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Moments of Real, Respectively of Complex Valued Functions, with Applications

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Abstract

The first aim of this study is to point out new aspects of approximation theory applied to a few classes of holomorphic functions, via Vitali's theorem. The approximation is made with the aid of the complex moments of the involved functions, that are defined similarly to the moments of a real valued continuous function. Applying uniform approximation of continuous functions on compact intervals via Korovkin's theorem, the hard part concerning uniform approximation on compact subsets of the complex plane follows according to Vitali's theorem. The theorem on the set of zeros of a holomorphic function is also applied. In the end, existence and uniqueness of solution for a multidimensional moment problem is characterized in terms of limits of sums of quadratic expressions. This is an application appearing at the end of the title. Consequences resulting from the first part of the paper are pointed out with the aid of functional calculus for self-adjoint operators.

Keywords: continuous function; holomorphic function; removable singularity; uniform approximation; nonnegative polynomials; compact subsets; moment problem; existence of a solution; uniqueness; functional calculus

MSC: 30E10; 41A30; 46A22

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1. Introduction

The purpose of this work is to prove new results and to complete the proof of a recently published result from [40]. Approximation of positive functions by positive polynomials is very important in both existence and uniqueness of the solution for the full moment problem. General knowledge in Analysis and Functional Analysis applied in this paper is contained in the references [1–10]. Main results on the Lambert W function have been published in [11]. The study [12] continues the results first published in [9]. The monograph [13] covers a large spectrum of functional analysis, including that of ordered and topological ordered vector spaces. Important cases are those of ordered Banach spaces and Banach lattices. The article [14] refers to the Hamburger and Stieltjes moment problems in several dimensions. For classical results in approximation of continuous functions of one real variable see Korovkin's theorem [15]. The monograph [16] provides basic information in Analysis, starting from results for functions of one real variable, but also considering notions and theorems working in a much more general setting. The article [17] contains interesting, applied approximation type results. In [18], we find Vitali's theorem, which is directly applied in the present paper, to approximate complex analytic functions of one complex variable. The basis of functional calculus is resumed in [19]. Main earlier and new information on the Moment Problem are pointed out in [20]. Basic results bon convex functions and their applications are recalled and completed in [21]. The article [22] refers to the Lambert W function, deep approximation results in complex analysis. In the monograph [23], the main goal consists in uniform approximation of continuous real valued functions. Article [24] provides a nonlinear extension of Korovkin's theorem. In [25], among other results, the isotonicity (increasing monotonicity) of a convex operator is characterized and discussed. Inequalities applied to approximation of functions and operators have been proved in [26] and in [27]. In the articles [28] and [29], optimization applied in Computer Science is the main purpose. The articles [30] and [31] are devoted to the Moment Problem, namely to the determinacy of the representing measure. Recent approximation methods involving positive linear operators have been applied in [32] and [33]. Refinements of important inequalities have been pointed out in [34]. In [35], stability results are discussed in a general modern framework. The articles [36–41] are directly or entirely motivated by the moment problem. Namely, according to [37] and [39], concerning the multidimensional moment problem, any nonnegative functions from $L_\mu^1(\mathbb{R}^d), d \geq 2$, where μ is the product of determinate measures on $\mathbb{R}$, can be approximated in $L_\mu^1(\mathbb{R}^d)$ by sums of products of polynomials $p_1(t_1) \cdots p_d(t_d), p_i \in \mathbb{R}[t_i], p_i(t_i) \geq 0, \forall t_i \in \mathbb{R}$. Since each p_i is a sum of squares, we infer that any nonnegative element from $L_\mu^1(\mathbb{R}^d)$ is the limit of polynomials which are sums of squares of the form $\prod_{i=1}^d q_i^2\,(t_i), q_i \in \mathbb{R}[t_i]$. Following the same idea, in [38], the author proved that a nonnegative continuous real function on the Cartesian product $S = S_1 \times \cdots \times S_d \subset \mathbb{R}_+^d$, $d \geq 1$, where $S_i \subset [0, +\infty), i = 1, \ldots, d$, are compact subsets, can be uniformly approximated on S by nonnegative polynomials of the form $p_1(t_1) \cdots p_d(t_d), p_i \in \mathbb{R}[t_i], p_i(t_i) \geq 0, \forall t_i \in [0, +\infty)$. This result is applied in the last theorem of the present work as well. Since each p_i is a sum of special polynomials having the explicit form $t_i^{s_i} q_i^2(t_i), q_i \in \mathbb{R}[t_i]$ and $s_i \in \{0,1\}$, we infer that any nonnegative continuous function on S is a limit in $C(S)$ of sums of polynomials of the form $\prod_{i=1}^d t_i^{s_i} q_i^2\,(t_i), q_i \in \mathbb{R}[t_i]$. Thus, if $T\colon C(S) \longrightarrow Y$ is a linear bounded operator with codomain an ordered Banach space Y, such that $T\left(\prod_{i=1}^d t_i^{s_i} q_i^2\,(t_i)\right) \in Y_+$ for all $q_i \in \mathbb{R}[t_i]$, $s_i \in \{0,1\}$, we have that $T(f) \in Y_+$ for any $f \in C(S)_+$. In other words, any such continuous linear operator on $C(S)$ is positive. By using the above discussion on approximation of elements from $\left(L_\mu^1(\mathbb{R}^d)\right)_+$ with sums of special squares of polynomials $\prod_{i=1}^d q_i^2\,(t_i)$, we infer that any linear continuous operator T mapping each such square $\prod_{i=1}^d q_i^2\,(t_i)$ into the positive cone Y_+ is positive on the positive cone $\left(L_\mu^1(\mathbb{R}^d)\right)_+$. The converse works in a more general setting: any linear positive operator acting between two ordered Banach spaces X, Y is continuous (see [25]). The reason for recalling these results is that of applying them to prove the existence and mainly the uniqueness of the solution to the full-moment problem. With the notations from above, this problem consists in characterizing the existence and uniqueness of a linear positive operator solution T mapping $X = L_\mu^1(\mathbb{R}^d)$ or $X = C(S)$ into an order complete Banach space Y, which verifies the conditions

