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Algebraic Learning in Finite Ring Continuum

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Abstract

The Finite Ring Continuum (FRC) models physical structure as emerging from a sequence of finite arithmetic shells of order $q = 4t + 1$. While Euclidean shells $\mathbb{F}_p$ support reversible Schrödinger dynamics, causal structure arises only in the quadratic extension $\mathbb{F}_{p^2}$, where the finite-field Dirac equation is defined. This paper resolves the conceptual tension between the quadratic expansion $\mathbb{F}_p \to \mathbb{F}_{p^2}$ and the linear progression of symmetry shells by introducing an algebraic innovation–consolidation cycle. Innovation corresponds to the temporary access to Lorentzian structure in the quadratic extension; consolidation extracts a finite invariant family and encodes it into the arithmetic of the next shell via a uniform Gödel recoding procedure. We prove that any finite invariant set admits such a recoding, and we demonstrate the full mechanism through an explicit worked example for $p = 13$. The results provide a coherent algebraic explanation for how finite representational systems—biological, computational, and physical—can acquire, assimilate and preserve structure.

Keywords: finite fields, relational finitude; latent structure; minimal sufficiency; modular arithmetic; epistemic uncertainty; foundational representations; vector embeddings; algebraic geometry; cross-modal alignment

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1. Introduction

The Finite Ring Continuum (FRC) is a recently proposed algebraic framework in which space, time, matter, and dynamical laws emerge from the internal structure of finite arithmetic shells [1,2]. Each shell is realised as a finite ring $\mathbb{Z}_q$ with order $q = 4t + 1$, where the chronon $t \in \mathbb{N}$ serves as a discrete radial parameter determining the combinatorial and geometric content of the universe. When q is prime, the corresponding shell becomes symmetry-complete and is identified with the finite field $\mathbb{F}_p$, $p = 4t + 1$. These prime shells constitute the principal stages of the FRC architecture, hosting Euclidean geometry, frame transformations, and the finite encodings of continuous mathematical structures.

A central theme in the FRC programme is the *duality* between Euclidean and Lorentzian phases. In symmetry-complete shells $\mathbb{F}_p$, the internal geometry is Euclidean: the multiplicative group splits into two square classes, but a genuine Lorentzian signature is inaccessible for primes $p \equiv 1 \pmod 4$. The emergence of causal structure therefore requires adjoining a square root of a quadratic nonsquare, giving rise to the quadratic extension $\mathbb{F}_{p^2}$ [2]. This extension forms the Lorentzian layer of the shell, enabling the definition of Minkowski-type quadratic forms, null directions, and ultimately the finite-field Dirac equation [3]. The Euclidean and Lorentzian phases thus coexist within each prime shell, with the latter representing a temporary expansion of algebraic capacity beyond that available to the Euclidean domain.

Another distinctive aspect of the FRC is the replacement of continuous number systems by *finite Gödel-style encodings*. Integers, rationals, and reals are represented via framed residues in $\mathbb{F}_p$, capturing scale and orientation information through algebraic transformations internal to the shell [1]. These finite encodings allow the continuum-like behaviour of number systems to be implemented within the strictly finite arithmetic of the FRC.

Despite this progress, a conceptual tension remains at the heart of the theory. On the one hand, the quadratic extension $\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2}$ induces a *quadratic expansion* of the representational domain, multiplying the available state space by a factor of p. On the other hand, the progression of symmetry shells along the chronon axis follows a *linear* law $q = 4t + 1$, with only a modest increase in structural capacity from one shell to the next. The question therefore arises: *How can a finite universe reconcile large, temporary expansions of its algebraic workspace with the comparatively slow linear growth of its long-term structural complexity?*

The purpose of this paper is to resolve this tension by introducing a mathematically precise *innovation–consolidation cycle*. In this cycle, the quadratic extension $\mathbb{F}_{p^2}$ plays the role of an innovation domain: the system temporarily accesses a richer algebraic space in which new invariants and structural properties become expressible. From this expanded space, only a *finite invariant signature* is retained. These invariants are then encoded, via a uniform and shell-independent Gödel procedure, into the arithmetic alphabet of the next Euclidean shell. Consolidation thus compresses a large but finite body of Lorentzian information into a compact symbolic representation that can be transmitted across shells.

We formalise this process by introducing:

1. the **innovation operator**, which embeds Euclidean states into the Lorentzian domain and evolves them under the finite-field Dirac equation;
2. the **invariant extractor**, which maps Lorentzian states to a finite set of orbit- or norm-type invariants; and
3. the **consolidation operator**, which Gödel-encodes these invariants into the next symmetry shell.

Our main structural result (Proposition 1) shows that *any* finite set of Lorentzian invariants, irrespective of size or algebraic origin, admits an injective encoding into a finite tuple of residues of the next shell. We furthermore illustrate the construction with an explicit worked example for $p = 13$, demonstrating the entire innovation–consolidation pipeline in a fully computable setting.

