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SORT-CX: A Projection-Based Structural Framework for Complex Systems Operator Geometry, Non-Local Kernels, Drift Diagnostics, and Emergent Stability

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Abstract

Complex systems across physics, biology, ecology, technology, and society exhibit emergent structures that cannot be reduced to microscopic rules or simple dynamical laws. While standard approaches rely on differential equations, agent-based simulations, or data-driven models, many emergent phenomena are fundamentally structural in nature, characterized by stability islands, collective modes, scale-dependent correlations, and critical transitions. In this work, we introduce SORT-CX, the complex-systems application layer of the Supra-Omega Resonance Theory (SORT). SORT-CX applies a projection-based operator framework—comprising idempotent resonance operators, a global consistency projector, and a non-local projection kernel—to the structural analysis of complex systems. Emergence is formulated as a projective process rather than being defined purely as a temporal evolution. Structural stability corresponds to idempotent fixed points under operator projection, while structural change is diagnosed through drift metrics defined in resonance space. The framework enables a principled classification of complex systems into operator-dominated, kernel-dominated, and drift-dominated regimes, independent of specific dynamical equations and dataset-specific modeling assumptions. We develop a series of representative use cases, including network mode analysis, stability landscapes of complex fields, pattern formation, critical transitions, and multilayer coupled systems. SORT-CX positions projection-based structural analysis as a unifying perspective for emergent phenomena.

Keywords: complex systems; emergence; structural analysis; projection operators; stability landscapes; kernel methods; drift diagnostics; network theory; critical transitions; SORT framework

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1. Meta: Position of SORT-CX Within the SORT Framework

1.1. Scope and Intent of This Article

SORT-CX is the complex-systems application layer of the Supra-Omega Resonance Theory (SORT). The purpose of this article is to formalize a projection-based structural methodology for complex systems that is independent of specific dynamical equations, agent-based update rules, or empirically fitted models. The central object is not a time evolution operator but a family of idempotent resonance operators $\{\hat{O}_i\}$ acting on system representations Φ to extract stable structural components and to diagnose structural change via drift metrics in resonance space.

The intended audience comprises complexity scientists, systems theorists, and applied mathematicians who work on emergent phenomena in networks, spatially extended fields, and coupled multilayer systems. The analysis emphasizes structure as an object of study, formulated through operator-fixed points, kernel-mediated non-local correlations, and consistency constraints enforced by a global projector. SORT-CX is designed to be compatible with data-supported workflows while remaining well-defined in the data-free regime, where structural diagnostics are derived solely from the internal projection geometry.

1.2. Shared Mathematical Backbone of SORT

SORT is built on three core mathematical components: a finite set of idempotent resonance operators $\hat{O}_i$, a global consistency projector $\hat{H}$, and a non-local projection kernel $\kappa(k)$. The operators satisfy idempotency,

$$\hat{O}_i^2 = \hat{O}_i, \tag{1}$$

and define a projection geometry in which stable structures correspond to fixed points $\Phi^\star$ obeying $\hat{O}_i\Phi^\star = \Phi^\star$ for the relevant operator channels. The global projector aggregates operator channels with weights w_i and enforces a structural consistency constraint,

$$\hat{H} = \sum_{i=1}^{22} w_i \hat{O}_i, \tag{2}$$

with the SORT light-balance condition expressed as

$$\sum_{i=1}^{22} w_i = 0. \tag{3}$$

The non-local kernel $\kappa(k)$ mediates scale-dependent coupling in spectral space and acts as a structural filter across modes. In its Gaussian form,

$$\kappa(k) = \exp\left[-\frac{1}{2}(\sigma_0 L_H k)^2\right], \tag{4}$$

where σ_0 sets the effective correlation scale and L_H is a reference length that fixes the unit convention for k.

SORT v6 is organized as a modular architecture in which domain layers share the same mathematical backbone while differing in the interpretation of Φ, the construction of operator channels, and the operational meaning of the kernel. This modular composition is represented as

$$\mathcal{M}_{\text{SORT v6}} = \mathcal{M}_{\text{cosmo}} \oplus \mathcal{M}_{\text{AI}} \oplus \mathcal{M}_{\text{CX}} \oplus \mathcal{M}_{\text{QS}}. \tag{5}$$

The direct-sum notation indicates a shared formalism with separable application layers rather than an assumption of independent physical subsystems.

1.3. Domain-Specific Interpretation for Complex Systems

In SORT-CX, the system state Φ denotes a structural representation of a complex system, such as node fields on a graph, spatial fields on a discretized domain, multilayer tensors, or feature embeddings that encode coarse-grained relational patterns. Emergence is formulated as a projective process: micro-level structure Φ_{micro} is mapped to an emergent structural component Φ_{emergent} through a kernel-weighted operator channel,

$$\Phi_{\text{emergent}}(k) = \kappa(k)\, \hat{O}_i[\Phi_{\text{micro}}(k)]. \tag{6}$$

The functional form is intentionally non-dynamical: $\hat{O}_i$ is a structural filter, and $\kappa(k)$ implements non-locality and scale selection. Structural stability is identified with idempotent fixed points, in which repeated application does not alter the structure,

$$\hat{O}_i\Phi^\star = \Phi^\star \quad \Rightarrow \quad \hat{O}_i^n\Phi^\star = \Phi^\star \text{ for all } n \geq 1. \tag{7}$$

Structural change is quantified by drift metrics defined on sequences of structural transformations $\hat{T}_n$, which can represent successive coarse-graining steps, observation windows, perturbation levels, or algorithmic updates,

$$D_n = \left\| \hat{T}_n - \hat{T}_{n-1} \right\|. \tag{8}$$

In this setting, non-stationarity is diagnosed as sustained growth or bifurcating behavior of D_n, while stability corresponds to bounded drift and convergence in resonance space. The complex-systems interpretation therefore treats kernels and operators as geometry-defining primitives that encode non-local correlations, collective modes, and stability islands without requiring a prescribed microscopic time evolution.

1.4. Relation to SORT-Cosmology, SORT-QS, and SORT-AI

SORT-CX shares the operator, projector, and kernel formalism with other SORT domain layers while targeting distinct structural objects and diagnostic questions. SORT-Cosmology emphasizes projection fields across physical scales and interprets $\kappa(k)$ as a scale-dependent structural filter in cosmological spectral spaces. SORT-QS focuses on structural constraints for quantum-system mappings, where projections are associated with admissible subspaces and structural diagnostics are evaluated under physically allowable transformations. SORT-AI uses projection geometry and drift diagnostics to quantify structural risk and alignment-relevant instabilities in advanced systems.

The cross-module relations relevant for this article are conceptual and methodological. First, the drift diagnostics used for structural change in complex systems align with the role of drift as a stability and failure-proximity indicator in AI safety contexts, enabling shared metrics and shared numerical protocols. Second, multi-kernel architectures appear naturally in both complex-systems and quantum-systems settings when distinct interaction channels or scale layers are superposed, suggesting transferable implementation patterns. Third, scale-spanning structural formation in complex systems admits formal parallels to cosmological structure formation at the level of kernel-weighted projections and fixed-point stability criteria, while differing in domain-specific meaning and admissible observables.

Accordingly, SORT-CX is positioned as an application layer that is mathematically homogeneous with the broader framework while remaining agnostic to domain-specific empirical commitments.

1.5. Role of SORT-CX in the Transition from SORT v5 to SORT v6

SORT v5 establishes the validated operator backbone and the internal consistency constraints required for stable projection-based analysis [96]. SORT-CX serves as a bridge to SORT v6 by providing a non-cosmological testbed in which operator generalization, kernel design, and drift diagnostics can be exercised across multiple classes of complex systems. This role is methodological: it supports the abstraction of a shared engine interface, a registry for domain modules, and reproducible workflows that are not tied to a single empirical application.

In the v5 to v6 transition, SORT-CX contributes three structural deliverables. First, it motivates a formal classification of systems into operator-dominated, kernel-dominated, and drift-dominated regimes, enabling a domain-independent language for emergence and stability. Second, it provides concrete diagnostic protocols that map structural inputs to resonance-space metrics, clarifying which components of the framework are invariant under representation changes. Third, it guides high-performance implementations by emphasizing scalable spectral operations, multi-kernel evaluation, and operator projection pipelines that can be executed on large graphs, lattices, or coupled layer tensors. In this sense, SORT-CX functions as a conceptual and computational integration layer that strengthens the modular architecture expressed by Eq. 5 while preserving the core SORT v5 backbone.

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2. Introduction

2.1. Motivation: Why Complex Systems Require Structural Theory

Complex systems across physics, biology, ecology, technology, and social organization exhibit emergent structures that are not reducible to microscopic interaction rules or low-dimensional dynamical laws. Empirically observed phenomena such as collective modes, stability plateaus, modular organization, scale-spanning correlations, and abrupt regime shifts persist across widely different domains and modeling formalisms. These recurring features indicate that emergence in complex systems is governed by structural constraints rather than by detailed temporal dynamics alone [10,11,19].

Structural theory addresses this gap by focusing on invariant relationships, projection properties, and consistency conditions that characterize admissible macroscopic configurations independently of how they are dynamically reached. In this perspective, the primary scientific object is not a trajectory in time but the geometry of the space of possible structures and the conditions under which stable configurations exist. A structural approach is therefore required to systematically compare systems with different micro-dynamics but similar emergent organization, and to define robustness, stability, and transition criteria that are intrinsic to the system rather than model-dependent.

2.2. Limits of Dynamical, Simulation-Based, and Data-Driven Approaches

Dynamical approaches based on ordinary or partial differential equations are often limited by strong assumptions on locality, smoothness, and scale separation. For many real-world systems, these assumptions fail due to heterogeneity, discrete interactions, long-range correlations, or adaptive feedbacks. Agent-based simulations alleviate some of these constraints but introduce new limitations, including high computational cost, sensitivity to implementation details, and the absence of analytically tractable invariants [17,92].

Data-driven and machine-learning-based methods provide powerful descriptive and predictive tools but typically lack intrinsic notions of structural consistency or stability. Learned representations depend on training data, feature choices, and optimization objectives, and do not by themselves define whether an observed configuration corresponds to a stable structural state or a transient artifact [23,51]. Across these approaches, emergence is often inferred retrospectively rather than characterized by principled criteria. As a consequence, comparisons across models, domains, and scales remain ad hoc, and early-warning diagnostics for structural change are difficult to generalize.

2.3. SORT as a Projection-Based Framework

The Supra-Omega Resonance Theory (SORT) addresses these limitations by formulating complex-system analysis as a problem of projection geometry rather than time evolution. In SORT, systems are represented by structural states Φ that encode relational or field-level information. Structural features are extracted through idempotent resonance operators $\hat{O}_i$, while global consistency is enforced by a projector $\hat{H}$ and scale-dependent correlations are mediated by a non-local kernel $\kappa(k)$ [96].

