as it can be drawn and the theorem verified with a coin or other circular object) that it seems almost out of the question that the fact can have escaped detection. Even if geometers have overlooked it, someone must have noticed it in casually drawing circles. But if this were the case, it seems like a theorem of sufficient interest to receive some prominence in the literature, and therefore ought to be well known” (see [6]).

We will consider a theorem which start with an asymmetric hypothesis (see the Figure 2).



**Figure 2.** The Many Coins Problem starts with an asymmetry in the hypothesis.

This hypothesis will eventually lead to a mosaic of circles of the same radius (see the Figure 3).



**Figure 3.** The Many Coins Problem. The final mosaic.

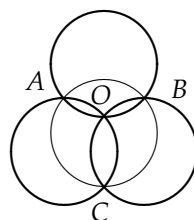
Section 3 deals with the (set-theoretical) Yang-Baxter equations ([1–3,7,8]). We will consider the symmetry or the asymmetry of the braid condition and the quantum Yang-Baxter equation. There exists an unifying equation for the braid condition and the quantum Yang-Baxter equation, and it will be denoted UYBC. Some solutions for UYBC will be given.

Finally, the concluding section attempts to connect the two above main topics, and add some clarification to the topics presented in the current paper.

## 2. On Tzitzeica-Johnson's Theorem and Pictorial Mathematics

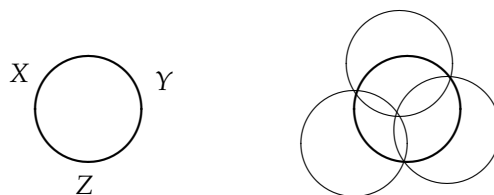
Florin Caragiu gave an invited talk at an International Workshop on Differential Geometry and Its Applications (UPG Ploiesti, 2015). In his talk he discussed about pictorial mathematics. This talk was published as a chapter in [9]. We are going to use his method in presenting the Tzitzeica-Johnson problem and its "relatives".

We consider the Tzitzeica-Johnson problem first ([10]). All the circles will have a fixed radius of length  $r$ . For three circles the intersection points of pairs of circles are denoted by  $A$ ,  $B$ ,  $C$  and the common point of intersection of the three circles is denoted by  $O$  (see the Figure 4).



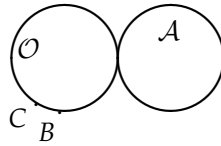
**Figure 4. The three coins problem** (Tzitzeica - 1908, Johnson - 1916). Under the above assumptions, there exists a circle with radius  $r$ , passing through the points  $A$ ,  $B$  and  $C$ .

There exist other similar theorems. For example, the dual of the Tzitzeica-Johnson theorem (see the Figure 5).



**Figure 5.** The dual of the Tzitzeica-Johnson theorem.

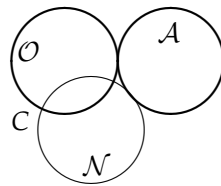
We consider a new theorem, which is related with "The three coins problem". We present it in a pictorial manner. All the circles have the same radius. The hypothesis of this theorem starts with an asymmetric picture (see the Figure 6).



**Figure 6. The Many Coins Problem (MCP).** We begin with 2 circles. The points B and C are taken on the first circle. This is an asymmetric picture.

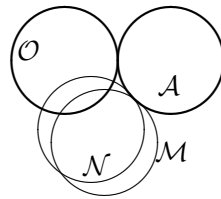
We will add more circles below.

We first add a circle through C (see the Figure 7). We call this circle  $\mathcal{N}$ .

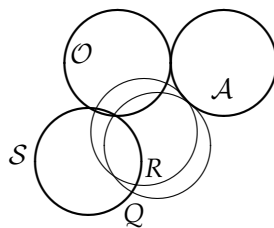


**Figure 7. MCP.** A third circle, tangent to the second circle, is drawn through the point C.

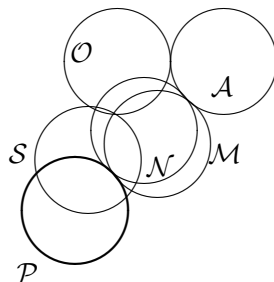
Because the first picture was asymmetric, the hypothesis continues to lead to asymmetry in the figures (see the Figures 8, 9 and 10).



**Figure 8. MCP.** A forth circle, tangent to the second circle, is drawn through the point B.

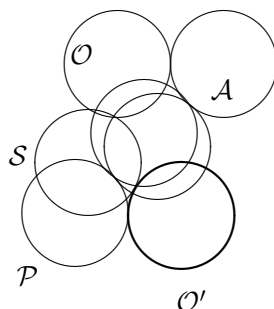


**Figure 9. MCP.** A fifth circle is drawn through the points B and C.



**Figure 10. MCP.** A sixth circle, tangent to the circles  $\mathcal{N}$  and  $\mathcal{M}$  is drawn.

The conclusion of the MCP is that the circle through Q and R is tangent to the circle  $\mathcal{P}$ . This can be seen in the following picture (see the Figure 11).



