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The Redshift Catastrophe—Structural Incompatibility Within FLRW Cosmology: A Formal Constraint on Early–Late Mapping

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The Redshift Catastrophe—Structural Incompatibility Within FLRW Cosmology: A Formal Constraint on Early–Late Mapping [†]

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[†] The term "Redshift Catastrophe" is chosen in deliberate analogy to the Ultraviolet Catastrophe, where the failure was not a data problem but structural: a mapping inconsistency between classical thermodynamic scaling laws and the spectral distribution, which required Planck's (initially) phenomenological correction (quanta). Here, the FLRW metric enforces a similarly rigid functional coupling between early and late-time geometry, and the resulting incompatibility is a failure of the mapping architecture itself rather than a statistical anomaly.

Abstract

This work establishes a functional-analytic obstruction within the standard cosmological framework. Concretely, we model the set of all possible FLRW expansion histories $H(z;\mathbf{p})$ as a finite-dimensional manifold immersed in the space of smooth functions. The process of extracting cosmological observables—the comoving distance D_A and the Alcock–Paczynski parameter F_{AP}—defines a smooth map Φ from this parameter manifold into an infinite-dimensional Banach space of constraints $\mathcal{C} = \mathbb{R} \times C([a,b]) \times \mathbb{R}^N$. Our central theorem shows that the image $\mathrm{Im}(\Phi)$ is a finite-dimensional submanifold of $\mathcal{C}$, and thus satisfies $\mu_C(\mathrm{Im}(\Phi)) = 0$ for any non-degenerate Borel measure $\mu_{\mathcal{C}}$ on $\mathcal{C}$. Physically, this means that a generic set of combined early and late-universe observations - represented by a point in $\mathcal{C}$ - has zero probability of lying exactly on the FLRW manifold. The resulting inevitable incoherence is not due to noise or systematics, but to the inherent geometric rigidity of the FLRW metric ansatz. This result formalises why joint early- and late-time cosmological constraints generically produce irreconcilable tensions within the standard framework.

Keywords: cosmological constant; finite-dimensional manifold; banach spaces; functional analysis

1. Introduction

We present a formal mathematical argument demonstrating that a flat Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology with a finite-dimensional parameter space cannot, in general, satisfy the combination of one global early-time integral constraint, one functional late-time constraint, and a set of discrete normalisation constraints. This result is structural and does not rely on any specific observational data.

The overdetermination of FLRW can be diagnosed empirically: several independent probes already hint at this implicitly through their mutually inconsistent preferences for the form of $H(z)$ [1–4]. However, none of these observations make the structural conflict explicit. A formal demonstration requires a statement about the dimensionality of the model's image relative to the space of constraints. For this reason, our argument is framed in terms of a finite-dimensional parameter manifold embedded in an infinite-dimensional Banach space of observables. An empirical analysis cannot establish the impossibility of an early–late FLRW mapping; the geometric argument can.

2. Observational Incompatibility

Empirical results from the Alcock-Paczyński (AP) sector [5,6] - in particular the extended tomography AP test applied to SDSS data [7,8] - show that using low-redshift geometric information alone, the preferred equation of state satisfies $w > -1$ specifically $\simeq -0.89$, in a tension of 4.2σ with the flat ΛCDM value $w = -1$ required by the Planck CMB measurement which fixes the early-time

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integral constraint $\mathcal{I}_*$ together with the acoustic scale θ_*. In a FLRW cosmology, the flat ΛCDM value forces a specific combination of (Ω_m, H_0) and pins the late–time curvature sector to behave as an exact cosmological constant. In particular, the CMB constraints sharply enforce $w = -1$ to within $\mathcal{O}(10^{-2})$.

However, prominent shape-based AP measurements [8] require a derivative history $H(z)$ whose functional form is incompatible with the $w = -1$ requirement. Matching the observed AP distortion at $z \simeq 0.5$ demands a deviation $\Delta H/H \sim \mathcal{O}(10^{-2})$ that corresponds precisely to the shift $w > -1$. If one enforces $w = -1$, the predicted AP ratio acquires the wrong redshift dependence; if one enforces $w > -1$, the early–time integral $\mathcal{I}_*$ fails, breaking consistency with the CMB acoustic scale. Because FLRW ties the local derivative constraint $H(z)$ and the global integral constraint $D_A(z)$ through the same density parameters, no single choice of $w(z)$ reconciles both simultaneously.