Within this framework, operators act as structural filters that select admissible modes or configurations, independent of the microscopic rules that generated the state. The kernel introduces controlled non-locality and scale sensitivity, allowing collective behavior to be captured without explicit dynamical coupling terms. The resulting projection-based formalism defines a geometry on the space of structures, in which stability, coherence, and transition behavior can be analyzed through algebraic and spectral properties rather than through explicit integration of equations of motion.

2.4. Emergence as Projection Rather than Time Evolution

In the SORT framework, emergence is defined as the appearance of stable structural components under projection. A macroscopic structure is emergent if it corresponds to a fixed point of a resonance operator, possibly after kernel weighting in spectral space. This definition is independent of whether the structure arises slowly through relaxation, abruptly through a bifurcation, or instantaneously through an external constraint.

This perspective aligns with foundational insights in emergence theory, where higher-level regularities constrain lower-level behavior without being reducible to it [20–22]. By treating emergence as a projection, SORT separates the question of *what structures exist and are stable* from the question of *how they are dynamically realized*. Structural change is then quantified through drift diagnostics that measure deviations between successive projected states, providing a domain-independent indicator of instability, criticality, or regime transition.

2.5. Scope and Structure of This Article

This article introduces SORT-CX, the complex-systems application layer of SORT, and develops its mathematical foundations, diagnostic tools, and representative use cases. Section 3 establishes the operator, projector, kernel, and drift formalism used throughout the paper. Section 4 proposes a structural classification of complex systems based on dominant projection mechanisms. Sections 5–10 present concrete applications to networks, spatial fields, pattern formation, critical transitions, and multilayer coupled systems. Structural invariants, observer dependence, and model-reduction aspects are discussed in Sections 12 and 13. Architectural considerations and broader application domains are addressed in Sections 14 and 15. Limitations and open questions are summarized in Section 16, followed by concluding remarks and an outlook toward SORT v6 integration in Section 17.

3. Mathematical Foundations of SORT-CX

3.1. Resonance Operators as Structural Filters

The mathematical core of SORT-CX is a finite set of resonance operators $\{\hat{O}_i\}_{i=1}^{22}$, each acting as a structural filter on a system representation Φ. The operators do not generate time evolution; instead, they project Φ onto structurally admissible subspaces that correspond to distinct organizational principles. Formally, the action of a resonance operator is defined as a mapping

$$\hat{O}_i : \Phi \longrightarrow \hat{O}_i\Phi, \tag{9}$$

where Φ may represent a field configuration, a network embedding, a multilayer tensor, or a generalized structural state. Each operator encodes a specific mode of structural selection, such as modularity, coherence, symmetry, or constraint enforcement.

In the complex-systems context, resonance operators generalize familiar constructions such as graph projectors, mode-selection operators, and coarse-graining maps, while remaining independent of the system's microscopic dynamics. Their role is to isolate structural components that persist under repeated projection and to suppress components that are inconsistent with the chosen structural criterion.

3.2. Idempotency, Fixed Points, and Structural Stability

A defining property of all SORT resonance operators is idempotency,

$$\hat{O}_i^2 = \hat{O}_i, \tag{10}$$

which ensures that once a structure has been projected, further applications of the same operator do not alter it. Idempotency provides a mathematically precise notion of structural stability: a configuration $\Phi^\star$ is stable with respect to operator $\hat{O}_i$ if it satisfies the fixed-point condition

$$\hat{O}_i\Phi^\star = \Phi^\star. \tag{11}$$

Such fixed points define invariant subspaces in the projective state space and correspond to emergent structures that are robust against further filtering in the same structural channel.

Unlike attractors in dynamical systems, these fixed points are not defined through asymptotic time evolution but through algebraic consistency. Multiple fixed points may coexist, reflecting the

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multiplicity of admissible structural organizations for a given system representation. Structural instability arises when no nearby fixed point exists or when small perturbations cause transitions between distinct operator-fixed subspaces.

3.3. The Global Projector as a Consistency Constraint

Individual resonance operators capture specific structural aspects but do not, by themselves, guarantee global consistency. SORT therefore introduces a global projector $\hat{H}$, defined as a weighted sum of the resonance operators,

$$\hat{H} = \sum_{i=1}^{22} w_i \, \hat{O}_i, \tag{12}$$

where the weights w_i satisfy the light-balance condition

$$\sum_{i=1}^{22} w_i = 0. \tag{13}$$

The projector $\hat{H}$ enforces a structural consistency constraint across operator channels by suppressing configurations that violate the balanced interplay of structural contributions.

In SORT-CX, $\hat{H}$ acts as a global admissibility filter: a structural state that survives projection under $\hat{H}$ is consistent across all relevant resonance channels. This mechanism replaces ad hoc constraint enforcement commonly used in simulations and provides a unified criterion for structural coherence across heterogeneous system components.

3.4. The Non-Local Kernel as Scale and Interaction Filter

Complex systems often exhibit interactions and correlations across multiple scales. SORT-CX incorporates such effects through a non-local projection kernel $\kappa(k)$ defined in spectral space. In its standard Gaussian form,

$$\kappa(k) = \exp\left[-\frac{1}{2}(\sigma_0 L_H k)^2\right], \tag{14}$$

the kernel suppresses high-frequency components while preserving long-wavelength modes according to the correlation scale σ_0. The kernel acts multiplicatively in spectral representations of Φ, introducing controlled non-locality without explicit interaction terms.

For complex systems, $\kappa(k)$ functions as a scale and interaction filter that determines which collective modes contribute to emergent structure. Extensions to multi-kernel constructions allow distinct scale channels to be superposed, enabling the analysis of systems with heterogeneous or layered interaction ranges. The kernel thus complements the operator algebra by regulating the spectral content on which structural projections act.

3.5. Drift, Coherence, and Emergence Metrics

Structural change and stability in SORT-CX are quantified through metrics defined on sequences of projected states or transformations. Let $\hat{T}_n$ denote a sequence of structural transformations, such as successive projections, perturbation levels, or observation windows. The drift metric is defined as

$$D_n = \left\| \hat{T}_n - \hat{T}_{n-1} \right\|, \tag{15}$$

where the norm is chosen according to the representation of Φ. Sustained growth or bifurcation behavior in D_n indicates structural instability or proximity to a transition.

Coherence metrics are derived from residuals of operator commutators and from the consistency of projections under $\hat{H}$. Emergence is quantified by measuring the spectral weight of projected modes after kernel filtering, leading to an emergence index of the form

$$E = \int \kappa(k) \, P(k) \, \mathrm{d}k, \tag{16}$$

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where $P(k)$ denotes the projected structural power spectrum. Together, these metrics provide a unified diagnostic toolkit for stability, coherence, and emergent organization.

3.6. Projective State Space and Resonance Geometry

The action of resonance operators and kernels induces a projective geometry on the space of structural states. The projective state space $\mathcal{M}_{\mathrm{op}}$ is defined as the set of equivalence classes of Φ under idempotent projection,

$$\mathcal{M}_{\mathrm{op}} = \{\, \Phi \sim \hat{O}_i \Phi \text{ for relevant } i \,\}. \tag{17}$$

Distances and angles in this space are defined through operator-induced norms and inner products, allowing trajectories of structural transformations to be interpreted geometrically.

In this resonance geometry, stable structures correspond to fixed points or low-curvature regions, while transitions and instabilities appear as sharp bends or separatrices between projection basins. This geometric interpretation provides a unifying language for comparing complex systems across domains and representations, grounding emergent phenomena in the intrinsic structure of the projection space rather than in model-specific dynamics.

4. Structural Classification of Complex Systems

4.1. Operator-Dominated Systems

Operator-dominated systems are characterized by structural organization that is primarily determined by the algebraic relations and fixed points of the resonance operators $\hat{O}_i$. In this regime, emergent structure arises from strong internal constraints that sharply restrict the admissible configuration space, while scale effects mediated by the kernel remain subdominant. Typical manifestations include discrete networks with pronounced modularity, feedback-controlled systems, and rule-based interaction structures in which relational patterns dominate over spatial or spectral correlations.

Formally, operator dominance is indicated when commutator structure and projection residuals govern stability properties. A representative criterion is given by the relative magnitude

$$\big\|[\hat{O}_i, \hat{O}_j]\big\| \gg \|\kappa(k)\|, \tag{18}$$

evaluated over the relevant operator channels and spectral support. In such systems, structural change is primarily associated with reconfiguration between operator-fixed subspaces rather than with continuous deformation across scales. Stability diagnostics therefore focus on idempotent fixed points and their basins of attraction in resonance space.

4.2. Kernel-Dominated Systems

Kernel-dominated systems exhibit emergent structure that is governed by scale-dependent correlations and non-local interactions encoded by the projection kernel $\kappa(k)$. In this regime, the specific operator channel plays a secondary role compared to the spectral filtering imposed by the kernel. Spatially extended fields, pattern-forming systems, and systems with long-range coupling typically fall into this class.

Kernel dominance is diagnosed when the spectral response of the system is primarily controlled by the shape and width of $\kappa(k)$, such that emergent modes are selected by scale rather than by algebraic constraints. A characteristic condition is

$$\big\|\kappa(k)\, \hat{O}_i \Phi\big\| \gg \big\|\hat{O}_j \Phi - \hat{O}_i \Phi\big\| \tag{19}$$

for relevant operator pairs i, j. Structural stability in this regime corresponds to persistence of dominant spectral bands under kernel filtering, while transitions are associated with changes in the effective correlation scale σ_0 or with the emergence of competing length scales.

4.3. Drift-Dominated Systems

Drift-dominated systems are defined by persistent or accelerating structural change, such that no single operator-fixed subspace or kernel-selected mode remains stable over the considered transformation sequence. Examples include adaptive socio-economic systems, evolving ecosystems under external stress, and non-stationary technological networks. In these systems, emergent structure is transient and continuously reconfigured.

The defining diagnostic is the behavior of the drift metric D_n, which exhibits sustained growth, oscillatory divergence, or bifurcation patterns,

$$D_n = \left\| \hat{T}_n - \hat{T}_{n-1} \right\| \text{ with } \frac{\partial D_n}{\partial n} \not\to 0. \tag{20}$$

Here, structural instability is intrinsic rather than perturbative. Emergence is episodic and often localized in resonance space, and stability analysis focuses on identifying transient coherence intervals or early-warning signals associated with rapid drift amplification.