**Figure 11. MCP. Conclusion.** The circle through Q and R is tangent to the circle  $\mathcal{P}$ .

**Proof of MCP.** We now sketch the proof to the The Many Coins Problem (MCP). It will be based on the Euler’s formula [11]. We will change the perspective a little bit in order to simplify the computations. Also, we will use the symmetries of the figure.

Our approach is to choose as the origin of a Cartesian system the center of the circle  $\mathcal{A}$ , and the x-axis to pass through the center of the circle  $\mathcal{P}$ .

The center of the circle  $\mathcal{O}$  will be  $2e^{i\beta}$ . The centers of the circle  $\mathcal{N}$  and  $\mathcal{M}$  will be  $2e^{i\alpha}$ , respectively  $2e^{-i\alpha}$ . The distance from the centers of the circles  $\mathcal{A}$  and  $\mathcal{P}$  will be  $4 \cos \alpha$ .

Suppose that the polar coordinates of C and B are  $\rho e^{i\theta}$  and  $\eta e^{i\theta'}$ . Then, the coordinates of the center of  $\mathcal{S}$  are  $\rho e^{i\theta} + \eta e^{i\theta'} - 2e^{i\beta}$  (because these coordinates form a rhombus).

By similar arguments, the coordinates of Q and R are  $2e^{i\alpha} + \eta e^{i\theta'} - 2e^{i\beta}$  and  $2e^{-i\alpha} + \rho e^{i\theta} - 2e^{i\beta}$ . The center of the circle  $\mathcal{O}'$  is  $2e^{-i\alpha} + 2e^{i\alpha} - 2e^{i\beta}$ .

Now, the distance between the center of the circle  $\mathcal{O}'$  and the center of the circle  $\mathcal{P}$  is:

$$|2e^{-i\alpha} + 2e^{i\alpha} - 2e^{i\beta} - 4 \cos \alpha| = 2.$$

□

### 3. Yang-Baxter Equations

The most intuitive interpretation of the Yang-Baxter equations is the simplest form of the Yang-Baxter equations. It is called the set-theoretical Yang-Baxter equation.

If  $X$  is a set, let  $R : X \times X \rightarrow X \times X$  be a function,  $R^{12} = R \times I$  and  $R^{23} = I \times R$ .

**Definition 3.1.** Using the above notation, the set-theoretical Yang-Baxter equation is the following:

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{1}$$

We now give more general definitions. We will be working over a field  $k$ . The tensor products will be defined over  $k$ . Let  $V$  be a vector space over  $k$ .

Let  $I : V \rightarrow V$  be the identity map of the space  $V$ .

We denote by  $\tau : V \otimes V \rightarrow V \otimes V$  the twist map defined by  $\tau(v \otimes w) = w \otimes v$ .

For  $R : V \otimes V \rightarrow V \otimes V$  a  $k$ -linear map, let  $R^{12} = R \otimes I$ ,  $R^{23} = I \otimes R : V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ .

We denote by  $R^{13}$  a linear map acting on the first and third component of  $V \otimes V \otimes V$ .

In fact,  $R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$ .

**Definition 3.2.** A **Yang-Baxter operator** is  $k$ -linear map  $R : V \otimes V \rightarrow V \otimes V$ , which satisfies the braid condition:

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \tag{2}$$

Usually, we also require that the map  $R$  is invertible.

**Remark 3.3.** If  $R$  satisfies (2) then both  $R \circ \tau$  and  $\tau \circ R$  satisfy the quantum Yang-Baxter equation:

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \tag{3}$$

**Remark 3.4.** We can see that the quantum Yang-Baxter equation is a symmetric equation. The braid condition is asymmetric. We now consider unifying equations for the quantum Yang-Baxter equation and the braid condition (UYBC):

$$R = XY, \quad R^{12} \circ X^{13} \circ R^{23} \circ Y^{12} = R^{23} \circ X^{13} \circ R^{12} \circ Y^{23}. \tag{4}$$

**Theorem 3.5.** The following new solution for UYBC is proposed below:

$$R = f \otimes g, \quad X = f \otimes I \text{ and } Y = I \otimes g. \text{ We require that } g^2 = g \text{ and } gfg = gf.$$

**Proof.** This can be checked directly. The left hand side reads:

$$R^{12} \circ X^{13} \circ R^{23} \circ Y^{12}(a \otimes b \otimes c) = R^{12} \circ X^{13} \circ R^{23}(a \otimes g(b) \otimes c) = R^{12} \circ X^{13}(a \otimes fg(b) \otimes g(c)) = R^{12}(f(a) \otimes fg(b) \otimes g(c)) = (f(f(a)) \otimes gfg(b) \otimes g(g(c))).$$

