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Article

What Is the Radius of Convergence in the Sequence Space $Seq(\mathbb{R})$?

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Abstract

Classical real analysis rigorously defines convergence via $\varepsilon - N$ criteria, yet it frequently regards the specific entry index N as a mere artifact of proof rather than an intrinsic property. This paper fills this quantitative void by developing a radius of convergence framework for the sequence space $\mathrm{Seq}(\mathbb{R})$. We define an index-based radius $\rho_a(\varepsilon)$ alongside a rescaled geometric radius $\rho_a^*(\varepsilon)$; the latter maps the unbounded index domain to a finite interval, establishing a structural analogy with spatial radii familiar in analytic function theory. We systematically analyze these radii within a seven-block partition of the sequence space, linking them to liminf-limsup profiles and establishing their stability under algebraic operations like sums, products, and finite modifications. The framework's practical power is illustrated through explicit asymptotic inversions for sequences such as Fibonacci ratios, prime number distributions, and factorial growth. By transforming the speed of convergence into a geometric descriptor, this approach bridges the gap between asymptotic limit theory and constructive analysis, offering a unified, fine-grained measure for both convergent and divergent behaviors.

Keywords: radius of convergence; real sequences; liminf–limsup radii; cauchy radius; sequence classification

MSC: 40A05(primary); 26A03; 26A15 (secondary)

> *"When the values successively attributed to the same variable approach indefinitely a fixed value, so as to end up by differing from it by as little as one could wish, this last is called the limit of all the others."*
>
> — Augustin–Louis Cauchy (1789–1857)

1. Introduction

1.1. Real-Valued Sequences and Convergence

Real-valued sequences remain one of the most basic but versatile objects in analysis. Classical mathematical literature emphasize the ε–N definition of convergence, the derived notions of limit inferior and limit superior, and the extension of limits to the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ [1]. These tools give a robust qualitative classification of limit behaviour—convergent, divergent to $\pm\infty$, or oscillatory—and they underpin the standard partition of $\mathrm{Seq}(\mathbb{R})$ into convergent and various divergent subclasses [2]. At the same time, the classical framework deliberately abstracts away *when* a sequence enters a prescribed error band around its limit and focuses instead on the asymptotic fact that such an entry eventually occurs.

Beyond this qualitative viewpoint, several quantitative refinements have been developed. Constructive and computable analysis introduce *moduli of convergence* or *Cauchy moduli*, functions $N(\varepsilon)$ that witness how far into the sequence one must go in order to guarantee an ε-accurate approximation of the limit [3–6]. Numerical analysis, in turn, describes the speed of convergence of iterative processes via rates and orders of convergence (e.g., "linear," "superlinear," or "quadratic" order), capturing how

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successive errors compare to each other rather than to a fixed ε-tube [7–9]. These perspectives illustrate that quantitative information about convergence is both mathematically rich and practically important.

1.2. Motivation

The present paper is motivated by a parallel line of work on *radii of continuity* for real-valued functions, where one assigns to each point x_0 and tolerance ε a maximal radius within which the function oscillation around $f(x_0)$ stays below ε [10]. Such radii encode local stability of functions in the metric space $F(\mathbb{R}, \mathbb{R})$ and lead to a fine-grained description of continuity that interacts naturally with algebraic operations and with classical moduli of continuity. From this vantage point, it is natural to ask whether an analogous "radial" description can be developed for sequences in $\mathrm{Seq}(\mathbb{R})$, with the index n playing the role of a one-dimensional "space" variable along the tail of the sequence.

In classical analysis, the phrase "radius of convergence" appears almost exclusively in the context of power series, where one studies the largest spatial radius $R \in [0, +\infty]$ such that the series $\sum c_n(z - a)^n$ converges whenever $|z - a| < R$ and diverges whenever $|z - a| > R$ [1]. This notion is attached to analytic functions in the complex plane rather than to bare sequences [11]; it is a geometric property of where a series converges, not of how quickly its coefficients or partial sums stabilize. By contrast, the moduli of convergence mentioned above provide index-based information for sequences but are rarely organized or studied through an explicit "radius" vocabulary. Thus, there is a conceptual gap between geometric radii in function spaces and quantitative convergence data in sequence spaces. The goal of this paper is to bridge this gap by introducing and systematically studying *radii of convergence for real-valued sequences* within the context of constructive mathematical analysis [3–5].

1.3. Organization of the Paper

This paper is organized as follows. In Section 2 we provide the necessary mathematical background for the subsequent sections. In Section 3 we introduce the notion of a radius of convergence for sequences, beginning with one-sided liminf and limsup radii and then extending the discussion to the two-sided radius of convergence, the geometric radius, and the Cauchy radius. We then study the stability of these radii under algebraic operations on sequences. Next, in Section 4 we present examples of computing the radius of convergence for two clusters of sequences: four convergent sequences and four divergent sequences with infinite limits. We conclude the paper with a brief discussion in Section 5.

2. Preliminaries

In this subsection we collect the basic notation and standard facts about real-valued sequences that will be used throughout the paper.

Definition 1 (Sequence space $Seq(\mathbb{R})$)**.** *We denote by $Seq(\mathbb{R})$ the space of all real-valued sequences $a = (a_n)_{n\in\mathbb{N}}$, $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, equipped with pointwise addition and scalar multiplication.*

Definition 2 (Limit inferior, limit superior, limit profile and limit)**.** *Let $a = (a_n)_{n\in\mathbb{N}} \in Seq(\mathbb{R})$. The limit inferior and limit superior of a are defined by $L_1(a) := \liminf_{n\to\infty} a_n \in \overline{\mathbb{R}}$, and $L_2(a) := \limsup_{n\to\infty} a_n \in \overline{\mathbb{R}}$, where in which $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The pair $(L_1(a), L_2(a))$ is referred as limit profile of a. When $L_1(a) = L_2(a) =: L(a) \in \mathbb{R}$, we write $\lim_{n\to\infty} a_n = L(a)$ and say that a is* convergent*.*

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Lemma 1 (Tail characterization of lim inf and lim sup)**.** *Let $a = (a_n)_{n\in\mathbb{N}} \in Seq(\mathbb{R})$, and let ℓ_N, u_N be the tail infimum and tail supremum defined by $\ell_N := \inf_{n\geq N} a_n$, and $u_N := \sup_{n\geq N} a_n$, respectively. Then $(\ell_N)_{N\in\mathbb{N}}$ is increasing, $(u_N)_{N\in\mathbb{N}}$ is decreasing, and*

$$\begin{aligned} L_1(a) &= \sup_{N\in\mathbb{N}} \ell_N = \lim_{N\to\infty} \ell_N, \\ L_2(a) &= \inf_{N\in\mathbb{N}} u_N = \lim_{N\to\infty} u_N, \\ L_1(a) &\leq L_2(a). \end{aligned}$$

In particular, $L_1(a), L_2(a) \in \mathbb{R}$ if and only if the monotone sequences (ℓ_N) and (u_N) converge to finite real limits.

Definition 3 (Cauchy sequence)**.** *A sequence $a = (a_n)_{n\in\mathbb{N}} \in Seq(\mathbb{R})$ is called* Cauchy *if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon \quad$ for all $m, n \geq N$.*

Theorem 1 (Cauchy Equivalency Criteria)**.** *In the complete space $\mathbb{R}$, every Cauchy sequence is convergent, and conversely every convergent sequence is Cauchy.*

Theorem 2 (Seven-block partition of $Seq(\mathbb{R})$ by lim inf and lim sup)**.** *For each $a \in Seq(\mathbb{R})$ let $L_1(a), L_2(a) \in \overline{\mathbb{R}}$ be its limit inferior and limit superior. Define*

$$\begin{aligned} A &:= \{a \in Seq(\mathbb{R}) : L_1(a) = -\infty,\ -\infty < L_2(a) < +\infty\}, \\ B &:= \{a \in Seq(\mathbb{R}) : -\infty < L_1(a) < L_2(a) < +\infty\}, \\ C &:= \{a \in Seq(\mathbb{R}) : -\infty < L_1(a) < L_2(a) = +\infty\}, \\ D &:= \{a \in Seq(\mathbb{R}) : L_1(a) = -\infty,\ L_2(a) = +\infty\}, \\ E &:= \{a \in Seq(\mathbb{R}) : L_1(a) = -\infty,\ L_2(a) = -\infty\}, \\ F &:= \{a \in Seq(\mathbb{R}) : L_1(a) = +\infty,\ L_2(a) = +\infty\}, \\ G &:= \{a \in Seq(\mathbb{R}) : -\infty < L_1(a) = L_2(a) < +\infty\}. \end{aligned}$$

Then the seven sets $A, \ldots, G$ are pairwise disjoint and

$$Seq(\mathbb{R}) = A \,\dot\cup\, B \,\dot\cup\, C \,\dot\cup\, D \,\dot\cup\, E \,\dot\cup\, F \,\dot\cup\, G. \tag{1}$$

In particular, every real-valued sequence belongs to exactly one of the blocks $A, \ldots, G$, according to the qualitative behaviour of its limit inferior and limit superior (unbounded on both sides, bounded but non-convergent, one-sided unbounded, diverging to $+\infty$ or $-\infty$, or convergent) [2].

Lemma 2 (Infimum of an intersection)**.** *Let $A, B \subseteq \mathbb{N}$ be nonempty sets with the following tail property:*

$$\forall N \in A,\ \forall M \in \mathbb{N},\ M \geq N \Longrightarrow M \in A,$$

$$\forall N \in B,\ \forall M \in \mathbb{N},\ M \geq N \Longrightarrow M \in B.$$

Then:

(i) *There exist integers $N_A, N_B \in \mathbb{N}$ such that $A = \{n \in \mathbb{N} : n \geq N_A\}$, $B = \{n \in \mathbb{N} : n \geq N_B\}$, and $A \cap B = \{n \in \mathbb{N} : n \geq \max\{N_A, N_B\}\}$.*

(ii) $\inf(A \cap B) = \max\{\inf A, \inf B\}$.

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Lemma 3 (Asymptotic inversion of $x \log x$). *Let $x\colon (0,\infty) \to (0,\infty)$ be a variable and put $y := x \log x$. Then, as $y \to \infty$,*

$$y \sim x \log x \qquad \Longrightarrow \qquad x \sim \frac{y}{\log y}. \tag{2}$$

3. Theory of Radius of Convergence for Sequences

In this section, we define the one-sided radius of convergence, two-sided radius of convergence, its geometric version and Cauchy radius of convergence. We, then study the the relationship between one-sided radius of convergence and the two-sided radius of convergence followed by the latter's stability under algebraic operations.