Learning-Theoretic Motivation. The innovation-consolidation cycle developed in this paper also exhibits a notable structural parallel with models of learning in biological and artificial systems. In predictive-coding and Bayesian brain theories [4,5], unexpected observations generate large prediction errors that trigger a temporary expansion of the internal representational space—often interpreted as the recruitment of new latent variables or explanatory causes [6,7]. This "surprise-driven" expansion is typically followed by consolidation, in which the enriched representation is compressed into a stable posterior, retaining only the structural regularities that improve future prediction [8,9].

In the algebraic setting of the Finite Ring Continuum, the quadratic extension $\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2}$ plays an analogous role: it temporarily enlarges the representational domain, enabling the system to access Lorentzian geometric features that are inaccessible within the Euclidean shell. The subsequent extraction and Gödel recoding of finite invariant families mirror the consolidation phase, producing a stable symbolic summary propagated to the next shell. This analogy is interpretive rather than biological, yet it provides a useful conceptual framework: both systems integrate new structure through alternating phases of expansion and compression within a finite representational substrate.

A complementary perspective is provided by the universal latent representation framework [10], which shows that diverse foundational models recover coordinate embeddings of a shared finite latent domain. The algebraic innovation-consolidation cycle developed here provides the structural mechanism by which such universal latent representations may arise, linking expressive expansion with invariant-preserving compression.

Conceptually, the results reveal how cumulative structure may emerge across shells in a universe governed by finite algebraic dynamics: innovation reveals new structural degrees of freedom, consolidation selects finitely many invariants, and shell progression provides a stable substrate for their propagation. This mechanism offers a unified algebraic perspective on the evolution of structure in the Finite Ring Continuum.

2. Background

The Finite Ring Continuum (FRC) is a discrete algebraic framework in which space, time and physical observables emerge from the internal structure of a succession of finite arithmetic shells. These symmetry shells are realised as finite rings $\mathbb{Z}_q$ with orders

$$q = 4t + 1, \tag{1}$$

indexed by the discrete radial parameter t, also referred to as the chronon. This section summarises the structural and dynamical aspects of FRC that are required for the development of the innovation–consolidation mechanism introduced later in the paper.

In the FRC formulation, each chronon $t \in \mathbb{N}$ determines a finite arithmetic shell $\mathbb{Z}_q$ with $q = 4t + 1$ [1,2]. When q is prime, the shell becomes *symmetry-complete* and is identified with the finite field $\mathbb{F}_p$, where $p = 4t + 1$. These prime shells serve as the fundamental stages of geometric and algebraic structure in FRC, hosting the full repertoire of frame transformations, scale relations, and symmetry operations.

A prime shell $\mathbb{F}_p$ carries an intrinsically Euclidean geometric interpretation: its multiplicative group decomposes into two square classes, but a Lorentzian signature is inaccessible within $\mathbb{F}_p$ itself for primes $p \equiv 1 \pmod 4$. As shown in [2], a genuine Minkowski-type form requires distinguishing a quadratic nonsquare, which is not available in the Euclidean shell. This motivates the introduction of the quadratic extension $\mathbb{F}_{p^2}$, which constitutes the Lorentzian layer of each symmetry shell.

The transition from the Euclidean shell $\mathbb{F}_p$ to its quadratic extension $\mathbb{F}_{p^2}$ is central to the causal and geometric structure of FRC. The extension is realised by adjoining a square root of a quadratic nonsquare $\nu \in \mathbb{F}_p$, yielding

$$\mathbb{F}_{p^2} \cong \mathbb{F}_p \oplus c\,\mathbb{F}_p, \qquad c^2 = \nu. \tag{2}$$

This enlarged algebra supports the Lorentzian quadratic form

$$Q_\nu(t, x, y, z) = -\nu\, t^2 + x^2 + y^2 + z^2, \tag{3}$$

which defines time-like, space-like, and null separations in the finite-field setting [2]. The Euclidean shell thus corresponds to an algebraically compressed phase of the symmetry structure, while the quadratic extension realises the full causal geometry.

Dynamics in FRC is likewise stratified into Euclidean and Lorentzian phases. Within a Euclidean shell $\mathbb{F}_p$, framed wavefunctions evolve according to a discrete Schrödinger equation, which is reversible, scale-periodic, and remains entirely internal to the Euclidean algebra [3]. This reversible dynamics expresses the conservation of information across Euclidean scale frames and reflects the absence of causal asymmetry in $\mathbb{F}_p$.

Upon lifting to the Lorentzian extension $\mathbb{F}_{p^2}$, one obtains the finite-field Dirac equation [3], constructed from the Clifford algebra $\mathrm{Cl}(1, 3; \mathbb{F}_{p^2})$ associated with Q_ν. The Lorentzian layer supports one-way causal propagation, null directions, and the algebraic analogue of relativistic mass-shell structure. Dirac evolution therefore represents the *innovation phase* of FRC: the system temporarily occupies a richer algebraic domain that exposes symmetry, geometric structure, and invariants inaccessible from within the Euclidean shell.