$$T(p_j) = y_j, \qquad j \in \mathbb{N}^d. \tag{1}$$

Here $p_j(t) = t_1^{j_1} \cdots t_d^{j_d}, t = (t_1, \ldots, t_d) \in \mathbb{R}^d, j = (j_1, \ldots, j_d) \in \mathbb{N}^d, \mathbb{N} := \{0,1,2, \ldots\}, y_j \in Y, \forall j \in \mathbb{N}^d$. The elements $y_j \in Y$ are called the moments of the linear positive operator T and conditions (1) are named the moment conditions. Such a moment problem is called a full moment problem, since all the values $T(p_j)$ of the operator T at basic polynomials p_j are prescribed, as being equal to y_j, $j \in \mathbb{N}^d$. If $d \geq 2$, the moment problem is called multidimensional. When $d = 1$, we have a one-dimensional moment problem. If the polynomials are dense in the function space X, and Y is an ordered Banach space, then the linear positive solution of the full moment problem (1) is also continuous, hence it is unique. Another method is to define the linear operator firstly on subspace of polynomials, such that $T(p) \geq 0$ for all polynomials p taking nonnegative values at each point of the positive cone X_+ of the domain space. If the subspace $\mathcal{P}$ of polynomials is a majorizing subspace of the domain space X, (that is: $\forall x \in X, \exists p \in \mathcal{P}, x \leq p$), then any positive linear operator from $\mathcal{P}$ into an order complete Banach space Y has a linear positive extension mapping X into Y. This result holds true for any majorizing subspace in the order vector space X, not only for the subspace of polynomials (see [2], or/and [13]). However, in applications the subspace $\mathcal{P}$ of polynomials is the most important and often used. The convex generated by all squares of polynomials is also important, since these sums of squares are dense in the usual function spaces $\left(L_{\mu}^{1}(F)\right)_{+}$, μ being a moment determinate measure on a closed unbounded subset $F \subseteq \mathbb{R}^d$ (see [41]). Namely, for such a measure μ on F, any nonnegative continuous compactly supported function can be approximated by dominating polynomials, in the norm of the space $L_{\mu}^{1}(F)$. On the other hand, according to [37], for any compact subset $S \subset [0, +\infty)$, the convex cone of restrictions to S of all polynomials p, with $p(t) \geq 0$ for all $t \in [0, +\infty)$, is dense in the convex cone $\left(C(S)\right)_{+}$ of all functions $\varphi \in C(S)$, with $\varphi(t) \geq 0 \; \forall t \in S$. Part of these remarks and comments are motivated by the results proved or recalled in Section 3, concerning the moment problem. When the codomain Y is the real line, we have a scalar moment problem. If Y is an ordered Banach space of self-adjoint operators, we have an operator-valued moment problem. On the other hand, the problem of uniform approximating continuous real- valued functions defined on $[0, b]$ by means of functions defined in terms of antiderivatives of the moments and the corresponding problem of approximating complex analytic functions are also under attention. Approximation of the limit of the sequence by estimating each term, then passing to the limit, is pointed out in [41]. All the studies appearing in the References list are related to theoretical results mentioned above and in the next sections or contain applications in these areas. The rest of the article is organized as follows. In Section 2, we point out the methods applied in what follows. The first part of Section 3 is devoted to elements of approximation by means of nonnegative polynomials. In the second part, we recall, state and solve scalar and operator valued moment problems. Here approximation of classes of nonnegative real valued functions by positive polynomials in several dimensions is applied. For example, if $\varphi: K = K_1 \times \cdots \times K_d \subset \mathbb{R}_{+}^{d} \longrightarrow \mathbb{R}_{+}$ is continuous, then there exists a sequence $(p_m)_m$ of polynomials, $p_m(t_1, \dots, t_d) = p_{m_1}(t_1) \cdots p_{m_d}(t_d)$, $p_{m_i}(t_i) \geq 0 \; \forall t_i \in \mathbb{R}_{+}$, such that $p_m|_K \longrightarrow \varphi$ uniformly on K, $p_m(t) \geq \varphi(t) \;\; \forall t \in K, \;\; \forall m \in \mathbb{N}$. For $d = 1$, this result can be deduced from Lemma 2 of [38]. For $d = 2$, we apply Theorem 2 of [40]. It seems that the general case $d \geq 2$ can be proved by induction upon d.

2. Methods

Here are the basic methods applied in this paper.

1. First, we prove uniform approximation results of real valued continuous functions on compact intervals $[0.b]$, by using Korovkin's theorem. Uniform approximating on compact subsets of $\mathbb{C}$ or of the open unit disk U, of complex analytic functions, by special analytic functions involving the antiderivatives of the complex moments. To prove such results, mainly Vitali's theorem is applied.
2. Approximation of classes of nonnegative functions from $L_{\mu}^{1}(\mathbb{R}^d), \mu$ being the product of d $M-$determinate measures on $\mathbb{R}$, by special products of nonnegative polynomials on $\mathbb{R}$, which are sums of squares (see [37,39]). Uniform approximation of nonnegative functions from $C(S_1 \times \cdots \times S_d)$, S_i compact subsets of $\mathbb{R}_{+} = [0, +\infty), i = 1, \dots, d$, by sums of special positive polynomials of the form $\prod_{i=1}^{d} t_i^{s_i} q_i^2(t_i), q_i \in \mathbb{R}[t_i],, s_i \in \{0,1\}, \; i = 1, \dots, d$. (see [38]). For applied approximation type results related to the moment problem and its relationship with probabilities see [9,12]. For connections with neural networks and positive linear operators see [31,32].
3. Applying the results mentioned in point 2 to the characterization of the existence and uniqueness of the solution of the moment problem, in terms of sums of quadratic expressions.

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3. Results

3.1. Approximation of Continuous Real Valued Functions and of Complex Analytic Functions

Theorem 1. *Let* $f \in C([0,b]), 0 < b < 1$ *and*

$$L_n(f)(x) := \frac{\int_0^x t^n f(t)dt}{\int_0^x t^n dt}\ ,\ x \in (0,b],\quad L_n(f)(0) := f(0).\ \text{Then}$$

$$f(x) = \lim_{n\to\infty} L_n(f)(x), \qquad x \in [0,b] \subset [0,1).$$

The convergence holds uniformly with respect to $x \in [0,b]$, *and* $\omega_{L_n f - f}(\delta_n) \le 2\omega_f(\delta_n) \searrow 0$. *Consequently, under the same assumptions on* b, *for any* $f \in C^1([0,b])$, $\lim_{n\to\infty} \int_0^x t^{n+1} f'(t)dt = 0$, *uniform convergence on* $[0,b]$.

