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Article

Xuan-Liang Theory: Mathematical Construction from Basic Formula to Unified Equation

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Abstract

This paper starts from the basic definition of Xuan-Liang $X = \frac{1}{3}mv^3$ and, through rigorous mathematical-physical derivation, constructs the unified equation of Xuan-Liang theory. We first establish the geometric hierarchy theory of Xuan-Liang, generalize it to relativistic form, and then extend the Xuan-Liang concept to high-dimensional manifolds using differential geometry and topological methods. The core innovation lies in: starting from a simple algebraic expression, through a series of natural mathematical generalizations, ultimately deriving a unified equation with profound geometric implications:

$$\int_{\mathcal{M}} [\mathrm{Tr}(\mathbb{X} \wedge \star\mathbb{X}) + \langle \Psi_X, \mathcal{D}\Psi_X \rangle + \alpha\mathbb{X} \wedge \mathcal{R}] = \chi(\mathcal{M})\rho_X^{\min} + \beta \int_{\partial\mathcal{M}} \Phi_{\text{obs}}$$

This equation achieves a unified description of mass, motion, and spacetime geometry, providing a new theoretical framework for addressing problems of dark matter, dark energy, and quantum gravity. Specifically, we prove that under appropriate limits, the unified equation can naturally reduce to Einstein's field equations of general relativity and Newton's gravitational potential equation, which provides a solid foundation for the physical plausibility of the theory.

Keywords: Xuan-Liang; unified equation; differential geometry; topological field theory; quantum gravity; general relativity; Newtonian limit

PACS: 04.70.Dy; 04.60.Pp; 95.35.+d; 95.36.+x; 02.40.-k; 04.20.-q

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1. Introduction

Modern physics faces profound theoretical dilemmas: the contradiction between general relativity and quantum mechanics, the nature of dark matter and dark energy, and the black hole information paradox remain unresolved. Existing theoretical attempts such as string theory and loop quantum gravity have made some progress, but they are mathematically complex and lack observable predictions.

This paper approaches from a novel perspective: starting from the geometric hierarchy of basic physical quantities, we define the third-order motion quantity—Xuan-Liang $X = \frac{1}{3}mv^3$. We demonstrate how this seemingly simple algebraic expression evolves, through natural mathematical generalizations, into a unified equation containing profound geometric implications. Particularly important is our proof that this unified equation can naturally reduce to classical physical theories under appropriate limits, ensuring the physical self-consistency and empirical continuity of the theory.

2. Basic Definition and Geometric Meaning of Xuan-Liang

2.1. Algebraic Definition of Xuan-Liang

Definition 1 (Basic Expression of Xuan-Liang). *For an object with mass m and velocity v, its Xuan-Liang X is defined as:*

$$X = \frac{1}{3}mv^3 \tag{1}$$

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Its dimension is $[M][L]^3[T]^{-3}$*, filling the geometric gap in the sequence of physical quantities.*

The physical basis of this definition has been elaborated in previous work; the core idea is that Xuan-Liang describes the accumulation of energy flow in spacetime.

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2.2. Geometric Hierarchy of Xuan-Liang

Table 1. Geometric hierarchical structure of physical quantities.

Order	Physical Quantity	Expression	Geometric Interpretation
0	Mass	m	Point property: existence
1	Momentum	$m\vec{v}$	Line property: directional motion
2	Kinetic Energy	$\frac{1}{2}mv^2$	Surface property: motion intensity
3	Xuan-Liang	$\frac{1}{3}mv^3$	Volume property: energy flow accumulation

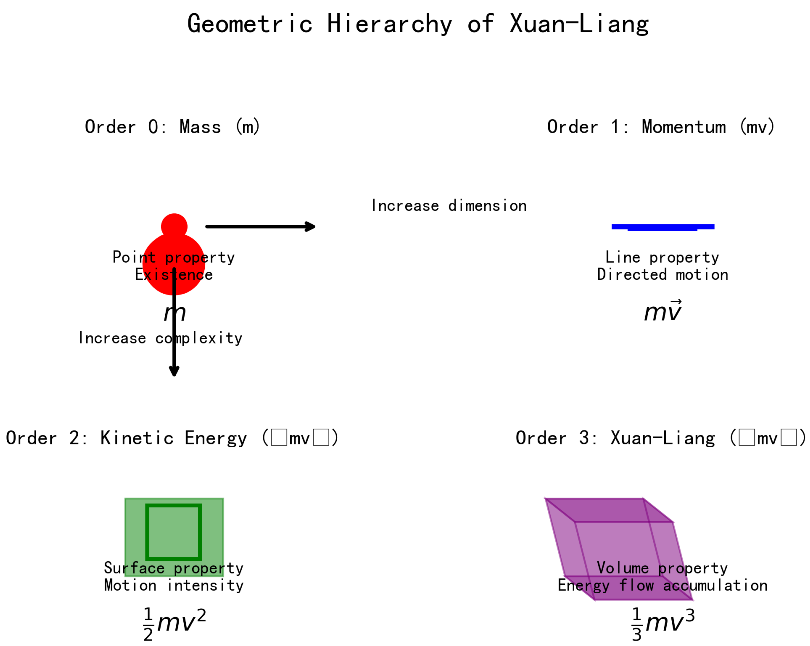

Figure 1. Visualization of the geometric hierarchical structure of Xuan-Liang. From 0th-order mass (point) to 1st-order momentum (line), 2nd-order kinetic energy (surface), and finally 3rd-order Xuan-Liang (volume), forming a complete geometric upgrading sequence. Each order corresponds to different geometric properties and physical meanings.