A distinctive feature of FRC is that real, rational, and complex number systems are not treated as infinite continua. Instead, they are represented through *finite Gödel-style encodings* built from residue classes in the finite fields $\mathbb{F}_p$ [1]. These elements encode relational scale, magnitude, and orientation information through algebraic transformations internal to the finite shell, and allow continuous mathematical structures to be realised as finite relational entities, as well as play a fundamental role in the interpretation of shell-to-shell evolution.

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3. Algebraic Structure of Innovation

The Finite Ring Continuum exhibits two distinct algebraic phases within each symmetry-complete prime shell $\mathbb{F}_p$: a Euclidean phase, in which the internal geometry remains algebraically compressed, and a Lorentzian phase, realised only after adjoining a quadratic nonsquare and passing to the extension $\mathbb{F}_{p^2}$. The purpose of this section is to formalise this transition as an *innovation step*—a temporary expansion of algebraic capacity that exposes structural degrees of freedom inaccessible within the Euclidean shell. This innovation structure forms the first half of the innovation–consolidation cycle developed in this work.

Let $p = 4t + 1$ be a symmetry-complete prime shell, and let H_p^E and H_p^L denote the Euclidean and Lorentzian state spaces, respectively. Throughout this paper, we model these as finite-dimensional modules,

$$H_p^E := \mathbb{F}_p^N, \qquad H_p^L := \mathbb{F}_{p^2}^N, \tag{4}$$

where N is fixed for the shell under consideration. The Lorentzian space H_p^L is related to H_p^E through scalar extension:

$$i_p : H_p^E \hookrightarrow H_p^L, \qquad i_p(\psi) := \psi \otimes 1, \tag{5}$$

which embeds Euclidean configurations into the richer algebraic environment of the quadratic extension.

The necessity of the Lorentzian layer is established in [2]: a Minkowski-type quadratic form cannot be realised within $\mathbb{F}_p$ when $p \equiv 1 \pmod 4$, since such primes do not admit an element whose square root lies inside the field. The quadratic extension

$$\mathbb{F}_{p^2} \cong \mathbb{F}_p \oplus c\,\mathbb{F}_p, \qquad c^2 = \nu, \tag{6}$$

with ν a quadratic nonsquare, is therefore the minimal structure in which a Lorentzian metric can exist.

The transition from H_p^E to H_p^L represents a genuine expansion of algebraic capacity. In the Euclidean shell, vectors consist only of components in $\mathbb{F}_p$, while in the quadratic extension each state acquires an additional component along the new basis element c. Explicitly, for each $\psi \in H_p^E$, the innovation space contains all elements of the form

$$\psi' = \psi + c\,\phi, \qquad \phi \in H_p^E. \tag{7}$$

Thus the representational domain expands from

$$H_p^E \quad \text{to} \quad H_p^E \oplus c\,H_p^E, \tag{8}$$

doubling its dimension over $\mathbb{F}_p$ and increasing its cardinality from p^N to p^{2N}. This is the algebraic content of the innovation phase: the system temporarily occupies a richer space of states in which new symbolic and geometric relations are available.

The quadratic expansion intimately parallels the standard finite-field construction used throughout the Dirac formalism in FRC [3]. There, the extension is required to define the Clifford algebra $\mathrm{Cl}(1,3;\mathbb{F}_{p^2})$, null directions, Lorentz boosts, and causal propagation. From the perspective of finite-field geometry, innovation corresponds to the activation of these additional structural degrees of freedom.

Let $D_p : H_p^L \to H_p^L$ denote the discrete Dirac evolution operator constructed in [3]. Although D_p is linear over $\mathbb{F}_{p^2}$, its effect on the embedded Euclidean states $i_p(H_p^E)$ is non-linear when viewed in the Euclidean frame. Indeed, the Dirac operator mixes Euclidean and Lorentzian components through the off-diagonal Clifford generators, creating features that cannot be represented by any linear operator acting solely on H_p^E.

This motivates the following definition.

Definition 1 (Innovation operator)**.** *The innovation operator associated with the prime shell $\mathbb{F}_p$ is the composite map*

$$A_p : H_p^E \xrightarrow{i_p} H_p^L \xrightarrow{D_p} H_p^L. \tag{9}$$

The operator A_p generates, from an initially Euclidean state, a collection of Lorentzian features whose algebraic content is not expressible within the Euclidean shell. In this sense, A_p plays the role of a non-linear "activation" operator: it maps a compressed Euclidean representation into a richer domain in which new invariants may be defined and extracted.

This interpretation is consistent with both the causal geometry of FRC [2] and the role of Dirac evolution as a generator of irreducible Lorentzian structure [3]. The innovation phase is therefore characterised mathematically by the temporary use of the quadratic extension to expose algebraic information hidden within the Euclidean shell.

The image $A_p(H_p^E)$ describes the entire collection of states that may arise from Euclidean initial data through Lorentzian expansion. This set is typically much larger than H_p^E itself since it includes components proportional to the newly introduced basis element c. The innovation output therefore constitutes a large, but finite, reservoir of potential invariants—a reservoir that will be compressed in the consolidation phase developed in Section 5.

In the full innovation–consolidation cycle, innovation corresponds to temporary access to $\mathbb{F}_{p^2}$-level structure; consolidation selects a finite invariant signature from this expanded space and recodes it into the next Euclidean shell. The present section formalises the algebraic half of this cycle.