3.1. One-Sided Liminf and Limsup Radii for a Single Sequence

We start our investigation by focusing on the limit profile $(L_1(a), L_2(a))$ of a given sequence $a = (a_n)_{n\in\mathbb{N}}$ and its associated radii, and their potential relationship:

Definition 4 (Liminf and limsup radii). *Let $a = (a_n)_{n\in\mathbb{N}}$ be a real sequence with associated tail infimum (l_n) and tail supremum (u_n) and let $\varepsilon > 0$.*

(i) *We define the* liminf radius *of a at level ε by:*

$$\begin{aligned}
\rho_{L_1}(a;\varepsilon) &:= \inf(T_1(a,\varepsilon)) : && (3)\\
T_1(a,\varepsilon) &:= \Big\{N \in \mathbb{N} :\ \forall n \geq N,\ l_n < -\varepsilon\Big\}, \quad \text{if } L_1(a) = -\infty && (4)\\
&:= \Big\{N \in \mathbb{N} :\ \forall n \geq N,\ |l_n - L_1(a)| < \varepsilon\Big\}, \quad \text{if } L_1(a) \neq \pm\infty && (5)\\
&:= \Big\{N \in \mathbb{N} :\ \forall n \geq N,\ l_n > \varepsilon\Big\} \quad \text{if } L_1(a) = +\infty. && (6)
\end{aligned}$$

(ii) *We define the* limsup radius *of a at level ε by:*

$$\begin{aligned}
\rho_{L_2}(a;\varepsilon) &:= \inf(T_2(a,\varepsilon)) : && (7)\\
T_2(a,\varepsilon) &:= \Big\{N \in \mathbb{N} :\ \forall n \geq N,\ u_n < -\varepsilon\Big\}, \quad \text{if } L_2(a) = -\infty && (8)\\
&:= \Big\{N \in \mathbb{N} :\ \forall n \geq N,\ |u_n - L_2(a)| < \varepsilon\Big\}, \quad \text{if } L_2(a) \neq \pm\infty && (9)\\
&:= \Big\{N \in \mathbb{N} :\ \forall n \geq N,\ u_n > \varepsilon\Big\} \quad \text{if } L_2(a) = +\infty. && (10)
\end{aligned}$$

(iii) *We refer to the pair $(\rho_{L_1}(a;.), \rho_{L_1}(a;.))$ as the* radii profile *of the sequence a.*

Remark 1. *By the standard properties of* lim inf *and* lim sup, *the sets $T_i(a,\varepsilon)$ are non-empty for every $\varepsilon > 0$, hence the corresponding radii $\rho_{L_i}(a;\varepsilon) \quad (i = 1,2)$ are finite integers.*

Remark 2. *As a direct result of Definition 4 and Theorem 2, there are seven different methods for the computation of radii profile given the limit profile.*

Theorem 3 (Relations between liminf and limsup radii when the two–sided limit exists). *Let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence with associated limit profile $(L_1(a), L_2(a))$ and radii profile $(\rho_{L_1}(a;.), \rho_{L_2}(a;.))$. Assume $L_1(a) = L_2(a) = L(a)$. Then:*

1. *If $L(a) = -\infty$, then for every $\varepsilon > 0$, $\rho_{L_1}(a;\varepsilon) \leq \rho_{L_2}(a;\varepsilon)$.*
2. *If $L(a) \in \mathbb{R}$, then in general there is no universal inequality between these radii and for given $\varepsilon > 0$ each of the three orderings $\rho_{L_1}(a;\varepsilon) < \rho_{L_2}(a;\varepsilon)$, $\quad \rho_{L_1}(a;\varepsilon) > \rho_{L_2}(a;\varepsilon)$, $\quad \rho_{L_1}(a;\varepsilon) = \rho_{L_2}(a;\varepsilon)$ can occur (see the proof for explicit examples).*
3. *If $L(a) = +\infty$, then for every $\varepsilon > 0$, $\rho_{L_2}(a;\varepsilon) \leq \rho_{L_1}(a;\varepsilon)$.*

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Proof. By Lemma 1 the tail infimum and tail supremum satisfy $\ell_N \leq u_N$ for all N, with $(\ell_N)_N$ increasing and $(u_N)_N$ decreasing. We treat the three cases separately.

(1) Case $L(a) = -\infty$. Here $L_1(a) = L_2(a) = -\infty$, and by Definition 4 it yields that if $N \in T_2(a, \varepsilon)$, then $u_n < -\varepsilon$ for all $n \geq N$, hence $\ell_n \leq u_n < -\varepsilon$ for all $n \geq N$, so $N \in T_1(a, \varepsilon)$. Thus $T_2(a, \varepsilon) \subseteq T_1(a, \varepsilon)$ and taking infimum from both sides it follows that $\rho_{L_1}(a; \varepsilon) \leq \rho_{L_2}(a; \varepsilon)$.

To see that the inequality can be strict, take $a_n := -n$. Then $\ell_N = -\infty$ and $u_N = -N$, so $T_1(a, \varepsilon) = \mathbb{N}$ and $T_2(a, \varepsilon) = \{N \in \mathbb{N} : N > \varepsilon\}$. Consequently, $\rho_{L_1}(a; \varepsilon) = 1 < \lceil \varepsilon \rceil = \rho_{L_2}(a; \varepsilon)$.

(2) Case $L(a) \in \mathbb{R}$. Then $L_1(a) = L_2(a) = L(a)$. We now show that no universal inequality between ρ_{L_1} and ρ_{L_2} holds in this case, by providing convergent sequences that realize all three orderings for a fixed $\varepsilon \in (0, 1)$.

First, define

$$a_n^{(1)} := \begin{cases} 1/k, & n = 2^k \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $a_n^{(1)} \to 0$. Every tail contains zeros, so $\ell_N = 0$ for all N, whence $T_1(a^{(1)}, \varepsilon) = \mathbb{N}$ and $\rho_{L_1}(a^{(1)}; \varepsilon) = 1$. On the other hand, $u_N \searrow 0$, so there exists a least $N_2 > 1$ with $u_n < \varepsilon$ for all $n \geq N_2$, giving $\rho_{L_2}(a^{(1)}; \varepsilon) = N_2 > 1$. Thus

$$\rho_{L_1}(a^{(1)}; \varepsilon) < \rho_{L_2}(a^{(1)}; \varepsilon).$$

Second, define

$$a_n^{(2)} := \begin{cases} -1/k, & n = 2^k \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Again $a_n^{(2)} \to 0$. Every tail contains zeros and negative spikes, so $u_N = 0$ for all N, and hence $T_2(a^{(2)}, \varepsilon) = \mathbb{N}$ and $\rho_{L_2}(a^{(2)}; \varepsilon) = 1$. The tail infima satisfy $\ell_N \uparrow 0$, so there exists a least $N_1 > 1$ with $|\ell_n| < \varepsilon$ for all $n \geq N_1$, and thus $\rho_{L_1}(a^{(2)}; \varepsilon) = N_1 > 1$. Hence

$$\rho_{L_2}(a^{(2)}; \varepsilon) < \rho_{L_1}(a^{(2)}; \varepsilon).$$

Finally, for the constant sequence $a_n^{(3)} \equiv L$ we have $\ell_N = u_N = L \in \mathbb{R}$ for all N, so $T_1(a^{(3)}, \varepsilon) = T_2(a^{(3)}, \varepsilon) = \mathbb{N}$ and $\rho_{L_1}(a^{(3)}; \varepsilon) = \rho_{L_2}(a^{(3)}; \varepsilon) = 1$. These three examples show that all three orderings between $\rho_{L_1}(a; \varepsilon)$ and $\rho_{L_2}(a; \varepsilon)$ can occur.

(3) Case $L(a) = +\infty$. Here $L_1(a) = L_2(a) = +\infty$, and by Definition 4 it yields that if $N \in T_1(a, \varepsilon)$, then $\ell_n > \varepsilon$ for all $n \geq N$, hence $u_n \geq \ell_n > \varepsilon$ for all $n \geq N$, so $N \in T_2(a, \varepsilon)$. Thus $T_1(a, \varepsilon) \subseteq T_2(a, \varepsilon)$ and taking infimum on both sides, it follows that $\rho_{L_2}(a; \varepsilon) \leq \rho_{L_1}(a; \varepsilon)$.

To see that the inequality can be strict, take $a_n := n$. Then $u_N = +\infty$ and $\ell_N = N$, so $T_2(a, \varepsilon) = \mathbb{N}$ and $T_1(a, \varepsilon) = \{N \in \mathbb{N} : N > \varepsilon\}$, and hence $\rho_{L_2}(a; \varepsilon) = 1 < \lceil \varepsilon \rceil = \rho_{L_1}(a; \varepsilon)$.

This completes the proof. □

Conjecture 1 (Limit profile vs. Radii profile). *Let $a = (a_n)_{n \in \mathbb{N}}$ be a sequence with associated limit profile $(L_1(a), L_2(a))$ and radii profile $(\rho_{L_1}(a; .), \rho_{L_2}(a; .))$. Then:*

$$L_1(a) = L_2(a) \iff \rho_{L_1}(a; .) = \rho_{L_2}(a; .). \tag{11}$$

Counterexamples. We present two counterexamples each for one direction of implication (11):

(a) $L_1(a) \neq L_2(a) \& \rho_{L_1} = \rho_{L_2}$:

Let $a = (a_n)_{n\in\mathbb{N}}$ be defined by $a_n = \frac{(-1)^n+1}{2}$, $(n \in \mathbb{N})$. Then $L_1(a) = 0$ and $L_2(a) = 1$. Next, for any $\varepsilon > 0$ and $n \in \mathbb{N}$ we have $|l_n - L_1(a)| = |0-0| = 0 < \varepsilon$, and $|u_n - L_1(a)| = |1-1| = 0 < \varepsilon$. Consequently:

$$\rho_{L_1}(a;\varepsilon) = \rho_{L_2}(a;\varepsilon) = 1 \qquad \text{for all } \varepsilon > 0.$$

(b) $L_1(a) = L_2(a) \& \rho_{L_1} \neq \rho_{L_2}$:

Let $a = (a_n)_{n\in\mathbb{N}}$ be defined by $a_n = 1 - \frac{1}{n}$, $(n \in \mathbb{N})$. Then $L_1(a) = 1$ and $L_2(a) = 1$. On the other hand, for given $0 < \varepsilon < 1$:

$$\begin{aligned} \rho_{L_1}(a;\varepsilon) &= \inf(\Big\{N \in \mathbb{N} : \ \forall n \geq N,\ |1 - \frac{1}{n} - 1| < \varepsilon\Big\}) = \lceil \frac{1}{\varepsilon} \rceil, \\ \rho_{L_2}(a;\varepsilon) &= \inf(\Big\{N \in \mathbb{N} : \ \forall n \geq N,\ |1-1| < \varepsilon\Big\}) = 1. \end{aligned}$$

Roughly speaking, there is no simple characterization beyond such tail-level regularity.