Results from the TDCOSMO Collaboration [4], provide an independent, JWST-calibrated confirmation that this early–late discrepancy persists even under maximally conservative systematics. The authors note that their results cross the 5σ threshold, a consensus-level admission that the contradiction is not an artefact of imprecise data. They further remark that *"there is no obvious leading contender to reconcile the measurements in tension,"* an acknowledgement that parameter-level adjustments within ΛCDM have been exhausted. Consequently, despite refined kinematics, extended samples, and hierarchical modelling, the inferred late-time value of H_0 remains incompatible with the Planck-calibrated early-time geometry.

The recent ACT DR6 results strengthens this situation [3]. ACT+BAO likelihoods tightly confine the late-time curvature to be indistinguishable from a cosmological constant, sharply restricting any fluid-based freedom in $w(z)$. Quantitatively, the ACT results enforce $w(z) \approx -1$ with negligible allowed evolution, because any deviation in the curvature sector alters the CMB–to–BAO distance ladder and is immediately excluded. Thus the entire class of exotic-fluid parametrisations - $w(z)$ models, CPL forms, phantom branches, early or dynamical dark energy, unified dark fluids - lose the degrees of freedom required to adjust the AP shape without simultaneously violating the observed curvature constraint.

In combination, the observational situation is therefore overdetermined under the FLRW mapping.

3. Geometric Origin of the Incompatibility

The observational conflict described above is not a statistical fluctuation but a manifestation of a structural obstruction. Its origin lies in a simple but far-reaching geometric fact: the FLRW model maps a *finite*-dimensional parameter space into an *infinite*-dimensional constraint space of observables. Once the early–time and late–time quantities are fixed, the model possesses no remaining functional freedom to alter the AP shape. The result is therefore not merely a tension but a failure of the mapping architecture itself. We now make this proposal more precise [9–11].

Let P be the finite-dimensional parameter space of flat FLRW cosmology (e.g. (Ω_m, H_0, w)). The model defines a smooth map

$$\Phi : P \longrightarrow \mathcal{C}, \tag{1}$$

where the constraint space $\mathcal{C}$ contains all observational quantities entering CMB, chronometer, and AP analyses. Crucially,

$$\mathcal{C} = \mathbb{R} \times C([a,b]) \times \mathbb{R}^N, \tag{2}$$

so that $\mathcal{C}$ contains at least one function-space component, making it infinite-dimensional.

By contrast, the image $\mathrm{Im}(\Phi)$ is a smooth manifold of dimension at most $\dim P \leq 7$. Therefore, a tiny finite-dimensional surface is embedded inside a vast infinite-dimensional Banach space of possible observational outcomes.

This is the structural origin of the incompatibility. The CMB integral constraints and the AP derivative constraints do not demand *points* in $\mathcal{C}$; they demand *functions* with specific shapes across a redshift interval. The FLRW map, with its finite degrees of freedom, cannot generically reproduce these

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shapes once early–late sectors are simultaneously imposed. The obstruction is therefore functional, not statistical.

To make this explicit, let

$$C_{\mathrm{Pl}} \subset \mathcal{C} \qquad \text{and} \qquad C_{\mathrm{AP}} \subset \mathcal{C} \tag{3}$$

denote the observational regions compatible with Planck and AP respectively. When ACT fixes the late–time curvature sector to a cosmological-constant form, these two regions select *different* functional behaviours of the expansion history. Because $\mathrm{Im}(\Phi)$ has dimension $\dim P$ while these constraints live in a space with infinitely many directions of variation, the image manifold cannot generically intersect both sets simultaneously.

The obstruction is therefore not that the manifold passes "far" from one region while approaching the other. The deeper problem is that

$$\mathrm{Im}(\Phi) \subset \mathcal{C} \quad \text{is finite-dimensional,} \tag{4}$$

$$C_{\mathrm{Pl}}, C_{\mathrm{AP}} \subset \mathcal{C} \quad \text{are infinite-dimensional constraint surfaces.} \tag{5}$$

and the functional requirements imposed by C_{Pl} and C_{AP} point in *different* directions in the ambient space. FLRW cannot move in those directions because those directions do not exist within $\mathrm{Im}(\Phi)$.

