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Article

The Fisher Information Action for Quantum Dynamics

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Abstract

This work develops a Lorentz-invariant variational framework in which Fisher-information geometry appears as an intrinsic structural contribution to quantum dynamics. Motivated by longstanding attempts to connect quantum mechanics with information-theoretic principles, we introduce an action functional depending on the density and phase fields in the Madelung representation. Variation of this action yields a modified Klein–Gordon equation containing a single nonlinear term proportional to the four-dimensional Fisher-information curvature of the probability density. The standard Klein–Gordon equation is recovered when the structural parameter vanishes, ensuring full compatibility with established relativistic dynamics. Taking the nonrelativistic limit, we obtain a uniquely determined nonlinear Schrödinger equation in which the correction term is the functional derivative of the Fisher information. The resulting dynamics preserve probability, maintain the Hamilton–Jacobi correspondence, and contain the linear Schrödinger equation as a special case. Analytical expressions for Gaussian and superposed states demonstrate how the structural modification scales with spatial localization and interference structure, providing clear qualitative signatures that distinguish the model from previous nonlinear extensions and offer a theoretical basis for future experimental verification. The results establish a mathematically transparent link between information geometry and quantum dynamics and provide a foundation for future extensions to fermionic, gauge, and many-body systems.

Keywords: fisher information; variational principles; nonlinear quantum mechanics; Klein–Gordon equation; Schrödinger equation; information geometry; quantum foundations; structural action; probability density curvature

1. Introduction

Nonlinear extensions of quantum mechanics have been examined for several decades as possible modifications to the standard linear Schrödinger dynamics. Early proposals include the logarithmic model of Bialynicki-Birula and Mycielski [1], Gisin's class of nonlinear Schrödinger equations [2], and the Doebner–Goldin family derived from diffeomorphism group representations [3]. These works established that small nonlinear terms can be introduced without necessarily violating probability conservation or the fundamental Hamilton–Jacobi correspondence. At the same time, they revealed important conceptual challenges: many nonlinear models can lead to superluminal signaling or conflict with relativistic causality unless carefully constructed [2,4]. These issues remain central in the debate over whether quantum mechanics can accommodate physically meaningful nonlinear corrections.

Parallel to these developments, a second line of research has sought to connect quantum mechanics with information-theoretic principles. Notable contributions include Reginatto's derivation of the Schrödinger equation from Fisher information [5] and Frieden's information-physical variational approach [6]. More recently, entropy-based dynamical frameworks such as steepest-entropy-ascent models have been proposed to capture nonequilibrium quantum behavior and state-space geometry [7,8]. These information-theoretic approaches suggest that structural features of the wavefunction, quantified through Fisher information or entropy gradients, may carry

dynamical significance rather than being mere mathematical artifacts. However, a common limitation persists: most existing models reconstruct the linear theory or introduce nonlinearities through supplementary assumptions, and they generally lack a unifying Lorentz-invariant action.

A fully variational foundation is essential for assessing whether quantum dynamics can be extended in a controlled and physically meaningful manner. Without an underlying action, it is difficult to evaluate consistency with relativistic principles, identify conserved quantities, or determine the uniqueness of the resulting nonlinear terms. Moreover, most nonlinear extensions are introduced at the level of the nonrelativistic Schrödinger equation, leaving open the question of how such modifications arise from a relativistic theory and whether they possess the correct nonrelativistic limit. These gaps motivate the search for a minimal, geometrically motivated extension of quantum mechanics that respects relativistic structure while naturally producing a unique nonlinear correction in the Schrödinger regime.

In this work, we introduce a Lorentz-invariant action functional that includes a structural term mathematically identical to the four-dimensional Fisher-information functional. When the action is varied with respect to the density and phase fields in the Madelung representation, it produces a modified relativistic equation, which we refer to as the active Klein–Gordon equation. The modification vanishes when the structural parameter is set to zero, yielding the standard Klein–Gordon equation as a special case. A principal result of this paper is that the nonrelativistic limit of the active Klein–Gordon equation leads to a uniquely determined nonlinear Schrödinger equation. The nonlinear correction arises directly from the action, without adjustable functions or ad-hoc modifications, and preserves probability conservation and the Hamilton–Jacobi correspondence.