3.2. Two-Sided Radius of Convergence and Cauchy Radius

We continue our investigation by focusing on the two-sided radius of convergence, the Cauchy (uniform) radius of a given sequence $a = (a_n)_{n\in\mathbb{N}}$, their potential relationship with each other and to the one-sided radii:

Definition 5 (Radius of convergence of a convergent sequence). *Let $a = (a_n)_{n\in\mathbb{N}}$ be a convergent real sequence with limit $L(a) \in \overline{\mathbb{R}}$. For $\varepsilon > 0$ we define the* radius of convergence *of a at level ε by:*

$$\rho_a(\varepsilon) := \inf(R(a,\varepsilon)) : \tag{12}$$

$$R(a,\varepsilon) := \Big\{N \in \mathbb{N} : \ \forall n \geq N,\ a_n < -\varepsilon\Big\}, \quad \text{if } L(a) = -\infty \tag{13}$$

$$:= \Big\{N \in \mathbb{N} : \ \forall n \geq N,\ |a_n - L(a)| < \varepsilon\Big\}, \quad \text{if } L(a) \neq \pm\infty \tag{14}$$

$$:= \Big\{N \in \mathbb{N} : \ \forall n \geq N,\ a_n > \varepsilon\Big\} \quad \text{if } L(a) = +\infty. \tag{15}$$

Thus $\rho_a(\varepsilon)$ is the smallest index after which the entire tail of a remains within the ε–tube around its limit L (Figure 1).

Remark 3. *Given the convergent sequences a, we may view $\rho_{L_1}(a;\varepsilon)$ and $\rho_{L_2}(a;\varepsilon)$ from Definition 4 as one-sided radii relative to the common limit $L(a) = L_1(a) = L_2(a)$, while $\rho_a(\varepsilon)$ in Definition 5 is the two-sided radius.*

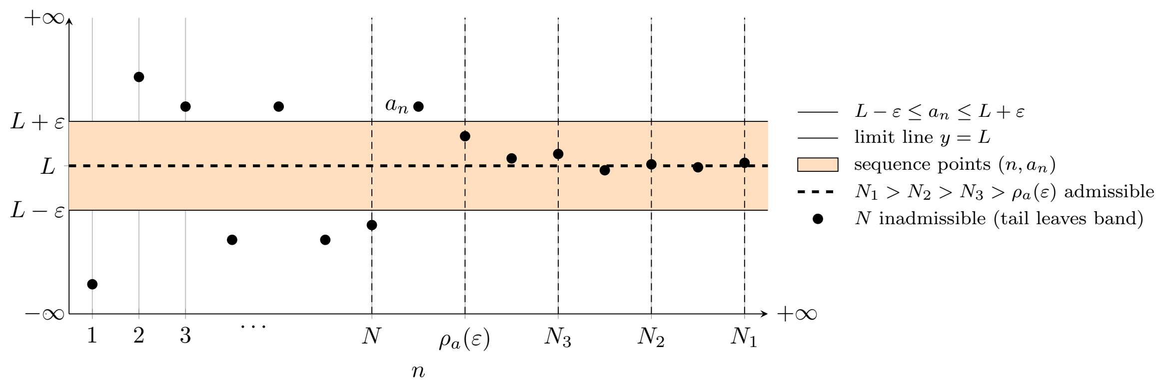

Figure 1. Radius of convergence $\rho_a(\varepsilon)$ for a convergent sequence $(a_n)_{n\in\mathbb{N}}$. The shaded horizontal strip is the ε-band around the limit $L = L(a)$. The earliest admissible index is $\rho_a(\varepsilon)$; for $N < \rho_a(\varepsilon)$ the tail still leaves the band (inadmissible), while for $N_1 > N_2 > N_3 > \rho_a(\varepsilon)$ the tail from each such index onward is contained in the band.

Definition 6 (Geometric radius)**.** *To emphasize the analogy with classical notions of radius (such as the radius of a ball in a metric space or the radius of convergence of a power series), one may equivalently work with the rescaled radius:*

$$\rho_a^*(\varepsilon) := \frac{1}{\rho_a(\varepsilon) - 1}, \qquad \varepsilon > 0, \tag{16}$$

with the convention $1/0 := +\infty$*(Figure 2).*

Remark 4. *The transformation* $\rho_a \mapsto \rho_a^*$ *is strictly decreasing on* $(1, +\infty]$ *and invertible via*

$$\rho_a(\varepsilon) = 1 + \frac{1}{\rho_a^*(\varepsilon)}, \qquad \varepsilon > 0, \tag{17}$$

so ρ_a *and* ρ_a^* *encode exactly the same information about the convergence speed of a. In particular, larger values of* $\rho_a^*(\varepsilon)$ *correspond to smaller entry indices* $\rho_a(\varepsilon)$*, so that* $\rho_a^*(\varepsilon)$ *behaves qualitatively like a geometric radius around the limiting point (Figure 2).*

Theorem 4 (Block-wise two-sided radius of convergence)**.** *Let* $a \in Seq(\mathbb{R})$ *belong to one of the seven blocks* $A, \ldots, G$ *of Theorem 2, and let* $\rho_a(\varepsilon)$ *be the two-sided radius of convergence of a at level* $\varepsilon > 0$ *(Definition 5). Then* $\rho_a(\varepsilon)$ *is well defined and finite for every* $\varepsilon > 0$ *if and only if* $a \in E \cup F \cup G$*; for* $a \in A \cup B \cup C \cup D$ *the two-sided radius is not defined, and on the convergent blocks we have that* $\varepsilon \mapsto \rho_a(\varepsilon)$ *is nonincreasing on G and nondecreasing on* $E \cup F$*.*

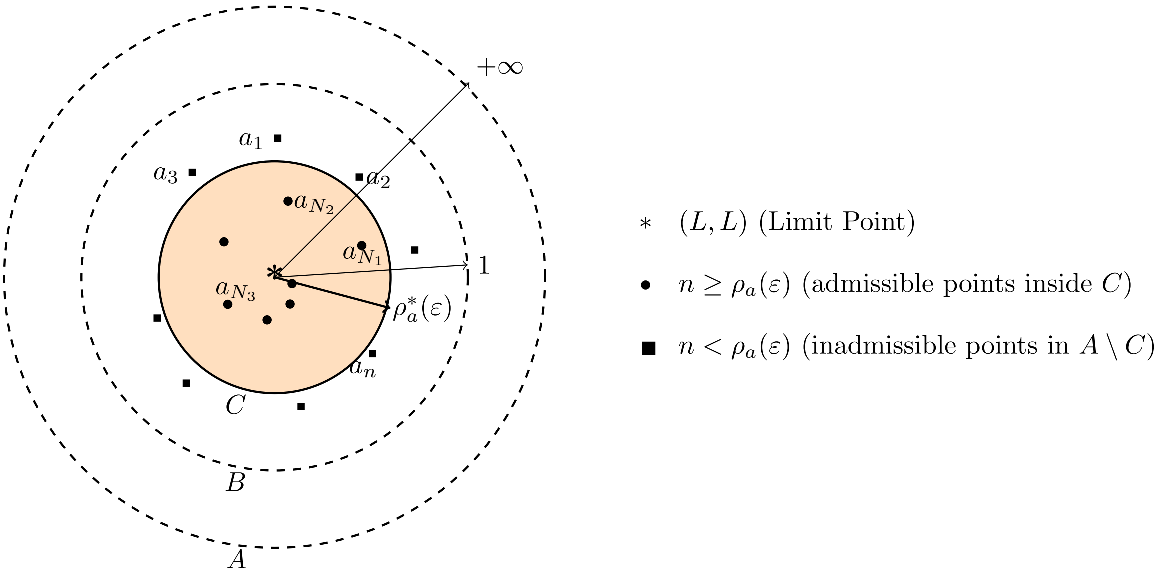

Figure 2. Geometric representation of the rescaled radius of convergence. The white reference disk A has centre $(L, L) : L = L(a)$ and radius $+\infty$ (dashed boundary). For a fixed $\varepsilon > 0$, the geometric radius $\rho_a^*(\varepsilon) = 1/(\rho_a(\varepsilon) - 1)$ determines the inner disk $C = \{x : \|x - (L, L)\| \leq \rho_a^*(\varepsilon)\}$ (orange region), with $0 < \rho_a^*(\varepsilon) \leq 1$ whenever $\rho_a(\varepsilon) \geq 2$. In this schematic we take $\rho_a^*(\varepsilon) = 2/3$ for clarity: the early terms a_1, a_2, a_3 and the closest outsider a_n (squares) lie in the annulus $A \setminus C$, while the tail terms $a_{N_1}, a_{N_2}, a_{N_3}$ and the remaining spiral points (dots) converge to the starred limit point (L, L) inside the orange disk.

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Proof. By Definition 5 the quantity $\rho_a(\varepsilon)$ is defined only when a converges in $\overline{\mathbb{R}}$ to a limit $L(a)$, and then

$$R(a,\varepsilon) = \begin{cases} \{N \in \mathbb{N} : \forall n \geq N,\ a_n < -\varepsilon\}, & L(a) = -\infty, \\ \{N \in \mathbb{N} : \forall n \geq N,\ |a_n - L(a)| < \varepsilon\}, & L(a) \in \mathbb{R}, \\ \{N \in \mathbb{N} : \forall n \geq N,\ a_n > \varepsilon\}, & L(a) = +\infty, \end{cases} \qquad \rho_a(\varepsilon) := \inf(R(a,\varepsilon)).$$

By Definition 2 and Theorem 2, the sequence a has an extended limit $L(a) \in \overline{\mathbb{R}}$ if and only if $L_1(a) = L_2(a)$, i.e. if and only if a lies in one of the convergent blocks E, F, G; on the divergent blocks A, B, C, D we have $L_1(a) < L_2(a)$, so no extended limit exists and $\rho_a(\varepsilon)$ is left undefined there.