Thus no point of $\mathrm{Im}(\Phi)$ can satisfy both constraint sets simultaneously. Joint-likelihood fits [8] compute, in effect, the point of the finite-dimensional image manifold closest to two non-intersecting infinite-dimensional constraint regions. The resulting parameter vector is therefore not a consistent cosmology but the projection of incompatible functional requirements onto an insufficiently flexible manifold.

4. Statement of the Incompatibility Theorem

The structural origin of the early–late tension can be expressed in a single geometric statement:

$$\mu_C(\mathrm{Im}(\Phi)) \; = \; 0. \tag{6}$$

This equality encapsulates the full content of the incompatibility theorem. The symbols appearing in it refer to the following objects.

(1) The Constraint Space C.

All possible combinations of cosmological observations are treated as points in the infinite-dimensional Banach space

$$C \; = \; \mathbb{R} \times C([a,b]) \times \mathbb{R}^N, \tag{7}$$

whose components correspond respectively to (a) an early-time integral constraint such as the CMB acoustic scale D_*, (b) a continuous Alcock–Paczyński function $F_{\mathrm{AP}}(z)$ defined on a redshift interval $[a,b]$, and (c) a finite collection of late-time distance measurements $(D_1, \ldots, D_N)$. The functional component $C([a,b])$ ensures that C is infinite-dimensional.

(2) The FLRW Map Φ.

A cosmological parameter vector $p \in P$ $(\Omega_m, H_0, w_0, \ldots)$ is sent by the FLRW model to its predicted observational triple: $\Phi : P \longrightarrow C$.

Since P is finite-dimensional (with dimension $n \lesssim 6$), its image $\mathrm{Im}(\Phi) = \{\Phi(p) : p \in P\}$ is an n-dimensional submanifold of the infinite-dimensional space C. This submanifold represents *all* observational outcomes compatible with any FLRW cosmology.

(3) The Measure μ_C.

The symbol μ_C denotes any reasonable measure on C (Gaussian, quasi-invariant, Wiener, etc.). A key geometric fact is that finite-dimensional submanifolds have measure zero in infinite-dimensional Banach spaces and follows from the fact that Gaussian (or, more generally, quasi-invariant) measures on such spaces assign zero measure to finite-dimensional affine subspaces [12,13]. This is the functional-analytic analogue of a line having zero area in a plane.

(4) Consequence.

Since $\mathrm{Im}(\Phi)$ is finite-dimensional while C is infinite-dimensional, the FLRW prediction set occupies *zero measure* inside the space of all possible observational constraints: $\mu_C(\mathrm{Im}(\Phi)) = 0$. Thus the observational outcomes permitted by the FLRW metric form a geometrically negligible subset of constraint space. Equivalently, the chance that a randomly chosen triple $(D_*, F_{\mathrm{AP}}(z), D_1, \ldots, D_N)$ lies on the FLRW image manifold is *identically zero*.

5. Framework, Definitions, and Postulates

We now show formally, using the manifold structure of the FLRW parameter map Φ, that the observational inconsistency reflects a genuine geometric incompatibility: no point of the image manifold $\mathrm{Im}(\Phi)$ can satisfy both constraint regions simultaneously, and the inconsistency cannot therefore be attributable to statistical noise or insufficient data alone.

Mathematical Background

The functional-analytic setting adopted here traces back to Banach's foundational formulation of complete normed spaces [14]. Measure-theoretic properties of infinite-dimensional Banach spaces, including the fact that finite-dimensional submanifolds have Gaussian measure zero, follow the standard modern treatment in Bogachev [12].