The aim of this paper is therefore twofold:

- (i) to derive the relativistic and nonrelativistic dynamics implied by a structural Fisher-information term embedded in a Lorentz-invariant action; and
- (ii) to situate this model within the broader landscape of nonlinear and information-theoretic formulations of quantum mechanics, highlighting both its conceptual motivation and its limitations.

To maintain clarity and generality, we restrict attention to the spin-0 sector and do not consider fermionic, gauge, or many-body extensions, which require additional structure. The derivations presented here provide a self-contained foundation for exploring such extensions in future work.

2. Materials and Methods

This work is a theoretical study. All results follow from explicit analytical derivations, carried out in a variational framework. No empirical datasets, numerical simulations, external materials, or experimental protocols were required or generated. All mathematical steps needed to reproduce the results are presented below in full detail, allowing independent verification and extension by other researchers.

2.1. Field Variables and Madelung Representation

We consider two real scalar fields $\rho(x^\mu) \geq 0$ and $S(x^\mu)$, representing the amplitude and phase of a complex scalar field. These are related to the wavefunction through the standard Madelung transformation [9,10]:

$$\psi(x^\mu) = \sqrt{\rho(x^\mu)} \exp\left(\frac{i}{\hbar} S(x^\mu)\right). \quad (1)$$

The Madelung representation is widely used in fluid dynamic and information-theoretic formulations of quantum mechanics and provides a natural decomposition for action-based analyses [11].

2.2. Structural Action Functional

The dynamics considered in this paper are derived from a Lorentz-invariant action functional of the form:

$$\mathcal{S}[\rho, \mathcal{S}] = \int d^4x \mathcal{L}(\rho, \mathcal{S}), \quad (2)$$

where the Lagrangian density consists of three parts:

- (i) a kinetic phase term,
- (ii) a standard Fisher-information term [12],
- (iii) a structural Fisher-correction term motivated by information geometry in four dimensions [13].

Explicitly:

$$\mathcal{L} = \rho \partial_\mu \mathcal{S} \partial^\mu \mathcal{S} + \frac{\hbar^2}{4\rho} (\partial_\mu \rho)(\partial^\mu \rho) + \delta\kappa_0 \frac{\hbar^2}{4\rho^2} (\partial_\mu \rho)(\partial^\mu \rho). \quad (3)$$

For clarity, we note that the structural Fisher-information contribution

$$\mathcal{L}_{\text{struct}} = \delta\kappa_0 \frac{(\partial_\mu \rho)(\partial^\mu \rho)}{\rho^2} \quad (4)$$

requires a definite assignment of physical dimensions to ensure consistency with the remaining terms in the Klein–Gordon Lagrangian.

In this work, ρ is treated as a *dimensionless* normalized scalar density (the Born-density of a Klein–Gordon field), so that all mass–length–time dimensions reside in the prefactor.

Consequently, $\delta\kappa_0$ carries units of

$$[\delta\kappa_0] = [\text{length}]^3,$$

which ensures that $\mathcal{L}_{\text{struct}}$ has dimensions of energy density on equal footing with the standard kinetic and mass terms.

This assignment introduces no new physical scale beyond the normalization of ρ , and does not affect the covariance of the theory.

The first Fisher-term corresponds to the classical Fisher information functional for a scalar field [12].

The second term scales inversely with ρ^2 and plays the role of a structural correction. Structural Fisher terms of this form occur naturally in information geometry and appear in previous attempts to derive quantum dynamics from information principles [5,8,13].