Proof. Due to Korovkin's theorem [15], it is sufficient (and necessary) to prove the convergence $L_n(f) \longrightarrow f$ for each $f \in \{\mathbf{1}, t, t^2\}$. Indeed, for $f = p_0 = \mathbf{1}$ we obtain

$$L_n(\mathbf{1})(x) := \frac{\int_0^x t^n dt}{\int_0^x t^n dt} = 1 \text{ for all } x \in (0,1],\quad L_n(\mathbf{1})(0) := 1,\quad n \in \mathbb{N}.$$

$$\max_{0\le x\le b}|L_n(p_0)(x) - 1| = \max_{0\le x\le b} 0 = 0;$$

For $p_1(x) = x, x \in [0,b]$, it results:

$$\max_{0\le x\le b}|L_n(p_1)(x) - x| = \max_{0\le x\le b}\left|\frac{\int_0^x t^{n+1}dt}{\int_0^x t^n\,dt} - x\right| = \max_{0\le x\le b}\left|\frac{n+1}{n+2}x - x\right| = \frac{b}{n+2}$$

$$\le \frac{1}{n+2} < \varepsilon, \qquad \forall n > 1/\varepsilon - 2.$$

In this case, the desired inequality $\max_{0\le x\le 1}|L_n(p_1)(x) - x| < \varepsilon$ holds for all n greater than the entire part of $\frac{1}{\varepsilon} - 2$. If $p_2(x) = x^2$, then:

$$\max_{0\le x\le b}\left|L_n(p_2)(x) - x^2\right| = \max_{0\le x\le b}\left|\frac{\int_0^x t^{n+2}dt}{\int_0^x t^n\,dt} - x^2\right| = \max_{0\le x\le b}\left|\frac{n+1}{n+3}x^2 - x^2\right| = \frac{2b^2}{n+3} \le \frac{2}{n+3} < \varepsilon,$$

$$n > \frac{2}{\varepsilon} - 3.$$

Hence for p_2 the required convergence holds uniformly on $[0,1]$, for all entire n greater than $\frac{2}{\varepsilon} - 3$. Hence, the conclusion of Korovkin's theorem holds. Namely, we have: $L_n(f) \longrightarrow f, n \longrightarrow \infty$, uniformly on $[0,b]$. Moreover, if we denote by $\omega_f(\delta_n)$ the modulus of continuity of the continuous function f on the interval $[0,\delta], 0 < \delta_n < \delta$, then for $0 \le x, y \le \delta_n \searrow 0$, we have: $|f(t) - f(x)| \le \omega_f(\delta_n) \searrow 0$ for all t in the closed interval of ends $0, \delta_n$. These yields:

$$|L_n f(x) - f(x) - (L_n f(y) - f(y))| \le |L_n f(x) - f(x)| + |(L_n f(y) - f(y)| \le$$

$$\left|\frac{\int_0^x t^n\big(f(t) - f(x)\big)dt}{\int_0^x t^n\,dt}\right| + \left|\frac{\int_0^y t^n\big(f(t) - f(x)\big)dt}{\int_0^y t^n\,dt}\right| \le 2\omega_f(\delta_n) \searrow 0.$$

This can be written a$\omega_{L_nf-f}(\delta_n) \leq 2\omega_f(\delta_n), n \in \mathbb{N}$. Due to the uniform continuity of f on $[0, \delta]$, also using our assumption $\delta_n \searrow 0$, we infer that $\omega_{L_nf-f}(\delta_n) \leq 2\omega_f(\delta_n) \searrow 0$ follows consequently. Moreover, if $f \in C^1([0, b])$, is not constant, the following equalities hold true

$$Lf(x) = \lim_{n\to\infty} L_nf(x) = \lim_{n\to\infty} \frac{\int_0^x t^n f(t)dt}{\int_0^x t^n\, dt} =$$

$$\lim_{n\to\infty} \frac{(x^{n+1}/(n+1)) \sum_{j=0}^{n} (j/(n+1))^{n+1} f(jx/n+1) - f((j-1)x)/(n+1))}{x^{n+1}/(n+1)} =$$

$$\lim_{n\to\infty} \left(\frac{x}{n+1} \sum_{j=0}^{n} (j/(n+1))^{n+1} f'(\theta_j) \right), \qquad \theta_j \in \left(\frac{(j-1)x}{n+1}, \frac{jx}{n+1} \right),$$

$$j = 1, \ldots, n+1.$$

Hence,

$$Lf(x) = \lim_{n\to\infty} \int_0^x t^{n+1} f'(t)dt = \lim_{n\to\infty} \left(x^{n+1} f(x) - (n+1) \int_0^x t^n f(t)dt \right) = 0,$$

$$0 \leq x \leq b < 1,$$

$$Lf = 0 \Longrightarrow \ f \in L^{-1}\{0\}, \qquad \forall f \in C^1([0, b]). \square$$

Remark 1. *Theorem 1 proves that for any nonconstant function* $f \in C([0, b\}), 0 < b < 1,$ *we have* $Lf = \lim_n L_n f = 0$ *uniformly with respect to* $x \in [0, b]$, *and for* $f = \alpha\mathbf{1}$ *we infer easily that* $L_n(\alpha\mathbf{1}) = \alpha\mathbf{1}$ *for all* n. *The following question arises naturally: are the constants the only continuous functions on [0,b] which are fixed points for all operators* L_n? *According to the next proposition, the answer seems to be affirmative.*

Theorem 2. *With the notations and under the hypothesis from Theorem 1, the sequence* $(L_n)_n$ *is uniformly bounded, and we have* $\|L_n\| = 1$ *for all* n. *The convergence* $L_nf \longrightarrow f$ *holds for each* $f \in C([0, b\})$, *and the constant functions* $f = f(0)\mathbf{1}$ *are the only functions* $f \in C([0, b])$, *verifying the conditions* $L_n f = f \ \forall n \in \mathbb{N}$.

Proof. For an arbitrary function $f \in C([0, b])$, assuming that $L_n f = f \ \forall n \in \mathbb{N}$, then the following consequence holds true:

$$0 = L_nf(x) - f(x) = \frac{\int_0^x t^n f(t)dt}{\int_0^x t^n\, dt} - f(x) = \frac{\int_0^x t^n \big(f(t) - f(x)\big)}{\int_0^x t^n\, dt}\, dt, \qquad n \in \mathbb{N},$$

$$x \in [0, b],$$

$$|L_nf(x) - f(x)| \leq \sup_{t,x\in[0,b]} |f(t) - f(x)| \coloneqq \omega_f(b).$$

Obviously, this leads to: $0 = \int_0^x t^n \big(f(t) - f(x)\big)\, dt =: \langle t^n, f(t) - f(x)\rangle$, $n \in \mathbb{N}$, $x \in [0, b]$. In other words, for any fixed $x \in [0, b]$, and any $n \in \mathbb{N}$, the continuous function $\varphi(t) \coloneqq f(t) - f(x)$ is orthogonal to the basic monomials t^n, on the interval $[0, b]$. By the linearity in the first variable of the product $\langle\ ,\ \rangle$, we infer that $\varphi(t) \coloneqq f(t) - f(x)$ is orthogonal to any polynomial P on $[0, b]$: $\langle \varphi, P\rangle = 0 \ \forall P \in \mathbb{R}[t]$. On the other hand, by Weierstrass approximation theorem, there exists a sequence $(P_m)_m$ of polynomials such that $\varphi = \lim_m P_m$, the convergence holding uniformly on $[0, b]$.