If $a \in G$, then $L(a) = L \in \mathbb{R}$ and, for $0 < \varepsilon_1 < \varepsilon_2$,

$$R(a,\varepsilon_1) = \{N : \forall n \geq N,\ |a_n - L| < \varepsilon_1\} \subseteq \{N : \forall n \geq N,\ |a_n - L| < \varepsilon_2\} = R(a,\varepsilon_2),$$

whence $\rho_a(\varepsilon_1) = \inf(R(a,\varepsilon_1)) \geq \inf(R(a,\varepsilon_2)) = \rho_a(\varepsilon_2)$, so $\varepsilon \mapsto \rho_a(\varepsilon)$ is nonincreasing on G.

If $a \in E$ (so $L(a) = -\infty$), then for $0 < \varepsilon_1 < \varepsilon_2$ we have

$$R(a,\varepsilon_2) = \{N : \forall n \geq N,\ a_n < -\varepsilon_2\} \subseteq \{N : \forall n \geq N,\ a_n < -\varepsilon_1\} = R(a,\varepsilon_1),$$

so $\rho_a(\varepsilon_1) \leq \rho_a(\varepsilon_2)$, i.e. ρ_a is nondecreasing on E. The case $a \in F$ with $L(a) = +\infty$ is analogous, using the sets $\{n \geq N : a_n > \varepsilon\}$, and yields the same monotonicity conclusion. Collecting the block-wise information gives Table 1. □

Table 1. Block-wise behaviour of the two-sided radius of convergence $\rho_a(\varepsilon)$.

Block	Limit profile $(L_1(a), L_2(a))$	Radius profile of $\rho_a(\varepsilon)$
A	$L_1(a) = -\infty,\ -\infty < L_2(a) < +\infty$	not defined (no extended limit)
B	$-\infty < L_1(a) < L_2(a) < +\infty$	not defined (no extended limit)
C	$-\infty < L_1(a) < L_2(a) = +\infty$	not defined (no extended limit)
D	$L_1(a) = -\infty,\ L_2(a) = +\infty$	not defined (no extended limit)
E	$L_1(a) = L_2(a) = -\infty$	$\rho_a(\varepsilon) < \infty$ for all $\varepsilon > 0$; ρ_a nondecreasing in ε
F	$L_1(a) = L_2(a) = +\infty$	$\rho_a(\varepsilon) < \infty$ for all $\varepsilon > 0$; ρ_a nondecreasing in ε
G	$L_1(a) = L_2(a) = L \in \mathbb{R}$	$\rho_a(\varepsilon) < \infty$ for all $\varepsilon > 0$; ρ_a nonincreasing in ε

Definition 7 (Cauchy radius)**.** *Let $a = (a_n)_{n\in\mathbb{N}}$ be a real sequence and $\varepsilon > 0$. We define the* Cauchy radius *of a at level ε by:*

$$\rho_a^C(\varepsilon) := \inf(C(a,\varepsilon)) : \tag{18}$$

$$C(a,\varepsilon) := \Big\{N \in \mathbb{N} :\ \forall m,n \geq N,\ |a_n - a_m| < \varepsilon\Big\}. \tag{19}$$

Remark 5. *As the direct result of Definition 7 and Theorem 1:*

$$\rho_a^C(\varepsilon) = +\infty \iff L(a) = \pm\infty. \tag{20}$$

Theorem 5 (Two–sided radius via liminf / limsup radii)**.** *Let $a = (a_n)_{n\in\mathbb{N}} \in Seq(\mathbb{R})$ and suppose its extended limit exists, i.e. $L_1(a) = L_2(a) =: L(a) \in \overline{\mathbb{R}}$. For $\varepsilon > 0$ let $R(a,\varepsilon), T_1(a,\varepsilon), T_2(a,\varepsilon)$ be as in Definitions 4 and 5. Then:*

$$R(a,\varepsilon) = T_1(a,\varepsilon) \cap T_2(a,\varepsilon), \tag{21}$$

$$\rho_a(\varepsilon) = \max\big\{\rho_{L_1}(a;\varepsilon), \rho_{L_2}(a;\varepsilon)\big\}. \tag{22}$$

Proof. First, we proof the equality (21). We distinguish the cases $L(a) \in \mathbb{R}$, $L(a) = +\infty$, and $L(a) = -\infty$.

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(a) Finite limit $L \in \mathbb{R}$. If $N \in R(a,\varepsilon)$, then for every $n \geq N$ we have $L - \varepsilon < a_k < L + \varepsilon$ for all $k \geq n$, hence $L - \varepsilon \leq \ell_n \leq u_n \leq L + \varepsilon$, so $|\ell_n - L| < \varepsilon$ and $|u_n - L| < \varepsilon$. Thus $N \in T_1(a,\varepsilon) \cap T_2(a,\varepsilon)$. Conversely, if $N \in T_1(a,\varepsilon) \cap T_2(a,\varepsilon)$, then for each $n \geq N$, $L - \varepsilon < \ell_n \leq a_n \leq u_n < L + \varepsilon$, so $|a_n - L| < \varepsilon$ and $N \in R(a,\varepsilon)$. Accordingly, $R(a,\varepsilon) = T_1(a,\varepsilon) \cap T_2(a,\varepsilon)$.

(b) Infinite limit $L(a) = +\infty$. Here, for any n we have $\ell_n > \varepsilon$ iff $a_k > \varepsilon$ for all $k \geq n$. Hence $R(a,\varepsilon) = T_1(a,\varepsilon)$. Moreover, $\ell_n \leq u_n \quad (n \in \mathbb{N})$ implies $T_1(a,\varepsilon) \subseteq T_2(a,\varepsilon)$, so $R(a,\varepsilon) = T_1(a,\varepsilon) \cap T_2(a,\varepsilon)$.

(c) Infinite limit $L(a) = -\infty$. This case is analogous to (b) with all inequalities reversed; again one obtains $R(a,\varepsilon) = T_1(a,\varepsilon) \cap T_2(a,\varepsilon)$.

Second, we prove equality (21). In all three cases the sets $R(a,\varepsilon)$, $T_1(a,\varepsilon)$, $T_2(a,\varepsilon)$ are nonempty tails of $\mathbb{N}$. By Lemma 2:

$$\begin{aligned}\rho_a(\varepsilon) &= \inf(R(a,\varepsilon)) = \inf\big(T_1(a,\varepsilon) \cap T_2(a,\varepsilon)\big) \\ &= \max\big\{\inf(T_1(a,\varepsilon)), \inf(T_2(a,\varepsilon))\big\} = \max\{\rho_{L_1}(a;\varepsilon), \rho_{L_2}(a;\varepsilon)\}.\end{aligned}$$

This completes the proof. □

Theorem 6 (Comparison of two–sided and Cauchy radii). *Let $a = (a_n)_{n\in\mathbb{N}}$ be a convergent real sequence with limit $L(a) \in \mathbb{R}$, two–sided radius $\rho_a(.)$ and Cauchy radius $\rho_a^C(.)$. Then, for every $\varepsilon > 0$ and every $t \in (0,1)$ we have:*

$$\rho_a^C(\varepsilon) \quad \leq \quad \max\big\{\rho_a(t\varepsilon), \rho_a\big((1-t)\varepsilon\big)\big\}. \tag{23}$$

Proof. Fix $\varepsilon > 0$ and $t \in (0,1)$, and let $N \in R(a,t\varepsilon) \cap R(a,(1-t)\varepsilon)$. Then for every $k \geq N$ we have $|a_k - L(a)| < t\varepsilon$ and $|a_k - L(a)| < (1-t)\varepsilon$. If $m,n \geq N$, the triangle inequality gives $|a_n - a_m| \leq |a_n - L(a)| + |a_m - L(a)| < t\varepsilon + (1-t)\varepsilon = \varepsilon$, so $N \in C(a,\varepsilon)$. This proves the set inclusion:

$$R(a,t\varepsilon) \cap R\big(a,(1-t)\varepsilon\big) \subseteq C(a,\varepsilon). \tag{24}$$

Since $R(a,\delta)$ and $C(a,\varepsilon)$ are tail sets in $\mathbb{N}$, by an application of Lemma 2 on inequality (24) we have:

$$\rho_a^C(\varepsilon) = \inf(C(a,\varepsilon)) \leq \inf\big(R(a,t\varepsilon) \cap R(a,(1-t)\varepsilon)\big) = \max\big\{\rho_a(t\varepsilon), \rho_a((1-t)\varepsilon)\big\}.$$

This completes the proof. □

Corollary 1. *Under the assumptions of the Theorem 6, for every $\varepsilon > 0$ we have:*

$$\rho_a^C(\varepsilon) \quad \leq \quad \rho_a(\varepsilon/2). \tag{25}$$

Remark 6. *The inequality (23) in Theorem 6 (or inequality (25) in Corollary 1) can be strict. As an example, consider the sequence $a = (\frac{1}{n})_{n\in\mathbb{N}}$. An straightforward calculation shows that: $\rho_a^C(\varepsilon) = \rho_a(\varepsilon) = \lceil \frac{1}{\varepsilon} \rceil$, for all $\varepsilon > 0$. Take, $\epsilon = \frac{1}{2}$. Then, $\rho_a^C(\frac{1}{2}) = 3 < 5 = \rho_a(\frac{1}{4})$.*

3.3. Stability Under Algebraic Operations

We now collect the main structural properties of the radius of convergence, in terms of its assigned sequence a and other features as follows:

Theorem 7 (Stability of the radius of convergence). *Let $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ be convergent real sequences with finite limits $L(a)$ and $L(b)$, respectively. Denote by ρ_a, ρ_b their radii of convergence in the sense of Definition 5. Then the following assertions hold:*

(i) ***Monotonicity and characterization of convergence.*** *For every $0 < \varepsilon_1 < \varepsilon_2$,*

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$$\rho_a(\varepsilon_2) \le \rho_a(\varepsilon_1). \tag{26}$$

(ii) ***Tail invariance under finite modification.*** *If $c = (c_n)$ is a sequence with $c_n = a_n$ for all $n \ge N_0$ for some $N_0 \in \mathbb{N}$, then c converges to $L(a)$ and*

$$\rho_c(\varepsilon) = \max\{1,\ \rho_a(\varepsilon) - (N_0 - 1)\} \qquad (\varepsilon > 0). \tag{27}$$

(iii) ***Affine transformations.*** *Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$, and define $c = (c_n)_{n\in\mathbb{N}} : \quad c_n := \alpha a_n + \beta$. Then $c_n \to \alpha L(a) + \beta$ and*

$$\rho_c(\varepsilon) = \rho_a\left(\frac{\varepsilon}{|\alpha|}\right) \qquad (\varepsilon > 0). \tag{28}$$