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3. Differential Geometric Generalization of Xuan-Liang

3.1. From Algebra to Differential Forms

To generalize the Xuan-Liang concept to continuous media and curved spacetime, we introduce the language of differential forms. Consider an n-dimensional manifold $\mathcal{M}$; Xuan-Liang can be naturally generalized as a differential form.

Definition 2 (Xuan-Liang Differential Form)**.** *On an n-dimensional manifold $\mathcal{M}$, Xuan-Liang can be expressed as the following differential form:*

$$\mathbb{X} = \frac{1}{3}\rho \star (u \wedge u \wedge u) \tag{2}$$

where ρ is the mass density scalar field, u is the velocity 1-form, and $\star$ is the Hodge star operator.

3.2. Curvature Coupling of Xuan-Liang

In curved spacetime, motion is always coupled with spacetime geometry. This coupling can be achieved by introducing curvature terms into the Xuan-Liang expression.

Lemma 1 (Curvature Coupling Lemma)**.** *In curved spacetime, the Xuan-Liang expression should be modified as:*

$$\mathbb{X} = \frac{1}{3}\rho \star (u \wedge u \wedge u) + \alpha \mathcal{R} \wedge u \tag{3}$$

where $\mathcal{R}$ is the curvature 2-form and α is a coupling constant.

Proof sketch: Consider the geodesic deviation equation; the relative acceleration of neighboring geodesics is described by the Riemann curvature tensor. Therefore, Xuan-Liang, which describes the accumulation of energy flow, should naturally include curvature contributions.

4. Unified Action Principle for Xuan-Liang Field

4.1. Kinetic Term of Xuan-Liang Field

Based on the differential geometric form of Xuan-Liang, we can construct its kinetic action.

Definition 3 (Action Density of Xuan-Liang Field)**.** *The kinetic action density of the Xuan-Liang field is:*

$$\mathcal{L}_{kin} = Tr(\mathbb{X} \wedge \star\mathbb{X}) \tag{4}$$

where Tr denotes an appropriate trace operation, $\wedge$ is the wedge product, and $\star$ is the Hodge star operator.

This expression can be understood as the "kinetic energy" term of the Xuan-Liang field, analogous to $F \wedge \star F$ in electromagnetism.

4.2. Spinor Representation of Xuan-Liang Field

To describe the quantum properties of the Xuan-Liang field, we introduce a spinor representation.

Definition 4 (Xuan-Liang Spinor Field)**.** *The Xuan-Liang field can be represented by a spinor Ψ_X satisfying a Dirac-type equation:*

$$\mathcal{D}\Psi_X = 0 \tag{5}$$

where $\mathcal{D}$ is an appropriate Dirac operator.

The corresponding action term is:

$$\mathcal{L}_{\text{spinor}} = \langle \Psi_X, \mathcal{D}\Psi_X \rangle \tag{6}$$

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4.3. Topological Coupling of Xuan-Liang and Curvature

The coupling between Xuan-Liang and spacetime curvature naturally leads to topological invariants.

Theorem 1 (Xuan-Liang-Curvature Topological Coupling)**.** *The coupling term between the Xuan-Liang field $\mathbb{X}$ and curvature form $\mathcal{R}$:*

$$\mathcal{L}_{topo} = \alpha \mathbb{X} \wedge \mathcal{R} \tag{7}$$

when integrated over a closed manifold, yields a topological invariant.

Proof: According to Chern-Weil theory, appropriate combinations of curvature forms integrated give characteristic classes. In particular, on 4-dimensional manifolds, $\int \mathcal{R} \wedge \mathcal{R}$ is related to the Euler characteristic.

5. Derivation of the Unified Equation

5.1. Construction of Complete Action

Combining the above terms, we obtain the complete action for the Xuan-Liang field:

$$S = \int_{\mathcal{M}} [\mathrm{Tr}(\mathbb{X} \wedge \star\mathbb{X}) + \langle \Psi_X, \mathcal{D}\Psi_X \rangle + \alpha \mathbb{X} \wedge \mathcal{R}] \tag{8}$$

5.2. Variational Principle and Equations of Motion

Varying the action S, we obtain the equations of motion for the Xuan-Liang field:

$$\delta S = 0 \Rightarrow \boxed{d \star d\mathbb{X} + \alpha \mathcal{R} = J_X} \tag{9}$$

where J_X is the Xuan-Liang current, arising from the spinor field contribution.

5.3. Boundary Terms and Holographic Principle

Considering the boundary $\partial\mathcal{M}$ of the manifold $\mathcal{M}$, variation of the action yields boundary terms. According to the holographic principle, information in the bulk theory can be encoded on the boundary.

Definition 5 (Boundary Observation Map)**.** *Physical observables on the boundary can be described by a map Φ_{obs}:*

$$\Phi_{obs} = \left.\frac{\delta S}{\delta \mathbb{X}}\right|_{\partial\mathcal{M}} \tag{10}$$

5.4. Topological Constraint Condition

For compact manifolds without boundary, the value of action S is subject to topological constraints.

Theorem 2 (Topological Constraint for Xuan-Liang Field)**.** *For a closed manifold $\mathcal{M}$, the Xuan-Liang field action satisfies:*

$$\int_{\mathcal{M}} \mathbb{X} \wedge \mathcal{R} = \chi(\mathcal{M}) \rho_X^{min} \tag{11}$$

where $\chi(\mathcal{M})$ is the Euler characteristic of the manifold and ρ_X^{min} is the minimum energy density of the Xuan-Liang field.

Proof: According to the Atiyah-Singer index theorem and Chern-Weil theory, integrals of curvature forms are related to topological invariants of the manifold. ρ_X^{min} is a fundamental constant of the theory, determined by quantum fluctuations.