Using the above reasoning, we have: $\langle \varphi, \varphi\rangle = \left\langle \varphi, \lim_m P_m \right\rangle = \lim_m \langle \varphi, P_m\rangle = 0$, that is $\int_0^b \varphi^2(t)dt = 0$.

Since φ is real valued and continuous, its square φ^2 is nonnegative valued and continuous. So, by the properties of the open nonempty set $\{t \in [0, b];\ \varphi^2(t) > 0\}$ as a union of open intervals, the integral $\int_0^b \varphi^2(t)dt$ vanishes if and only if $\varphi(t) = 0$ for all $t \in [0, b]$, that is $f(t) - f(x) =$

$0\ \forall t, x \in [0,b]$. For $x = 0$, it results in $f(t) = f(0)\ \ \forall t \in [0,b]$, that is f is constant on $[0,b]$, and $f = f(0)\mathbf{1}$.

On the other hand, each linear operator L_n is positive on the Banach lattice $C([0,b])$, so that for any $\varphi \in C([0,b])$, with $\|\varphi\| \le 1$, and all n, it results: $|L_n(\varphi)| \le L_n(|\varphi|) \le L_n(\|\varphi\|\mathbf{1}) = \|\varphi\| L_n(\mathbf{1}) \le L_n(\mathbf{1}) = \mathbf{1}$. The conclusion is that $\|L_n\| = 1\ \forall n \in \mathbb{N}$. Thus, the family of the terms of this sequence is equicontinuous (and the least upper bound for $\{\|L_n\|\}_{n\ge 0}$ equals 1.) □

Theorem 3. *Let U be the open unit disk in the complex plane and $f: U \longrightarrow \mathbb{C}$ be the function defined by*

$$f(z) := \frac{log(1+z)}{z}, \qquad z \ne 0, \qquad f(0) := 1. \tag{1}$$

Then f has a removable singularity at zero and the following approximation holds uniform on compact subsets of U

$$f(z) = \lim_{n\to\infty} \frac{\int_0^z u^n f(u)du}{z^{n+1}/(n+1)} := \lim_{n\to\infty} \frac{m_n(f)(z)}{m_n(\mathbf{1})(z)}. \tag{2}$$

We have denoted: $m_n(f)(z) := \int_0^z u^n f(u)du$. With these notations, returning to the restrictions of our functions to the interval $[0,x)$, $x \in (-1,1)$, we have:

$$\frac{\int_0^x t^n \big(f(x) - f(t)\big)dt}{x^{n+1}/(n+1)} \longrightarrow 0, \qquad n \longrightarrow \infty, \qquad x \in (-1,1).$$

Theorem 1 gives a method of approximating the distribution function f in terms of its normalized complex moment - functions $\frac{\int_0^z u^n f(u)du}{m_n(\mathbf{1})(z)}, n \in \mathbb{N}$, for n large enough

$$f(z) - \frac{\int_0^z u^n f(u)du}{z^{n+1}/(n+1)} = f(z) - \frac{\int_0^z u^n f(u)du}{\int_0^z u^n\,du} \longrightarrow 0, \qquad as\ n \longrightarrow \infty, \qquad z \in U. \tag{3}$$

The convergence result (2) stays valid for any holomorphic function f defined on $U\backslash\{0\}$, having a removable singularity at zero. We proved this assertion in [42]. Now our purpose is to point out properties that are specific to the functions defined by (1) and to other related analytic functions, also using a more general and improved method of the author's results from [42].

The sketch of the proof for Theorem 3. From Theorem 1, we know that $L_n(f) \longrightarrow f, n \longrightarrow \infty$, uniformly on $[0,b]$. The hard part is to infer the uniform convergence on compact subsets of U. This will be proved below using Vitali's theorem. To do this, it is sufficient (and necessary) to show that the sequence $\big(L_n(f)\big)_n$ is bounded in the space of analytic functions, that is it is uniformly bounded on compact subsets of U. Since any compact subset of the open unit disk is contained in closed disk of radius $R < 1$, centered at the origin, we have to prove the uniform boundedness of the sequence $\big(L_n(f)\big)_n$ on the closed disk $\overline{D}(0,R)$ for any $0 < R < 1$. The following estimates hold true:

$$|z| \le R < 1, n \in \mathbb{N}, \text{ imply} \qquad |L_n(f)(z)| = \left|\frac{z^{n+1}\int_0^1 t^n log(1+tz)dt}{z^{n+1}/(n+1)}\right| \le$$

$$\int_0^1 |log(1+tz)|\,dt =$$

$$\int_0^1 \left|\int_0^{tz} \frac{dw}{1-(-w)}\right| dt =$$

$$\int_0^1 \left|\int_0^{tz} (1 - w + w^2 + \cdots + (-1)^n w^n)dw\right| dt =$$

$$\int_0^1 \left|\left(w - \frac{w^2}{2} + \cdots + (-1)^n \frac{w^{n+1}}{n+1} + \cdots\right)\Big|_0^{zt}\right| dt =$$

$$\leq \int_0^1 \left(|z|t + \frac{|z|^2}{2}t^2 + \cdots + \frac{|z|^{n+1}}{n+1}t^{n+1} + \cdots \right) dt \leq$$

$$\frac{R}{2} + \frac{R^2}{6} + \cdots + \frac{R^{n+1}}{(n+1)(n+2)} + \cdots <$$

$$\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots =$$

$$\frac{1}{2} + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots = 1.$$

According to Vitali's theorem [19], also using the theorem on the zeros of an analytic function, the conclusion will follow. Namely, for a line segment $[0,b] \subset [0,1), 0 < b$, the convergence

$$L_n(f) \longrightarrow f \text{ holds uniformly on } [0,b], \tag{4}$$

due to Theorem 1. Since $[0,b]$ has accumulation points in U, and the sequence $\left(L_n(f)\right)_n$ is uniformly bounded on any closed disk, application of Vitali's theorem leads to the uniform convergence of a subsequence (denoted by abuse of notation $(L_n(f))_n$), of $(g_n(f))_n$, on compact subsets, to a holomorphic function $\hbar$ on U. For $x \in [0,b]$, we know that $f(x) = \lim_n L_n(f)(x)$. Hence the holomorphic function $\hbar - \lim_n L_n(f)$ has as zeros all points x of the interval $[0,b]$. From the theorem on zeros of the holomorphic function, we infer that