Definition 1 (FLRW Parameter Space)**.** *We consider flat FLRW cosmologies specified by a parameter vector* $\mathbf{p} \in \mathbb{R}^n$*. For each* $\mathbf{p}$*, the expansion history* $H(z; \mathbf{p})$ *is a smooth function for* $z \geq 0$*. The space of admissible FLRW histories is thus an* n*-dimensional manifold of functions.*

Definition 2 (Comoving Angular Diameter Distance)**.** *For a given expansion history* $H(z)$*, define*

$$D_A[H](z) = \frac{1}{1+z} \int_0^z \frac{c\, dz'}{H(z')}. \tag{8}$$

This functional depends on the entire inverse expansion history on $[0, z]$*.*

Definition 3 (Alcock–Paczynski Functional)**.** *Define the AP distortion functional*

$$F_{AP}[H](z) = \frac{H(z)\, D_A[H](z)}{c}. \tag{9}$$

This observable mixes local information about $H(z)$ *with integrated information about its past behaviour.*

We now state three abstract postulates representing the structure of early- and late-time geometric constraints.

Postulate 1 (Early-Time Integral Constraint)**.** *There exists a constant* D_* *such that*

$$\mathcal{I}_*[H] \equiv D_A[H](z_*) = D_*, \tag{10}$$

where $z_* \gg 1$ *(e.g., the last-scattering epoch). This is a single, global constraint on the definite integral of* $1/H(z)$*.*

Postulate 2 (Late-Time Functional Constraint)**.** *There exists a target function* $F^*_{AP}(z)$ *defined on a nontrivial interval* $[a,b]$ *with* $0 < a < b \ll z_*$ *such that*

$$F_{AP}[H](z) = F^*_{AP}(z) \quad \textit{for all } z \in [a,b]. \tag{11}$$

In the idealised limit of fine redshift resolution and negligible uncertainty, this imposes an uncountable family of pointwise constraints on $H(z)$*.*

Postulate 3 (Late-Time Discrete Normalization Constraints)**.** *There exist redshifts* $z_1, \ldots, z_N$ *with* $z_i \ll z_*$ *and constants* D_i *such that*

$$D_A[H](z_i) = D_i, \qquad i = 1, \ldots, N. \tag{12}$$

These impose N additional discrete constraints on the distance scale at low redshift.

6. Theorem: FLRW Structural Incompatibility

Theorem 1 (FLRW Structural Incompatibility)**.** *Let* $H(z;\mathbf{p})$ *be a flat FLRW expansion history with* $\mathbf{p} \in \mathbb{R}^n$*. Under Postulates 1–3, the system of constraints*

$$\begin{aligned} \mathcal{I}_*[H(\cdot;\mathbf{p})] &= D_*, \\ F_{AP}[H(\cdot;\mathbf{p})](z) &= F^*_{AP}(z) \quad \forall z \in [a,b], \\ D_A[H(\cdot;\mathbf{p})](z_i) &= D_i \quad (i = 1, \ldots, N) \end{aligned} \tag{13}$$

is generically overdetermined.

In particular, unless the triple $(D_*, F^*_{AP}, \{D_i\})$ *lies on an n-dimensional submanifold of the infinite-dimensional constraint space* $\mathbb{R} \times C([a,b]) \times \mathbb{R}^N$*, the preimage of this triple under the FLRW map is empty.*

Proof. Consider the map

$$\Phi : \mathbf{p} \longmapsto \Big(\mathcal{I}_*[H(\cdot;\mathbf{p})],\ F_{AP}[H(\cdot;\mathbf{p})]\big|_{[a,b]},\ \{D_A[H(\cdot;\mathbf{p})](z_i)\}_{i=1}^N\Big) \tag{14}$$

with domain $\mathbb{R}^n$ and codomain

$$\mathcal{C} \equiv \mathbb{R} \times C([a,b]) \times \mathbb{R}^N, \tag{15}$$

where $C([a,b])$ denotes the Banach space of continuous functions on $[a,b]$.

The image $\mathrm{Im}(\Phi)$ is (at most) an n-dimensional submanifold of the infinite-dimensional space $\mathcal{C}$.

- **Postulate 1** fixes one component in the $\mathbb{R}$ factor.
- **Postulate 2** selects one element in the $C([a,b])$ factor.
- **Postulate 3** fixes N points in the $\mathbb{R}^N$ factor.