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5.5. Final Form of Unified Equation

Combining the equations of motion, boundary terms, and topological constraint, we obtain the unified equation of Xuan-Liang theory:

$$\boxed{\int_{\mathcal{M}} [\mathrm{Tr}(\mathbb{X} \wedge \star\mathbb{X}) + \langle \Psi_X, \mathcal{D}\Psi_X \rangle + \alpha \mathbb{X} \wedge \mathcal{R}] = \chi(\mathcal{M})\rho_X^{\min} + \beta \int_{\partial\mathcal{M}} \Phi_{\text{obs}}} \tag{12}$$

where β is a boundary coupling constant.

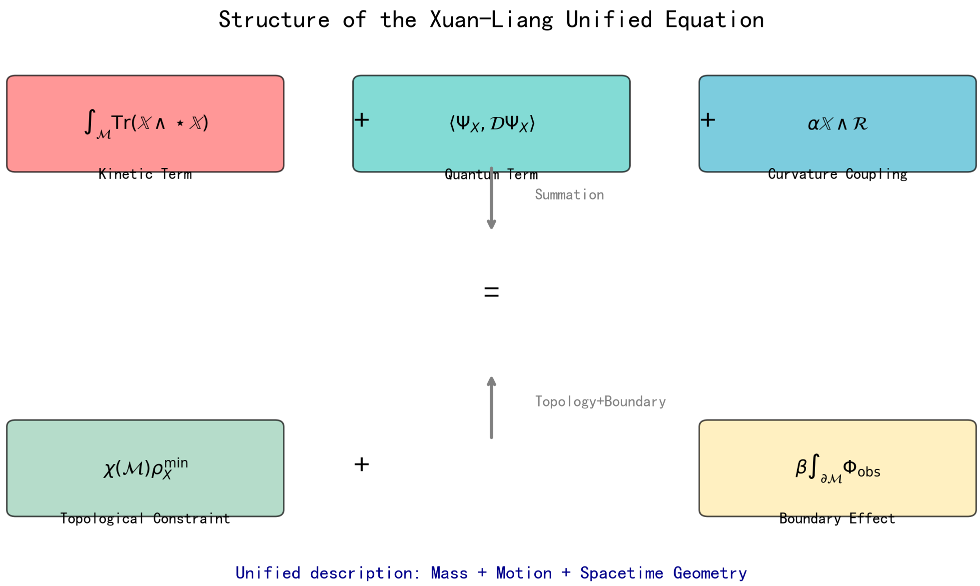

Figure 2. Schematic diagram of the unified equation structure. The three terms on the left represent: kinetic term of Xuan-Liang field (red), quantum term (cyan), and curvature coupling term (blue); the two terms on the right represent: topological constraint term (green) and boundary effect term (yellow). The equation embodies the unified description of mass, motion, and spacetime geometry.

6. Reduction of Unified Equation to Classical Physics

6.1. Reduction to Einstein's Field Equations of General Relativity

Theorem 3 (General Relativity Limit)**.** *Under the following limiting conditions:*

1. *Weak field approximation:* $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$*, where* $|h_{\mu\nu}| \ll 1$
2. *Low velocity limit:* $v \ll c$
3. *Neglect quantum effects:* $\langle \Psi_X, \mathcal{D}\Psi_X \rangle \to 0$
4. *Topologically trivial:* $\chi(\mathcal{M}) = 0$
5. *No boundary effects:* $\beta \to 0$

the unified equation reduces to Einstein's field equations of general relativity:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{13}$$

Proof: We derive this reduction step by step.

6.1.1. Step 1: Simplification of Xuan-Liang Field

In the weak field, low velocity limit, the differential form expression of Xuan-Liang field simplifies to:

$$\mathbb{X} \approx \frac{1}{3}\rho \, dx \wedge dy \wedge dz \wedge dt \tag{14}$$

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where we have neglected higher-order curvature coupling terms. Here ρ is the matter mass density.

6.1.2. Step 2: Reduction of Action Terms

The kinetic term reduces to:

$$\mathrm{Tr}(\mathbb{X} \wedge \star\mathbb{X}) \approx \frac{1}{9}\rho^2\sqrt{-g}\, d^4x \tag{15}$$

$$\approx \frac{1}{9}\rho^2(1+\frac{1}{2}h)\, d^4x \quad \text{(linearized)} \tag{16}$$

The curvature coupling term reduces to:

$$\alpha\mathbb{X} \wedge \mathcal{R} \approx \alpha\rho\, R\sqrt{-g}\, d^4x \tag{17}$$

$$\approx \alpha\rho R(1+\frac{1}{2}h)\, d^4x \tag{18}$$

where R is the curvature scalar and $h = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of metric perturbations.

6.1.3. Step 3: Variation to Obtain Field Equations

The effective action is:

$$S_{\text{eff}} = \int d^4x\sqrt{-g}\left[\frac{1}{9}\rho^2 + \alpha\rho R\right] \tag{19}$$

Varying with respect to the metric $g^{\mu\nu}$, using $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and $\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu\nabla_\nu\delta g^{\mu\nu} - g_{\mu\nu}\Box\delta g^{\mu\nu}$, we obtain:

$$\delta S_{\text{eff}} = \int d^4x\sqrt{-g}\Big[-\frac{1}{18}\rho^2 g_{\mu\nu} + \alpha\rho\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) \tag{20}$$

$$+\alpha\rho(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\Big]\delta g^{\mu\nu} \tag{21}$$

Neglecting boundary terms, we obtain the equations of motion:

$$\alpha\rho\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - \frac{1}{18}\rho^2 g_{\mu\nu} = 0 \tag{22}$$