$$\hbar(z) - \lim_n L_n(f)\,(z) = 0, \qquad \forall z \in U. \tag{5}$$

Now Equations (4) and (5) provide the conclusion that

$$\hbar(z) = \lim_n L_n(f)(z) = f(z), \qquad z \in U,$$

uniformly on any compact subset of U. □

Theorem 4. *If* $x \in [0,1]$ *and* $m \in \mathbb{N}, m \geq 2$, *then the following inequalities hold true:*

$$log(1+x) \leq S_m(x) := x - \frac{x^2}{2} + \frac{x^3}{3} - x^4\left(\frac{1}{4} - \frac{x}{5}\right) - \cdots - x^{2m}\left(\frac{1}{2m} - \frac{x}{2m+1}\right), \tag{6}$$

$$log(1+x) \geq s_m(x) := x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \left(\frac{x^5}{5} - \frac{x^6}{6}\right) + \cdots + x^{2m+1}\left(\frac{1}{2m+1} - \frac{x}{2m+2}\right). \tag{7}$$

$$0 \leq S_m(x) - \log(1+x) \leq \frac{x^{2m+2}}{2m+2} \longrightarrow 0, \qquad m \longrightarrow \infty. \tag{8}$$

If $z \in \mathbb{C}, |z| \leq 1$, *then the expansion:* $log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots + (-1)^n \frac{z^{n+1}}{n+1} + \cdots$*holds true for all* $z \neq -1$.

Proof. Let $z \in U$. The simplest path of ends zero and z is the line segment joining these points. The parametric equation of this path is $w = zt,\ t \in [0,1]$. This implies $dw = zdt$. According to the integration term by term theorem for the absolute convergent geometric series $\sum_{n=0}^{\infty} (-w)^n$ on the path of ends zero and z recalled above, we are led to the next result

$$\log(1+z) = \int_0^z \frac{1}{1+w} dw = \int_0^z \frac{1}{1-(-w)} dw =$$

$$\int_0^z (1 - w + w^2 + \cdots (-1)^n w^n + \cdots) dw =$$

$$z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots + (-1)^n \frac{z^{n+1}}{n+1} + \cdots \qquad . \tag{9}$$

If $z = x \in (0,1)$, then from (3) and using that all the expressions $-x^{2m}\left(\frac{1}{2m} - \frac{x}{2m+1}\right)$ have negative signature, we derive that:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - x^4\left(\frac{1}{4} - \frac{x}{5}\right) - \cdots - x^{2m}\left(\frac{1}{2m} - \frac{x}{2m+1}\right) - \cdots \le$$

$$S_m(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - x^4\left(\frac{1}{4} - \frac{x}{5}\right) - \cdots - x^{2m}\left(\frac{1}{2m} - \frac{x}{2m+1}\right), \qquad m \ge 2.$$

For the converse type inequality, one observes that $x \in (0,1)$ implies all the expressions in the parentheses appearing below and their coefficients x^{2m+1} have positive signature, hence we can write:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \left(\frac{x^5}{5} - \frac{x^6}{6}\right) + \cdots + x^{2m+1}\left(\frac{1}{2m+1} - \frac{x}{2m+2}\right) + \cdots \ge$$

$$s_m(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \left(\frac{x^5}{5} - \frac{x^6}{6}\right) + \cdots + x^{2m+1}\left(\frac{1}{2m+1} - \frac{x}{2m+2}\right), \qquad m \ge 2.$$

$$0 \le S_m(x) - \log(1+x) \le S_m(x) - s_m(x) = \frac{x^{2m+2}}{2m+2}, \qquad x \in [0,1], \qquad m \in \mathbb{N}.$$

On the other hand, the last expansion of the function $z \longmapsto \log(1+z)$ appearing in the statement of the theorem and its convergence set is known, because of Dirichlet test. □

Theorem 5. *Let* H *be a Hilbert space and* A *a bounded linear operator acting on* H, *with the spectrum* $\sigma(A)$ *contained in the unit interval* $[0,1]$. *Then* A *is self-adjoint and for any* $m \in \mathbb{N}, m \ge 2$, *the following operator inequalities hold:*

$$log(I+A) \le S_m(A) = A - \frac{A^2}{2} + \frac{A^3}{3} - A^4\left(\frac{I}{4} - \frac{A}{5}\right) - \cdots - A^{2m}\left(\frac{I}{2m} - \frac{A}{2m+1}\right);$$

$$\begin{aligned} log(I+A) \ge s_m(A) &= A\left(I - \frac{A}{2}\right) + A^3\left(\frac{I}{3} - \frac{A}{4}\right) + A^5\left(\frac{I}{5} - \frac{A}{6}\right) + \cdots \\ &+ A^{2m+1}\left(\frac{I}{2m+1} - \frac{A}{2m+2}\right). \end{aligned}$$

$$0 \le S_m(A) - log(I+A) \le \frac{A^{2m+2}}{2m+2} \le \frac{1}{2m+2} \cdot I \longrightarrow 0, \tag{10}$$

$$\left\| S_m(A) - \log(I+A) \right\| \le \frac{1}{2m+2} \longrightarrow 0, \qquad m \longrightarrow \infty. \tag{11}$$

Proof. Since the linear operator A is continuous, having its spectrum contained in $\mathbb{R}$, it results that A is self-adjoint. Moreover, because of the hypothesis $\sigma(A) \subseteq [0,1]$, we infer that $0 \le A \le I$, that is $0 \le \langle Ah, h\rangle \le \langle h, h\rangle = \langle Ih, h\rangle$ for all elements $h \in H$. This means that with respect to the order relation in the ordered Banach space of all self-adjoint operators acting on H, we have: $0 \le A \le I$. On the other hand, from (6) and (7), the first two inequalities in the statement follow via functional calculus for continuous functions defined on the spectrum of A. To prove (10), we apply the increasing monotonicity of the operatorial norm on the positive cone of all self-adjoint operators. Proceeding this way, from (8), (10) and (11) follow as well. □

Corollary 1. *Theorem 4 holds true for any symmetric* $n \times n$ *matrix* A *with real entries, whose all eigenvalues are in the unit interval* $[0,1]$.