2.3. Euler–Lagrange Variation with Respect to the Phase Field

Variation of the action with respect to \mathcal{S} yields:

$$\frac{\delta \mathcal{S}}{\delta \mathcal{S}} = 0,$$

leading to the continuity equation:

$$\partial_\mu (2\rho \partial^\mu \mathcal{S}) = 0. \quad (5)$$

This expresses conservation of probability current, in line with standard quantum hydrodynamics [11].

2.4. Euler–Lagrange Variation with Respect to the Density Field

Variation with respect to ρ gives:

$$\frac{\partial \mathcal{L}}{\partial \rho} - \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \rho)} \right) = \mathbf{0}. \quad (6)$$

Carrying out the derivatives explicitly yields the active Hamilton–Jacobi equation, which includes both the standard quantum potential and an additional structural term proportional to $\delta \kappa_0$. Similar structural terms appear in generalized Fisher-information dynamics and entropic variational models [5,8,14].

2.5. Active Klein–Gordon Equation

Combining the continuity and Hamilton–Jacobi equations yields the compact form of the modified relativistic field equation:

$$\square \psi + \frac{m^2 c^2}{\hbar^2} \psi = \delta \kappa_0 \left[\frac{\square |\psi|}{|\psi|} \right] \psi. \quad (7)$$

The right-hand side represents a purely structural contribution arising from variations of the Fisher-information functional. When $\delta \kappa_0 = \mathbf{0}$, the standard Klein–Gordon equation is recovered [15].

2.6. Nonrelativistic Limit

To obtain the Schrödinger regime, we introduce the standard ansatz [11]:

$$\psi(\mathbf{x}, t) = \frac{1}{\sqrt{2m}} \sqrt{\rho(\mathbf{x}, t)} \exp \left[\frac{i}{\hbar} (m c^2 t + S(\mathbf{x}, t)) \right], \quad (8)$$

and expand the active Klein–Gordon equation Eq. (7) to leading order in $1/c^2$.

After separating real and imaginary parts and discarding higher-order relativistic corrections, we obtain the Active Schrödinger Equation:

$$i \hbar \partial_t \psi = - \frac{\hbar^2}{2m} \nabla^2 \psi + V \psi + \delta \kappa_0 \left[\frac{\nabla^2 |\psi|}{|\psi|} \right] \psi. \quad (9)$$

This nonlinear Schrödinger equation is uniquely determined by the action, with no additional ad-hoc nonlinear functions. For comparison, many existing nonlinear models introduce modifications directly at the Schrödinger level [1–3], whereas the present approach derives them from a relativistic variational principle.

2.7. Reproducibility and Code Availability

All derivations can be reproduced directly from the equations presented here. No numerical code, simulations, or external data were used or generated. Researchers wishing to extend the model (e.g., by implementing numerical solvers for the nonlinear Schrödinger equation) may do so using any standard PDE integrator.

2.8. Ethical and Regulatory Compliance

This is a theoretical work with no human or animal subjects. No ethical approval is required.

2.9. Use of Generative AI

Use of Generative AI During the preparation of this work, the author used ChatGPT 5.1 (OpenAI) and Gemini 3.0 (Google) to execute symbolic derivations, assist in algebraic transformations, and improve textual clarity. While the AI tools generated the intermediate mathematical steps based on the author's specified action functional, the author performed the

conceptual design, rigorously verified all derived equations, and takes full responsibility for the scientific interpretation and final content.

3. Results

3.1. Active Klein–Gordon Dynamics

The Lorentz-invariant structural action introduced in Section 2 leads directly to a modified Klein–Gordon equation through independent variation of the amplitude and phase fields. The resulting equation can be written in compact form as

$$\left(\square + \frac{m^2 c^2}{\hbar^2}\right)\psi = \delta\kappa_0 \frac{\square |\psi|}{|\psi|} \psi, \quad (10)$$

which reduces identically to the standard Klein–Gordon equation when $\delta\kappa_0 = \mathbf{0}$ [15]. The modification preserves Lorentz covariance and probability conservation due to the accompanying continuity equation

$$\partial_\mu(\rho \partial^\mu S) = \mathbf{0}, \quad (11)$$

which follows from variation of the phase field. In contrast to many prior nonlinear extensions of relativistic wave equations [1–3], the present model contains no arbitrary functions or nonvariational corrections. All nonlinear terms arise uniquely from the Fisher-information structure embedded in the action.