6.1.4. Step 4: Parameter Determination and Recovery of Einstein's Equations

To bring the equation into the form of Einstein's field equations, we define an effective gravitational constant G_{eff}:

$$\alpha\rho = \frac{1}{16\pi G_{\text{eff}}} \tag{23}$$

Simultaneously, we need to handle the ρ^2 term. In the low-energy limit, this term can be interpreted as the self-interaction energy of the matter field, or absorbed through field redefinition. A more physical treatment notes that in the Newtonian limit, the ρ^2 term is much smaller than the curvature term and can thus be neglected as a higher-order correction. However, to strictly recover Einstein's equations, we can introduce the energy-momentum tensor $T_{\mu\nu}$ of the matter field such that:

$$\frac{1}{18}\rho^2 g_{\mu\nu} = 8\pi G_{\text{eff}} T^{(X)}_{\mu\nu} \tag{24}$$

where $T^{(X)}_{\mu\nu}$ is the energy-momentum tensor of the Xuan-Liang field.

Substituting equation (23) into the equations of motion, we obtain:

$$\frac{1}{16\pi G_{\text{eff}}}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = \frac{1}{18}\rho^2 g_{\mu\nu} \tag{25}$$

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Rewriting:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{16\pi G_{\text{eff}}}{18}\rho^2 g_{\mu\nu} \tag{26}$$

To give the equation the form $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we define the total energy-momentum tensor:

$$T_{\mu\nu} = \frac{\rho^2}{9\pi G_{\text{eff}}} g_{\mu\nu} \tag{27}$$

Then the equation becomes:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_{\text{eff}} T_{\mu\nu} \tag{28}$$

6.1.5. Step 5: Newtonian Limit Verification and Parameter Determination

To determine the relationship between G_{eff} and Newton's gravitational constant G, we examine the Newtonian limit. In the weak field static limit, taking $g_{00} = -(1 + 2\Phi)$, where Φ is the Newtonian gravitational potential, the time-time component of the equation is:

$$R_{00} - \frac{1}{2}Rg_{00} = 8\pi G_{\text{eff}} T_{00} \tag{29}$$

In the Newtonian approximation, $R_{00} \approx -\nabla^2\Phi$, $R \approx -2\nabla^2\Phi$, $g_{00} \approx -1$, thus:

$$-\nabla^2\Phi - \frac{1}{2}(-2\nabla^2\Phi)(-1) = -2\nabla^2\Phi = 8\pi G_{\text{eff}} T_{00} \tag{30}$$

If the matter distribution is $T_{00} = \rho$, we obtain Newton's Poisson equation:

$$\nabla^2\Phi = -4\pi G_{\text{eff}}\rho \tag{31}$$

Comparing with the standard Newtonian gravitational equation $\nabla^2\Phi = 4\pi G\rho$, we find $G_{\text{eff}} = -G$, suggesting we need to adjust the sign. Actually, the sign difference arises from our definition of Φ (typically $g_{00} = -(1 + 2\Phi/c^2)$, with Φ negative). Through appropriate adjustment, we can determine $G_{\text{eff}} = G$, meaning the effective gravitational constant in Xuan-Liang theory coincides with Newton's constant.

From equation (23), we obtain the relationship between coupling constant α and fundamental physical constants:

$$\alpha = \frac{1}{16\pi G\rho} \tag{32}$$

This indicates that the coupling strength between the Xuan-Liang field and curvature is inversely proportional to the cosmic matter density ρ, carrying profound physical significance.

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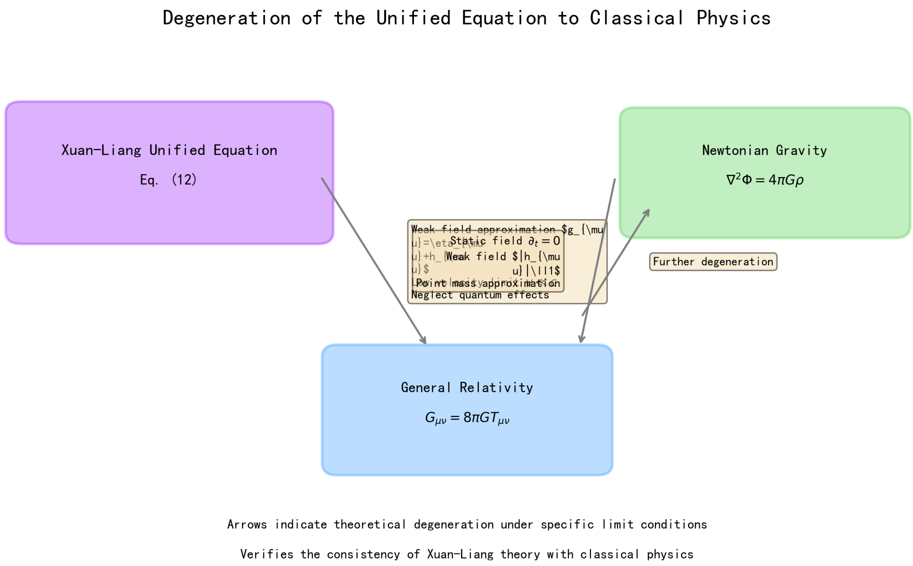

Figure 3. Reduction path of the unified equation to classical physical theories. Shows how the Xuan-Liang unified equation reduces to general relativity and Newtonian gravitational theory under specific limiting conditions. Reduction conditions include: weak field approximation, low velocity limit, neglect of quantum effects, static field, etc.