4. Finite Invariant Extraction

The innovation phase described in Section 3 temporarily lifts Euclidean states $H_p^E = \mathbb{F}_\mathsf{p}^N$ into the quadratic extension $H_p^L = \mathbb{F}_{p^2}^N$, where Lorentzian geometry and the Dirac operator are defined. The Lorentzian shell contains a much richer algebraic structure than the Euclidean shell; however, only a *finite* portion of this structure is ultimately retained during the consolidation step. This section formalises the process of extracting such finite invariant structures from the innovation output.

Let Q_ν be the Lorentzian quadratic form on $\mathbb{F}_{p^2}^4$,

$$Q_\nu(t,x,y,z) = -\nu t^2 + x^2 + y^2 + z^2, \tag{10}$$

where ν is a quadratic nonsquare in $\mathbb{F}_\mathsf{p}$, and let $G_p := O(Q_\nu, \mathbb{F}_{p^2})$ denote its finite orthogonal group. The group G_p acts on the Lorentzian state space $H_p^L \subseteq \mathbb{F}_{p^2}^N$ via linear transformations preserving Q_ν.

The orbit structure of G_p partitions H_p^L into finitely many equivalence classes. Such orbits represent algebraically meaningful properties of Lorentzian states, including causal types (time-like, null, space-like), norm values, and stabilisers. Finite orthogonal groups over $\mathbb{F}_{p^2}$ are well understood [11,12], and their orbit decomposition is discrete and finite by construction.

In the innovation–consolidation cycle, these orbits—or functions of them—form the natural candidates for finite invariant signatures.

Let $A_p : H_p^E \to H_p^L$ be the innovation operator defined in Section 3. To extract a finite structural summary from the innovation output, we introduce a map

$$J_p : H_p^L \longrightarrow I_p, \tag{11}$$

where I_p is a finite set.

Definition 2 (Lorentzian invariant family)**.** *A set I_p is called a Lorentzian invariant family if:*

1. *I_p is finite,*
2. *I_p is equipped with a well-defined action of G_p, and*
3. *$J_p(g \cdot \psi) = J_p(\psi)$ for all $g \in G_p$ and all $\psi \in H_p^L$.*

The definition allows considerable flexibility. For example, J_p may return:

- the Lorentzian norm $Q_v(\psi)$,
- the orbit index of ψ under G_p,
- a tuple of norm values for components of ψ,
- a combinatorial signature derived from Dirac evolution, or
- any algebraically defined coarse-graining of the above.

What matters for consolidation is not the specific form of J_p, but the *finiteness* of I_p.

For any prime $p = 4t + 1$, the Lorentzian state space $H_p^L = \mathbb{F}_{p^2}^N$ contains p^{2N} possible states. By contrast, a typical invariant family I_p extracted from the quadratic extension may be dramatically smaller. For instance, if J_p returns only the Lorentzian norm, then

$$I_p = \mathbb{F}_\mathsf{p}, \tag{12}$$

which has cardinality p. Other invariant families may have cardinality $\Theta(p)$, $\Theta(p^2)$, or anything in between, depending on the structural information retained.

This large disparity between the innovation space and the invariant space is a key feature of FRC dynamics: innovation explores the richer algebra of $\mathbb{F}_{p^2}$, while consolidation retains only a compressed finite summary. Section 5 will show how any such finite invariant family can be transferred to the next symmetry shell.

Combining the innovation operator A_p with the invariant extractor J_p, we obtain a map

$$S_p := J_p \circ A_p : H_p^E \to I_p, \tag{13}$$

which associates to every Euclidean initial state a finite structural signature derived from the Lorentzian innovation phase.

Remark 1. *The map S_p captures the algebraic "emergent features" produced during innovation. Its output will later serve as the input to consolidation, where it is embedded into the finite alphabet of the next symmetry shell. Unlike the Dirac dynamics, which may explore a domain of size p^{2N}, the map S_p always returns an element of a* finite set I_p*, independently of the size of the extension.*

The extraction of finite invariant data thus provides the conceptual and algebraic bridge between innovation and consolidation.

5. Consolidation and Gödel Recoding

The innovation phase temporarily expands the algebraic domain from the Euclidean shell $\mathbb{F}_p$ into the quadratic extension $\mathbb{F}_{p^2}$, thereby exposing a large finite collection of emergent features. Section 4 introduced a finite invariant family I_p that summarises the Lorentzian structure relevant at the shell indexed by the prime $p = 4t + 1$. Consolidation now refers to the process of transferring this finite structural information into the next symmetry shell of the Finite Ring Continuum.

This section formalises consolidation as an instance of *finite Gödel recoding*, in which a finite alphabet is embedded into the arithmetic of the next shell. The mathematical content is straightforward, but it plays a central conceptual role in the innovation–consolidation cycle: a possibly large but finite invariant structure arising from $\mathbb{F}_{p^2}$ is compressed into an algebraically stable representation available in the next Euclidean shell.