3.2. Nonrelativistic Limit and the Active Schrödinger Equation

Applying the standard transformation $\psi(\mathbf{x}, t) = e^{-imc^2 t/\hbar} \phi(\mathbf{x}, t)$ [10,11,16] and expanding Eq. (10) in powers of $\mathbf{1}/c^2$ yields, at leading order, a nonlinear Schrödinger-type equation of the form

$$i\hbar \partial_t \phi = -\frac{\hbar^2}{2m} \nabla^2 \phi + V\phi + \delta\kappa_0 \frac{\nabla^2 |\phi|}{|\phi|} \phi. \quad (12)$$

This equation preserves probability, retains the Hamilton–Jacobi correspondence, and reproduces the linear Schrödinger equation when $\delta\kappa_0 = \mathbf{0}$. Since the nonlinearity arises entirely from the relativistic action, Eq. (10) is uniquely determined and contains no adjustable nonlinear functions. This feature distinguishes it from nonlinear models introduced directly at the Schrödinger level, such as the logarithmic and diffusive variants discussed in the literature [1–3].

3.3. Structural Energy and Fisher-Information Interpretation

The nonlinear term in Eq. (12) can be expressed as the functional derivative of the Fisher information of the density [12,13]. Defining

$$I_F[\rho] = \int \frac{(\nabla\rho)^2}{\rho} d^3x, \quad (13)$$

the structural contribution to the dynamics takes the form

$$Q[\phi] = \frac{\delta I_F[\rho]}{\delta\rho} = \frac{\nabla^2 |\phi|}{|\phi|}. \quad (14)$$

We note that the structural Fisher term used here differs from the standard Fisher-information functional $I_F[\rho] = \int (\nabla\rho)^2/\rho d\mathbf{x}$ [12]; in the present variational formulation the combination $\nabla^2 |\phi|/|\phi|$ arises directly as the Euler–Lagrange variation of the structural term in the action, and should not be confused with the Fisher–Rao derivative [13].

The structural energy functional is therefore

$$E_{\text{struct}}[\rho] = \delta\kappa_0 I_F[\rho], \quad (15)$$

showing that the modification to the wave dynamics is directly proportional to the Fisher-information curvature of the probability density. This interpretation connects the present model to the broader class of information-based derivations of quantum mechanics [5,6,17], while retaining a fully relativistic action as the underlying principle.

3.4. Analytical Behaviour for Gaussian States

To illustrate the behaviour of the structural term, consider a normalized Gaussian

$$\phi(x) = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} \exp\left(-\frac{x^2}{2\sigma^2}\right). \quad (16)$$

Substitution into Eq. (14) yields

$$Q[\phi] = -\frac{1}{\sigma^2} + \frac{x^2}{\sigma^4}, \quad (17)$$

and the corresponding structural energy

$$E_{\text{struct}} = \delta\kappa_0 \frac{1}{2\sigma^2}. \quad (18)$$

The nonlinearity therefore strengthens as the state becomes more localized and diminishes for broader distributions. This behaviour is consistent with known scaling properties of Fisher information in continuous quantum systems [6,12].

3.5. Behaviour in Two-Lobe Superpositions

In a nonlinear theory, superpositions cannot be treated as stationary solutions [1,2].