(iv) ***Sums.*** *Define $s = (s_n)_{n\in\mathbb{N}} : \quad s_n := a_n + b_n$ and $L(s) := L(a) + L(b)$. Then, $s_n \to L(s)$ and*

$$\rho_s(\varepsilon) \le \max\left\{\rho_a\left(\frac{\varepsilon}{2}\right), \rho_b\left(\frac{\varepsilon}{2}\right)\right\} \qquad (\varepsilon > 0). \tag{29}$$

(v) ***Products.*** *Define $p = (p_n)_{n\in\mathbb{N}} : \quad p_n := a_n b_n$ (Hadamard Product) and $L(p) := L(a)L(b)$. For $\varepsilon > 0$ set*

$$r(\varepsilon) := \min\left\{1,\ \frac{\varepsilon}{1 + |L(a)| + |L(b)|}\right\}.$$

Then $p_n \to L(p)$ and

$$\rho_p(\varepsilon) \le \max\{\rho_a(r(\varepsilon)),\ \rho_b(r(\varepsilon))\} \qquad (\varepsilon > 0). \tag{30}$$

(vi) ***Quotients.*** *Let $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ be convergent real sequences with finite limits $L(a)$ and $L(b) \ne 0$. Define $q = (q_n)_{n\in\mathbb{N}} : \quad q_n := \frac{a_n}{b_n}$, and $L(q) := \frac{L(a)}{L(b)}$. Then $q_n \to L(q)$ and, for every $\varepsilon > 0$,*

$$\rho_q(\varepsilon) \le \max\left\{\rho_a\left(\frac{\varepsilon|L(b)|^2}{2(|L(a)| + |L(b)|)}\right), \rho_b\left(\frac{\varepsilon|L(b)|^2}{2(|L(a)| + |L(b)|)}\right)\right\}. \tag{31}$$

Proof. (i) If $0 < \varepsilon_1 < \varepsilon_2$, then $R(a, \varepsilon_1) \subseteq R(a, \varepsilon_2)$ by definition. Now, taking infimum from both sides it follows that $\rho_a(\varepsilon_2) \le \rho_a(\varepsilon_1)$.

(ii) If $c_n = a_n$ for all $n \ge N_0$ and $a_n \to L(a)$, then clearly $c_n \to L(a)$. Moreover, by Definition 5,

$$R(c, \varepsilon) = \{N \in \mathbb{N} : \forall n \ge N,\ |c_n - L(a)| < \varepsilon\} = \{N \in \mathbb{N} : \forall n \ge N,\ |a_{n+N_0-1} - L(a)| < \varepsilon\}.$$

Hence, $R(c, \varepsilon) = \{N \in \mathbb{N} : N + N_0 - 1 \in R(a, \varepsilon)\}$, and taking infima gives

$$\rho_c(\varepsilon) = \inf(R(c, \varepsilon)) = \inf\{N : N + N_0 - 1 \in R(a, \varepsilon)\} = \rho_a(\varepsilon) - (N_0 - 1).$$

Since $\rho_c(\varepsilon) \ge 1$, we conclude

$$\rho_c(\varepsilon) = \max\{1,\ \rho_a(\varepsilon) - (N_0 - 1)\}.$$

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(iii) For $\alpha \neq 0$ and $\beta \in \mathbb{R}$ we have

$$|c_n - (\alpha L(a) + \beta)| = |\alpha||a_n - L(a)|.$$

Thus $|c_n - (\alpha L(a) + \beta)| < \varepsilon$ if and only if $|a_n - L(a)| < \varepsilon/|\alpha|$, and the stated identity for ρ_c follows by taking infima over N.

(iv) Fix $\varepsilon > 0$ and put $\varepsilon_k := \varepsilon/2$ for $1 \leq k \leq 2$. If $n \geq N := \max\{\rho_1(\varepsilon_1), \rho_2(\varepsilon_2)\}$, then we have $|a_n - L(a)| < \varepsilon/2$, and $|b_n - L(b)| < \varepsilon/2$, respectively. Hence

$$\left|s_n - L(s)\right| \leq |a_n - L(a)| + |b_n - L(b)| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $N \in R(s, \varepsilon)$ and therefore $\rho_s(\varepsilon) \leq N$, as claimed.

(v) Fix $\varepsilon > 0$ and define $r(\varepsilon)$ as in the statement. Let $N := \max\{\rho_a(r(\varepsilon)), \rho_b(r(\varepsilon))\}$. Then for every $n \geq N$ we have

$$|a_n - L(a)| < r(\varepsilon) \leq 1, \qquad |b_n - L(b)| < r(\varepsilon) \leq 1.$$

In particular $|b_n| \leq |L(b)| + 1$ for all $n \geq N$, and we may write

$$a_n b_n - L(a)L(b) = (a_n - L(a))(b_n - L(b)) + L(a)(b_n - L(b)) + L(b)(a_n - L(a)).$$

Hence, for $n \geq N$,

$$\begin{aligned}
|a_n b_n - L(a)L(b)| &\leq |a_n - L(a)||b_n - L(b)| + |L(a)||b_n - L(b)| + |L(b)||a_n - L(a)| \\
&\leq r(\varepsilon)^2 + (|L(a)| + |L(b)|)r(\varepsilon) \\
&\leq (1 + |L(a)| + |L(b)|)\, r(\varepsilon) \\
&\leq \varepsilon,
\end{aligned}$$

using $r(\varepsilon) \leq 1$ in the third line and the definition of $r(\varepsilon)$ in the last line. Thus $N \in R(p, \varepsilon)$ and the claimed inequality for ρ_p follows.

(vi) Fix $\varepsilon > 0$. Since $L(b) \neq 0$, there exists $\eta > 0$ such that $|b_n - L(b)| < \eta$ implies $|b_n| > \frac{1}{2}|L(b)|$. For such n, we can write

$$\left|\frac{a_n}{b_n} - \frac{L(a)}{L(b)}\right| \leq \frac{|a_n - L(a)|}{|b_n|} + |L(a)|\left|\frac{1}{b_n} - \frac{1}{L(b)}\right| \leq \frac{|a_n - L(a)|}{\frac{1}{2}|L(b)|} + \frac{|L(a)|\,|b_n - L(b)|}{|L(b)|(\frac{1}{2}|L(b)|)}.$$

Hence a sufficient condition for $\left|\frac{a_n}{b_n} - \frac{L(a)}{L(b)}\right| < \varepsilon$ is that both:

$$|a_n - L(a)| < r, \quad |b_n - L(b)| < r, \quad r := \frac{\varepsilon|L(b)|^2}{2(|L(a)| + |L(b)|)}. \tag{32}$$

Let $N_a := \rho_a(r)$ and $N_b := \rho_b(r)$. By Definition 2.13, for all $n \geq N_a$ we have $|a_n - L(a)| < r$, and for all $n \geq N_b$ we have $|b_n - L(b)| < r$. Therefore, for all $n \geq \max\{N_a, N_b\}$, inequalities (32) hold and consequently $\left|\frac{a_n}{b_n} - \frac{L(a)}{L(b)}\right| < \varepsilon$. Hence $\max\{N_a, N_b\} \in R(q, \varepsilon)$, and taking infima gives

$$\rho_q(\varepsilon) \leq \max\{\rho_a(r),\, \rho_b(r)\},$$

which proves (31).

This completes the proof. □

Remark 7. *The inequalities (29)- (31) in Theorem 7 can be strict. As examples, it is sufficient to consider the following examples:*

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*For the **sum** case, take $a_n = 1/n$ and $b_n = -1/n$, so $s_n = a_n + b_n \equiv 0$ and hence $L(a) = L(b) = L(s) = 0$. Solving $1/n < \delta$ gives $\rho_a(\delta) = \rho_b(\delta) = \lceil 1/\delta \rceil$ whenever $1/\delta \notin \mathbb{N}$, so for any $0 < \varepsilon \le 1$ with $2/\varepsilon \notin \mathbb{N}$ we have $\rho_a(\varepsilon/2) = \rho_b(\varepsilon/2) = \lceil 2/\varepsilon \rceil \ge 3$. Because s_n is constant, $\rho_s(\varepsilon) = 1$ for every $\varepsilon > 0$. Thus $\rho_s(\varepsilon) = 1 < \lceil 2/\varepsilon \rceil = \max\{\rho_a(\varepsilon/2), \rho_b(\varepsilon/2)\}$ for such ε, so inequality (29) is strict.*

*For the **product** case, take $a_n = 1 + 1/n$ and $b_n = n/(n+1)$, so $p_n = a_n b_n \equiv 1$ and $L(a) = L(b) = L(p) = 1$. From $|a_n - 1| = 1/n < \delta$ we get $\rho_a(\delta) = \lceil 1/\delta \rceil$ when $1/\delta \notin \mathbb{N}$, and from $|b_n - 1| = 1/(n+1) < \delta$ we obtain $\rho_b(\delta) = \lceil 1/\delta \rceil - 1$ in the same generic case. For (26), with $0 < \varepsilon \le 1$ and $r(\varepsilon) = \varepsilon/3$ chosen so that $1/r(\varepsilon) \notin \mathbb{N}$, both radii satisfy $\rho_a(r(\varepsilon)) = \lceil 1/r(\varepsilon) \rceil$ and $\rho_b(r(\varepsilon)) = \lceil 1/r(\varepsilon) \rceil - 1$, hence $\max\{\rho_a(r(\varepsilon)), \rho_b(r(\varepsilon))\} \ge 3$. Since $p_n \equiv 1$ gives $\rho_p(\varepsilon) = 1$, we have $\rho_p(\varepsilon) = 1 < \max\{\rho_a(r(\varepsilon)), \rho_b(r(\varepsilon))\}$ for these ε, so (30) is strict.*

*For the **quotient** case, take $a_n = b_n = 1 + 1/n$, so $q_n = a_n/b_n \equiv 1$ and $L(a) = L(b) = L(q) = 1$. As in the previous example, solving $1/n < \delta$ yields $\rho_a(\delta) = \rho_b(\delta) = \lceil 1/\delta \rceil$ whenever $1/\delta \notin \mathbb{N}$. In (31) the inner radius is $\delta_* = \varepsilon/4$, so for $0 < \varepsilon \le 1$ with $4/\varepsilon \notin \mathbb{N}$ we get $\rho_a(\delta_*) = \rho_b(\delta_*) = \lceil 4/\varepsilon \rceil \ge 5$. Because $q_n \equiv 1$ gives $\rho_q(\varepsilon) = 1$, we obtain $\rho_q(\varepsilon) = 1 < \max\{\rho_a(\delta_*), \rho_b(\delta_*)\}$ for such ε, showing that inequality (31) is strict.*

The next theorem shows that, for convergent series viewed through the lens of their partial-sum sequences, the radius of convergence is essentially the asymptotic inverse of the tail (remainder) decay rate.