Let $p = 4t + 1$ be a symmetry-complete prime shell. The next shell along the chronon axis is

$$q_{t+1} = 4(t + 1) + 1 = p + 4. \tag{14}$$

If q_{t+1} is prime, it is itself symmetry-complete and given by the finite field $\mathbb{F}_{p'}$ with $p' = q_{t+1}$. If q_{t+1} is composite, it is realised as the finite ring $\mathbb{Z}_{q_{t+1}}$ (or its decomposition into prime-power components).

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In either case, the next shell provides a finite alphabet suitable for representing the invariant structures extracted at level p.

For clarity of exposition—and because it is the case most relevant to shell-to-shell symmetry inheritance—we focus here on the prime case $p' = q_{t+1}$, noting explicitly that the composite case presents no additional difficulty for the constructions below.

Recall that the invariant extractor

$$J_p : H_p^L \to I_p \tag{15}$$

returns a value in the finite set I_p. Since I_p is finite, it admits an injective encoding into a finite tuple of residues of the next shell. This is the formal mathematical expression of consolidation.

We now state the key structural result.

Proposition 1 (Gödel Recoding of Finite Invariants). *Let $p = 4t + 1$ be a symmetry-complete prime shell and let I_p be any finite invariant family extracted during the innovation phase. Let $q_{t+1} = 4(t + 1) + 1$ denote the order of the next shell, and let R_{t+1} denote either the finite field $\mathbb{F}_{q_{t+1}}$ (if q_{t+1} is prime) or the ring $\mathbb{Z}_{q_{t+1}}$ (if composite). Then there exists an integer $M \geq 1$ and an injective map*

$$C_p : I_p \hookrightarrow R_{t+1}^M \tag{16}$$

that is definable in the ring language of R_{t+1}.

Proof. Since I_p is finite, let $n = |I_p|$. Choose any ordering $I_p = \{i_1, \dots, i_n\}$ and encode each i_k by its index $k - 1 \in \{0, \dots, n - 1\}$. Write $n - 1$ in base q_{t+1} and let $M := \lceil \log_{q_{t+1}} n \rceil$. Represent the index $k - 1$ by its base-q_{t+1} expansion in R_{t+1}^M. This yields an injective encoding $C_p : I_p \hookrightarrow R_{t+1}^M$. The map is definable by a uniform Gödel-numbering procedure, which uses only the arithmetic of R_{t+1}. □

This result shows that no matter how large the innovation output may be, the finite invariant signature I_p is always amenable to a compact encoding in the next shell. The number M of required symbols depends only on $|I_p|$, not on the size of the quadratic extension or the complexity of the innovation operator.

We may now combine the invariant extractor J_p with the Gödel encoder C_p to define the consolidation map.

Definition 3 (Consolidation operator). *Let $S_p = J_p \circ A_p$ be the innovation–invariant map. The consolidation operator is the map*

$$U_p := C_p \circ S_p : H_p^E \longrightarrow R_{t+1}^M, \tag{17}$$

which assigns to every Euclidean initial state a finite code in the next symmetry shell.

The map U_p realises the full innovation–consolidation cycle at the algebraic level:

$$H_p^E \xrightarrow{\text{innovation}} H_p^L \xrightarrow{\text{invariant extraction}} I_p \xrightarrow{\text{consolidation}} R_{t+1}^M. \tag{18}$$

In this interpretation, innovation serves as a discovery phase, temporarily expanding the representational domain, while consolidation produces a stable symbolic summary that seeds the next Euclidean shell.

Although Proposition 1 holds for arbitrary finite I_p, physical considerations suggest that invariant families arising from Lorentzian dynamics may have cardinalities comparable to the size of the Euclidean shell.

The following conjecture articulates this expectation.

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Conjecture 1 (Intermediate Consolidation)**.** *For symmetry-complete primes $p = 4t + 1$, the physically relevant invariant families I_p extracted from Lorentzian innovation satisfy*

$$|I_p| = \mathcal{O}(p), \tag{19}$$

so that in particular $I_p \hookrightarrow \mathbb{F}_{q_{t+1}}$ whenever $q_{t+1} = p + 4$ is prime.

The conjecture is consistent with the behaviour of many natural finite-field invariants (such as norms, causal classes, and orbit-type signatures), which typically grow linearly in p rather than quadratically in p^2. Its verification or refinement is deferred to future work, as it requires a systematic classification of invariant families associated with the Dirac operator on $\mathbb{F}_{p^2}$.

Consolidation as defined above does not rely on the conjecture. Rather, the conjecture identifies an additional structural economy that may govern the evolution of symmetry shells in the Finite Ring Continuum.

6. Numerical Example: The Case $p = 13$

To illustrate the innovation–consolidation cycle in a concrete and entirely computable setting, we now work out an explicit example for the symmetry-complete prime shell

$$p = 13 = 4 \cdot 3 + 1. \tag{20}$$

The choice $p = 13$ is minimal among primes of the form $4t + 1$ that permit a nontrivial quadratic nonsquare, and it provides a clean demonstration of the quadratic extension, invariant extraction, and shell-to-shell recoding.