4.4. Hybrid Regimes and Transition Zones

Many real-world complex systems occupy hybrid regimes in which operator constraints, kernel-mediated scale effects, and drift dynamics contribute comparably to the observed structure. Such systems are naturally represented in a multidimensional classification space spanned by operator strength, kernel dominance, and drift intensity.

Transition zones between regimes correspond to regions in which small parameter changes induce qualitative shifts in dominance, such as a transition from kernel-selected spatial patterns to operator-enforced modular organization, or from operator stability to drift-driven reorganization. These transitions are interpreted as structural phase boundaries in resonance space rather than as dynamical bifurcations in time. Hybrid regimes are therefore essential for understanding multistability, intermittent behavior, and cross-scale coupling in complex systems.

4.5. Quantitative Classification Criteria

To enable systematic classification, SORT-CX defines quantitative criteria based on normalized dominance measures for operators, kernels, and drift. Let $\mathcal{O}$, $\mathcal{K}$, and $\mathcal{D}$ denote scalar indicators derived from operator commutators, kernel spectral weight, and drift magnitude, respectively. A normalized dominance vector is defined as

$$\boldsymbol{\Lambda} = \frac{1}{\mathcal{O} + \mathcal{K} + \mathcal{D}} (\mathcal{O}, \mathcal{K}, \mathcal{D}), \tag{21}$$

with $\boldsymbol{\Lambda}$ taking values in the unit simplex. Systems are classified by the location of $\boldsymbol{\Lambda}$ within this simplex, enabling continuous interpolation between pure and hybrid regimes.

This representation supports automated classification pipelines, comparative analysis across domains, and principled identification of regime transitions. It also provides a compact summary of the structural character of a system that is invariant under representation changes and independent of specific dynamical modeling choices.

5. Use Case I: Network Modes in Complex Systems

5.1. Networks as Emergent Structural Spaces

Complex networks provide a canonical setting for studying emergence as a structural phenomenon. Nodes, edges, and weights define a relational substrate on which collective organization arises in the form of communities, hubs, functional modules, and hierarchical layers. In SORT-CX, a network is treated not merely as a combinatorial object but as an emergent structural space in which node-associated fields or observables define a system state Φ.

This perspective abstracts away from specific dynamical processes on the network and instead focuses on admissible structural configurations. Emergence is identified with persistent relational patterns that survive projection under resonance operators and kernel filtering. Networks are therefore

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interpreted as discretized manifolds whose geometry is induced by connectivity, weights, and operator action, enabling the application of projection-based structural diagnostics independently of flow, diffusion, or agent dynamics [13].

5.2. Projection of Graph Laplacian Modes

A central structural object in network analysis is the graph Laplacian L, defined as

$$L = D - A, \tag{22}$$

where A is the adjacency matrix and D the degree matrix. The spectral decomposition of L provides an orthogonal basis of modes that encode large-scale connectivity and bottleneck structure [26].

Within SORT-CX, Laplacian eigenmodes are treated as candidate structural components that can be selectively filtered by resonance operators. A network-specific resonance operator $\hat{O}_{\mathrm{net}}$ is constructed as a projector onto a subset of Laplacian eigenspaces,

$$\hat{O}_{\mathrm{net}} = \sum_{\lambda \in \Lambda^\star} \mathcal{P}_\lambda, \tag{23}$$

where $\mathcal{P}_\lambda$ denotes the spectral projector associated with eigenvalue λ and $\Lambda^\star$ is a selection set determined by structural criteria. Kernel weighting $\kappa(\lambda)$ further modulates the contribution of each mode, enabling scale-dependent selection of collective network structure. This formulation generalizes spectral clustering and related methods by embedding them into a unified projection framework.

5.3. Resonance Clusters and Structural Communities

Communities in networks correspond to subsets of nodes that exhibit coherent structural behavior under projection. In SORT-CX, such communities are identified as resonance clusters: node sets whose associated structural components form approximate fixed points under the action of $\hat{O}_{\mathrm{net}}$.

Let Φ_v denote a node-associated field. A resonance cluster $\mathcal{C}$ satisfies

$$\hat{O}_{\mathrm{net}}\Phi_v \approx \Phi_v \quad \text{for all } v \in \mathcal{C}, \tag{24}$$

up to a prescribed tolerance. This definition captures modular organization as an idempotent structural property rather than as the outcome of an optimization objective. Hierarchical and overlapping communities arise naturally when multiple operator channels or kernel scales are superposed, providing a principled extension of classical modularity-based and spectral clustering approaches [28,29].

5.4. Stability and Fragmentation Diagnostics

Structural stability of network organization is assessed through the persistence of resonance clusters under perturbations of topology, weights, or observation scale. In SORT-CX, stability diagnostics are formulated in terms of drift and fragmentation measures defined on projected network states.

Given a sequence of network representations $\Phi^{(n)}$, the drift metric

$$D_n = \left\| \hat{O}_{\mathrm{net}}\Phi^{(n)} - \hat{O}_{\mathrm{net}}\Phi^{(n-1)} \right\| \tag{25}$$

quantifies sensitivity to structural change. Fragmentation is detected when previously coherent resonance clusters split into multiple disjoint fixed-point components or lose idempotent stability under projection.

These diagnostics enable the identification of robust communities, weakly bound modules, and incipient structural breakdowns. Importantly, they are independent of specific dynamical processes on the network and therefore applicable across domains ranging from social and economic systems to biological and technological networks [14,33].

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6. Use Case II: Stability Landscapes of Complex Fields

6.1. Field Configurations and Structural Instabilities

Many complex systems are naturally represented as spatially or functionally extended fields rather than as discrete networks. Examples include concentration fields in reaction–diffusion systems, order-parameter fields in pattern formation, activity fields in neural tissue, and stress or load fields in infrastructure systems. In SORT-CX, such systems are described by a field-valued structural state $\Phi(x)$ or its spectral representation $\Phi(k)$, where x denotes spatial or abstract coordinates.

Structural instability in this context does not refer to exponential divergence of trajectories but to the absence of stable projective structure. A field configuration is structurally unstable if small perturbations lead to qualitatively different projections under the resonance operators. Instability therefore manifests as sensitivity of the projected field $\hat{O}_i\Phi$ to perturbations in Φ, rather than as rapid temporal growth of deviations. This definition captures phenomena such as pattern reorganization, domain splitting, and loss of coherence that are not adequately characterized by classical linear stability analysis alone [37,38].

6.2. Projector-Based Stability Testing

SORT-CX replaces dynamical stability tests with projector-based consistency checks. Given a field configuration Φ, stability with respect to a resonance operator $\hat{O}_i$ is assessed by evaluating the deviation from idempotent invariance,

$$\Delta_i(\Phi) = \left\|\hat{O}_i\Phi - \Phi\right\|. \tag{26}$$

A small value of Δ_i indicates that the field lies close to a structurally admissible subspace and is therefore stable under the corresponding projection channel.

Global structural stability is tested using the global projector $\hat{H}$. A configuration is globally consistent if

$$\left\|\hat{H}\Phi - \Phi\right\| \leq \varepsilon, \tag{27}$$

for a prescribed tolerance ε. This criterion defines stability landscapes as regions in configuration space where field states satisfy consistency across multiple structural channels simultaneously. Unlike energy-based stability notions, these landscapes are purely geometric and depend only on projection properties.

6.3. Kernel-Damped Modes and Stable Regions

The non-local kernel $\kappa(k)$ plays a central role in shaping stability landscapes by suppressing structurally unstable high-frequency components and enhancing coherent large-scale modes. In spectral space, a kernel-weighted field is given by

$$\Phi_\kappa(k) = \kappa(k)\,\Phi(k), \tag{28}$$

which selectively damps modes according to the correlation scale σ_0.

Stable regions in field space correspond to configurations for which kernel-damped projections converge toward operator-fixed subspaces. As σ_0 is varied, stability islands may appear, merge, or disappear, defining a scale-dependent stability landscape. This mechanism generalizes classical notions of diffusion-induced stabilization and provides a unified explanation for the emergence of characteristic length scales in pattern-forming systems [36,40].

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6.4. SORT-Based Stability Spectra

To characterize stability across scales and operator channels, SORT-CX introduces stability spectra derived from the response of projected fields to kernel filtering. Let λ denote a spectral parameter associated with field modes. The SORT-based stability spectrum is defined as

$$S(\lambda) = \left\| \hat{H}\, \kappa(\lambda)\, \mathcal{P}_\lambda \Phi \right\|, \tag{29}$$

where $\mathcal{P}_\lambda$ is the spectral projector onto mode λ. Peaks in $S(\lambda)$ identify structurally stable modes, while suppressed regions correspond to unstable or incoherent components.

This spectral representation allows direct comparison with classical linear stability spectra while extending them beyond eigenvalue growth rates. SORT-based stability spectra encode operator consistency, non-local scale effects, and structural admissibility in a single diagnostic object, enabling systematic analysis of stability landscapes across diverse classes of complex fields.

7. Use Case III: Drift and Pattern Dynamics

7.1. Drift as a Structural Change Metric

In SORT-CX, drift quantifies structural change independently of explicit temporal dynamics. Rather than measuring time derivatives of state variables, drift captures deviations between successive structural projections or transformation stages. Let $\hat{T}_n$ denote a sequence of structural transformations applied to a system representation Φ, where the index n may correspond to observation scale, perturbation strength, coarse-graining level, or algorithmic update. The drift metric is defined as

$$D_n = \left\| \hat{T}_n - \hat{T}_{n-1} \right\|. \tag{30}$$

This definition treats drift as a geometric quantity in projective state space rather than as a temporal rate. Small or vanishing drift indicates structural stationarity, while sustained or accelerating drift signals loss of coherence, proximity to instability, or ongoing reorganization.

Drift provides a unifying measure for non-stationary behavior across domains, including evolving networks, adaptive fields, and socio-technical systems. Because it is defined purely in terms of projection operators and norms, it remains invariant under changes in microscopic dynamics or modeling formalism.

7.2. Local and Global Drift Diagnostics

Drift diagnostics can be evaluated locally or globally, depending on the resolution at which structural change is analyzed. Local drift measures are defined on subregions of the system representation, such as node neighborhoods in networks or spatial patches in continuous fields. For a localized projection $\hat{T}_n^{(x)}$, the local drift is

$$D_n^{(x)} = \left\| \hat{T}_n^{(x)} - \hat{T}_{n-1}^{(x)} \right\|, \tag{31}$$

revealing spatially heterogeneous or component-specific structural change.

Global drift aggregates these local contributions into a system-level diagnostic,

$$D_n^{\mathrm{glob}} = \int D_n^{(x)}\, \mathrm{d}x, \tag{32}$$

or, in discrete settings, as a weighted sum over components. Comparison between local and global drift profiles enables the identification of localized precursors to large-scale transitions, as well as the distinction between coherent global reorganization and fragmented, region-specific change.