Theorem 8 (Inverse Tail Principle)**.** *Let $s = \sum_{n=1}^{\infty} a_n$ be a convergent series with finite sum $S \in \mathbb{R}$. Define the partial sums and remainders by $s_n = \sum_{k=1}^{n} a_k, \quad n \ge 1$, and $R_n = S - s_n = \sum_{k=n+1}^{\infty} a_k, \quad n \ge 1$. Assume there exists a strictly decreasing, continuous, invertible function $f : (0,\infty) \to (0,\infty)$ such that*

$$|R_n| \sim f(n) \qquad (n \to \infty). \tag{33}$$

Then, as $\varepsilon \to 0^+$,

(i) the two-sided radius of convergence of the partial-sum sequence $s = (s_n)$ satisfies

$$\rho_s(\varepsilon) \sim f^{-1}(\varepsilon), \tag{34}$$

(ii) the geometric radius of convergence satisfies

$$\rho_s^*(\varepsilon) \;\sim\; \frac{1}{f^{-1}(\varepsilon)}. \tag{35}$$

Proof. By the definition of the two-sided radius for a convergent sequence,

$$\begin{aligned} \rho_s(\varepsilon) &= \inf\Big\{ N \in \mathbb{N} : \ \forall n \ge N,\ |s_n - S| < \varepsilon \Big\} \\ &= \inf\Big\{ N \in \mathbb{N} : \ \forall n \ge N,\ |R_n| < \varepsilon \Big\}. \end{aligned} \tag{36}$$

From $|R_n| \sim f(n)$, for every $\eta \in (0,1)$ there exists $N_\eta \in \mathbb{N}$ such that for all $n \ge N_\eta$,

$$(1-\eta) f(n) \ \le\ |R_n| \ \le\ (1+\eta) f(n). \tag{37}$$

Fix $\eta \in (0,1)$ and $\varepsilon > 0$ small enough so that $f^{-1}(\varepsilon/(1+\eta)) \ge N_\eta$. Set

$$N_+(\varepsilon,\eta) \;=\; \left\lceil f^{-1}\!\left(\frac{\varepsilon}{1+\eta}\right) \right\rceil, \qquad N_-(\varepsilon,\eta) \;=\; \left\lfloor f^{-1}\!\left(\frac{\varepsilon}{1-\eta}\right) \right\rfloor. \tag{38}$$

If $n \ge N_+(\varepsilon,\eta)$, then $f(n) \le f(N_+(\varepsilon,\eta)) \le \varepsilon/(1+\eta)$, hence

$$|R_n| \ \le\ (1+\eta) f(n) \ \le\ (1+\eta)\frac{\varepsilon}{1+\eta} \ =\ \varepsilon, \tag{39}$$

so $N_+(\varepsilon, \eta)$ is admissible and therefore

$$\rho_s(\varepsilon) \;\leq\; N_+(\varepsilon, \eta). \tag{40}$$

On the other hand, if $n \leq N_-(\varepsilon, \eta)$, then $f(n) \geq f(N_-(\varepsilon, \eta)) \geq \varepsilon/(1-\eta)$, hence

$$|R_n| \;\geq\; (1-\eta) f(n) \;\geq\; (1-\eta)\frac{\varepsilon}{1-\eta} \;=\; \varepsilon, \tag{41}$$

so no $n \leq N_-(\varepsilon, \eta)$ can be admissible, which implies

$$\rho_s(\varepsilon) \;>\; N_-(\varepsilon, \eta). \tag{42}$$

Combining,

$$N_-(\varepsilon, \eta) \;<\; \rho_s(\varepsilon) \;\leq\; N_+(\varepsilon, \eta). \tag{43}$$

Since $f^{-1}(\varepsilon) \to \infty$ as $\varepsilon \to 0^+$, the floor/ceiling terms are negligible at the scale of $f^{-1}(\varepsilon)$, and letting $\varepsilon \to 0^+$ followed by $\eta \to 0^+$ yields

$$\frac{\rho_s(\varepsilon)}{f^{-1}(\varepsilon)} \;\to\; 1, \tag{44}$$

which proves (i).

For (ii), note that $\rho_s(\varepsilon) \to \infty$ as $\varepsilon \to 0^+$, and therefore

$$\begin{aligned} \rho_s^*(\varepsilon) \;&=\; \frac{1}{\rho_s(\varepsilon) - 1} \;=\; \frac{1}{\rho_s(\varepsilon)} \cdot \frac{1}{1 - \frac{1}{\rho_s(\varepsilon)}} \\ &\sim\; \frac{1}{\rho_s(\varepsilon)} \;\sim\; \frac{1}{f^{-1}(\varepsilon)} \qquad (\varepsilon \to 0^+), \end{aligned} \tag{45}$$

as claimed. □

Corollary 2 (Radius classes induced by tail decay)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with sum $S \in \mathbb{R}$, and define $s_n = \sum_{k=1}^{n} a_k$, and $R_n = S - s_n = \sum_{k=n+1}^{\infty} a_k$. Let $\rho_s(\varepsilon)$ and $\rho_s^*(\varepsilon)$ denote the two-sided and geometric radii of the partial-sum sequence $s = (s_n)$. Assume $\varepsilon \to 0^+$. Then the following three radius classes hold:*

Class 1: Polynomial (sublinear) convergence. Condition: *there exist constants $C > 0$ and $p > 1$ such that*

$$|R_n| \sim C\, n^{-(p-1)} \qquad (n \to \infty). \tag{46}$$

Inversion / Radius law:

$$\rho_s(\varepsilon) \;\sim\; \left(\frac{C}{\varepsilon}\right)^{\frac{1}{p-1}}, \tag{47}$$

$$\rho_s^*(\varepsilon) \;=\; \frac{1}{\rho_s(\varepsilon) - 1} \sim \frac{1}{\rho_s(\varepsilon)} \sim \left(\frac{\varepsilon}{C}\right)^{\frac{1}{p-1}}. \tag{48}$$

Interpretation: *these are "slow" series; to gain one more decimal digit of precision, the required number of terms increases by a power factor.*

Class 2: Geometric (linear) convergence. Condition: *there exist constants $C > 0$ and $r \in (0,1)$ such that*

$$|R_n| \sim C\, r^n \qquad (n \to \infty). \tag{49}$$

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Inversion / Radius law:

$$\rho_s(\varepsilon) \sim \frac{\ln(C/\varepsilon)}{\ln(1/r)} \sim \frac{1}{\ln(1/r)} \ln\left(\frac{1}{\varepsilon}\right), \tag{50}$$

$$\rho_s^*(\varepsilon) = \frac{1}{\rho_s(\varepsilon)-1} \sim \frac{1}{\rho_s(\varepsilon)} \sim \frac{\ln(1/r)}{\ln(1/\varepsilon)}. \tag{51}$$

Interpretation: *these are "fast" series; reducing ε by a fixed factor requires adding only a constant number of terms.*

Class 3: Superlinear (factorial-type) convergence. Condition: *the tail is factorially small, e.g.*

$$|R_n| \sim \frac{C}{(n+1)!} \qquad (n \to \infty). \tag{52}$$

Inversion / Radius law: *using asymptotic inversion (via Stirling),*

$$\rho_s(\varepsilon) \sim \frac{\ln(C/\varepsilon)}{\ln\big(\ln(C/\varepsilon)\big)} \sim \frac{\ln(1/\varepsilon)}{\ln\big(\ln(1/\varepsilon)\big)}, \tag{53}$$

$$\rho_s^*(\varepsilon) = \frac{1}{\rho_s(\varepsilon)-1} \sim \frac{1}{\rho_s(\varepsilon)} \sim \frac{\ln\big(\ln(1/\varepsilon)\big)}{\ln(1/\varepsilon)}. \tag{54}$$

Interpretation: *these are "ultra-fast" series; the radius grows very slowly, so the "cost" of increased precision is minimal.*

Proof. Each class is an application of the Inverse Tail Principle with a specific asymptotic tail model

$$|R_n| \sim f(n), \tag{55}$$

followed by explicit (asymptotic) inversion of f.

Class 1. Take $f(n) = Cn^{-(p-1)}$. Solving $Cn^{-(p-1)} = \varepsilon$ yields

$$n = \left(\frac{C}{\varepsilon}\right)^{\frac{1}{p-1}} = f^{-1}(\varepsilon), \tag{56}$$

so $\rho_s(\varepsilon) \sim f^{-1}(\varepsilon)$, and again

$$\rho_s^*(\varepsilon) \sim \frac{1}{\rho_s(\varepsilon)}. \tag{57}$$

Class 2. Take $f(n) = Cr^n$. Solving $Cr^n = \varepsilon$ gives

$$n = \frac{\ln(C/\varepsilon)}{\ln(1/r)} = f^{-1}(\varepsilon), \tag{58}$$

hence $\rho_s(\varepsilon) \sim f^{-1}(\varepsilon)$. Moreover,

$$\rho_s^*(\varepsilon) = \frac{1}{\rho_s(\varepsilon)-1} \sim \frac{1}{\rho_s(\varepsilon)}. \tag{59}$$

Class 3. Take $f(n) = C/(n+1)!$. Using Stirling

$$(n+1)! \sim \sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1}, \tag{60}$$

the relation $C/(n+1)! \asymp \varepsilon$ is equivalent to

$$\ln\Big(\frac{C}{\varepsilon}\Big) \ \sim\ (n+1)\ln(n+1), \tag{61}$$

whose asymptotic solution satisfies

$$n \ \sim\ \frac{\ln(C/\varepsilon)}{\ln\big(\ln(C/\varepsilon)\big)} \ =\ f^{-1}(\varepsilon). \tag{62}$$

Thus $\rho_s(\varepsilon) \sim f^{-1}(\varepsilon)$, and

$$\rho_s^*(\varepsilon) = \frac{1}{\rho_s(\varepsilon)-1} \sim \frac{1}{\rho_s(\varepsilon)}. \tag{63}$$

□

4. Examples and Explicit Computations

We present the calculation of radius of convergence for several key classical sequences as follows:

4.1. Radii of Convergence for Classical Convergent Sequences

Example 1 ($n^{1/n} \to 1$)**.** *Let $a = (a_n)_{n\in\mathbb{N}}$ be given by*

$$a_n := n^{1/n} = \exp\Big(\frac{\log n}{n}\Big), \qquad n \in \mathbb{N}.$$

Then $L(a) = 1$. For $n \geq 3$ we have $0 < x_n := \frac{\log n}{n} \leq 1$, and the elementary estimate $0 < e^x - 1 \leq 2x$ for $0 < x \leq 1$ yields

$$0 < a_n - 1 \ = \ e^{x_n} - 1 \leq 2x_n = 2\,\frac{\log n}{n}, \qquad n \geq 3. \tag{64}$$