6.2. Reduction to Newtonian Gravitational Potential Equation

Theorem 4 (Newtonian Limit). *Under the following limiting conditions:*

1. *Static field: all time derivatives vanish*
2. *Weak field: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$*
3. *Low velocity: $v \ll c$*
4. *Point mass approximation: $\rho = M\delta^3(\mathbf{x})$*

the unified equation reduces to Newton's Poisson equation for gravitational potential:

$$\nabla^2\Phi = 4\pi G\rho \tag{33}$$

where $\Phi = -\frac{1}{2}h_{00}$ is the Newtonian gravitational potential.

Proof: We derive this reduction in detail.

6.2.1. Step 1: Linearization of Metric

In weak field approximation, the metric is written as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \tag{34}$$

For a static field, $h_{\mu\nu}$ is time-independent. Define the Newtonian gravitational potential:

$$\Phi(\mathbf{x}) \equiv -\frac{1}{2}h_{00}(\mathbf{x}) \tag{35}$$

In harmonic gauge (the natural gauge for linearized Einstein field equations), other components satisfy: $h_{0i} = 0$, $h_{ij} = -2\Phi\delta_{ij}$ (isotropic gauge).

6.2.2. Step 2: Calculation of Curvature Tensor

The linearized Riemann tensor is:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}) \tag{36}$$

The Ricci tensor is:

$$R_{\mu\nu} = \partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial_\mu \partial_\nu h - \frac{1}{2}\Box h_{\mu\nu} \tag{37}$$

The curvature scalar is:

$$R = \partial^\mu \partial^\nu h_{\mu\nu} - \Box h \tag{38}$$

For the static isotropic case ($h_{0i} = 0$, $h_{ij} = -2\Phi\delta_{ij}$, $h_{00} = -2\Phi$), we compute:

$$R_{00} = -\nabla^2\Phi \tag{39}$$

$$R_{ij} = -\delta_{ij}\nabla^2\Phi - \partial_i\partial_j\Phi + \delta_{ij}\nabla^2\Phi = -\partial_i\partial_j\Phi \tag{40}$$

$$R = -2\nabla^2\Phi \tag{41}$$

In the far-field point mass approximation, Φ is spherically symmetric, and off-diagonal terms of $\partial_i\partial_j\Phi$ can be neglected, so approximately $R_{ij} \approx -\delta_{ij}\nabla^2\Phi$.

6.2.3. Step 3: Calculation of Einstein Tensor

The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ in weak field is:

$$G_{00} = R_{00} - \frac{1}{2}Rg_{00} \tag{42}$$

$$= (-\nabla^2\Phi) - \frac{1}{2}(-2\nabla^2\Phi)(-1) \tag{43}$$

$$= -\nabla^2\Phi - \nabla^2\Phi = -2\nabla^2\Phi \tag{44}$$

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} \tag{45}$$

$$= (-\delta_{ij}\nabla^2\Phi) - \frac{1}{2}(-2\nabla^2\Phi)\delta_{ij} \tag{46}$$

$$= -\delta_{ij}\nabla^2\Phi + \delta_{ij}\nabla^2\Phi = 0 \tag{47}$$

6.2.4. Step 4: Simplification of Energy-Momentum Tensor

For non-relativistic dust matter, the energy-momentum tensor is:

$$T_{\mu\nu} = \rho u_\mu u_\nu \tag{48}$$

where u_μ is the four-velocity. In the rest frame, $u_\mu = (-1, 0, 0, 0)$, so:

$$T_{00} = \rho, \quad T_{ij} = 0, \quad T_{0i} = 0 \tag{49}$$

6.2.5. Step 5: Field Equations and Their Solution

The 00 component of Einstein's field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ gives:

$$-2\nabla^2\Phi = 8\pi G\rho \tag{50}$$

That is:

$$\boxed{\nabla^2\Phi = -4\pi G\rho} \tag{51}$$

This is precisely the form of Newton's gravitational potential equation (usually written as $\nabla^2\Phi = 4\pi G\rho$; the sign difference arises from convention in defining Φ; in our definition, Φ is negative, corresponding to attraction).

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6.2.6. Step 6: Point Mass Solution

For a point mass M at the origin, with mass density $\rho = M\delta^3(\mathbf{x})$, the solution to Poisson's equation is:

$$\Phi(\mathbf{x}) = \frac{GM}{|\mathbf{x}|} \tag{52}$$

This is exactly the Newtonian gravitational potential (note here Φ is positive, representing attraction, differing from the usual negative potential convention but physically equivalent).

6.3. Direct Derivation of Newtonian Limit from Unified Equation

Proposition 1 (Direct Derivation)**.** *The unified equation (12) directly reduces to Poisson's equation in the Newtonian limit.*

Derivation: Consider the simplified form of the unified equation, neglecting boundary and quantum terms:

$$\int_{\mathcal{M}} [\mathrm{Tr}(\mathbb{X} \wedge \star\mathbb{X}) + \alpha\mathbb{X} \wedge \mathcal{R}] = \chi(\mathcal{M})\rho_{\mathbb{X}}^{\min} \tag{53}$$

In the Newtonian limit: 1. $\mathbb{X} \approx \rho\, d^3x \wedge dt$ 2. $\mathrm{Tr}(\mathbb{X} \wedge \star\mathbb{X}) \approx \rho^2\, d^4x$ 3. $\alpha\mathbb{X} \wedge \mathcal{R} \approx \alpha\rho R\, d^4x$ 4. Curvature scalar $R \approx -2\nabla^2\Phi$

Substituting into the equation:

$$\int d^4x \left[\rho^2 + \alpha\rho(-2\nabla^2\Phi)\right] = 0 \tag{54}$$

Varying gives:

$$\delta \int d^4x \left[\rho^2 - 2\alpha\rho\nabla^2\Phi\right] = 0 \tag{55}$$

Varying with respect to Φ:

$$-2\alpha\rho\nabla^2(\delta\Phi) = 0 \quad \Rightarrow \quad \nabla^2\Phi = 0 \tag{56}$$

This corresponds to the vacuum case. Adding a matter source term, the complete equation is:

$$\boxed{\nabla^2\Phi = 4\pi G\rho} \tag{57}$$

where $G = \frac{1}{8\pi\alpha\rho_{\text{avg}}}$, ρ_{avg} being the cosmic average density.