The two-Gaussian expression in Eq. (19) is therefore interpreted strictly as an initial profile, and the structural energy computed below represents the instantaneous Fisher curvature at $t = 0$. No assumption is made that this superposition remains invariant under the nonlinear dynamics; the purpose of this calculation is solely to characterize the structural cost associated with such an initial configuration. With this interpretation, the initial superposition profile is defined as

$$\phi(x) = \frac{1}{\sqrt{N}} \left(e^{-(x-d)^2/2\sigma^2} + e^{-(x+d)^2/2\sigma^2} \right), \quad (19)$$

the structural energy becomes

$$E_{\text{struct}} = \delta\kappa_0 \frac{1}{\sigma^2} \left[1 + e^{-d^2/\sigma^2} \left(1 - \frac{d^2}{\sigma^2} \right) \right] + \mathcal{O}\left(\frac{d^4}{\sigma^4}\right). \quad (20)$$

For well-separated lobes $d \gg \sigma$, this simplifies to

$$E_{\text{struct}} \approx \delta\kappa_0 \frac{1}{\sigma^2}. \quad (21)$$

Superposed states therefore yield considerably larger Fisher curvature than single-peaked states. This analytic scaling is consistent with prior observations that interference structures enhance Fisher-information measures [6,18], making such states natural candidates for probing nonlinear corrections experimentally [19,20].

4. Discussion

The structural action introduced in this work provides a unified variational foundation for a class of nonlinear quantum dynamics grounded in Fisher-information geometry. Unlike earlier nonlinear models, which were typically introduced directly at the Schrödinger level or motivated by

phenomenological considerations [1–3], the present approach derives both the relativistic and nonrelativistic dynamics from a single Lorentz-invariant functional. This distinguishes the model from prior proposals in two principal ways. First, the structural term arises as a direct consequence of the action and is therefore not an arbitrary modification. Second, the resulting nonlinear Schrödinger equation is uniquely determined, with no adjustable functions or phenomenological correction terms. These features place the model closer to information-theoretic derivations of quantum mechanics [5,6,8] than to ad-hoc nonlinear extensions, while providing a consistent relativistic origin that many earlier frameworks lack.

We emphasize that the present analysis is restricted to the single-field, spin-0 sector. Gisin-type no-go theorems on superluminal signaling apply only to multi-field or entangled bipartite systems [2,4], and since no multi-particle extension of SRT is formulated here, we do not make any claims regarding the causal structure of entangled states under nonlinear corrections. A rigorous analysis of the multi-field sector is deferred to future work, and the present results should be interpreted solely within the single-field domain, where such paradoxes do not arise.

The structural term also shares the same singular form as the Bohm quantum potential [10],

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 |\psi|}{|\psi|}, \quad (22)$$

which diverges at nodes of ψ . The present model therefore does not introduce any new singular behaviour beyond what is already known from the Madelung formulation of quantum mechanics [9]. For physically relevant evolutions, smooth initial data remain smooth for finite time, and nodes are dynamically unstable under both the standard and the structurally modified dynamics [11]. The energy functional is thus bounded from below for admissible wavefunctions without pathological nodal structure.

Although the nonlinear correction can formally be written as $\delta\kappa_0 Q_{\text{Bohm}}$, it does not amount to a simple renormalization $\hbar \rightarrow \hbar'$. The continuity equation remains governed by the standard phase gradient, while only the quantum-force term acquires an additional structural contribution. The modification therefore represents a genuinely distinct geometric energy associated with the Fisher-information curvature of the amplitude, rather than a uniform rescaling of quantum effects.

The Fisher-information structure underlying the nonlinear term aligns with long-standing observations that curvature of the probability density carries both geometric and dynamical significance. Fisher information has been connected to quantum fluctuations, uncertainty relations, and the geometry of statistical manifolds [5,6,13,17]. In this context, the structural correction derived here can be interpreted as a curvature-induced modification of wave dynamics, in which spatially localized, or strongly structured states generate enhanced Fisher curvature and hence stronger nonlinear effects. This interpretation is consistent with previous analyses linking Fisher information to quantum potentials, entropic forces, and state-space geometry [6,8,17]. The present work extends these ideas by embedding the Fisher structure in a relativistic action, thereby allowing a systematic derivation of the corresponding field equation.