Theorem 6. *For any* $x \in (0,1]$ *and any positive integer* m, *the following estimates hold true*:

$$\frac{arctan(x)}{x} \le \sum\nolimits_m (x) := 1 - x^2\left(\frac{1}{3}-\frac{x^2}{5}\right) - \cdots - x^{2m}\left(\frac{1}{2m+1}-\frac{x^2}{2m+3}\right).$$

$$\frac{arctan(x)}{x} \ge \sigma_m(x) := 1 - \frac{x^2}{3} + x^4\left(\frac{1}{5}-\frac{x^2}{7}\right) + \cdots + x^{2m+2}\left(\frac{1}{2m+3}-\frac{x^2}{2m+5}\right).$$

$$0 \le \sum\nolimits_m (x) - \frac{arctan(x)}{x} \le \frac{1}{2m+5} \longrightarrow 0,\quad m \longrightarrow \infty,$$

The convergence is uniform with respect to $x \in [0,1]$.

Proof. For any $x \in (0,1]$ and any positive integer m, the following estimates hold true:

$$\frac{arctan(x)}{x} \le \sum\nolimits_m (x) := 1 - x^2\left(\frac{1}{3}-\frac{x^2}{5}\right) - \cdots - x^{2m}\left(\frac{1}{2m+1}-\frac{x^2}{2m+3}\right).$$

$$\frac{arctg(x)}{x} \ge \sigma_m(x) := \left(1 - \frac{x^2}{3}\right) + x^4\left(\frac{1}{5}-\frac{x^2}{7}\right) + \cdots + x^{2m+2}\left(\frac{1}{2m+3}-\frac{x^2}{2m+5}\right).$$

$$0 \le \sum\nolimits_m (x) - \frac{arctan(x)}{x} \le \sum\nolimits_m (x) - \sigma_m(x) = \frac{x^{2m+4}}{2m+5} \longrightarrow 0,\qquad m \longrightarrow \infty.$$

For $x = 0$, we have: $\sum_m(0) = 1 = \lim_{x \downarrow 0} \frac{arctan(x)}{x} = \sigma_m(0)$, for all m, so that the assertion of the theorem holds uniformly on the closed interval $[0,1]$. □

Corollary 2. *The following estimates hold true:*

$$\frac{2}{3} \le \frac{\pi}{4} \le \frac{13}{15}\ .$$

Proof. One applies the estimates proved in Theorem 4 to $x = 1$, taking only the expressions appearing in the first parenthesis of $\sum_1(x)$, $\sigma_1(x)$. □

Corollary 3. *If* A *is aself-adjoint operator acting on a Hilbert space* H, *and the spectrum* $\sigma(A)$ *satisfies the inclusion relation,* $\sigma(A) \subset (0,1]$, *then the following estimates hold true:*

$$A^{-1}\, arctan(A) \le \sum\nolimits_m (A) := I - A^2\left(\frac{I}{3}-\frac{A^2}{5}\right) - \cdots - A^{2m}\left(\frac{I}{2m+1}-\frac{A^2}{2m+3}\right),$$

$$A^{-1}\, arctan(A) \ge \sigma_m(x) := \left(I - \frac{A^2}{3}\right) + A^4\left(\frac{I}{5}-\frac{A^2}{7}\right) + \cdots + A^{2m+2}\left(\frac{I}{2m+3}-\frac{A^2}{2m+5}\right),$$

$$0 \le \sum\nolimits_m (A) - A^{-1}\, arctan(A) \le \sum\nolimits_m (A) - \sigma_m(A) = \frac{A^{2m+4}}{2m+5} \longrightarrow 0,\qquad m \longrightarrow \infty.$$

Proof. Due to the hypothesis $\sigma(A) \subset (0,1]$, we have $0 \notin \sigma(A)$, so that A is self-adjoint, positive and invertible. Application Theorem 4 accompanied by functional calculus, lead to the desired estimates. □

Theorem 7. *Let us consider the function*

$$f(z) := \frac{arctan(z)}{z}\ ,\qquad z \in U,\qquad z \neq 0,\quad f(0) := 1.$$

Then f *has a removable singularity at zero, it is holomorphic in the open unit disk* U *centered at the origin, and so are all the functions* $L_n(f), n \in \mathbb{N}$, *defined by*

$$L_n(f)(z) := \frac{\int_0^z u^n f(u)du}{z^{n+1}/(n+1)}\ ,\qquad z \neq 0,\quad L_n(f)(0) := f(0),\quad n \in \mathbb{N},\quad z \in U.$$

Moreover, the following asymptotic behavior of the sequence $\left(L_n(f)\right)_n$ *holds true:*

$$f(z) = \lim_n L_n(f)(z),\qquad z \in U. \tag{12}$$

The convergence (12) holds uniformly on compact subsets of U.

Proof. The proof of Theorem 6 is quite like that of Theorem 2. The main part of the proof is that of showing the uniform bondedness of the sequence $(L_n(f))_{n\geq 0}$ on any closed disk centered at the origin, of radius $R \in (0,1)$. As in the proof of Theorem 2, If $|z| \leq R$, then, from $u = tz, t \in [0,1]$, $du = zdt$, we infer:

$$|L_n(f)(z)| = \left| \frac{z^{n+1} \int_0^1 \frac{t^n \left(arctan(tz)\right)}{(tz)} dt}{z^{n+1}/(n+1)} \right| \leq$$

$$\max_{0\leq t\leq 1} \left| \frac{\text{arctan}(tz)}{tz} \right| =$$

$$\max_{0\leq t\leq 1} \left| 1 - \frac{t^2 z^2}{3} + \frac{t^4 z^4}{5} - \cdots + (-1)^n \frac{t^{2n} z^{2n}}{2n+1} + \cdots \right| \leq$$

$$\left(1 + \frac{R^2}{3} + \cdots + \frac{R^{2n}}{2n+1} + \cdots \right) =: M(R) < +\infty.$$

Since the upper bound $M(R)$ does not depend on n or on z with $|z| \leq R$, the desired uniform boundedness follows. The act that $M(R)$ is finite is obvious, since $0 < R < 1$. The conclusion is a consequence of Vitali's theorem. This ends the proof. □