Together, these postulates specify a target point

$$\mathbf{T} = \big(D_*,\ F^*_{AP}\big|_{[a,b]},\ (D_1, \ldots, D_N)\big) \in \mathcal{C}. \tag{16}$$

For $\mathbf{T}$ to be realized within the FLRW framework, it must lie on the finite-dimensional submanifold $\mathrm{Im}(\Phi)$. However, in an infinite-dimensional space, any finite-dimensional submanifold has **measure zero** [12]

Therefore, the FLRW family is generically incapable of satisfying Postulates 1–3 simultaneously. ☐

6.1. Error-Tolerant Version of the Incompatibility

In practice, observations do not specify a single constraint triple $\mathbf{T} \in \mathcal{C}$, but an allowed neighbourhood determined by measurement uncertainties. We now show that the measure-zero obstruction persists even when we allow a finite tolerance.

Let $(\mathcal{C}, \|\cdot\|)$ be the separable Banach constraint space as above, endowed with a non-degenerate Gaussian measure $\mu_{\mathcal{C}}$ (for example, the product of a Gaussian measure on the function-space component $C([a,b])$ and Euclidean Gaussian measures on the finite-dimensional factors) [12]. Let $\Phi\colon P \to \mathcal{C}$ be the FLRW parameter map, where P is a finite-dimensional smooth manifold.

For $\varepsilon > 0$ define the ε–thickening of the image

$$\mathrm{Im}_{\varepsilon}(\Phi) \;:=\; \{\, X \in \mathcal{C} \;|\; \mathrm{dist}(X, \mathrm{Im}(\Phi)) < \varepsilon \,\}, \quad \mathrm{dist}(X, \mathrm{Im}(\Phi)) := \inf_{p \in P} \|X - \Phi(p)\|. \tag{17}$$

Theorem 2 (Error-tolerant incompatibility)**.** *Assume $\Phi\colon P \to \mathcal{C}$ is C^1 with $\dim P < \infty$. Then for every $\varepsilon > 0$,*

$$\mu_{\mathcal{C}}\big(\mathrm{Im}_{\varepsilon}(\Phi)\big) = 0. \tag{18}$$

*In particular, for $\mu_{\mathcal{C}}$–almost every constraint triple $\mathbf{T} \in \mathcal{C}$ and every fixed $\varepsilon > 0$, there is **no** parameter vector $p \in P$ for which the FLRW predictions lie within the error tolerance:*

$$\|\Phi(p) - \mathbf{T}\| < \varepsilon. \tag{19}$$

Sketch of proof. Since P is finite-dimensional and Φ is C^1, the image $\mathrm{Im}(\Phi)$ is contained in a countable union of finite-dimensional C^1 embedded submanifolds of $\mathcal{C}$ (by taking a countable atlas of P and using the separability of $\mathcal{C}$). For a Gaussian measure on an infinite-dimensional separable Banach space, any finite-dimensional C^1 submanifold has measure zero [12].

Furthermore, the ε–tubular neighbourhood of such a submanifold can be covered by a countable union of Lipschitz images of $\mathbb{R}^k$ for some finite k (using a partition of unity and local parametrizations of the manifold and its normal bundle). Since Lipschitz images of $\mathbb{R}^k$ are also $\mu_{\mathcal{C}}$–null (as they are still finite-dimensional in measure-theoretic sense under the Gaussian measure), the thickening $\mathrm{Im}_{\varepsilon}(\Phi)$ remains a $\mu_{\mathcal{C}}$–null set.

Now let

$$E_{\varepsilon} := \big\{\mathbf{T} \in \mathcal{C} \;\big|\; \exists p \in P \text{ with } \|\Phi(p) - \mathbf{T}\| < \varepsilon\big\}. \tag{20}$$

By definition, $E_{\varepsilon} = \mathrm{Im}_{\varepsilon}(\Phi)$, so

$$\mu_{\mathcal{C}}\big(E_{\varepsilon}\big) = 0. \tag{21}$$

Hence, for $\mu_{\mathcal{C}}$–almost every $\mathbf{T} \in \mathcal{C}$, no choice of parameters p achieves

$$\|\Phi(p) - \mathbf{T}\| < \varepsilon. \tag{22}$$

□

Operationally, Theorem 2 states that inflating the observational target from a single triple $\mathbf{T}$ to any fixed-width error region (a ball of radius ε in $\mathcal{C}$) does not rescue the FLRW map: the set of observational triples that can be matched even *within* the uncertainties still has measure zero in the full constraint space. Measurement errors enlarge the target, but a finite-dimensional cosmology remains too small to hit it except on a $\mu_{\mathcal{C}}$–null set.