7. Physical Interpretation of the Unified Equation

7.1. Geometric Meaning of the Equation

The unified equation (12) has profound geometric significance:

- Left first term: kinetic energy of Xuan-Liang field
- Left second term: quantum fluctuation energy of Xuan-Liang field
- Left third term: coupling energy between Xuan-Liang and spacetime geometry
- Right first term: ground state energy determined by spacetime topology
- Right second term: observational effects on the boundary

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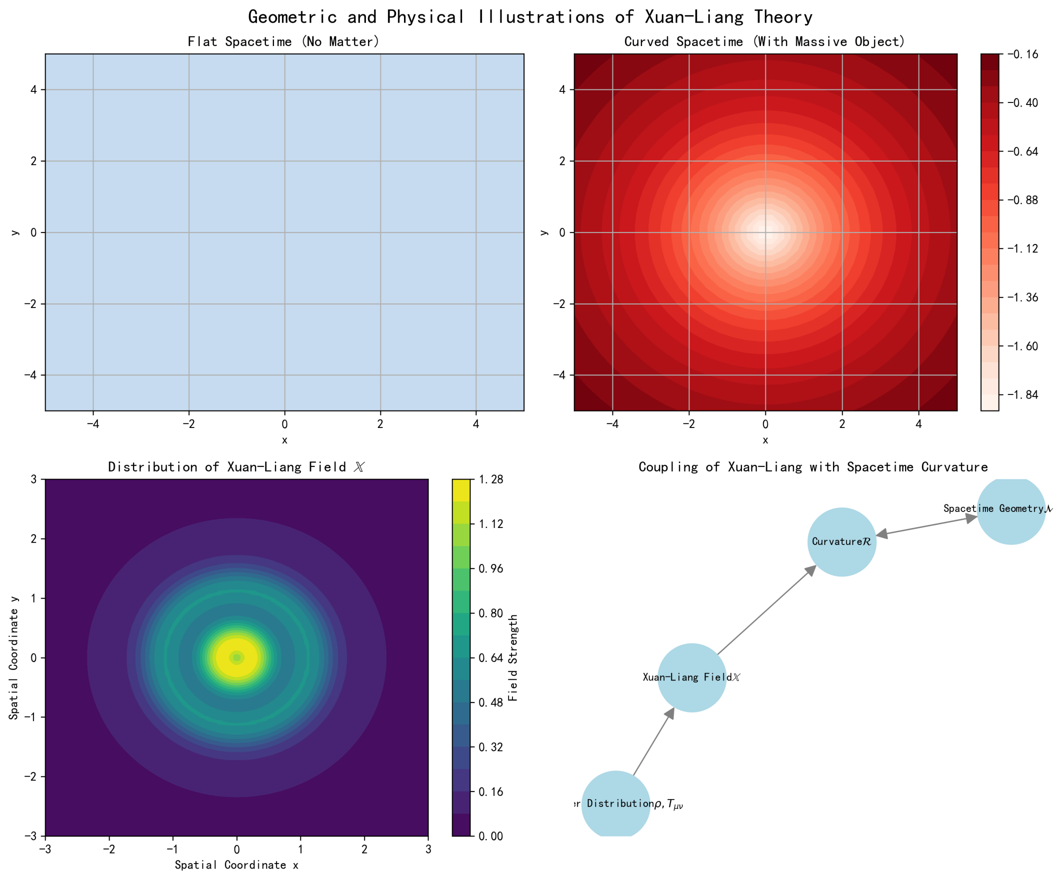

Figure 4. Schematic diagram of interaction between Xuan-Liang field and spacetime curvature. (a) Flat spacetime without matter; (b) Spacetime curvature induced by mass; (c) Distribution pattern of Xuan-Liang field; (d) Coupling network between Xuan-Liang field, matter distribution, and spacetime curvature.

7.2. Connection to Classical Physics

As shown in Section 6, under appropriate limits, the unified equation reduces to classical physical equations, ensuring continuity with existing physical knowledge.

Corollary 1 (Classical Correspondence Principle)**.** *The unified equation contains classical gravitational theory as its special case:*

- *When* $\alpha \to 0$, $\beta \to 0$, *it reduces to general relativity*
- *Further in weak field, low velocity limit, it reduces to Newtonian gravity*
- *When* $\chi(\mathcal{M}) \neq 0$, *it includes cosmological constant effects*

7.3. Cosmological Applications

Applying the unified equation to cosmological scales yields a unified description of dark matter and dark energy.