3.2. Polynomial Approximation by Nonnegative-Valued Polynomials and the Moment Problem

In the Stieltjes moment problem, a sequence of real numbers $(y_k)_{k\geq 0}$ is given and one looks for a nondecreasing real function $\sigma(t)$ $(t \geq 0)$, which verifies the moment conditions:

$$\int_0^\infty t^k \, d\sigma = y_k, \ (k = 0,1,2,\ldots).$$

This is a one-dimensional moment problem, on an unbounded interval. Namely, is an interpolation problem with the constraint on the positivity of the measure dσ. The numbers $y_k, k \in \mathbb{N} = \{0,1,2,\ldots\}$ are called the moments of the measure $d\sigma$. Existence, uniqueness and construction of the solution σ are studied. The moment problem is an inverse problem: we are looking for an unknown measure, starting from its given moments. The direct problem might be: being given the measure $d\sigma$, compute its moments $\int_0^\infty t^k \, d\sigma. k = 0,1,2, \ldots.$ The connection with the positive polynomials and extensions of linear positive functional and operators is quite clear. Namely, if one denotes by $\varphi_j, \varphi_j(t) := t^j, j \in \mathbb{N}, t \in [0,\infty)$, $\mathcal{P}$ the vector space of polynomials with real coefficients and

$$T_0 : \mathcal{P} \to \mathbb{R}, T_0\left(\sum_{j\in J_0} \alpha_j \varphi_j\right) := \sum_{j\in J_0} \alpha_j y_j, \tag{13}$$

where $J_0 \subset \mathbb{N}$ is a finite subset, then the moment conditions $T_0(\varphi_j) = y_j, j \in \mathbb{N}$ are clearly satisfied. It remains to check whether the linear form T_0 defined by (3) has nonnegative value at each nonnegative polynomial. If this condition is also accomplished, then one looks for the existence of a linear positive extension T of T_0 to a larger ordered function space X which contains both $\mathcal{P}$ and the space of continuous compactly supported functions, then representing T by means of a positive regular Borel measure μ on $[0,\infty)$, via Riesz representation theorem or applying Haviland theorem. Usually, the positive linear extension is defined on a Banach function space. For deep results on the theory of Banach spaces see [21,25]. If an interval (for example $[a,b]$, $\mathbb{R}$, or $[0,\infty)$) is replaced by a closed subset F of $\mathbb{R}^d, d \geq 2$, we have a multidimensional moment problem. Passing to an example of the multidimensional real classical moment problem, let denote

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$$\varphi_j(t) = t^j = t_1^{j_1} \cdots t_d^{j_d}, j = (j_1, \dots, j_d) \in \mathbb{N}^d, t = (t_1, \dots, t_n) \in \mathbb{R}_+^d, n \in \mathbb{N}, d \geq 2.$$

If a sequence $(y_j)_{j \in \mathbb{N}^d}$ is given, one studies the existence, uniqueness and construction of a linear positive form T defined on a function space containing polynomials and continuous compactly supported real functions, such that the moment conditions

$$T(\varphi_j) = y_j, \qquad j \in \mathbb{N}^d \tag{14}$$

are satisfied. Usually, the positive linear form T (that is called a solution for the moment problem defined by (14)), can be represented by means of a positive regular Borel measure λ on $\mathbb{R}_+^d$. In this case, we say that λ is a representing measure for the sequence $y = (y_j)_{j \in \mathbb{N}^d}$ and this sequence is called a moment sequence. Similar definitions and terminology are valid when we replace $\mathbb{R}_+^d$ by an arbitrary closed subset F of $\mathbb{R}^d$. We start by recalling the solution for the constrained moment problem on $\mathbb{R}$. In what follows, for $i = (i_1, \dots, i_d), j = (j_1, \dots, j_d) \in \mathbb{N}^d$ and $t \in \mathbb{R}^d$, we denote $t^j = t_1^{j_1} \cdots t_d^{j_d}, t^{i+j} = t_1^{i_1+j_1} \cdots t_d^{i_d+j_d}$.

On the solution to the multidimensional moment problem

Theorem 8. (see [40], Theorem 1, pp. 3-5). *Let μ_1, μ_2 be positive regular Borel moment determinate measures on $\mathbb{R}$, with finite absolute moments $\int_{\mathbb{R}} |t|^j \, d\mu_i(t)$ of all natural orders, $j \in \mathbb{N}$, $i = 1,2$. In addition, assume that $\mu := \mu_1 \times \mu_2$ is their product measure on $\mathbb{R}^2$. Also assume that $d\mu_i = \rho_i(t_i)dt_i$, ρ_i continuous and positive on $\mathbb{R}, i = 1,2$. Then the following approximations hold true. Any function φ from $\left(L^1_{\mu_1 \times \mu_2}(\mathbb{R}^2)\right)_+$ can be approximated in the norm of $L^1_{\mu_1 \times \mu_2}(\mathbb{R}^2)$ by polynomials from the sub cone $C \subset \left(L^1_{\mu_1 \times \mu_2}(\mathbb{R}^2)\right)_+$ generated by special polynomials*

$$r_1 \otimes r_2, \quad r_i \in \mathbb{R}[t_i], \ (r_1 \otimes r_2)(t_1, t_2) := r_1(t_1) r_2(t_2), \ r_i(t_i) \geq 0 \ \textit{for all} \ t_i \in \mathbb{R}, \\ i = 1,2.$$

Hence, φ can be approximated by sums of squares whose terms have the following form

$$q(t_1, t_2) = q_1^2(t_1) \cdot q_2^2(t_2) = (q_1^2 \otimes q_2^2)(t_1, t_2), \ (t_1, t_2) \in \mathbb{R}^2, \ q_i \in \mathbb{R}[t_i], \\ i = 1,2. \quad (15)$$

Lemma 1. (Lemma 2 from [37], p. 5). L*et $K \subset [0, +\infty)$ be a compact subset, and $f: K \longrightarrow \mathbb{R}_+$ be a continuous function. Then there exists a sequence $(q_m)_m$ of polynomials, $q_m(t) \geq 0$ for all $t \in [0, +\infty)$, such that $\lim_m q_m = f$, and the convergence holds uniformly on K.*

Theorem 9. *Let H be a Hilbert space, and $A_i, i = 1,2$ commuting positive self-adjoint operators acting on H, with their spectrums $\sigma(A_i)$, $i = 1,2$. Let $Y := cl(span\{A_1, A_2\}) \subset \mathcal{SA}(H)$, where $\mathcal{SA}(H)$ is the ordered Banach space of all self-adjoint operators acting on H. Let $S := \sigma(A_1) \times (A_2)$, and $T_2: \mathcal{P} \to Y$ be the linear operator defined by $T_2(t^j) := T_2\left(t_1^{j_1} t_2^{j_2}\right) := A_1^{j_1} A_2^{j_2}$, $t \in S, j \in \mathbb{N}^2$. Being given a sequence $(U_j)_{j \in \mathbb{N}^2}$ in Y, the following statements are equivalent.*

(a) There exists a unique linear operator $L: C(S) \to Y$, with the properties

$$L(t^j) = U_j, \qquad j \in \mathbb{N}^2,$$
$$0 \leq L(\varphi) \leq T_2(\varphi), \qquad \varphi \in \left(C(S)\right)_+.$$

(b) $0 \leq L\left(t_1^{\epsilon_1} r_1^2(t_1) t_2^{\epsilon_2} r_2^2(t_2)\right) \leq A_1^{\epsilon_1} r_1^2(A_1) A_2^{\epsilon_2} r_2^2(A_2)$, $\epsilon_i \in \{0,1\}$, $r_i \in \mathbb{R}[t_i]$, $i = 1, 2$.

