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A New Mechanics That Is Symmetrical to the Present Classical Mechanics

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Abstract

This paper presents the fundamental formulas of a new classical mechanics that is symmetrical to the present classical mechanics. Non-Euclidean geometries tell us that one of the postulates in an axiomatic system may take different forms. Modified postulates can lead to new axiomatic systems. The classical mechanics is an axiomatic systems and the Newton's three laws are three postulates. From the three laws, together with the necessary definitions of physical concepts and propositions, all contents of classical mechanics can be derived. Newton's second law can be simply stated as: force is the cause of acceleration. The author thinks that it can be modified as: force is the cause of deceleration. This results in a new axiomatic system called new classical mechanics. The most distinctive feature of the new mechanics is that the direction of momentum is opposite to that of velocity, and the kinetic energy is negative, i.e., a negative sign is attached to the PKE. Therefore, the new classical mechanics can be called negative kinetic energy (NKE) one, while the existing classical mechanics can be called PKE (PKE) one. These two parts can be collectively referred to as a while classical mechanics, which includes both PKE and NKE parts. The formulas of these two parts have symmetry with respect to positive and negative kinetic energy. The PKE classical mechanics describes the motion of macroscopic matter that we can observe, while the NKE classical mechanics describes the motion of macroscopic matter that we cannot observe, i.e., the motion of dark matter. Our universe has symmetry with respect to PKE and NKE, which is also the symmetry with respect to matter and dark matter. The basic equations of quantum mechanics describing the motion of micro-particles also have symmetry with respect to PKW and NKE, which has been elaborated in the author's previous work. The theory presented in this paper describe the motion of macroscopic NKE matter.

Keywords: new mechanics; classical mechanics; axiomatic system; Newton's second law; modified Newton's second law; symmetry; negative kinetic energy; dark matter

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1. Introduction

1.1. Axiomatic Systems in Geometry

Euclid assumed five postulates of geometry. Starting from these five postulates and defining some definitions and propositions, he strictly proved many theorems and derived a series of conclusions, thus establishing the geometry named after him. Euclidean geometry is an axiomatic system, and the entire Euclidean geometry is logically rigorous and self-consistent [1].

The fifth postulate can be simply stated as: given a straight line l and a point P outside l, there is one and only one straight line passing through P that is parallel to l. If these two lines are parallel, they remain parallel to infinity and never intersect. This postulate is also known as the "parallel postulate".

Lobachevsky challenged Euclid's fifth postulate. He found that the fifth postulate can also be modified to: given a straight line l and a point P outside l, at least two straight lines can be drawn through P that are parallel to l, and these two lines and the line l never intersect. This new fifth postulate, together with the original four postulates, forms a new set of five postulates. Starting from these new five postulates, together with appropriate definitions and propositions, the results of another type of geometry are derived, called Lobachevskian geometry. This is also an axiomatic system of geometry, and the entire Lobachevskian geometry is logically rigorous and self-consistent.

Later, Riemann proposed a third form of the fifth postulate: if a straight line is drawn through a point P outside a given straight line *l* line, then these two parallel lines inevitably intersect. In other words, there cannot be two completely parallel lines. Thus, Riemannian geometry was founded. Riemannian geometry is also an axiomatic system, which itself is logically rigorous and self-consistent without any flaws.

Lobachevskian geometry and Riemannian geometry are collectively referred to as non-Euclidean geometry.

The five postulates of Euclidean geometry are all based on the common sense of our daily life. Their correctness can be immediately verified with a paper and a pen, and the conclusions of Euclidean geometry can be verified in daily life.

However, the fifth postulates of Lobachevskian geometry and Riemannian geometry violate people's common sense. Therefore, the contents of non-Euclidean geometry is counterintuitive. Knowledge that violates common sense is not easily accepted at the very beginning.

But things that people have not seen in daily life are not necessarily non-existent. People's common sense is the experience accumulated from things involved in the scope of our daily life. In a larger scope or other aspects that we have not touched in daily life, there may be situations that violate our common sense, such as on the cosmic scale and the micro scale.

Later, people gradually learned that Euclidean geometry is the geometry of flat space, and the space of our daily life is a flat one. But the flat space we live in is only a local part of celestial bodies. On the cosmic scale, space may be curved. Non-Euclidean geometry describes the geometry of curved space. For example, the Earth's surface conforms to Riemannian geometry, and the geometry of general relativity is non-Euclidean geometry.

In mathematics, in addition to geometry, there are other axiomatic systems.

1.2. Axiomatic Systems in Physics

There are also axiomatic systems in physics, which can be distinguished into two cases.

One case is the application of axiomatic systems in mathematics to physical systems. A typical example is the application of non-Euclidean geometry in physics. If there is no upper limit to velocity, Galilean spacetime (i.e., Euclidean spacetime) can be used; if there is a finite upper limit to velocity, Minkowski spacetime is used [2]. Some people have tried to establish other axiomatic systems in the field of physics. Axiomatic systems for kinematics in Minkowski four-dimensional spacetime have been established [3–5]. Minkowski spacetime has been axiomatized in the style of Hilbert's axiomatization of Euclidean geometry, even in several versions [6]. The synchronization process of particle motion and light beam propagation is considered as Lobachevskian parallel lines in velocity space [7]. Multiple axiomatic systems for special relativity (SR) have been constructed over the past century [8]. An axiomatic system for the spacetime background of general relativity has also been proposed [9]. The above are axiomatic systems related to spacetime, which seem to merely involve kinematic parts, such as relative motion and parallax.

There are few studies on establishing axiomatic systems in dynamics. Only the construction of Lobachevskian geometry for super-conformal mechanics has been seen [10].

The other case is that there are axiomatic systems in dynamics in physics itself.