7.3. Coherent and Oscillatory Patterns

Beyond monotonic growth or decay, drift dynamics often exhibit coherent or oscillatory patterns that encode higher-order structural organization. Coherent drift corresponds to synchronized variation

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across components, indicating coordinated structural adaptation. Oscillatory drift patterns arise when the system alternates between distinct projection basins without settling into a single fixed point.

Such behavior is captured by analyzing the temporal or parametric structure of the drift sequence $\{D_n\}$, including its autocorrelation and spectral content. An oscillatory drift regime is characterized by

$$D_{n+m} \approx D_n \quad \text{for some period } m > 0, \tag{33}$$

reflecting cyclic transitions between structurally admissible states. These patterns generalize limit cycles in dynamical systems to a purely structural setting and are particularly relevant for systems exhibiting recurrent reorganization without catastrophic breakdown.

7.4. Structural Order Transitions

Structural order transitions correspond to qualitative changes in the organization of a system as detected through drift diagnostics. Such transitions occur when drift behavior changes regime, for example from bounded to unbounded growth, from coherent to fragmented patterns, or from oscillatory to stationary behavior. In SORT-CX, these transitions are interpreted as crossings between regions of projective state space associated with different operator-fixed subspaces.

A transition point n_c is identified when a drift-sensitive order parameter exhibits non-analytic behavior,

$$\left.\frac{\partial D_n}{\partial n}\right|_{n=n_c} \text{discontinuous or divergent.} \tag{34}$$

This formulation parallels phase-transition concepts while remaining independent of thermodynamic assumptions or explicit time evolution. Structural order transitions detected through drift provide early-warning indicators for regime shifts, loss of resilience, or reconfiguration events across a wide range of complex systems.

8. Use Case IV: Pattern Formation and Mode Selection

8.1. Pattern Formation as Eigenmode Projection

Pattern formation in complex systems is traditionally described as the amplification of specific modes under dynamical instabilities. In SORT-CX, pattern formation is reformulated as a projection problem in which admissible spatial or relational patterns correspond to eigenmodes selected by resonance operators. Let $\mathcal{L}$ denote a linear structural operator associated with the system representation, such as a Laplacian or generalized coupling operator. Its spectral decomposition defines a basis of candidate modes $\{\varphi_\lambda\}$.

A pattern emerges when a resonance operator $\hat{O}_i$ projects the system state onto a subspace spanned by a subset of these eigenmodes,

$$\hat{O}_i = \sum_{\lambda \in \Lambda^\star} \mathcal{P}_\lambda, \tag{35}$$

where $\mathcal{P}_\lambda$ denotes the projector onto eigenmode φ_λ. Patterns are therefore identified as idempotent projections rather than as transient dynamical states. This formulation captures stationary, oscillatory, and spatially structured patterns within a single algebraic framework and does not rely on explicit instability growth rates.

8.2. Kernel-Induced Mode Selection

While resonance operators define admissible pattern classes, the non-local kernel $\kappa(k)$ determines which modes dominate at a given scale. Kernel-induced mode selection operates by weighting eigenmodes according to their spectral parameter k, leading to an effective projected state

$$\Phi_{\text{pattern}}(k) = \kappa(k)\,\hat{O}_i\Phi(k). \tag{36}$$

Modes whose characteristic scales match the kernel width set by σ_0 are preferentially amplified, while incompatible scales are suppressed.

This mechanism generalizes classical wavelength selection in reaction–diffusion systems by decoupling scale selection from specific reaction kinetics. Multi-kernel constructions allow simultaneous activation of multiple pattern scales, producing hierarchical or composite patterns that are stable under projection but would require fine-tuned dynamics in traditional formulations.

8.3. Stability and Destabilization of Pattern Classes

In SORT-CX, pattern stability is defined by idempotent invariance under repeated projection. A pattern class represented by $\Phi^\star$ is stable if

$$\hat{H}\,\kappa(k)\,\Phi^\star = \Phi^\star, \tag{37}$$

within a prescribed tolerance. Destabilization occurs when variations in kernel parameters, operator weights, or structural perturbations cause the projected state to exit the corresponding fixed-point subspace.

Transitions between pattern classes are interpreted as structural bifurcations in projective state space rather than as dynamical instabilities. This perspective naturally accommodates abrupt pattern switching, coexistence of multiple stable patterns, and intermittent pattern breakdowns, all of which are observed in complex systems but are difficult to capture with single-equation dynamical models.

8.4. Representative Examples Across Domains

The projection-based formulation of pattern formation applies across a wide range of domains. In biological morphogenesis, classical Turing patterns correspond to kernel-selected eigenmodes of diffusion operators, reinterpreted here as operator-fixed pattern classes [36,40]. In chemical systems, oscillatory and stationary reaction–diffusion patterns arise from different kernel and operator combinations without invoking detailed kinetics [38].

In ecological and environmental systems, spatial vegetation patterns and resource distributions can be analyzed as stable projection outcomes under scale-dependent kernels. In socio-technical contexts, recurring activity or usage patterns correspond to relational eigenmodes of interaction networks filtered by operator constraints. Across these examples, SORT-CX provides a unified structural language for pattern formation that is independent of domain-specific dynamics while remaining compatible with established phenomenology.

9. Use Case V: Criticality, Phase Transitions, and Tipping Points

9.1. Drift Gradients as Early Structural Indicators

Critical behavior in complex systems is preceded by characteristic changes in structural stability that can be detected prior to macroscopic breakdown. In SORT-CX, early-warning signals are formulated in terms of drift gradients rather than temporal variance or autocorrelation. Let D_n denote the drift metric defined in Eq. 30. Structural proximity to a critical regime is indicated by amplification of the drift gradient,

$$G_n = \frac{\partial D_n}{\partial n}, \tag{38}$$

evaluated along a control sequence n representing scale, perturbation strength, or structural parameter variation.

An increasing or diverging G_n signals loss of projective stability and shrinking basins of operator-fixed subspaces. Unlike classical early-warning indicators that rely on stochastic fluctuations, drift gradients are defined deterministically through projection geometry and therefore remain meaningful in data-scarce or noise-free settings. This formulation provides a structural analogue to critical slowing down while avoiding assumptions about underlying dynamics [43,44].

9.2. Kernel Scaling and Structural Criticality

Structural criticality in SORT-CX is closely linked to the scaling behavior of the non-local kernel $\kappa(k)$. As the correlation scale parameter σ_0 is varied, the effective spectral support of the kernel changes, altering which modes contribute to emergent structure. Critical regimes occur when small variations in σ_0 produce large changes in projected structure.

This sensitivity is quantified by the kernel scaling response,

$$\chi_\kappa = \frac{\partial}{\partial \sigma_0} \left\| \kappa(k)\, \hat{H}\Phi \right\|, \tag{39}$$

which plays a role analogous to susceptibility in statistical physics. Divergence or sharp peaks in χ_κ indicate structural criticality, characterized by scale-free behavior and competition between multiple projection channels. In this regime, the system exhibits heightened responsiveness to perturbations without requiring fine-tuned dynamical parameters.

9.3. Instability Manifolds and Critical Surfaces

Critical transitions in SORT-CX are geometrically represented by instability manifolds in projective state space. These manifolds separate regions associated with distinct operator-fixed subspaces and define boundaries beyond which structural consistency cannot be maintained. Let $\mathcal{M}_{\mathrm{op}}$ denote the projective state space introduced in Eq. 17. A critical surface $\Sigma_c \subset \mathcal{M}_{\mathrm{op}}$ is defined by the condition

$$\left\| \hat{H}\Phi - \Phi \right\| = \varepsilon_c, \tag{40}$$

where ε_c marks the onset of global inconsistency.

Crossing Σ_c corresponds to a qualitative reorganization of structure rather than to a smooth deformation. Instability manifolds may have codimension one or higher, allowing for multiple competing transition pathways. This geometric representation unifies diverse notions of criticality, including bifurcations, percolation thresholds, and resilience loss, within a single projective framework.

9.4. Structural Interpretation of Tipping Points

In SORT-CX, tipping points are interpreted as projection catastrophes in which no admissible fixed point remains within the current structural basin. A tipping point is reached when incremental changes in control parameters force the system across a critical surface Σ_c, resulting in an abrupt transition to a distinct projection basin or to a fragmented structural state.

Formally, a tipping event at index n_c is identified by the joint conditions

$$\lim_{n \to n_c^-} D_n < \infty \quad \text{and} \quad \lim_{n \to n_c^+} D_n \gg D_{n_c^-}, \tag{41}$$

indicating a discontinuous increase in drift. This formulation captures irreversible regime shifts observed in ecological, climatic, and socio-technical systems without invoking specific dynamical mechanisms [46,50].

The structural interpretation emphasizes that tipping points arise from the exhaustion of structural consistency rather than from extreme external forcing. As a result, SORT-CX provides a principled basis for anticipating critical transitions through projection-based diagnostics and for comparing tipping behavior across heterogeneous complex systems.

10. Use Case VI: Multilayer and Coupled Systems

10.1. Coupled Networks and Interdependent Systems

Many real-world complex systems consist of multiple interacting subsystems whose structures cannot be reduced to a single network or field representation. Examples include interdependent infrastructure networks, socio-technical systems, ecological–economic couplings, and multilayer communication or transport systems. Such systems are characterized by distinct layers with their own internal organization, coupled through cross-layer dependencies that fundamentally alter stability and resilience properties [73,86].

In SORT-CX, each layer ℓ is represented by its own structural state $\Phi^{(\ell)}$, while interdependencies are encoded through coupling operators that act across layers. Emergence in coupled systems is therefore understood as a property of the joint projection space rather than of any individual layer. Structural failure or reorganization may propagate across layers even when each layer remains locally stable in isolation, highlighting the necessity of a genuinely multilayer structural framework.

10.2. Multilayer Operator Projections

Multilayer systems are analyzed by extending resonance operators to act on the direct sum of layer-specific state spaces. Let $\hat{O}_i^{(\ell)}$ denote a resonance operator acting on layer ℓ. A multilayer operator is defined as

$$\hat{O}_i^{\mathrm{ML}} = \bigoplus_{\ell} \hat{O}_i^{(\ell)} + \hat{O}_i^{\mathrm{cross}}, \tag{42}$$

where $\hat{O}_i^{\mathrm{cross}}$ encodes structural constraints induced by interlayer coupling. This construction allows both intra-layer organization and cross-layer coherence to be enforced within a single projection.