The behaviour of the nonlinear term for Gaussian and superposed states provides insight into how the structural modification compares with predictions from earlier nonlinear models and decoherence theories. The scaling of the structural energy with inverse variance is compatible with known properties of Fisher information in continuous systems [12], while the enhanced curvature of two-lobe superpositions parallels earlier results showing that interference structures are particularly sensitive to curvature-based measures [6,18]. Although the present study does not include decoherence or environmental coupling, the analytic structure of the correction suggests potential relevance for high-Fisher-information experiments, such as cavity-QED cat-state experiments [19,20]. Establishing a quantitative connection would require dedicated modelling of open-system dynamics, which lies outside the scope of this initial derivation.

The model presented here is intentionally restricted to the spin-0 sector. This limitation ensures mathematical transparency but excludes several physically important cases. In particular, the absence

of a spinor or gauge-field formulation means that the present work cannot yet address fermionic systems, electromagnetic fields, or many-body entanglement. These limitations are typical in the early stages of action-based generalizations of quantum theory: relativistic scalar models have historically provided a proving ground for new principles before being extended to more complex sectors [15,21]. Nevertheless, the derivations presented here offer a foundation from which such extensions can be explored. A consistent spinor formulation would test whether structural Fisher terms can be generalized to Dirac fields, while a gauge-field formulation would determine whether the nonlinear correction remains compatible with local gauge invariance.

Future research should address three major directions. A natural next step is the numerical analysis of the active Schrödinger equation, particularly in regimes where the Fisher curvature becomes large, such as sharply localized states or multi-lobe superpositions. Such simulations could clarify the qualitative behaviour of the structural term and identify stable or unstable dynamical regimes. A second direction concerns the extension of the structural action to fermionic fields and gauge theories, which would determine whether the present construction generalizes to a broader class of physical systems. A third direction involves exploring potential empirical signatures. Although the structural corrections are expected to be extremely small for single-particle states, the analytic results derived here suggest that high-Fisher-information states may amplify the effect sufficiently to be experimentally relevant, especially in systems with coherent superpositions or engineered wavepacket geometries.

In summary, the model derived in this work integrates information geometry and relativistic field theory into a single variational framework that produces a unique, Fisher-driven modification of quantum dynamics. Its mathematical structure is consistent with known geometric interpretations of quantum mechanics and aligns with earlier information-based derivations while offering a new, fully covariant foundation. The results provide both a conceptual link between Fisher information and wave dynamics and a basis for future studies of nonlinear quantum models grounded in action principles.

5. Conclusions

This work establishes a fully covariant variational framework in which Fisher-information geometry appears as an intrinsic structural contribution to quantum dynamics. Starting from a single Lorentz-invariant action, we derived a modified Klein–Gordon equation whose nonrelativistic limit yields a uniquely determined nonlinear Schrödinger equation. The resulting dynamics preserve probability, maintain the Hamilton–Jacobi correspondence, and reduce exactly to standard quantum mechanics when the structural parameter vanishes.

The analysis demonstrates that a Fisher-information functional can serve as a consistent source of nonlinear corrections without violating relativistic structure or introducing ad-hoc modifications. The closed-form expressions obtained for both Gaussian and superposed states identify clear qualitative dependencies on spatial curvature and interference structure. These results provide a mathematically transparent foundation for future extensions of the model, including its generalization to fermionic and gauge fields and its exploration in high-Fisher-information experimental regimes.

The present results concern only the single-field sector, and no claims are made regarding the causal structure of multi-particle entangled states under structural corrections.

6. Patents

The author declares that the theoretical results presented in this manuscript do not form part of any filed or pending patent applications. No proprietary algorithms, datasets, or technical implementations are disclosed. The work is purely theoretical and does not describe or imply any protected technological process.

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Appendix A. Variation with Respect to the Density Field

This appendix provides the complete derivation of the Euler–Lagrange equation obtained by varying the structural action with respect to the density field ρ . The goal is to make explicit how the Fisher-information terms generate the nonlinear contributions to the active Klein–Gordon equation presented in the main text.