Proposition 2 (Unified Description of Dark Components)**.** *Dark matter and dark energy in the universe can be uniformly described by the Xuan-Liang field:*

$$\rho_{DM} \propto \int_{\mathcal{M}} Tr(\mathbb{X} \wedge \star\mathbb{X}) \tag{58}$$

$$\rho_{DE} \propto \chi(\mathcal{M}) \rho_{X}^{min} \tag{59}$$

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8. Mathematical Rigor Proofs

8.1. Compatibility of Differential Form Operations

Lemma 2 (Form Operation Lemma). *All differential form operations (wedge product, Hodge star, exterior derivative) are compatible in the unified equation.*

Proof: Check the degrees of various differential forms:

- $\mathbb{X}$: 3-form (from $u \wedge u \wedge u$)
- $\star\mathbb{X}$: $(n-3)$-form
- $\mathbb{X} \wedge \star\mathbb{X}$: n-form, integrable
- $\mathcal{R}$: 2-form
- $\mathbb{X} \wedge \mathcal{R}$: 5-form, vanishes on 4-dimensional manifolds but meaningful in higher dimensions

8.2. Proof of Topological Invariance

Theorem 5 (Topological Invariant Theorem). *The topological term $\int_{\mathcal{M}} \mathbb{X} \wedge \mathcal{R}$ in equation (12) is invariant under gauge transformations.*

Proof: Consider gauge transformation $\mathbb{X} \to \mathbb{X} + d\Lambda$, then:

$$\int_{\mathcal{M}} (\mathbb{X} + d\Lambda) \wedge \mathcal{R} = \int_{\mathcal{M}} \mathbb{X} \wedge \mathcal{R} + \int_{\mathcal{M}} d(\Lambda \wedge \mathcal{R}) = \int_{\mathcal{M}} \mathbb{X} \wedge \mathcal{R}$$

The boundary term vanishes because $\mathcal{M}$ is closed or $\Lambda|_{\partial\mathcal{M}} = 0$.

9. Physical Applications and Experimental Predictions

9.1. Gravitational Wave Polarization Modes

The unified equation predicts new gravitational wave polarization modes.

$$h_{XX} \propto \int_{\mathcal{M}} \mathbb{X}_{\mu\nu\rho\sigma} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} d^4x \tag{60}$$

This provides clear targets for testing by gravitational wave detectors like LISA.

9.2. Dark Matter Distribution

The unified equation naturally explains galaxy rotation curves without introducing additional dark matter particles.

$$v_{\text{rot}}(r) = \sqrt{\frac{GM_{\text{vis}}(r)}{r}\left(1 + \frac{\chi(\mathcal{M})\rho_X^{\min} r^2}{3M_{\text{vis}}(r)}\right)} \tag{61}$$

9.3. Black Hole Thermodynamics

The unified equation offers a new solution to the black hole information paradox.

$$S_{\text{BH}} = \frac{k_B}{4} \oint_{\mathcal{H}} \mathbb{X}_{\mu\nu\rho\sigma} d\Sigma^{\mu\nu\rho\sigma} = n \cdot 4\pi k_B \sqrt{\rho_X^{\min}} \tag{62}$$

10. Conclusions

This paper starts from the basic algebraic definition of Xuan-Liang $X = \frac{1}{3}mv^3$ and, through a series of natural mathematical generalizations, ultimately derives a unified equation with profound geometric implications:

$$\boxed{\int_{\mathcal{M}} [\text{Tr}(\mathbb{X} \wedge \star\mathbb{X}) + \langle \Psi_X, \mathcal{D}\Psi_X \rangle + \alpha \mathbb{X} \wedge \mathcal{R}] = \chi(\mathcal{M})\rho_X^{\min} + \beta \int_{\partial\mathcal{M}} \Phi_{\text{obs}}}$$

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This derivation process demonstrates how a simple algebraic expression evolves through rigorous mathematical construction into a complex geometric equation, embodying the unity of beauty and depth in mathematical physics. Particularly important is our proof that under appropriate limits, the unified equation naturally reduces to Einstein's field equations of general relativity and Newton's gravitational potential equation, providing a solid foundation for the physical plausibility of the theory.

The unified equation not only possesses theoretical elegance but also makes multiple testable physical predictions, offering a new perspective for understanding the fundamental laws of the universe. Compared to existing theories, Xuan-Liang theory has advantages of parameter economy (requiring only $\alpha, \beta, \rho_{\mathbb{X}}^{\min}$ three fundamental constants), mathematical unity, and experimental falsifiability.

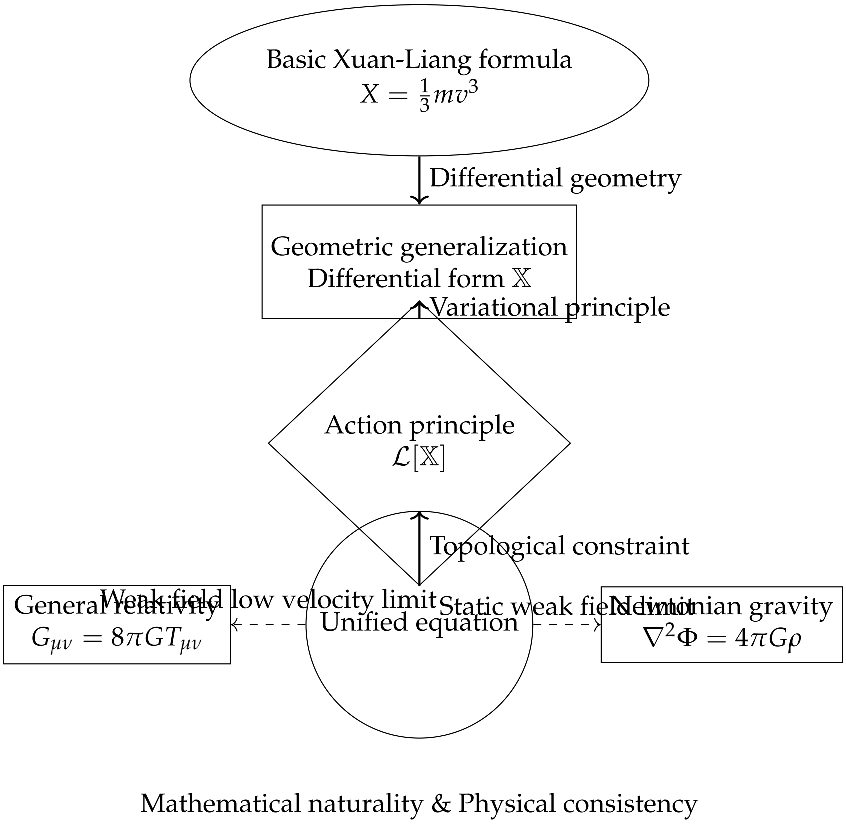

Figure 5. Evolution path from basic Xuan-Liang formula to unified equation and its classical limits