7. Remarks

The incompatibility already follows from Postulates 1 and 2. Postulate 3 further restricts the compatible set, but is not required for the structural overdetermination.

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Finite Data Sampling The conclusion of overdetermination persists if Postulate 2 is weakened to require agreement only at a finite set of M redshifts. In this case the constraint space reduces to $\mathbb{R}^{1+M+N}$, while the FLRW parameter space remains n-dimensional. Whenever $1 + M + N > n$ - the situation for all modern surveys - the system is overdetermined even without invoking the idealised functional limit.

Fine-Tuning Interpretation The theorem implies that a simultaneous solution to Postulates 1-3 requires the target triple $(D_*, F^*_{AP}, \{D_i\})$ to lie on the n-dimensional submanifold $\mathrm{Im}(\Phi)$ within an infinite-dimensional constraint space. Such alignment constitutes a fine-tuning condition of co-dimension $\infty - n$. In the absence of this special tuning, the system has no solution within the FLRW family. This clarifies that the structural incompatibility is not a statistical anomaly but a generic feature of the FLRW mapping.

On the status of Λ and Dark Energy Although Λ represents an unobserved physical component with negative pressure - dark energy, it contributes only a single scalar parameter to the cosmological map. Its inclusion therefore, leaves the image of the FLRW model finite-dimensional, and thus of measure zero. In particular, Λ may alter the dynamics of the expansion but does not introduce a functional degree of freedom capable of relaxing the rigid early–late coupling responsible for the incompatibility.

Interpretation The theorem does not assert that FLRW fails for particular data, nor that current tensions arise from statistical fluctuations. It states that the rigidity of the current FLRW map - a finite-dimensional parameterisation attempting to satisfy both an early-time integral constraint and a late-time functional derivative constraint - makes compatibility a non-generic event of measure zero. The incompatibility is geometric, not statistical. (*See Appendix A & B for further clarifications*)

8. Outlook

Formulating the early–late tension within a functional-analytic framework clarifies both the origin of the incompatibility and suggests a route to its resolution. The obstruction proved here follows from a single geometric fact: the FLRW model defines a smooth map $\Phi\colon P \to \mathcal{C}$, from a finite-dimensional parameter manifold P into the infinite-dimensional constraint space $\mathcal{C} = \mathbb{R} \times C([a, b]) \times \mathbb{R}^N$. The image $\mathrm{Im}(\Phi)$ is therefore a finite-dimensional submanifold of $\mathcal{C}$ and, for generic observational data $(D_*, F^*_{AP}, \{D_i\}) \in \mathcal{C}$, has no preimage under Φ. This places the tension on structural rather than statistical grounds.

A crucial implication is that adding further FLRW parameters cannot resolve the incompatibility: such parameters merely reposition points within the same finite-dimensional image $\mathrm{Im}(\Phi)$, because they alter the numerical inputs to the map without changing its functional form. By contrast, resolving the obstruction requires *structural* degrees of freedom - that modify the map itself rather than its coordinates.

Accordingly, restoring compatibility requires enlarging the image of Φ by extending the function space, $\widetilde{P} = P \times \mathbb{R}^m$, and defining a correspondingly deformed map $\widetilde{\Phi}\colon \widetilde{P} \to \mathcal{C}$ whose additional elements genuinely alter the functional form of the mapping while still arising from a single coherent expansion history.

A structural function or functions may be introduced phenomenologically at first as effective geometric degrees of freedom that relax the classical rigidity of FLRW while preserving its early-universe limit. Their physical interpretation - whether geometric, dynamical, or topological - may emerge only once a consistent structural extension has been identified.

Functional analysis not only diagnoses the incompatibility but also supplies the geometric constraint that any viable extended FLRW theory must satisfy: it must introduce, at minimum, additional *structural* degrees of freedom required to overcome the mapping obstruction established here.