Operator-fixed points of $\hat{O}_i^{\mathrm{ML}}$ correspond to structurally consistent configurations across all layers. Importantly, such fixed points need not coincide with fixed points of the individual $\hat{O}_i^{(\ell)}$, reflecting the emergence of genuinely collective structure that only exists in the coupled system. This framework generalizes classical multilayer network formalisms by embedding them into an operator-algebraic projection structure [74,75].

10.3. Multi-Kernel Interference and Scale Coupling

Different layers in a coupled system often operate on distinct characteristic scales. SORT-CX accommodates this by associating each layer with its own kernel $\kappa^{(\ell)}(k)$. The effective multilayer kernel arises from the superposition and interference of these layer-specific kernels,

$$\kappa^{\mathrm{ML}}(k) = \sum_{\ell} w_{\ell}\, \kappa^{(\ell)}(k), \tag{43}$$

with weights w_ℓ reflecting coupling strength or relevance.

Multi-kernel interference leads to scale coupling effects in which structural modes in one layer activate or suppress modes in another. Resonant amplification occurs when kernel supports overlap, while destructive interference suppresses incompatible scales. This mechanism explains cross-scale cascades and emergent synchronization phenomena observed in interdependent systems, without requiring explicit dynamical coupling terms.

10.4. Stability and Resilience in Coupled Resonance Spaces

Stability and resilience in multilayer systems are assessed by evaluating structural consistency in the combined resonance space. A coupled configuration $\{\Phi^{(\ell)}\}$ is globally stable if it satisfies

$$\hat{H}\, \hat{O}_i^{\mathrm{ML}}\, \kappa^{\mathrm{ML}}(k)\, \Phi = \Phi, \tag{44}$$

within a prescribed tolerance. Violation of this condition in any subset of layers can trigger system-wide reorganization due to interlayer dependencies.

Resilience is quantified by the ability of the coupled system to absorb perturbations in one layer without inducing excessive drift in others. Drift diagnostics extended to multilayer projections reveal whether disturbances remain localized or propagate across layers, providing a structural explanation for cascading failures and robustness trade-offs [76,85].

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By formulating multilayer and interdependent systems within a unified projection framework, SORT-CX enables systematic comparison of resilience properties across domains and offers principled diagnostics for identifying structurally vulnerable coupling configurations.

11. Use Case VII: Cross-Scale Systemic Risk in Multilayer Systems

11.1. Systemic Risk as a Structural Phenomenon

Systemic risk in complex systems arises when coupled subsystems interact across scales in a manner that enables local perturbations to propagate and amplify globally. Such risks are not adequately captured by component-level failure probabilities or by single-layer stability analysis. In SORT-CX, systemic risk is formulated as a structural property of multilayer projection spaces, emerging from the interaction of resonance operators, kernel scales, and drift dynamics across layers.

Rather than modeling specific shock scenarios, SORT-CX identifies configurations in which structural coherence is fragile under cross-scale coupling. Risk is therefore defined as proximity to projection-inconsistent regimes in which small structural deviations can induce large-scale reorganization. This framing applies uniformly to climate–energy–economic couplings, financial–technological–regulatory systems, and bio–social–infrastructure networks.

11.2. Cross-Scale Coupling and Multilayer Projection

Let $\Phi^{(\ell)}$ denote the structural state of layer ℓ in a multilayer system. Cross-scale coupling is represented by a joint projection,

$$\Phi_{\text{multi}} = \hat{H} \sum_{\ell} \sum_{i} \hat{O}_i^{(\ell)} \, \kappa^{(\ell)}(k) \, \Phi^{(\ell)}, \tag{45}$$

where $\hat{O}_i^{(\ell)}$ are layer-specific resonance operators and $\kappa^{(\ell)}(k)$ encode scale-dependent interactions within and across layers. The global projector $\hat{H}$ enforces consistency across the combined resonance space.

Systemic risk emerges when admissible projections at the single-layer level become incompatible under joint projection. Such incompatibilities manifest as elevated drift, loss of coherence, or fragmentation in the combined structural state, even when each layer appears individually stable.

11.3. Emergent Mode Locking and Risk Amplification

A characteristic feature of cross-scale systemic risk is emergent mode locking, in which dominant structural modes from different layers align and reinforce each other. In SORT-CX, mode locking corresponds to the convergence of operator-fixed subspaces across layers,

$$\hat{O}_i^{(\ell_1)} \Phi_{\text{multi}} \approx \hat{O}_j^{(\ell_2)} \Phi_{\text{multi}}, \tag{46}$$

over a finite range of kernel scales. While such alignment can enhance coherence and efficiency, it also reduces structural diversity and increases sensitivity to perturbations.

When mode locking occurs near instability manifolds, small cross-layer perturbations can trigger rapid structural transitions. SORT-CX detects these conditions through joint analysis of drift gradients, coherence measures, and kernel-scale sensitivity, providing early structural indicators of amplification-prone regimes.

11.4. Structural Early-Warning Signals and Governance Relevance

SORT-CX offers a unified set of early-warning diagnostics for systemic risk that are independent of domain-specific dynamics. Elevated cross-layer drift, narrowing of kernel-robust stability regions, and sharp coherence transitions signal loss of structural resilience before observable failures occur. These indicators generalize early-warning concepts from critical transition theory to multilayer, non-dynamical settings [43,44].

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Because the diagnostics operate at the level of structural admissibility, they remain applicable when detailed models are unavailable, incompatible, or contested. This makes SORT-CX particularly suitable for comparative risk assessment across alternative system architectures, policy-relevant abstractions, or scenario ensembles.

11.5. Relation to Existing Use Cases

This use case integrates and extends the preceding analyses of networks (Section 5), stability landscapes (Section 6), drift dynamics (Section 7), and multilayer systems (Section 10). It does not introduce new mathematical primitives, but demonstrates that the existing SORT-CX formalism is intrinsically suited to cross-scale systemic risk analysis.

By making this capability explicit, SORT-CX positions itself as a structural framework for identifying, comparing, and monitoring systemic risk in coupled complex systems, addressing a central challenge in contemporary complexity science and governance-relevant modeling.

12. Structural Invariants and Symmetries

12.1. Projective Invariants from Idempotency

A central feature of SORT-CX is the existence of structural invariants that arise directly from the idempotent nature of resonance operators. For an idempotent operator $\hat{O}_i$ satisfying $\hat{O}_i^2 = \hat{O}_i$, the projected subspace is invariant under repeated application of the operator. Any state component $\Phi^\star$ that fulfills

$$\hat{O}_i \Phi^\star = \Phi^\star \tag{47}$$

defines a projective invariant of the system. Such invariants are independent of the representation of the underlying system and persist under admissible perturbations that do not violate operator consistency.

These projective invariants generalize conserved quantities in dynamical systems. Rather than being tied to symmetries of equations of motion, they encode invariance under structural projection. As a result, they remain well-defined even in non-stationary, adaptive, or non-dynamical settings and provide a robust basis for cross-domain comparison of emergent organization.

12.2. Resonance Symmetries and Conservation Patterns

Resonance symmetries arise when sets of operators commute or close under a defined algebraic structure. If two resonance operators satisfy

$$[\hat{O}_i, \hat{O}_j] = 0, \tag{48}$$

their projections share a common invariant subspace. Such symmetries imply the existence of conservation patterns in structural space, where certain relational or spatial features remain unchanged across transformations.

Unlike classical symmetries associated with spatial or temporal transformations, resonance symmetries act in the space of structural descriptions. They capture invariance under changes of scale, representation, or observation protocol, provided these changes are compatible with the operator algebra. Conservation patterns induced by resonance symmetries explain the persistence of modular organization, scaling relations, or functional roles across otherwise heterogeneous system realizations.

12.3. Fixed Points and Structural Attractors

Fixed points of resonance operators play the role of structural attractors in SORT-CX. A structural attractor is defined as a configuration toward which projected states converge under repeated application of operators and kernel filtering. Formally, a state $\Phi^\star$ is a structural attractor if

$$\lim_{n\to\infty} \left\| \hat{O}_i^n \, \kappa(k) \, \Phi - \Phi^\star \right\| = 0. \tag{49}$$

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These attractors characterize long-lived or recurrent structural organization without reference to temporal dynamics. Multiple attractors may coexist, separated by instability manifolds as described in Section 9.

Structural attractors provide a natural language for multistability and regime coexistence in complex systems. Transitions between attractors correspond to changes in projection dominance or kernel scaling rather than to classical bifurcations, enabling a unified interpretation of abrupt and gradual structural change.

12.4. Topological Invariants and Structural Signatures

Beyond algebraic invariants, SORT-CX admits topological invariants that characterize the global structure of projected state spaces. When structural states are embedded into suitable topological spaces, features such as connected components, cycles, or higher-dimensional holes become invariant under continuous deformations induced by projection.

Let $\mathcal{S}$ denote the space of admissible projected states. Topological invariants $\mathcal{I}(\mathcal{S})$ are preserved under operator-consistent transformations,

$$\mathcal{I}(\mathcal{S}) = \mathcal{I}(\hat{O}_i \mathcal{S}), \tag{50}$$

and provide structural signatures that distinguish qualitatively different organizational regimes. Such invariants connect SORT-CX to methods from topological data analysis, where persistent features across scales are interpreted as robust structural information [87,88].

Topological structural signatures complement operator- and kernel-based diagnostics by capturing global organizational properties that are insensitive to local perturbations, offering an additional layer of robustness for the analysis of complex systems.

13. Observer Role and Structural Model Reduction

13.1. Observer-Dependent Projection Choices

In SORT-CX, the observer enters the formalism through the choice of projection operators, kernels, and evaluation norms. These choices determine which structural aspects of a system are rendered visible and which are suppressed. The observer does not influence the underlying system state Φ, but selects a projection map $\hat{O}_i$ and kernel $\kappa(k)$ that define the effective structural description.

This dependence is explicit and controlled. Different observers, or the same observer at different analysis stages, may apply different projection schemes to the same system without contradiction. Structural statements are therefore always conditional on a specified projection context. This formulation avoids implicit observer bias by making the role of observational choice explicit and mathematically encoded, while preserving objectivity at the level of projective invariants and fixed points that remain stable across admissible projection families.

13.2. Structural Observables Versus Dynamical Variables

SORT-CX distinguishes structural observables from dynamical variables. Dynamical variables describe instantaneous system configurations or trajectories, while structural observables are defined as quantities invariant or stable under projection. A structural observable $\mathcal{S}_i$ is constructed as

$$\mathcal{S}_i = \left\| \hat{O}_i \, \kappa(k) \, \Phi \right\|, \tag{51}$$

and encodes information about coherence, modularity, scale dominance, or consistency that is independent of temporal parametrization.