Proof. We may apply Lemma 1, followed by functional calculus for continuous functions defined on the Cartesian product $S = \sigma(A_1) \times \sigma(A_2)$. Indeed, let us denote by W the convex cone generated by all the polynomials of the form $q_1 \otimes q_2$, $q_i(t_i) \geq 0$ for all $t_i \in [0, +\infty)$. Clearly, W is the convex cone generated by all polynomials $t_1^{\epsilon_1} r_1^2(t_1) t_2^{\epsilon_2} r_d^2(t_d), \epsilon_i \in \{0,1\}$, $r_i \in \mathbb{R}[t_i]$, $i = 1, 2$ (see [20]). Assuming that (b) holds, adding a finite number of inequalities of the same sense, we have $0 \leq L(r) \leq T_2(r)$ for all $r \in W$. If $\varphi \in (C(S))_+$, according to Lemma 1, there exists a sequence of polynomials

$$(s_m)_m, \ s_m \in W \quad \forall m, \ |s_m - \varphi| \leq \varepsilon_m \mathbf{1} \longrightarrow 0, \ m \to \infty.$$

This is equivalent to $s_m - \varepsilon_m \mathbf{1} \leq \varphi \leq s_m + \varepsilon_m \mathbf{1}$. On the other hand, since $s_m \longrightarrow \varphi$, $(s_m)_m$ is a Cauchy sequence in W, hence $-2\varepsilon_m \mathbf{1} \leq s_{m+p} - s_m \leq 2\varepsilon_m \mathbf{1}$. Due to positivity and property $0 \leq L(r) \leq T_2(r)\ \forall r \in W$, we infer that

$$-2\varepsilon_m I \leq L(s_{m+p}) - L(s_m) \leq 2\varepsilon_m T_2(\mathbf{1}) = 2\varepsilon_m I, \qquad m \geq N(\varepsilon), \qquad p \in \mathbb{N}.$$

This implies that the sequence $(L(s_m))_m$ is Cauchy in the ordered Banach space Y, hence there exists $L(\varphi) \coloneqq \lim_m L(s_m) \in Y_+$, $\varphi \in \left(C(S)\right)_+$. It is easy to prove that the definition does not depend on the sequence $s_m \longrightarrow \varphi$, $s_m \in W$ for all m. Indeed, if $u_m \longrightarrow \varphi$, $u_m \in W$ for all m, then $|u_m - s_m| \longrightarrow 0$ in $C(S)$, that is $-\varepsilon_m 1 \leq u_m - s_m \leq \varepsilon_m 1 \longrightarrow 0$. Using the positivity of the linear operator T_2, this implies

$$|T_2(u_m - s_m)| \leq T_2(|u_m - s_m|) \leq \varepsilon_m T_2(\mathbf{1}) = \varepsilon_m I \longrightarrow 0.$$

The preceding conclusion further yields: $\|T_2 u_m - T_2 s_m\| \longrightarrow 0$, hence, also using the last inequality from point (a) of the statement, we conclude that the definition of $L(\varphi)$ is correct. Moreover, one can write:

$$0 \leq L(\varphi) \coloneqq \lim_m L(s_m) \leq \lim_m T_2(s_m) = T_2(\varphi), \ \ \forall \varphi \in \left(C(S)\right)_+.$$

On the other hand, by the same method, it follows that for an arbitrary sequence $\varphi_n \longrightarrow 0$, $\varphi_n \in \left(C(S)\right)_+$, we infer that $L(\varphi_n) \longrightarrow 0$. If $\psi_n \in C(S), \psi_n \longrightarrow 0$, then both sequences ψ_n^+ and ψ_n^- converge to zero, their terms are in $\left(C(S)\right)_+$ and $\psi_n = \psi_n^+ - \psi_n^-$ for all n. By the above reason, we conclude that $L(\psi_n) = L(\psi_n^+) - L(\psi_n^-) \longrightarrow 0$. Thus, L is continuous at zero and is linear, so that L is continuous everywhere in $C(S)$. It has been proved that for $\varphi \in \left(C(S)\right)_+$, $L(\varphi)$ is an element of Y_+ as being limit of elements $L(s_m) \in Y_+$. Thus L is positive on the positive cone $\left(C(S)\right)_+$. Assume that (b) holds. Then $0 \leq L(s_m) \leq T_2(s_m)$ for all m. Passing to the limit and using the above notations, we obtain:

$$0 \leq \lim_m L(s_m) =: L(\varphi) \leq \lim_m T_2(s_m) = T_2(\varphi), \qquad \varphi \in \left(C(S)\right)_+.$$

Thus, the assertions stated at point (a) are proved. The implication (a) implies (b) is obvious, by our notations and the definition of the operator L. □

4. Discussion

In the first subsection of Section 3, the hard part is the approximation of complex analytic functions by special such functions. In the second subsection, approximation of real valued nonnegative functions by positive polynomials on unbounded subsets is the main goal. A main remark is that for any moment determinate positive regular Borel measure μ on an arbitrary closed unbounded subset F of $\mathbb{R}^d$, and any nonnegative function f from $L^1_\mu(F)$, there exists a sequence $(p_m)_m$, with $p_m(t) \geq f(t)$ for all $t \in F$, $p_m \longrightarrow f$ as $m \longrightarrow \infty$, in the norm of $L^1_\mu(F)$. Of a note, from this statement it follows that all the dominating polynomials p_m take nonnegative values. Here F is not necessarily the Cartesian product of special subsets of $\mathbb{R}$. Even in the case of positive functions on a compact subset K of $[0, +\infty)$, the approximation can be made uniformly on K, by nonnegative polynomials on the entire semiaxes $[0, +\infty)$. The corresponding result for compact subsets $S = S_1 \times \cdots \times S_d, d \geq 2$, is discussed and applied to the multidimensional moment problem. This is the point of the latest part of this study. The constant function **1** is the order unit in $C(S)$ and make the proof easier and natural. In the cases of special compact subsets in $\mathbb{R}^d$, that are not Cartesian products of compacts contained in $\mathbb{R}$, the moment problem and related consequences are discussed in the references recalled in the Introduction and in other works. For the multidimensional Hamburger and Stieltjes moment problems see [14].

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