The function $f(x) := 2(\log x)/x$ is strictly decreasing on $[3,\infty)$ with $\lim_{x\to\infty} f(x) = 0$. Hence for every sufficiently small $\varepsilon > 0$ there exists a unique real number $N_\varepsilon \geq 3$ such that $f(N_\varepsilon) = \varepsilon$, and for all integers $n \geq N_\varepsilon$ we have $|a_n - 1| \leq \varepsilon$. In particular, $R(a,\varepsilon) \supseteq \{N \in \mathbb{N} : N \geq N_\varepsilon\}$, implying:

$$\rho_a(\varepsilon) \ \leq\ \Big\lceil \frac{4}{\epsilon}\log\frac{4}{\epsilon}\Big\rceil, \qquad (0 < \varepsilon < \frac{1}{2}). \tag{65}$$

Using the expansion $e^x - 1 \sim x$ as $x \to 0$, we have

$$a_n - 1 = e^{(\log n)/n} - 1 \sim \frac{\log n}{n},$$

so solving $(\log n)/n \approx \epsilon$ yields the well-known asymptotic profile

$$N_\varepsilon \ \sim\ \frac{1}{\varepsilon}\log\frac{1}{\varepsilon}, \qquad (\varepsilon \downarrow 0), \tag{66}$$

and therefore

$$\rho_a(\varepsilon) \ \sim\ \frac{1}{\varepsilon}\log\frac{1}{\varepsilon}, \qquad (\varepsilon \downarrow 0). \tag{67}$$

Example 2 ($(1+\frac{1}{n})^n \to e$)**.** *Let $a = (a_n)_{n\in\mathbb{N}}$ be given by*

$$a_n := \Big(1+\frac{1}{n}\Big)^n, \qquad n \in \mathbb{N}.$$

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It is classical that $a_n \nearrow e$, *hence* $L(a) = e$. *For* $x > 0$ *we have*

$$x - \frac{x^2}{2} \quad \leq \quad \log(1+x) \leq x. \tag{68}$$

With $x = 1/n$ *this yields*

$$1 - \frac{1}{2n} \quad \leq \quad n\log\Big(1 + \frac{1}{n}\Big) \quad \leq \quad 1. \tag{69}$$

Exponentiating and using $e^{1-\frac{1}{2n}} = e\, e^{-\frac{1}{2n}}$ *we obtain*

$$\begin{aligned} 0 < e - a_n \quad &= \quad e - \exp\Big(n\log\Big(1 + \frac{1}{n}\Big)\Big) \\ &\leq \quad e - e^{1-\frac{1}{2n}} = e(1 - e^{-\frac{1}{2n}}) \leq \frac{e}{2n}, \end{aligned} \tag{70}$$

where we used $1 - e^{-t} \leq t$ *for* $t \geq 0$ *in the last step. Hence for every* $\varepsilon > 0$ *and every integer* $N \geq e/(2\varepsilon)$ *we have* $|a_n - e| \leq e/(2n) \leq \varepsilon$ *for all* $n \geq N$. *In the notation of Definition 5, we have* $R(a,\varepsilon) \supseteq \Big\{N \in \mathbb{N} : N \geq \frac{e}{2\varepsilon}\Big\}$, *implying:*

$$\rho_a(\varepsilon) \quad \leq \quad \Big\lceil \frac{e}{2\varepsilon} \Big\rceil, \qquad (\varepsilon > 0). \tag{71}$$

Moreover, from the classical expansion $e - a_n = \frac{e}{2n} + O(n^{-2})$ *we infer the asymptotic profile*

$$\rho_a(\varepsilon) \quad \sim \quad \frac{e}{2\varepsilon}, \qquad (\varepsilon \downarrow 0). \tag{72}$$

Example 3 (Fibonacci ratios $F_{n+1}/F_n \to \varphi$)**.** *Let* $(F_n)_{n\geq 0}$ *be the Fibonacci numbers and consider*

$$a_n := \frac{F_{n+1}}{F_n}, \qquad n \geq 1.$$

Using Binet's formula [12]:

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}, \qquad \varphi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2},$$

we obtain

$$a_n = \frac{\varphi^{n+1} - \psi^{n+1}}{\varphi^n - \psi^n} = \varphi\,\frac{1 - r^{n+1}}{1 - r^n}, \qquad r := \frac{\psi}{\varphi}.$$

Since $|r| = \varphi^{-2} < 1$, *the ratios converge to* $\varphi = L(a)$ *and*

$$a_n - \varphi \quad = \quad \varphi\,\frac{(1 - r^{n+1}) - (1 - r^n)}{1 - r^n} = \varphi\,\frac{r^n(1-r)}{1 - r^n}. \tag{73}$$

Taking absolute values and using $|1 - r^n| \geq 1 - |r|^n \geq 1 - |r|$ *gives*

$$|a_n - \varphi| \quad \leq \quad \frac{\varphi|1-r|}{1-|r|}\,|r|^n =: C\,|r|^n, \qquad C = \frac{\varphi(3-\varphi)}{\varphi - 1} = \varphi + 2. \tag{74}$$

Thus, for any $\varepsilon > 0$, *every integer*

$$N \geq \frac{\log(C/\varepsilon)}{-\log|r|}$$

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satisfies $|a_n - \varphi| \le \varepsilon$ *for all* $n \ge N$*. Consequently*

$$\rho_a(\varepsilon) \quad \le \quad \left\lceil \frac{\log((\varphi+2)/\varepsilon)}{-\log(2-\varphi)} \right\rceil, \qquad \varepsilon > 0, \tag{75}$$

because $|r| = 2 - \varphi$*. Since* (73) *shows* $|a_n - \varphi| \sim \varphi(1-r)|r|^n$*, we obtain the logarithmic asymptotic*

$$\rho_a(\varepsilon) \quad \sim \quad \frac{\log(1/\varepsilon)}{-\log(2-\varphi)}, \qquad (\varepsilon \downarrow 0). \tag{76}$$

Example 4 (Leibniz partial sums for $\pi/4$)**.** *Define the sequence of partial sums of the Leibniz series by*

$$a_n := \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1}, \qquad n \in \mathbb{N}.$$

Then $a_n \to L(a) = \pi/4$ *as* $n \to \infty$*. Since this is an alternating series with monotonically decreasing terms* $b_k := 1/(2k+1) \downarrow 0$*, the alternating series test yields the sharp remainder bound*

$$\left|a_n - \tfrac{\pi}{4}\right| \quad \le \quad b_n = \frac{1}{2n+1}, \qquad n \in \mathbb{N}, \tag{77}$$

and the right-hand side is strictly decreasing in n*. Consequently, for* $\varepsilon > 0$ *we have*

$$\begin{aligned} R(a,\varepsilon) &= \left\{N \in \mathbb{N} : \forall n \ge N,\ \tfrac{1}{2n+1} \le \varepsilon\right\} \\ &= \left\{N \in \mathbb{N} : \tfrac{1}{2N+1} \le \varepsilon\right\}. \end{aligned} \tag{78}$$

Solving the inequality $1/(2N+1) \le \varepsilon$ *gives* $2N+1 \ge 1/\varepsilon$*, that is,*

$$N \quad \ge \quad \frac{1}{2\varepsilon} - \frac{1}{2}. \tag{79}$$

Therefore the radius of convergence of (a_n) *at level* ε *is*

$$\rho_a(\varepsilon) \quad = \quad \max\left\{1,\ \left\lceil \frac{1}{2\varepsilon} - \frac{1}{2} \right\rceil \right\}, \qquad \varepsilon > 0, \tag{80}$$

and in particular

$$\rho_a(\varepsilon) \quad \sim \quad \frac{1}{2\varepsilon}, \qquad (\varepsilon \downarrow 0). \tag{81}$$

4.2. Radii of Convergence for Classical $+\infty$ Divergent Sequences

Example 5 (Geometric progression 2^n)**.** *Let* $a = (a_n)_{n \in \mathbb{N}}$ *be defined by*

$$a_n := 2^n, \qquad n \in \mathbb{N}.$$

Then $a_n \nearrow +\infty$*, so* $L(a) = +\infty$ *in Definition* 5*. Since* (a_n) *is strictly increasing, for any* $\varepsilon > 0$ *the condition*

$$\forall n \ge N,\ a_n > \varepsilon$$

is equivalent to $a_N > \varepsilon$*. Hence*

$$R(a,\varepsilon) \quad = \quad \{N \in \mathbb{N} : 2^N > \varepsilon\}, \qquad \varepsilon > 0. \tag{82}$$

Solving $2^N > \varepsilon$ *for* N *gives* $N > \log_2 \varepsilon$*, so the smallest admissible index is*

$$\rho_a(\varepsilon) \quad = \quad \inf(R(a,\varepsilon)) = \max\left\{1,\ \lceil \log_2 \varepsilon \rceil \right\}, \qquad \varepsilon > 0. \tag{83}$$

In particular,

$$\rho_a(\varepsilon) \quad \sim \quad \log_2 \varepsilon = \frac{\log \varepsilon}{\log 2}, \qquad (\varepsilon \to +\infty), \tag{84}$$

where "$\sim$" denotes asymptotic equivalence, not an algebraic equality.