This separation resolves a common ambiguity in complex-systems analysis, where dynamical variability is often conflated with structural change. In SORT-CX, a system may exhibit strong dynamical fluctuations while remaining structurally stable, or conversely display minimal dynamical variation while undergoing significant structural reorganization. Structural observables provide a principled basis for identifying the latter case, which is often inaccessible to trajectory-based analysis.

13.3. SORT as a Structural Model-Reduction Framework

Model reduction in SORT-CX is achieved through projection rather than through truncation of dynamical equations. By selecting a reduced set of resonance operators and kernel scales, the observer defines a lower-dimensional structural representation that preserves essential organization while discarding structurally irrelevant detail. Let $\{\hat{O}_{i_1}, \ldots, \hat{O}_{i_m}\}$ denote a reduced operator set. The reduced structural state is

$$\Phi_{\text{red}} = \hat{H} \sum_{a=1}^{m} \hat{O}_{i_a}\, \kappa(k)\, \Phi. \tag{52}$$

This construction guarantees that the reduced model remains globally consistent and structurally admissible.

Unlike conventional reduction techniques that rely on time-scale separation or linearization, SORT-based reduction is invariant under changes in dynamics and applies equally to stationary, adaptive, and non-dynamical systems. The resulting reduced representations retain direct interpretability in terms of structural features and support systematic comparison across models, domains, and observational choices.

14. Architecture and Engine Integration

14.1. SORT-CX Components and Design Principles

SORT-CX is designed as a modular extension layer within the broader SORT framework, adhering to strict separation between mathematical core, domain adapters, and evaluation interfaces. Its primary components comprise resonance operator modules, kernel evaluation modules, projection and drift diagnostics, and classification utilities. Each component operates on abstract structural states Φ and interacts exclusively through well-defined projection interfaces.

The design principles emphasize composability, algebraic transparency, and domain independence. Resonance operators are implemented as idempotent mappings with explicit algebraic properties, kernels are defined as parameterized spectral filters, and diagnostics are formulated as normed geometric quantities. This architecture ensures that SORT-CX analyses remain invariant under representation changes and that extensions introduce no hidden dynamical assumptions.

14.2. Registry and Extension Mechanisms

Extensibility in SORT-CX is managed through a formal registry that enumerates available operators, kernels, and diagnostic functions. Each registered component is uniquely identified, versioned, and associated with a precise mathematical specification. New operators or kernels may be introduced provided they satisfy the core SORT consistency requirements, including idempotency conditions for operators and normalization constraints for kernels.

The registry mechanism enables controlled expansion of the framework across application domains without altering the core engine. Domain-specific extensions, such as network Laplacian operators or spatial field kernels, are encapsulated as registered modules that can be composed with existing components. This approach prevents fragmentation of the framework while supporting systematic growth toward SORT v6.

14.3. Integration into the SORT Core Engine

SORT-CX integrates into the SORT core engine by reusing the global projector $\hat{H}$, the kernel infrastructure $\kappa(k)$, and the transformation and drift pipelines introduced in SORT v5. From an architectural perspective, SORT-CX acts as a specialized analysis layer that invokes core projection routines while supplying domain-adapted operator sets and interpretation logic.

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This integration guarantees consistency across SORT-CX, SORT-Cosmology, SORT-QS, and SORT-AI. Structural states produced or analyzed in one domain can be passed through the same core engine and re-evaluated under different operator or kernel selections. As a result, cross-domain comparisons and joint analyses are supported without duplication of mathematical infrastructure or loss of interpretability.

14.4. Data Interfaces and Reproducibility

Reproducibility in SORT-CX is ensured through standardized data interfaces that separate raw system representations from projection configuration and evaluation output. Structural states Φ, operator selections, kernel parameters, and diagnostic settings are serialized independently and referenced by immutable identifiers.

All projection results are deterministic given a fixed configuration, enabling exact replication of analyses across platforms and implementations. This design supports transparent benchmarking, systematic sensitivity studies with respect to projection choices, and long-term comparability of results across SORT versions. By embedding reproducibility at the architectural level, SORT-CX provides a robust foundation for cumulative structural analysis of complex systems.

14.5. Validation and Structural Benchmarks

To ensure methodological robustness and external assessability, SORT-CX incorporates a validation layer that defines minimal structural benchmarks and explicit failure conditions. The objective is not numerical performance optimization, but the verification of internal consistency, comparability to established methods, and the principled identification of regimes in which the framework does not yield meaningful diagnostics.

Validation in SORT-CX is formulated at the level of projection geometry and diagnostic behavior. For each major use case, a class of minimal benchmark scenarios is defined in which the expected structural outcome is known a priori. These benchmarks serve to test whether operator projections, kernel filtering, and drift diagnostics behave consistently with their formal definitions. Representative benchmark classes include synthetically generated networks with controlled modularity and imposed structural drift, reduced field configurations with analytically stable projection fixed points, and toy pattern-formation settings in which admissible modes are fixed by construction. The benchmarks are intentionally low-dimensional and interpretable, ensuring that deviations can be attributed unambiguously to projection artifacts rather than to numerical complexity.

Structural comparison is performed against established baseline methods that capture related but narrower notions of structure. These include spectral techniques such as principal component analysis and graph Laplacian embeddings, classical spectral clustering and community-detection methods, change-point detection schemes for non-stationary systems, and Lyapunov-based stability indicators where applicable. The comparison is qualitative and diagnostic in nature: SORT-CX is evaluated with respect to whether it recovers comparable structural partitions, stability regions, or transition indicators, while additionally providing information about operator consistency, kernel scale effects, and drift geometry that is absent in the baselines.

An explicit component of the validation layer is the identification of failure modes. SORT-CX is designed to remain silent in regimes where projection-based structural analysis is ill-defined or misleading. Such regimes include systems dominated by high-frequency noise with no coherent projection fixed points, representations in which operator channels cannot be meaningfully constructed, and situations in which drift metrics diverge due to incompatible observation scales rather than genuine structural change. Declaring non-applicability in these cases is treated as a successful validation outcome, as it demonstrates that the framework does not over-interpret structure where none is present.

All validation and benchmark procedures are integrated into the SORT core engine as optional diagnostic passes that can be executed independently of domain-specific pipelines. This design ensures

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that validation remains reproducible, comparable across applications, and extensible as new operator channels or kernel constructions are introduced in the transition toward SORT v6.

15. Applications Beyond Classical Complex Systems

15.1. Social and Collective Systems

Social and collective systems exhibit emergent organization that is weakly constrained by individual-level rules and strongly shaped by relational structure, norms, and information flow. Examples include opinion formation, cultural clustering, collective decision-making, and coordination phenomena. In SORT-CX, such systems are represented as relational structures whose emergent properties arise from projection onto coherent interaction patterns rather than from explicit behavioral dynamics.

Resonance operators capture persistent collective modes such as consensus blocks, polarization axes, or role-based groupings, while kernels encode effective interaction ranges across social distance or communication layers. Structural drift diagnostics reveal slow reconfiguration of collective organization even when observable behavior appears stationary. This perspective aligns with established results in social dynamics while providing a representation-invariant framework for comparing heterogeneous collective systems [77,80].

15.2. Economic and Financial Networks

Economic and financial systems are characterized by dense coupling, feedback loops, and multi-scale dependencies that challenge equilibrium-based modeling. SORT-CX treats markets, production networks, and financial linkages as structural systems whose stability depends on projection consistency rather than on local optimization.

Operator projections identify dominant structural modes such as sectoral coupling, systemic risk clusters, or liquidity channels. Kernel filtering captures scale-dependent aggregation, distinguishing local shocks from system-wide structural stress. Drift-based diagnostics detect gradual erosion of structural resilience prior to crises, offering a complementary viewpoint to volatility-based and stochastic indicators. This formulation enables comparative analysis of financial architectures without assuming rational agents or equilibrium dynamics [78,79].

15.3. Ecological and Biological Systems

Ecological and biological systems exhibit multistability, nonlinear feedback, and cross-scale coupling between organisms, populations, and environments. SORT-CX represents such systems as coupled structural fields or interaction networks whose emergent organization is governed by projection constraints.

Resonance operators isolate stable interaction motifs, trophic structures, or functional groupings, while kernels encode spatial dispersal, interaction ranges, or resource gradients. Structural stability landscapes provide a unified description of resilience, regime shifts, and recovery behavior that is independent of specific population dynamics. This approach complements classical ecological theory by emphasizing structural admissibility over dynamical trajectories [52,56,81].

15.4. Infrastructure and Cyber-Physical Systems

Modern infrastructure systems integrate physical assets, information networks, and control logic, forming tightly coupled cyber-physical structures. Examples include power grids, transportation systems, communication networks, and distributed control architectures. In such systems, failures often propagate structurally rather than dynamically, through dependency and coupling patterns.

SORT-CX models infrastructure as a multilayer structural system in which resonance operators enforce consistency across physical and digital layers, and kernels capture spatial and operational scales. Drift diagnostics identify emerging fragility due to incremental reconfiguration, load redistribution, or coupling intensification. This structural perspective supports resilience assessment and vulnerability

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analysis without reliance on detailed failure simulations, aligning with contemporary studies of interdependent infrastructure risk [84–86].

16. Limitations and Open Questions

16.1. Structure Versus Dynamics

SORT-CX deliberately prioritizes structural admissibility and projection geometry over explicit time evolution. While this choice enables representation-invariant analysis and cross-domain comparability, it also implies that SORT-CX does not, by itself, predict temporal trajectories or transient responses. Structural stability does not necessarily imply dynamical stability, and structurally admissible configurations may be dynamically inaccessible under specific evolution rules.

This separation is intentional and reflects a conceptual boundary rather than a deficiency. SORT-CX addresses the question of which structures are possible and consistent, not how quickly or through which path they are realized. Bridging structural predictions with detailed dynamical behavior requires external dynamical models, with SORT-CX providing constraints and admissible targets rather than full temporal descriptions.

16.2. Relation to PDEs, Agent Models, and Network Theory

SORT-CX is not a replacement for established modeling approaches such as partial differential equations, agent-based models, or classical network theory. Instead, it operates at a complementary level of abstraction. PDEs specify local evolution laws, agent models encode decision rules and interactions, and network theory formalizes connectivity patterns. SORT-CX abstracts from these details and evaluates the resulting structures through projection consistency.

An open question concerns the systematic mapping between these modeling paradigms and SORT-CX projections. While specific correspondences have been illustrated in use cases, a general translation framework remains to be developed. In particular, identifying minimal conditions under which a given dynamical or agent-based model converges toward a SORT-admissible structural attractor is an unresolved problem with implications for model validation and reduction.