The field $\mathbb{F}_{13}$ consists of residues modulo 13. Its six quadratic residues are

$$\{1, 3, 4, 9, 10, 12\},$$

and its six nonsquares are

$$\{2, 5, 6, 7, 8, 11\}.$$

The presence of nonsquares confirms (as shown in [2]) that $\mathbb{F}_{13}$ cannot host a Lorentzian quadratic form internally; a causal structure requires adjoining a square root of a nonsquare.

Choose a quadratic nonsquare, e.g. $\nu = 2$. The quadratic extension is then

$$\mathbb{F}_{13^2} := \mathbb{F}_{13}[X]/(X^2 - 2), \tag{21}$$

and we write c for the residue class of X. Every element of the extension can be uniquely written as

$$z = a + cb, \qquad a, b \in \mathbb{F}_{13}, \qquad c^2 = 2. \tag{22}$$

This gives the explicit decomposition

$$\mathbb{F}_{13^2} \cong \mathbb{F}_{13} \oplus c\,\mathbb{F}_{13},$$

doubling the dimension over $\mathbb{F}_{13}$ and expanding the state-space cardinality from 13^N to 13^{2N}. This is the algebraic manifestation of the innovation step (Section 3).

To construct an explicit finite invariant family I_{13}, we use the standard norm map from $\mathbb{F}_{13^2}$ to $\mathbb{F}_{13}$ [11], given by

$$N : \mathbb{F}_{13^2} \longrightarrow \mathbb{F}_{13}, \qquad N(z) := z \cdot \bar{z}, \tag{23}$$

where the conjugation is defined by

$$\overline{a + cb} := a - cb, \tag{24}$$

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the unique nontrivial field automorphism of $\mathbb{F}_{13^2}$.

Explicitly, for $z = a + cb$ we compute

$$N(a + cb) = (a + cb)(a - cb) = a^2 - (c^2)b^2 = a^2 - 2b^2 \pmod{13}. \tag{25}$$

Thus the invariant extractor is

$$J_{13} := N, \qquad I_{13} := \mathrm{Im}(N) \subseteq \mathbb{F}_{13}.$$

It is a classical fact that the norm map

$$N : \mathbb{F}_{p^2}^{\times} \to \mathbb{F}_p^{\times}$$

is surjective for any prime p [11]. In particular, for $p = 13$,

$$N(\mathbb{F}_{13^2}^{\times}) = \mathbb{F}_{13}^{\times} = \{1, 2, \ldots, 12\}.$$

Since $N(0) = 0$, the full image of N is

$$I_{13} = \mathbb{F}_{13}, \qquad |I_{13}| = 13.$$

Thus an innovation space of size $169 = 13^2$ collapses, under invariant extraction, into a set of only 13 possible values. This is a concrete demonstration of the consolidation principle: innovation generates a rich collection of Lorentzian features, but the invariant extractor retains only a finite and highly compressed signature.

For $p = 13$, the next shell along the chronon axis is

$$q_{t+1} = 4(t + 1) + 1 = 17,$$

which is prime. Thus the next symmetry-complete shell is $\mathbb{F}_{17}$.

Since $|I_{13}| = 13 < 17$, an injective Gödel encoding

$$C_{13} : I_{13} \hookrightarrow \mathbb{F}_{17}$$

exists trivially. For example, identifying both fields with their canonical integer representatives, we may define

$$C_{13}(x) := x, \qquad x \in \{0, 1, \ldots, 12\}, \tag{26}$$

viewing x as an element of $\mathbb{F}_{17}$. This is an injective and definable embedding of the invariant family into the alphabet of the next symmetry shell.

Although in this particular example the invariant set I_{13} already fits inside $\mathbb{F}_{13}$, the explicit recoding into $\mathbb{F}_{17}$ illustrates the general consolidation mechanism established in Proposition 1. More elaborate invariant families (e.g. orbit-type signatures or causal classifications) may require the additional representational capacity of $\mathbb{F}_{17}$ or even a finite tuple $\mathbb{F}_{17}^M$.

To summarize, the example $p = 13$ demonstrates the entire innovation-consolidation cycle:

Innovation: Euclidean states in $\mathbb{F}_{13}$ are lifted to the quadratic extension $\mathbb{F}_{13^2}$, enabling Lorentzian structure.

Invariant extraction: The Lorentzian norm $N(a + cb) = a^2 - 2b^2 \pmod{13}$ produces a finite invariant set $I_{13} = \mathbb{F}_{13}$.

Consolidation: The invariant set is embedded into the next symmetry shell via the injective code $C_{13} : I_{13} \hookrightarrow \mathbb{F}_{17}$.

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This fully explicit construction provides a minimal demonstration of how finite invariant families emerge from the quadratic extension and how they may be recoded into successive symmetry shells in FRC.

7. Discussion

The innovation–consolidation cycle developed in this work provides an algebraically precise interpretation of how information is processed within the Finite Ring Continuum. The key structural elements of this cycle are:

1. the temporary expansion of representational capacity through the quadratic extension $\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2}$;
2. the extraction of a finite invariant signature I_p from the Lorentzian domain; and
3. the recoding of these invariants into the arithmetic alphabet of the next symmetry shell.