Example 6 (The Fibonacci sequence F_n)**.** *Let $(F_n)_{n\geq 0}$ be the Fibonacci numbers with $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, and define*

$$a_n := F_n, \qquad n \in \mathbb{N}.$$

Then $a_n \nearrow +\infty$, so $L(a) = +\infty$. By Binet's formula,

$$F_n \quad = \quad \frac{\varphi^n - \psi^n}{\sqrt{5}}, \qquad \varphi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}, \tag{85}$$

and since $|\psi| < 1$ we have for all $n \geq 1$

$$\frac{\varphi^n - 1}{\sqrt{5}} \quad \leq \quad F_n \leq \frac{\varphi^n + 1}{\sqrt{5}}. \tag{86}$$

Because (F_n) is strictly increasing, the set $R(a, \varepsilon)$ is

$$R(a,\varepsilon) \quad = \quad \{N \in \mathbb{N} : F_N > \varepsilon\}, \qquad \varepsilon > 0. \tag{87}$$

For $\varepsilon \geq 1$ the lower bound in (86) implies that every N with

$$\frac{\varphi^N - 1}{\sqrt{5}} > \varepsilon$$

is admissible. Equivalently,

$$\varphi^N \quad > \quad \sqrt{5}\,\varepsilon + 1 \quad \Longrightarrow \quad N \;\geq\; \frac{\log(\sqrt{5}\,\varepsilon + 1)}{\log \varphi}, \tag{88}$$

and therefore

$$\rho_a(\varepsilon) \quad \leq \quad \max\left\{1, \left\lceil \frac{\log(\sqrt{5}\,\varepsilon + 1)}{\log \varphi} \right\rceil\right\}, \qquad \varepsilon > 0. \tag{89}$$

Conversely, from the upper bound in (86) we obtain, for $\varepsilon > 1/\sqrt{5}$,

$$F_N > \varepsilon \quad \Longrightarrow \quad \frac{\varphi^N + 1}{\sqrt{5}} > \varepsilon \Longrightarrow \varphi^N > \sqrt{5}\,\varepsilon - 1, \tag{90}$$

whence

$$N \quad \geq \quad \frac{\log(\sqrt{5}\,\varepsilon - 1)}{\log \varphi}. \tag{91}$$

Thus for sufficiently large ε,

$$\left\lceil \frac{\log(\sqrt{5}\,\varepsilon - 1)}{\log \varphi} \right\rceil \quad \leq \quad \rho_a(\varepsilon) \leq \left\lceil \frac{\log(\sqrt{5}\,\varepsilon + 1)}{\log \varphi} \right\rceil. \tag{92}$$

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Both bounds in (92) are asymptotic to $(\log \varepsilon)/\log \varphi$*, so*

$$\rho_a(\varepsilon) \quad \sim \quad \frac{\log \varepsilon}{\log \varphi}, \qquad (\varepsilon \to +\infty), \tag{93}$$

again in the asymptotic sense only.

Example 7 (The prime numbers p_n)**.** *Let* p_n *denote the* n*-th prime and set*

$$a_n := p_n, \qquad n \in \mathbb{N}.$$

Then $a_n \nearrow +\infty$*, so* $L(a) = +\infty$ *and*

$$R(a,\varepsilon) \quad = \quad \{N \in \mathbb{N} : \ p_N > \varepsilon\}, \qquad \varepsilon > 0. \tag{94}$$

Hence $\rho_a(\varepsilon)$ *equals the number of primes not exceeding* ε*, plus one. Let* $\pi(x) := \#\{p \le x : \ p \text{ prime}\}$ *be the prime counting function. Then*

$$\rho_a(\varepsilon) \quad = \quad \pi(\varepsilon) + 1. \tag{95}$$

By the prime number theorem (PNT) [13],

$$\pi(x) \quad \sim \quad \frac{x}{\log x}, \qquad (x \to \infty), \tag{96}$$

and (96) is equivalent *to the well–known asymptotic* $p_n \sim n \log n$ *for the* n*-th prime. Substituting* $x = \varepsilon$ *into (96) and using (95) yields*

$$\rho_a(\varepsilon) \quad \sim \quad \frac{\varepsilon}{\log \varepsilon}, \qquad (\varepsilon \to +\infty). \tag{97}$$

Note that (97) is a consequence *of the PNT and its standard asymptotic inversion, not an exact algebraic formula.*

Example 8 (Factorials $n!$)**.** *Let* $a = (a_n)_{n \in \mathbb{N}}$ *be given by*

$$a_n := n!, \qquad n \in \mathbb{N}.$$

Then $a_n \nearrow +\infty$*, so* $L(a) = +\infty$ *and, as in the previous examples,*

$$R(a,\varepsilon) \quad = \quad \{N \in \mathbb{N} : \ N! > \varepsilon\}, \qquad \varepsilon > 0. \tag{98}$$

Thus $\rho_a(\varepsilon)$ *is the smallest* N *with* $N! > \varepsilon$*. To understand its growth, we use Stirling's formula*

$$n! \quad \sim \quad \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad (n \to \infty), \tag{99}$$

which implies

$$\log(n!) \quad = \quad n \log n - n + O(\log n). \tag{100}$$

Set $n := \rho_a(\varepsilon)$ *and write heuristically* $n! \approx \varepsilon$*. Taking logarithms and using (100), we obtain*

$$\log \varepsilon \quad \sim \quad \rho_a(\varepsilon)\, \log \rho_a(\varepsilon), \qquad (\varepsilon \to +\infty). \tag{101}$$

Equation (101) *is an* asymptotic *relation of the form* $y \sim x \log x$*, and its inversion must therefore be understood in the asymptotic sense of Lemma* 3*, not as an exact algebraic division. Applying Lemma* 3 *with* $y := \log \varepsilon$ *and* $x := \rho_a(\varepsilon)$ *yields*

$$\rho_a(\varepsilon) \sim \frac{\log \varepsilon}{\log \log \varepsilon}, \qquad (\varepsilon \to +\infty). \tag{102}$$

Thus the radius function for the factorial sequence has the standard inverse–$x \log x$ *growth: it is "almost logarithmic" in* ε*, with a* $\log \log \varepsilon$ *correction in the denominator. The exact inverse of* $x \mapsto x \log x$ *can be written using the Lambert W–function, but only the leading asymptotics are needed here.*

Table 2 presents the summary of radius of convergence of above sequences for the asymptotic cases:

Table 2. Convergence and geometric radii for the classical sequences of Section 4(all relations are asymptotic).

#	Name of Sequence	Convergence radius $\rho_a(\varepsilon)$	Geometric radius $\rho_a^*(\varepsilon)$
1	$n^{1/n} \to 1$	$\sim \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \quad (\varepsilon \downarrow 0)$	$\sim \frac{\varepsilon}{\log(1/\varepsilon)} \quad (\varepsilon \downarrow 0)$
2	$(1+\frac{1}{n})^n \to e$	$\sim \frac{e}{2\varepsilon} \quad (\varepsilon \downarrow 0)$	$\sim \frac{2\varepsilon}{e} \quad (\varepsilon \downarrow 0)$
3	Fibonacci ratios $F_{n+1}/F_n \to \varphi$	$\sim \frac{\log(1/\varepsilon)}{-\log(2-\varphi)} \quad (\varepsilon \downarrow 0)$	$\sim \frac{-\log(2-\varphi)}{\log(1/\varepsilon)} \quad (\varepsilon \downarrow 0)$
4	Leibniz partial sums for $\pi/4$	$\sim \frac{1}{2\varepsilon} \quad (\varepsilon \downarrow 0)$	$\sim 2\varepsilon \quad (\varepsilon \downarrow 0)$
5	Geometric progression 2^n	$\sim \frac{\log \varepsilon}{\log 2} \quad (\varepsilon \to +\infty)$	$\sim \frac{\log 2}{\log \varepsilon} \quad (\varepsilon \to +\infty)$
6	Fibonacci numbers F_n	$\sim \frac{\log \varepsilon}{\log \varphi} \quad (\varepsilon \to +\infty)$	$\sim \frac{\log \varphi}{\log \varepsilon} \quad (\varepsilon \to +\infty)$
7	Prime numbers p_n	$\sim \frac{\varepsilon}{\log \varepsilon} \quad (\varepsilon \to +\infty)$	$\sim \frac{\log \varepsilon}{\varepsilon} \quad (\varepsilon \to +\infty)$
8	Factorials $n!$	$\sim \frac{\log \varepsilon}{\log \log \varepsilon} \quad (\varepsilon \to +\infty)$	$\sim \frac{\log \log \varepsilon}{\log \varepsilon} \quad (\varepsilon \to +\infty)$

5. Discussion

5.1. Summary of the Radius-of-Convergence Viewpoint in Seq(R)

In this paper we proposed a radius–of–convergence viewpoint for real sequences $a = (a_n) \in Seq(R)$ that complements the classical limit–based description. Starting from the liminf/limsup profile $(L_1(a), L_2(a))$, we introduced the one–sided liminf and limsup radii $\rho_{L_1}(a;\varepsilon)$ and $\rho_{L_2}(a;\varepsilon)$, the two–sided radius of convergence $\rho_a(\varepsilon)$, the rescaled geometric radius ρ_a^*, and the Cauchy radius ρ_a^C. These constructions provide quantitative thresholds for entering an ε–tube either around the limit interval $[L_1(a), L_2(a)]$ or around the circular area in $\mathbb{R}^2$, and they remain meaningful for finite and infinite limits alike. We established basic structural properties (monotonicity in ε, block–wise behavior across the seven–block partition, and stability under algebraic operations such as sums, scalar multiples, and products) and illustrated them on eight representative examples from convergent and $+\infty$–divergent clusters. Altogether, the radii offer a unified language to compare how fast different sequences converge, diverge, or oscillate inside the global space $Seq(R)$.

5.2. Relation to Classical Cauchy Convergence and lim inf / lim sup *Theory*

The new radii are tightly linked to standard tools such as Cauchy convergence and the lim inf / lim sup framework. For sequences with $L_1(a) = L_2(a) = L(a)$, the one–sided radii coincide and the two–sided radius $\rho_a(\varepsilon)$ is comparable to the Cauchy radius $\rho_a^C(\varepsilon)$, so that the finiteness of these radii for every $\varepsilon > 0$ recovers the usual Cauchy criterion. For general sequences, the liminf and limsup radii encode how quickly the tails approach the lower and upper envelope of the limit set, and our comparison results show how $\rho_a(\cdot)$ is controlled by $\rho_{L_1}(a;\cdot)$ and $\rho_{L_2}(a;\cdot)$, with explicit examples where the corresponding inequalities are sharp or strict. In this way, the radius–of–convergence viewpoint refines the qualitative information carried by lim inf and lim sup into a quantitative scale that still respects the classical Cauchy/limit dichotomy.

5.3. Future Work

The present study suggests several directions for further investigation. First, it would be natural to develop radii for transformed sequences and associated series, for example under linear filters, Cesàro means, discrete differentiation, or when passing from (a_n) to the partial sums $(\sum_{k=1}^{n} a_k)$, and to compare these radii with classical notions of convergence acceleration. Second, the interaction between the radii and the seven–block classification, together with the underlying graph structures on blocks, deserves a more systematic analysis; this includes tracking how radii behave along edges of the block graph and identifying "radius–preserving" or "radius–contracting" transitions between blocks. Third, it would be interesting to extend the framework beyond real sequences, for instance to vector–valued sequences in normed spaces and to random sequences, where one could study Cauchy and convergence radii in almost sure, in–probability, or L^p senses. We hope that these extensions will further clarify how radius–based descriptors fit into the broader landscape of convergence theory.

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