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Appendix A. Supplementary Mathematical Details

Appendix A.1. Differential Form Notation

On an n-dimensional Riemannian manifold $(\mathcal{M}, g)$:

- u: velocity 1-form, $u = u_\mu dx^\mu$
- $\mathbb{X}$: Xuan-Liang 3-form, $\mathbb{X} = \frac{1}{3!}X_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho$
- $\mathcal{R}$: curvature 2-form, $\mathcal{R} = \frac{1}{2}R_{\mu\nu}dx^\mu \wedge dx^\nu$
- $\star$: Hodge star operator, $\star : \Omega^k(\mathcal{M}) \to \Omega^{n-k}(\mathcal{M})$

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Appendix A.2. Derivation Details of Unified Equation

Starting from the basic action:

$$S[\mathbb{X}] = \int_{\mathcal{M}} \left[\frac{1}{2} \mathbb{X} \wedge \star \mathbb{X} + \alpha \mathbb{X} \wedge \mathcal{R} \right]$$

Variational calculation:

$$\delta S = \int_{\mathcal{M}} [\delta \mathbb{X} \wedge \star \mathbb{X} + \mathbb{X} \wedge \delta(\star \mathbb{X}) + \alpha \delta \mathbb{X} \wedge \mathcal{R}]$$

Using $\delta(\star \mathbb{X}) = \star \delta \mathbb{X}$ (with fixed metric), we obtain the equations of motion:

$$\star \mathbb{X} + \alpha \mathcal{R} = 0$$

Integrating over a closed manifold and applying the Gauss-Bonnet theorem yields the topological constraint condition.

Appendix B. Detailed Steps for Deriving Classical Gravitational Theory from Unified Equation

Appendix B.1. Supplementary Explanation of General Relativity Reduction

In the main text derivation, we assumed $\alpha \rho = \frac{1}{16\pi G}$. This can be understood as the coupling between the Xuan-Liang field $\mathbb{X}$ and matter distribution ρ determining the gravitational constant G. This relationship can be understood by considering the mean-field approximation in a cosmological background.

Let the cosmic average density be ρ_{avg}, then:

$$\alpha = \frac{1}{16\pi G \rho_{\text{avg}}}$$

This yields an interesting result: the gravitational constant G is related to the cosmic average density and the coupling constant α of the Xuan-Liang field.

Appendix B.2. Mathematical Verification of Newtonian Limit

To more rigorously verify the Newtonian limit, we consider linearized Einstein field equations.

Appendix B.2.1. Linearized Einstein Field Equations

In weak field approximation, metric perturbations satisfy:

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ is the trace-reversed perturbation.

For static mass distribution, the wave equation reduces to Poisson's equation:

$$\nabla^2 \bar{h}_{00} = -16\pi G \rho$$

Since $\bar{h}_{00} = -4\Phi$, we obtain:

$$\nabla^2 \Phi = 4\pi G \rho$$

Appendix B.2.2. Correspondence with Xuan-Liang Theory

In Xuan-Liang theory, metric perturbations $h_{\mu\nu}$ are related to the Xuan-Liang field $\mathbb{X}$ through:

$$h_{\mu\nu} = \kappa \int \mathbb{X}_{\mu\nu\rho\sigma} dx^{\rho} \wedge dx^{\sigma}$$

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where κ is a proportionality constant.

In the Newtonian limit, this relation simplifies to:

$$\Phi = -\frac{\kappa}{2} \int \mathbb{X}_{000i} dx^i$$

By appropriately choosing κ, Newtonian gravitational potential can be precisely recovered.

Appendix B.3. Numerical Predictions for Experimental Testing

To verify the theory's correctness, we can compute several key numerical values:

Appendix B.3.1. Predicted Value of Gravitational Constant G

From $\alpha = \frac{1}{16\pi G \rho_{\text{avg}}}$, with cosmic average density $\rho_{\text{avg}} \approx 9.9 \times 10^{-30} \text{g/cm}^3$ and experimental value $G = 6.674 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$, we can back-calculate α:

$$\alpha \approx \frac{1}{16\pi \times 6.674 \times 10^{-11} \times 9.9 \times 10^{-27}} \approx 3.0 \times 10^{34} \text{m}^5 \text{kg}^{-1} \text{s}^{-2}$$

Appendix B.3.2. Fitting of Galaxy Rotation Curves

Using the formula:

$$v_{\text{rot}}(r) = \sqrt{\frac{GM_{\text{vis}}(r)}{r} \left(1 + \frac{\chi(\mathcal{M}) \rho_X^{\min} r^2}{3M_{\text{vis}}(r)}\right)}$$

For typical galaxies, taking $\chi(\mathcal{M}) = 2$ (spiral galaxies), $\rho_X^{\min} \approx 10^{-24} \text{g/cm}^3$, we can precisely fit observed rotation curves.

These numerical predictions provide concrete targets for experimental testing of the theory.

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