16.3. Open Mathematical Questions

Several mathematical aspects of SORT-CX warrant further investigation. These include the classification of admissible resonance operator algebras, the topology and dimensionality of projective state spaces, and the precise conditions for existence and uniqueness of structural attractors. The behavior of instability manifolds and critical surfaces in high-dimensional projection spaces also remains incompletely characterized.

Another open question concerns the rigorous relationship between kernel scaling limits and universality classes of structural behavior. Establishing formal convergence theorems and bounds for drift metrics under projection sequences would strengthen the mathematical foundations of SORT-CX and clarify its relation to established theories of criticality and emergence.

16.4. Data-Free Versus Data-Supported Structural Analysis

SORT-CX enables data-free structural analysis by operating on abstract system representations without requiring empirical calibration. This feature is advantageous for theoretical exploration and comparative studies but limits direct empirical validation. In practical applications, structural analysis is often informed or constrained by data, raising questions about how empirical uncertainty propagates through projection diagnostics.

Developing principled methods for integrating data-supported estimates of structural states with projection-based analysis remains an open challenge. This includes defining confidence measures for structural observables, assessing robustness under incomplete or noisy data, and establishing protocols for updating projection choices as new information becomes available. Addressing these questions is essential for extending SORT-CX from a theoretical framework to a broadly applicable analytical tool.

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16.5. Robustness and Uncertainty

SORT-CX is explicitly designed to provide structural diagnostics that are robust with respect to noise, sampling variability, and representational abstraction. Robustness in this context does not refer to numerical stability of simulations, but to the persistence of structural conclusions under controlled variations of projection choices and observation parameters. This subsection clarifies the scope and limits of such robustness.

A primary source of sensitivity arises from the kernel correlation scale σ_0, which determines the effective interaction range and scale resolution of the analysis. Structural diagnostics such as stability regions, coherence measures, and drift gradients are evaluated across bounded intervals of σ_0 to assess scale robustness. Structural features that persist over a finite range of kernel scales are interpreted as robust, whereas features that appear only at finely tuned values of σ_0 are classified as scale-fragile and treated with caution.

A second source of uncertainty concerns the selection of resonance operator subsets. SORT-CX does not assume that a unique or complete operator set is available for all systems. Robustness is therefore assessed by evaluating diagnostics across operator subsets that satisfy the global consistency constraints defined by the projector $\hat{H}$. Structural conclusions that remain invariant under admissible operator subset variations are considered operator-robust, while strong dependence on a single operator channel signals limited interpretability.

A third sensitivity dimension arises from the choice of observational representation. Different embeddings, coarse-grainings, or measurement protocols may induce different structural states Φ for the same underlying system. SORT-CX addresses this by treating representation choice as an explicit part of the projection context. Robustness is evaluated by comparing diagnostic outcomes across representations that preserve relational or topological structure. Discrepancies are interpreted as representation-induced uncertainty rather than as intrinsic structural change.

Crucially, SORT-CX does not aim to produce point predictions or sharp numerical estimates. Its output consists of structural robustness statements, admissible regimes, and confidence regions in projection space. This distinction limits over-interpretation and aligns the framework with its intended role as a structural analysis tool rather than as a predictive dynamical model. By making robustness and uncertainty explicit, SORT-CX provides reviewers and practitioners with clear criteria for when structural conclusions are reliable and when they should be withheld.

17. Conclusion and Outlook

17.1. SORT-CX as a Unifying Structural Framework

SORT-CX establishes a projection-based structural framework that unifies the analysis of complex systems across domains, representations, and scales. By formulating emergence, stability, and transition behavior in terms of resonance operators, non-local kernels, and global consistency constraints, SORT-CX decouples structural organization from specific dynamical assumptions. This abstraction enables direct comparison between systems that differ widely in microscopic rules but share common structural features.

The core contribution of SORT-CX lies in its ability to identify admissible structural states, invariant patterns, and critical boundaries within a single mathematical language. As demonstrated throughout the use cases, networks, fields, multilayer systems, and adaptive structures can be analyzed within the same projective geometry, providing a coherent foundation for structural reasoning in complex systems science.

17.2. Implications for Complex Systems Science

The projection-based perspective introduced by SORT-CX has several implications for the study of complex systems. First, it shifts the focus from trajectory prediction to structural admissibility, reframing emergence as a geometric property rather than as a dynamical outcome. Second, it provides principled diagnostics for stability, drift, and criticality that are invariant under changes in modeling

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formalism. Third, it supports systematic model comparison and reduction by identifying which structural features are essential and which are incidental.

These implications suggest a complementary role for SORT-CX alongside established approaches. Rather than replacing dynamical, statistical, or data-driven methods, SORT-CX supplies a structural backbone against which such methods can be interpreted and constrained. This perspective contributes to ongoing efforts to develop general theories of emergence, resilience, and critical transitions that transcend domain-specific models.

17.3. Conceptual Role in the Transition from v5 to v6

Within the broader SORT program, SORT-CX plays a central conceptual role in the transition from version v5 to v6. While v5 established the mathematical core of resonance operators, kernels, and global projection, SORT-CX demonstrates how this core can be systematically applied to complex systems beyond its original cosmological and quantum contexts.

SORT-CX thus serves as a validation layer for the generality of the SORT formalism. It clarifies which elements of the framework are universal and which require domain-specific adaptation. Insights gained from SORT-CX inform the refinement of operator registries, kernel hierarchies, and diagnostic pipelines planned for SORT v6, ensuring that the next iteration of the framework is structurally grounded and extensible across application domains.

17.4. Outlook Toward High-Performance Structural Analysis

Future developments of SORT-CX will focus on scaling projection-based structural analysis to high-dimensional systems and large datasets. The operator and kernel formulations are naturally amenable to parallelization, spectral methods, and distributed computation, enabling high-performance evaluation of structural diagnostics across large networks, fine-grained spatial fields, and multilayer architectures.

High-performance structural analysis opens the possibility of real-time monitoring of drift, stability, and criticality in evolving systems, as well as large-scale comparative studies across ensembles of complex systems. As SORT transitions toward v6, the integration of SORT-CX with high-performance computing environments is expected to play a key role in advancing structural approaches to complexity from conceptual frameworks to operational analytical tools.

17.5. Decision Relevance and Governance

Beyond its role as an analytical framework, SORT-CX provides decision-relevant structural signals that are applicable in governance, oversight, and strategic assessment contexts. Its contribution lies not in prescribing actions or policies, but in identifying regimes of structural stability, fragility, and transition that are critical for informed decision-making under uncertainty.

First, SORT-CX functions as an early-warning system for structural degradation. Drift diagnostics, coherence loss, and proximity to critical surfaces indicate when a system is approaching a regime in which existing abstractions, controls, or assumptions may cease to be valid. Such signals are particularly relevant in domains where delayed recognition of structural change leads to disproportionate risk, including socio-technical infrastructures, financial systems, and large-scale computational models.

Second, SORT-CX enables structural validity checks for abstractions and models used in governance-relevant settings. By evaluating whether a reduced or simplified representation remains projectively consistent under $\hat{H}$, the framework provides a principled test of whether an abstraction preserves essential structural features. This is especially relevant when models are reused across contexts, scaled beyond their original design, or embedded into automated decision pipelines.

Third, SORT-CX offers a diagnostic layer for comparative assessment of alternative system designs or control strategies. Rather than ranking options by performance metrics, SORT-CX compares their structural resilience, sensitivity to scale, and robustness under operator and representation variation. These diagnostics support deliberative processes by clarifying trade-offs and identifying structurally vulnerable configurations without committing to a specific optimization objective.

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In this role, SORT-CX contributes interpretative signals rather than operational prescriptions. It is therefore well-suited as a supporting tool in governance frameworks that require transparency, robustness assessment, and early detection of structural risk, while remaining agnostic to policy choices and value judgments.

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Appendix A. Mathematical Details

Appendix A.1. Complete Operator Definitions and Weights

This appendix provides the formal mathematical specification of the structural elements introduced in Section 3, in particular the resonance operators discussed in Section 3.1. Let $\mathcal{H}$ denote the projective state space of structural configurations. Each resonance operator $\hat{O}_i : \mathcal{H} \to \mathcal{H}$ is defined as an idempotent projector,

$$\hat{O}_i^2 = \hat{O}_i, \tag{A1}$$

ensuring that its image defines a structurally admissible subspace.

In practical applications, multiple operators act simultaneously. Their combined effect is represented by a weighted superposition,

$$\hat{O}_{\text{eff}} = \sum_i w_i \hat{O}_i, \tag{A2}$$

with weights $w_i \in \mathbb{R}^+$ satisfying $\sum_i w_i = 1$. The weights encode relative structural relevance and do not modify the algebraic properties of the individual operators. Global admissibility of any such superposition is enforced by the global projector $\hat{H}$ introduced in Section 3.3, such that only configurations consistent with all active operators are retained.

Appendix A.2. Proof of Jacobi Consistency

The consistency of the resonance operator algebra, introduced conceptually in Section 3.2, requires that nested operator compositions do not lead to ordering ambiguities. This requirement is formalized by Jacobi consistency. For any triplet of resonance operators $\hat{O}_i, \hat{O}_j, \hat{O}_k$, the Jacobi identity reads

$$[\hat{O}_i, [\hat{O}_j, \hat{O}_k]] + [\hat{O}_j, [\hat{O}_k, \hat{O}_i]] + [\hat{O}_k, [\hat{O}_i, \hat{O}_j]] = 0. \tag{A3}$$

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Because each $\hat{O}_i$ is idempotent, any commutator vanishes on the intersection of their invariant subspaces. For non-commuting pairs, admissible compositions are restricted by the action of the global projector $\hat{H}$, which projects onto the common structurally consistent subspace. Applying $\hat{H}$ to each nested commutator yields zero, proving that the projected operator algebra satisfies Eq. A3.

This result guarantees that structural diagnostics and fixed-point constructions are independent of operator ordering and justifies the use of resonance operator sequences throughout the framework.

Appendix A.3. Light-Balance Condition

Global structural consistency in SORT-CX relies on a balance condition introduced implicitly in Section 3.3. Let $\mathcal{B}_i$ denote the signed structural contribution associated with operator $\hat{O}_i$ after kernel weighting. The light-balance condition is defined as

$$\sum_i \mathcal{B}_i = 0. \tag{A4}$$

This condition prevents net structural bias toward any single projection channel and ensures that the global projector $\hat{H}$ admits non-trivial fixed points. Operationally, $\mathcal{B}_i$ is computed from normalized trace or norm contributions of $\hat{O}_i\,\kappa(k)\,\Phi$.

Violation of Eq. A4 implies accumulation of unresolved structural excess and leads to divergent drift behavior as discussed in Section 7. The light-balance condition is therefore a necessary criterion for global stability and for the absence of structural anomalies within SORT-CX analyses.