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9. Conclusion

We have established a purely geometric obstruction within the FLRW framework: the model imposes a rigid coupling between the global integral of $1/H(z)$ and its derivative manifestation at low redshift through the functional $F_{AP}[H]$. We have demonstrated that the FLRW metric, the geometric foundation of the standard cosmological model, imposes a mathematical rigidity that is incompatible with the existence of three independent classes of geometric observations: the CMB acoustic scale (fixing the high-z integral), tomographic AP measurements (fixing the low-z derivative), and late-time distance indicators (fixing the amplitude). No function H(z) within this framework can satisfy all three constraints simultaneously. The incompatibility follows directly from the finite dimensionality of the FLRW parameter space relative to the infinite-dimensional constraint space of observables. A finite-dimensional image cannot, except on a set of measure zero, satisfy independent functional constraints drawn from distinct observational sectors. It follows that any consistent reformulation of the cosmological mapping must introduce at a minimum a functional degree of freedom to enlarge the image. This requirement should be regarded as a structural condition on any future extension of the standard model.

Our result formalises the current emerging anomalies in early-late FLRW mapping. It explains why no amount of greater precision in future surveys within an unextended FLRW framework will resolve the tension.

Appendix A. Clarifications on the Structural Incompatibility Theorem

The following notes address common questions or misconceptions that may arise in response to Theorem 1. Each item provides a concise clarification of the mathematical structure of the argument.

Q1. Is the theorem trivial? Is it merely the statement that a model with few parameters becomes overdetermined by enough data?

A. No. The theorem concerns the geometry of the map $\Phi : \mathbb{R}^n \to \mathbb{R} \times C([a,b]) \times \mathbb{R}^N$, which includes both a global integral functional and a continuous functional. The structural incompatibility arises from the fact that the FLRW image is an n-dimensional submanifold of an infinite-dimensional space. This is not a counting of data points but a geometric limitation of the FLRW mapping itself.

Q2. What if the AP functional is effectively low-dimensional? Could it reduce to a few principal components?

A. Even in the finite-sampling case, the constraint space becomes $\mathbb{R}^{1+M+N}$. Overdetermination arises whenever $1 + M + N > n$, which is the case for all modern datasets. Thus the structural mismatch persists without taking the idealised limit of a full functional constraint.

Q3. Are the constraints really independent? Could the early-time integral and late-time AP shape be related?

A. They are independent by construction. The early-time constraint $\mathcal{I}_*$ fixes a definite integral of $1/H(z)$ up to z_*, while $F_{AP}[H](z)$ depends only on H and its integral up to $z \ll z_*$. No functional identity links them for general FLRW histories; any such identity would have to be explicitly demonstrated.

Q4. Could FLRW be enlarged by adding curvature or more fluids? Do more parameters remove the tension?

A. Adding curvature increases n by at most one. Adding smooth dark-energy fluids enlarges $\mathbf{p}$ but does not alter the fact that FLRW backgrounds constitute a *finite-dimensional* family of functions. The theorem applies to the entire class of finite-parameter FLRW expansions. Avoiding it requires modifying the metric ansatz itself.

Q5. What about modified gravity? Do $f(R)$ or Horndeski theories increase the freedom of the background?

A. No. In any metric theory with FLRW symmetry, the background $H(z)$ is still determined by a finite number of parameters. Modified gravity does not generate an infinite-dimensional family of background expansions, so the theorem applies unchanged.

Q6. Does the theorem assume infinite precision? Is the functional constraint valid only in the zero-uncertainty limit?

A. No. The finite-sampling form of the AP constraint is treated explicitly in Remark 2. The structural incompatibility arises whenever the number of independent low-redshift constraints exceeds the dimension n of the FLRW parameter space. The idealised functional limit is used only to describe the geometry of the constraint space.

Q7. Could the universe lie exactly on the FLRW-compatible manifold? Is fine-tuning a valid escape?

A. This requires the target triple $(D_*, F^*_{AP}, \{D_i\})$ to lie on an n-dimensional submanifold of the infinite-dimensional space $\mathbb{R} \times C([a,b]) \times \mathbb{R}^N$. Such points form a set of measure zero and correspond to extreme fine-tuning. The theorem therefore, treats compatibility as non-generic.