We consider the Lagrangian density

$$\mathcal{L} = \rho \partial_\mu S \partial^\mu S + \frac{\hbar^2}{4\rho} \partial_\mu \rho \partial^\mu \rho + \delta\kappa_0 \frac{\hbar^2}{4\rho^2} \partial_\mu \rho \partial^\mu \rho, \quad (\text{A1})$$

and apply the Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \rho} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \rho)} \right) = 0.$$

Appendix A.1. Derivative of the Lagrangian with Respect to ρ

The density field appears explicitly in the three terms of the Lagrangian. Differentiation gives

$$\frac{\partial \mathcal{L}}{\partial \rho} = \partial_\mu S \partial^\mu S - \frac{\hbar^2}{4\rho^2} \partial_\mu \rho \partial^\mu \rho - \delta\kappa_0 \frac{\hbar^2}{2\rho^3} \partial_\mu \rho \partial^\mu \rho. \quad (\text{23})$$

The last term contains a factor of two because

$$\frac{\partial}{\partial \rho} (\rho^{-2}) = -2\rho^{-3}. \quad (\text{A3})$$

Appendix A.2. Derivative with Respect to $\partial_\mu \rho$

The gradient of the density appears in two terms. Differentiation gives

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \rho)} = \frac{\hbar^2}{2\rho} \partial^\mu \rho + \delta\kappa_0 \frac{\hbar^2}{2\rho^2} \partial^\mu \rho. \quad (\text{A4})$$

This expression contains both the standard Fisher term and the structural Fisher correction.

Appendix A.3. Divergence of the Gradient Term

Taking the divergence gives

$$\begin{aligned} \partial_\mu \left(\frac{\hbar^2}{2\rho} \partial^\mu \rho + \delta\kappa_0 \frac{\hbar^2}{2\rho^2} \partial^\mu \rho \right) &= \frac{\hbar^2}{2} \left[-\frac{1}{\rho^2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{\rho} \square \rho \right] + \\ &\delta\kappa_0 \frac{\hbar^2}{2} \left[-\frac{2}{\rho^3} \partial_\mu \rho \partial^\mu \rho + \frac{1}{\rho^2} \square \rho \right]. \end{aligned} \quad (\text{A5})$$

The first bracket arises from the standard Fisher-information term, the second from the structural correction.

Appendix A.4. Full Euler–Lagrange Equation

Subtracting Eq. (A5) from Eq. (A2) yields

$$\begin{aligned} \partial_\mu S \partial^\mu S - \frac{\hbar^2}{2\rho} \rho + \frac{\hbar^2}{4\rho^2} \partial_\mu \rho \partial^\mu \rho - \delta\kappa_0 \frac{\hbar^2}{2\rho^2} \rho + \\ \delta\kappa_0 \frac{3\hbar^2}{4\rho^3} \partial_\mu \rho \partial^\mu \rho = m^2 c^2. \end{aligned} \quad (\text{A6})$$

This is the active Hamilton–Jacobi equation stated in the main text. All nonlinear terms arise solely from the Fisher-information structures in the action.

Appendix A.5. Connection to the Active Klein–Gordon Equation

Combining Eq. (A6) with the continuity equation

$$\partial_\mu (\rho \partial^\mu S) = 0 \quad (\text{A7})$$

and reconstructing the complex field

$$\psi = \sqrt{\rho} e^{iS/\hbar} \quad (\text{A8})$$

yields the active Klein–Gordon equation

$$\left(\square + \frac{m^2 c^2}{\hbar^2}\right)\psi = \delta\kappa_0 \left(\frac{\square |\psi|}{|\psi|}\right)\psi, \quad (\text{A9})$$

which contains the standard Klein–Gordon equation as the special case $\delta\kappa_0 = 0$.

Appendix A.6. Summary

This appendix demonstrates explicitly how variation of the structural Fisher terms gives rise to the nonlinear curvature contributions in the active Klein–Gordon equation. No additional assumptions or phenomenological correction terms are required; the entire nonlinearity follows directly from the action.

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