Appendix A.4. Kernel Normalization and Convergence

The non-local kernel $\kappa(k)$, introduced in Section 3.4, must satisfy normalization and convergence requirements to ensure well-defined projections. Normalization is imposed as

$$\int \kappa(k)\,\mathrm{d}k = 1, \tag{A5}$$

which guarantees preservation of overall structural magnitude under kernel application. Convergence requires sufficient decay at large spectral parameter,

$$\lim_{|k|\to\infty} \kappa(k) = 0, \tag{A6}$$

ensuring suppression of incoherent high-frequency modes.

Under these conditions, repeated kernel-weighted projection converges toward operator-fixed subspaces,

$$\lim_{n\to\infty} \big(\hat{O}_i\,\kappa(k)\big)^n \Phi = \Phi^\star, \tag{A7}$$

for structurally admissible initial states Φ. This result establishes the mathematical basis for stability landscapes, drift diagnostics, and critical surfaces discussed in Sections 6 and 9.

Appendix B. Diagnostic Procedures

Appendix B.1. Drift Metrics and Computation

This appendix specifies the diagnostic procedures corresponding to the drift concepts introduced in Sections 3.5 and 7. Let $\Phi^{(n)}$ denote a sequence of structural representations indexed by an ordering parameter n, such as scale resolution, perturbation level, or projection iteration. The projected structural transformation is defined as

$$\hat{T}_n = \hat{H} \sum_i \hat{O}_i\,\kappa(k)\,\Phi^{(n)}. \tag{A8}$$

The drift metric is computed as the normed difference between successive projected states,

$$D_n = \big\|\hat{T}_n - \hat{T}_{n-1}\big\|. \tag{A9}$$

This quantity measures structural change independently of temporal parametrization. For localized diagnostics, the same construction applies to restricted subspaces or spatial domains, yielding local drift profiles as discussed in Section 7.2. Drift sequences $\{D_n\}$ form the primary input for stability, criticality, and transition analyses.

Appendix B.2. Stability Tests

Stability diagnostics in SORT-CX are based on projector invariance rather than on eigenvalue growth or Lyapunov exponents. A structural state Φ is tested for stability with respect to a resonance operator $\hat{O}_i$ by evaluating the idempotency deviation

$$\Delta_i = \left\| \hat{O}_i \, \Phi - \Phi \right\|. \tag{A10}$$

Global stability is assessed using the global projector $\hat{H}$,

$$\Delta_H = \left\| \hat{H} \, \Phi - \Phi \right\|. \tag{A11}$$

A configuration is deemed structurally stable if Δ_i and Δ_H remain below prescribed tolerances for all active operators. Stability landscapes introduced in Section 6 are constructed by evaluating these diagnostics across families of configurations or parameter values.

Appendix B.3. Coherence Analysis

Coherence quantifies the degree to which structural components align across operator channels and scales. For a given projected state $\Phi_{\text{proj}} = \hat{H} \sum_i \hat{O}_i \, \kappa(k) \, \Phi$, coherence is defined as

$$C = \frac{1}{N} \sum_i \frac{\left\| \hat{O}_i \, \Phi_{\text{proj}} \right\|}{\left\| \Phi_{\text{proj}} \right\|}, \tag{A12}$$

where N is the number of active operators. High coherence indicates alignment of operator-fixed subspaces, while low coherence signals fragmentation or competing structural modes.

Tracking coherence across projection sequences enables detection of gradual loss of organization, onset of multistability, or approach to critical surfaces as described in Section 9. Coherence diagnostics complement drift metrics by distinguishing coherent reorganization from incoherent structural decay.

Appendix B.4. Classification Decision Trees

Structural classification procedures operationalize the dominance criteria introduced in Section 4. For a given system representation, scalar indicators $\mathcal{O}$, $\mathcal{K}$, and $\mathcal{D}$ are computed from operator commutators, kernel-weighted spectral response, and drift magnitude, respectively. These indicators define the dominance vector

$$\mathbf{\Lambda} = \frac{1}{\mathcal{O} + \mathcal{K} + \mathcal{D}} (\mathcal{O}, \mathcal{K}, \mathcal{D}), \tag{A13}$$

which locates the system within the classification simplex.

Decision trees map regions of this simplex to qualitative regime labels, such as operator-dominated, kernel-dominated, drift-dominated, or hybrid. Thresholds and branching criteria are fixed a priori and remain invariant across domains. This procedure ensures reproducible and interpretable classification of complex systems based solely on structural diagnostics, independent of underlying dynamics or data-driven fitting.

Appendix C. Implementation Guide

Appendix C.1. Python Code Examples

This appendix outlines a reference implementation strategy for SORT-CX diagnostics in Python, consistent with the mathematical definitions introduced in Sections 3 and 14. A minimal implemen-

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tation operates on numerical representations of structural states Φ as vectors, matrices, or tensors, depending on domain. Resonance operators $\hat{O}_i$ are implemented as idempotent linear maps, kernels $\kappa(k)$ as spectral weight functions, and the global projector $\hat{H}$ as a consistency-enforcing projection.

A typical evaluation pipeline proceeds by computing the kernel-weighted projection $\hat{O}_i\,\kappa(k)\,\Phi$ for all active operators, aggregating results via the global projector, and evaluating diagnostics such as drift, coherence, and stability using norm-based measures defined in Appendix B. Numerical linear-algebra backends are sufficient for all core operations, and no domain-specific solvers are required. This design ensures portability and ease of integration into existing scientific workflows.

Appendix C.2. MOCK v3 Integration

SORT-CX integrates directly with the MOCK v3 execution environment described in the SORT v5 framework. In this context, structural states Φ correspond to deterministic outputs of the MOCK pipeline at specified resolution or parameter settings. Resonance operators and kernels are registered as configurable modules that operate on MOCK-generated data without modifying the underlying simulation logic.

Integration is achieved by inserting SORT-CX projections as post-processing stages in the MOCK workflow. This allows systematic evaluation of structural admissibility, drift, and stability across parameter sweeps or resolution hierarchies. Because all projections are deterministic and side-effect free, SORT-CX analyses can be executed repeatedly on identical MOCK outputs to verify reproducibility and sensitivity, supporting the transition toward extended analysis pipelines planned for SORT v6.

Appendix C.3. Reproducibility Checklist

To ensure reproducible SORT-CX analyses, the following elements must be specified and archived for each evaluation run. The structural state representation Φ and its provenance must be uniquely identified. The full list of active resonance operators, their weights, and kernel parameters must be recorded. Norm definitions, tolerances for stability tests, and drift evaluation protocols must be fixed and documented.

Given these inputs, all SORT-CX diagnostics are deterministic. Reproducibility therefore reduces to configuration control rather than stochastic averaging. This checklist aligns with the architectural principles outlined in Section 14.4 and ensures that structural results remain comparable across platforms, implementations, and SORT versions.

Appendix C.4. API Reference and Usage Patterns

The SORT-CX API is organized around a small set of core abstractions: structural states, resonance operators, kernels, projectors, and diagnostics. Each abstraction exposes a minimal interface for evaluation and composition. Typical usage patterns involve constructing an operator set, selecting kernel configurations, invoking projection routines, and querying diagnostic outputs such as drift sequences or stability measures.

Higher-level workflows combine these primitives into classification, comparison, or monitoring tasks. Because all components adhere to strict mathematical contracts, alternative implementations or optimized backends can be substituted without altering analytical semantics. This API design supports long-term extensibility and provides a stable foundation for integration of SORT-CX into larger analysis systems and high-performance environments.

Appendix D. Comparison with Existing Frameworks

Appendix D.1. Synergetics

Synergetics provides a foundational framework for understanding self-organization through order parameters, slaving principles, and instability hierarchies [1,2]. In synergetic systems, macroscopic order parameters emerge dynamically and constrain microscopic degrees of freedom via explicit time evolution near instabilities.

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SORT-CX differs conceptually by decoupling structural emergence from dynamical bifurcation mechanisms. Resonance operators in SORT-CX act as a priori structural filters rather than dynamically selected order parameters. The slaving principle is replaced by global projective consistency enforced through $\hat{H}$, and stability is defined through idempotent invariance rather than through proximity to dynamical instabilities. SORT-CX therefore generalizes synergetic ideas to settings where time evolution is inaccessible, irrelevant, or ill-defined, while remaining compatible with synergetic interpretations when explicit dynamics are present.

Appendix D.2. Complex Network Theory

Complex network theory focuses on structural descriptors such as degree distributions, clustering coefficients, modularity, and spectral properties of graph operators [14,33**?**]. These tools provide powerful descriptive and comparative metrics but are typically tied to specific representations and lack a unifying notion of structural admissibility across domains.

SORT-CX subsumes network-theoretic constructs by treating graph-based operators, such as Laplacians, as specific instances of resonance operators. Community structure, centrality patterns, and spectral modes correspond to operator-fixed subspaces under projection. Kernel weighting introduces controlled scale dependence beyond standard spectral cutoffs. Unlike classical network theory, SORT-CX provides a unified stability, drift, and criticality diagnostics applicable to networks, fields, and multilayer systems within the same formalism.

Appendix D.3. Dynamical Systems Theory

Dynamical systems theory characterizes system behavior through differential equations, phase-space trajectories, fixed points, limit cycles, and bifurcations [58,59]. Stability is defined through eigenvalue spectra, Lyapunov exponents, and invariant manifolds, all of which rely on explicit time parametrization.

SORT-CX departs from this paradigm by replacing trajectories with projection sequences and phase space with projective state space. Fixed points correspond to operator-invariant structures rather than to dynamical equilibria, and bifurcations are reinterpreted as crossings of instability manifolds in projection geometry. While dynamical systems theory excels at predicting temporal evolution, SORT-CX addresses complementary questions concerning which structural organizations are admissible and stable independent of specific dynamics.

Appendix D.4. Statistical Physics Approaches

Statistical physics explains emergent phenomena through ensemble behavior, phase transitions, and renormalization concepts [63–65]. Criticality is associated with diverging correlation lengths and scale invariance arising from microscopic interactions under coarse-graining.

SORT-CX provides a non-ensemble-based alternative in which criticality emerges from kernel scaling and projection instability rather than from thermodynamic limits. The kernel $\kappa(k)$ plays a role analogous to coarse-graining operators, but without reliance on probabilistic assumptions or equilibrium ensembles. Structural susceptibilities and drift gradients replace thermodynamic response functions, enabling analysis of critical-like behavior in deterministic, finite, or non-ergodic systems. This positions SORT-CX as a complementary structural framework to statistical physics, particularly suited for systems outside traditional equilibrium settings.

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