Taken together, these stages articulate a mechanism through which finite universes may generate, compress, and transmit structural information across successive symmetry shells.

Innovation as structural expansion. Innovation corresponds to an algebraic phase transition. The Euclidean shell $\mathbb{F}_p$ is unable to support Lorentzian geometry; the quadratic extension $\mathbb{F}_{p^2}$ is therefore the minimal enlargement required to introduce causal structure [2]. This extension is multiplicative in size: the domain expands from p^N to p^{2N}, and new directions become accessible through the adjoining of c, where $c^2 = \nu$ for a quadratic nonsquare ν.

The Dirac evolution defined on $\mathbb{F}_{p^2}$ [3] generates a set of features that cannot be represented within the Euclidean shell. From the perspective of information geometry, the innovation operator A_p acts as a non-linear feature map: Euclidean states acquire new components, symmetries, and invariants through their embedding into the Lorentzian domain.

In this way, innovation can be understood as a *structural expansion*, where latent degrees of freedom are revealed through a finite algebraic extension.

Consolidation as finite structural selection. While the innovation phase makes accessible an expanded representational workspace, the consolidation phase imposes a finite informational bottleneck. Only a small portion of the Lorentzian structure is retained, encoded in a finite invariant family I_p. The invariant extractor J_p serves as a coarse-graining map that reduces the large innovation space, of size p^{2N}, to a finite set whose cardinality depends only on algebraic orbits, causal classes, or other structural attributes.

The Gödel recoding theorem (Proposition 1) shows that any such finite invariant family can be embedded into the arithmetic of the next shell. This is a general mechanism for information transfer across shells: invariants survive while the full innovation state does not.

Consolidation therefore acts as a form of *structural selection*: from a temporarily enlarged algebraic space, the system retains only an irreducible signature that can be represented within the finite alphabet of the next Euclidean shell.

Shell progression and cumulative structure. In the Finite Ring Continuum, the shell order increases linearly with the chronon according to $q = 4t + 1$ [1]. This linear progression contrasts sharply with the quadratic expansion associated with innovation. The innovation–consolidation cycle resolves the apparent tension between these two growth laws: the universe may temporarily access an extended representational space, but only a finite combinatorial summary of this structure is passed forward to the next shell.

Under this perspective, the cumulative structure of the continuum—its symmetries, invariants, and internal reference frames—is not derived directly from the raw richness of the quadratic extension, but from the sequence of consolidated invariant families I_p encoded across shells.

This effect mirrors evolutionary principles observed in hierarchical information-processing systems, where each layer incorporates structural summaries of the one before it.

Interpretation of the worked example. The explicit example for $p = 13$ (Section 6) demonstrated this process in concrete form. The Lorentzian innovation space $\mathbb{F}_{13^2}$ contains 169 states per degree

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of freedom, yet the invariant extractor based on the field norm yields an invariant family of size 13. Although in this case the invariant set happens to fit into the same shell $\mathbb{F}_{13}$, the Gödel recoding into $\mathbb{F}_{17}$ illustrates the general mechanism for transferring invariant structure along the shell sequence.

In more elaborate scenarios, such as invariant families derived from orbit-type classification under the finite Lorentz group, or from discrete mass-shell structure of the Dirac operator, one expects larger invariant families that may require the arithmetic capacity of the next shell. The intermediate-scale behaviour suggested in Conjecture 1 reflects this expectation.

Parallels with Biological Learning. Although the Finite Ring Continuum is a purely algebraic construct, the innovation–consolidation cycle uncovered in this work displays a notable parallel with learning dynamics in biological and artificial systems. In predictive-coding and Bayesian models of cognition [4,5,9], a large prediction error (surprise) [8] triggers a temporary broadening of the internal model: new latent variables, feature directions, or explanatory causes become accessible, and past evidence is reinterpreted in this expanded representational space [6,7]. This expansion is followed by consolidation, in which the newly discovered structure is compressed into a stable representation such as a concept, category, or memory trace, reflecting a transition from a rich but unstable posterior to a compact prior that guides future inference.

The quadratic extension $\mathbb{F}_{p^2}$ plays an analogous role in FRC. Innovation exposes additional algebraic degrees of freedom not present in the Euclidean shell, enabling the formation of new invariants that capture structural relations unavailable at the Euclidean level. Consolidation then selects a finite invariant signature and recodes it into the arithmetic of the next symmetry shell, paralleling the compression of novel information into a stable cognitive representation. Thus, both biological and algebraic systems exhibit a common pattern:

$$\text{prior} \longrightarrow \text{expanded innovation space} \longrightarrow \text{compressed posterior}.$$

While this analogy is interpretive rather than biological, it provides a useful conceptual bridge for understanding how finite universes—and finite cognitive systems—can accumulate structure through alternating phases of expansion and compression.