A natural concern is whether a sufficiently tuned choice of cosmological parameters might restore consistency within the FLRW framework. This intuition arises from the usual statistical practice of adjusting the small number of FLRW parameters to accommodate diverse datasets. However, the incompatibility demonstrated in the preceding formal proof is not parametric but geometric. Once the CMB calibration fixes the global integral constraint, the FLRW ansatz determines the full late-time functional form of $H(z)$. The AP sector imposes an effectively infinite-dimensional constraint, and the image of a finite-dimensional parameter manifold cannot, except on a set of measure zero, intersect it. Consequently, no amount of parameter tuning can reconcile the two constraint classes; the obstruction arises from the rigidity of the FLRW mapping itself.

Q8. FLRW is successful empirically. Does that contradict the theorem?

A. No. FLRW can successfully describe *individual* observational sectors (e.g. early-time geometry or specific late-time probes). The theorem concerns the *joint* imposition of early-time integral constraints and late-time derivative constraints. It states that the FLRW map cannot generically satisfy both simultaneously.

Q9. What if the FLRW map has hidden degeneracies or unusual Jacobians? Could this enlarge its image?

A. No. A map from an n-dimensional manifold has image of dimension at most n, even if the map is non-injective. Non-injectivity cannot increase the dimensionality of the image. This is a standard result in differential geometry and ensures the theorem's robustness.

Q10. Isn't this just saying FLRW is rigid? Is the result philosophical rather than technical?

A. The rigidity is a mathematical property of the FLRW line element. The theorem formally demonstrates that this rigidity has a concrete consequence: the combined early-time, functional late-time, and discrete normalisation constraints generically have no common preimage in the finite-dimensional FLRW family. This is a technical geometric result, not a philosophical claim.

Appendix B. Conceptual Clarification of the Geometric Setting

Our argument employs geometric language that lies slightly outside the standard cosmology toolkit, which typically focuses on parametric fits, Fisher matrices, and likelihood surfaces. In that familiar context, a manifold is often visualized as a flexible two or three-dimensional surface embedded in $\mathbb{R}^3$. This appendix clarifies a key conceptual point that can arise when the argument is first encountered: why the FLRW image manifold and an observed constraint point cannot, in general, be made to intersect.

Dimensionality and Measure

The image of the FLRW map $\Phi : P \to \mathcal{C}$ has dimension equal to the number of cosmological parameters (at most $\sim$ 5–7 in typical models). The constraint space $\mathcal{C} = \mathbb{R} \times C([a,b]) \times \mathbb{R}^N$ is *infinite-dimensional* because of its function-space component $C([a,b])$.

In an infinite-dimensional Banach space, any finite-dimensional submanifold has **measure zero**. This is not a visual metaphor - it is a rigorous statement in geometry [12,13]. Therefore a randomly chosen point in such a space lies on a given finite-dimensional submanifold with probability zero.

Why the "Flexible Sheet" Intuition Fails

The intuitive picture of two finite-dimensional surfaces in $\mathbb{R}^3$ that can be bent or deformed to intersect does not apply here. The actual setting is fundamentally different:

- A **finite-dimensional** manifold (the image of FLRW parameters),
- embedded in an **infinite-dimensional** Banach space (the space of constraint triples $\mathcal{C}$).

The embedding map Φ is *fixed by the physics* of the FLRW metric and distance integrals; it is not a freely deformable geometric object. The impossibility of intersection for a generic constraint point is not assumed but *forced* by the structure of the problem:

1. The finite dimensionality of $\mathrm{Im}(\Phi)$,
2. The functional (infinite-dimensional) nature of the AP constraint,
3. The fixed acoustic scale calibration (the early-time integral),
4. The lack of additional functional degrees of freedom in the FLRW ansatz.

This clarifies why our theorem is **structural** rather than empirical. The mismatch arises not from noisy data or insufficient model complexity, but from the intrinsic geometric limitations of the FLRW framework when confronted with independent early and late-universe constraints. Consequently, when modern surveys provide high-dimensional (or functional) late-time constraints, the probability that these align exactly with a Planck-calibrated FLRW model is zero - not small, but mathematically zero. This is the essence of the Redshift Catastrophe.

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