The left hand side reads:

$$R^{23} \circ X^{13} \circ R^{12} \circ Y^{23}(a \otimes b \otimes c) = R^{23} \circ X^{13} \circ R^{12}(a \otimes b \otimes g(c)) = R^{23} \circ X^{13}(f(a) \otimes g(b) \otimes g(c)) = R^{23}(f^2(a) \otimes g(b) \otimes g(c)) = f^2(a) \otimes fg(b) \otimes g^2(c).$$

□

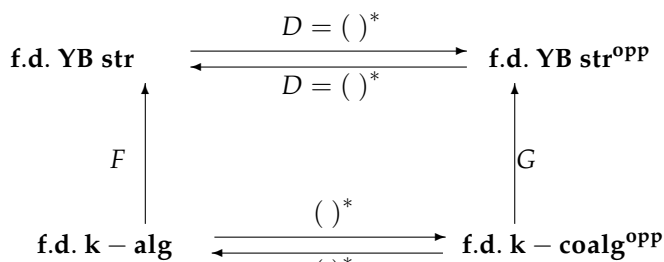
**Remark 3.6.** If  $g^2 = g$  and  $f = I$ , then  $R = I \otimes g$  is a solution for the quantum Yang-Baxter equation, the braid condition and QYBC.

### 4. Conclusions and Final comments

There exists an unifying equation for the braid condition and the quantum Yang-Baxter equation, and it was denoted UYBC. Some solutions for it were given.

The Yang-Baxter equations could be interpreted as a unifying framework for the duality of finite dimensional algebras and finite dimensional coalgebras ([12]). The duality between finite dimensional algebras and coalgebras to the category **f.d. YB str**.

This can be seen below, in the following diagram:



This idea of duality extension will be considered in geometric framework below. For the Tzitzeica-Johnson theorem there exists a dual theorem (see the Figure 12).



**Figure 12.** The Tzitzeica-Johnson’s theorem and its dual

The unification of the Tzitzeica-Johnson theorem and its dual will be given in terms of cones below.

**Theorem 4.1.** Let  $\hat{O}$  be a cone of vertex  $O$  and base the circle  $\mathcal{O}$ .

Let  $\hat{A}$ ,  $\hat{B}$  and  $\hat{C}$  be three cones such that  $O \in \mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$  and  $A, B, C \in \mathcal{O}$ . (So, these three cones are oriented in the opposite direction compared with the first cone.)

Let  $\hat{X}$ ,  $\hat{Y}$  and  $\hat{Z}$  be three cones such that  $A, B \in \mathcal{X}$ ,  $A, C \in \mathcal{Y}$  and  $B, C \in \mathcal{Z}$ . (The orientation of these cones is the same with the orientation of the first cone.) Also,  $X \in \mathcal{A} \cap \mathcal{B}$ ,  $Y \in \mathcal{A} \cap \mathcal{C}$  and  $Z \in \mathcal{B} \cap \mathcal{C}$ .

The conclusion is that there exists a cone of vertex  $T$  and base the circle  $\mathcal{T}$  such that  $X, Y, Z \in \mathcal{T}$  and  $T \in \mathcal{X} \cap \mathcal{Y} \cap \mathcal{Z}$ . Of course, the orientation of this cone is opposite to the orientation of the first cone.

**Proof.** The proof follows from applying both the Tzitzeica-Johnson theorem and its dual.

The Tzitzeica-Johnson theorem can be proved using complex numbers and the Euler's formula. We chose a Cartesian coordinate system with the origin  $O$ . The circles which intersect in  $O$  are given by the following equations:  $|z - e^{i\alpha}| = 1$ ,  $|z - e^{i\beta}| = 1$  and  $|z - e^{i\gamma}| = 1$ .

The intersections of these circles which differ from  $O$  are given by the following complex numbers:  $e^{i\alpha} + e^{i\beta}$ ,  $e^{i\beta} + e^{i\gamma}$  and  $e^{i\alpha} + e^{i\gamma}$ .

So, the fourth circle is given by the equation  $|z - e^{i\alpha} - e^{i\beta} - e^{i\gamma}| = 1$ ,

The dual of the Tzitzeica-Johnson theorem can also be proved using the Euler's formula. On the unit circle we pick three points:  $e^{i\alpha}$ ,  $e^{i\beta}$  and  $e^{i\gamma}$ .

The circles through two of these are given by the following equations:

$$|z - e^{i\alpha} - e^{i\beta}| = 1, |z - e^{i\alpha} - e^{i\gamma}| = 1 \text{ and } |z - e^{i\beta} - e^{i\gamma}| = 1.$$

The conclusion is that the last three circles pass through the point  $e^{i\alpha} + e^{i\beta} + e^{i\gamma}$ .

□

**Remark 4.2.** The dual of the above theorem is the theorem itself, because the dual of a cone is also a cone.

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