Relation to Universal Latent Representation. The innovation-consolidation framework developed in this work is closely aligned with the representational perspective formulated in [10], where we argue that independently trained foundational models across disparate modalities are shown to recover bijective coordinate charts of a single finite latent domain embedded in a symmetry-complete shell of the Finite Ring Continuum. The key mechanism underlying this universality is the alternation between expressive expansion (via nonlinear or multi-layer transformations) and representational compression into minimal sufficient statistics.

The present paper reveals that this alternation has a direct algebraic analogue in FRC. Innovation corresponds to the temporary enlargement of the representational domain via the quadratic extension $\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2}$, enabling the formation of Lorentzian invariants inaccessible within the Euclidean shell. Consolidation selects a finite invariant signature and recodes it into the next arithmetic shell, producing a stable, compressed representation of the newly discovered structure.

Viewed together, the two theories suggest that innovation-consolidation is not merely a mechanism internal to the FRC but also the structural principle by which finite learning systems—biological, computational, or algebraic—construct universal latent representations. Where ULR demonstrates that minimal sufficient embeddings coincide across modalities, the present work provides the underlying algebraic dynamics that generate, select, and propagate the representational invariants from one shell to the next.

Conceptual implications. The innovation–consolidation cycle provides a mechanism by which a finite universe can consistently generate and accumulate structure across discrete epochs. Innovation introduces new algebraic possibilities; consolidation retains only finite, symmetry-invariant signatures;

and shell progression provides a coherent arithmetic substrate on which these signatures can be encoded.

This mechanism offers a resolution to one of the conceptual challenges in the FRC programme: how large-scale, cumulative structure emerges from finite algebraic dynamics that evolve across discrete shells. It also opens potential connections with information-theoretic models of learning and adaptation, where systems repeatedly undergo phases of expansion and compression, integrating new structural information into a stable representational form.

Further exploration of invariant families, their growth rates, and their behaviour under Dirac evolution may yield deeper insights into the structural evolution of the Finite Ring Continuum.

8. Conclusions

This work introduced a formal innovation–consolidation cycle within the Finite Ring Continuum (FRC), providing a new algebraic perspective on how finite universes may generate, transform, and preserve structure across successive symmetry shells. This innovation phase was shown to arise naturally from the quadratic extension $\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2}$, the minimal enlargement required for Lorentzian geometry and Dirac evolution. This extension expands the representational capacity of a prime shell by a factor of p, revealing structural degrees of freedom that cannot be expressed within the Euclidean shell alone. The Dirac operator on $\mathbb{F}_{p^2}$ thus acts as a generator of emergent features: states acquire additional algebraic components through their projection into the Lorentzian domain.

From this expanded space, only a finite fraction of information is retained. The invariant extractor J_p compresses the Lorentzian innovation output into a finite signature I_p, typically far smaller than the innovation space and determined by orbit-types, causal classes, or other algebraic invariants. We proved that any such finite invariant family can be encoded into the next symmetry shell via a uniform Gödel recoding procedure. This establishes consolidation as a mathematically well-defined mechanism for transmitting structural information across shells, independent of the detailed form of the innovation operator.

The explicit example $p = 13$ demonstrated the entire cycle in a concrete and computable setting. A 169-element Lorentzian extension collapsed, under the norm invariant, into a 13-element signature, which was then injected into the next shell $\mathbb{F}_{17}$. Although this example represented an extreme case of compression, it illustrated clearly how innovation and consolidation interact within FRC.

The structural analogy with learning systems suggests that the innovation-consolidation cycle identified in the Finite Ring Continuum may reflect a general organisational principle of finite representational systems. Across biological [4,5,7], computational [8,9], physical [2,3], and more broadly algebraic domains, new structure is acquired through a transient expansion of the representational space and subsequently preserved through a selective compression into a stable form. In this perspective, the alternation of expansion and compression constitutes a unifying pattern by which finite systems integrate novelty while maintaining coherent long-term structure.

Conceptually, the results resolve a tension implicit in earlier FRC formulations: the quadratic growth of representational capacity associated with innovation and the linear shell progression $q = 4t + 1$. The innovation-consolidation cycle shows that these processes are complementary rather than contradictory. A universe may momentarily occupy a richer algebraic workspace, yet only a finite summary of the emergent structure propagates forward to subsequent shells. This mechanism offers a mathematically grounded explanation of how cumulative geometric and algebraic structure can arise from finite, discrete dynamics.

Several directions for future research emerge naturally. A systematic classification of invariant families associated with the Dirac operator may clarify whether the intermediate-scale behaviour conjectured in Conjecture 1 holds generically. It will also be valuable to explore how sequences of consolidated invariant families evolve across multiple shells, and whether such sequences converge, stabilise, or display new forms of algebraic organisation. Finally, the analogy between algebraic innovation in FRC and information processing in learning systems suggests deeper connections between finite-field geometry, representation theory, and epistemic dynamics.

Overall, the innovation-consolidation viewpoint provides a unifying principle for how finite representational systems—biological, computational, and physical—can acquire, assimilate and preserve structure. In each case, new information is introduced through a transient expansion of the representational domain and subsequently stabilised by a selective compression into invariant form. The Finite Ring Continuum offers a precise algebraic realization of this principle, linking learning dynamics with the structural evolution of a finite physical universe.

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