Classical mechanics is an axiomatic system. It has three postulates, namely Newton's three laws. Indeed, Newton wrote his book [11] according to the structure of an axiomatic system. Starting from Newton's three laws, together with some necessary definitions and propositions, such as velocity, acceleration, mass, momentum, angular momentum, energy, kinetic energy, potential energy, mechanical energy, force, conservative force, inertia, reference frame, etc., all contents of Newtonian mechanics can be obtained. Classical mechanics can be divided into two parts: Newtonian mechanics and special relativistic mechanics. The former applies to the case where the motion velocity of objects is much less than the speed of light, and there is Galilean transformation between different inertial frames. The latter applies to the case where the velocity is close to the speed of light, and Lorentz transformation is used between different inertial frames. For Newtonian mechanics, propositions such as Galilean transformation need to be added. For special relativistic mechanics, propositions such as the constancy of the light speed, the principle of relativity, and Lorentz transformation need to be added. Classical mechanics is a logically rigorous and self-consistent system, which is an axiomatic system.

Compared with axiomatic systems in mathematics, the advantages of the axiomatic system of classical mechanics are that its physical concepts are related to the motion of objective matter, and the conclusions derived from the postulates are generally expressed by mathematical formulas, which describe the motion of objectives. Therefore, according to Newton's three laws, quantitative research on the motion state of objects can be carried out. Among Newton's three laws, the second law is particularly important because specific quantitative calculations all start from this law.

Electromagnetism is also an axiomatic system. Based on Coulomb's law and the Lorentz transformation of special relativity, together with some definitions and propositions, such as charge, current, electric field, magnetic field, electric flux, etc., all contents of electromagnetism can be obtained [12]. Therefore, it can be said that electromagnetism has two postulates: Coulomb's law that describes force between two point charges and the Lorentz transformation of four-dimensional spacetime between two inertial reference frames. Starting from these two postulates, an axiomatic system is formed, which is logically rigorous and self-consistent.

1.3. There Can Be New Contents Concerning Classical Mechanics

The example of non-Euclidean geometry tells us that in an axiomatic system, some postulates can have different forms. As long as one of the postulates is modified into another form, a new set of postulates can be constituted. Starting from this new set of postulates, together with appropriate definitions and propositions, a new axiomatic system can be constructed, showing new mathematical contents. The new axiomatic system is required to be logically rigorous and self-consistent.

The author believes that classical mechanics can be developed along this line of thought, and the resultant can be called new classical mechanics: the way is modifying the second of Newton's three laws into another form. Newton's second law can be briefly stated as: force is the cause of acceleration. Now, this law is modified to be: force is the cause of deceleration. The new Newton's second law, together with the original first and third laws, constitutes three postulates. Starting from these three postulates, together with appropriate definitions and propositions, a new classical mechanics is formed, which is an axiomatic system. This new axiomatic system is logically rigorous and self-consistent.

The axiomatic system of classical mechanics has physical contents. Compared with the axiomatic system in geometry, the new classical mechanics reveals two characteristics. The first characteristic is that the physical concepts and formulas of the existing classical mechanics and the new classical mechanics show a kind of symmetry. The second characteristic is that the new axiomatic system reflects new physical contents.

The first characteristic is the exploration of a symmetry. The symmetry is exhibited everywhere in the whole classical mechanics. One example is that force producing acceleration and force producing deceleration show symmetry. Another example is the expressions of kinetic energy. It is well-known that when an object with mass m moves in velocity v, it is of a kinetic energy $K = mv^2/2$, which is always non-negative and will be called positive kinetic energy (PKE) hereinafter. It can be said that classical mechanics describes the behavior of PKE objects after being subjected to force. The most prominent feature of the motion of matter described by the new classical mechanics is that they have negative kinetic energy (NKE) with the expression $K_{(-)} = -mv^2/2$, i.e., a negative sign is attached to the PKE. The new classical mechanics describes the motion behavior of objects with NKE after being subjected to force. From now on, we will refer to the classical mechanics so far as PKE classical mechanics, and the new one as NKE classical mechanics. The two are collectively referred to as classical mechanics. That is to say, according to the author's view, the PKE classical mechanics and NKE classical mechanics, which are symmetric about each other, combine into the whole classical mechanics. The comprehensive symmetry between the two parts of the classical mechanics will be reflected in the main text starting from Section 2 of this paper.

The second characteristic is that the new axiomatic system should reflect new physical contents. It has become a common sense that the PKE classical mechanics describes the motion of visible macroscopic matter in the universe. The concept of NKE is not easily accepted. The NKE classical mechanics should also have physical contents, that is, it should also describe the motion of a type of matter in the universe. Before clearly pointing out what kind of matter the NKE energy describes, we briefly review the emergence and use of the concept of NKE. In fact, the concept of NKE has long appeared and has been applied in some cases.

The concept of NKE first appeared in 1929. Kudar [13] used the Schrödinger equation to calculate the problem of a free particle passing through a region where its energy is lower than the height of the potential barrier. This is the famous tunneling effect in quantum mechanics. Although the tunneling effect is now well known and is introduced in quantum mechanics textbooks, the concept of NKE involved in this problem is hardly discussed. However, since then, the concept of NKE has been mentioned intermittently.

In the discussion of measurement errors in quantum mechanics, the treatment of NKE states is involved [14]. In Rydberg atom scattering, incident electrons can have NKE [15]. The reverse of the wave vector in quantum mechanics can also lead to NKE [16]. The concept of NKE has also been used in supersymmetry theory [17]. Since the research on dark matter and dark energy has become a hot topic, NKE has been regarded as a possible model of dark energy or a term in the dark energy model [18,19]. Some people also believe that NKE is an important component in black holes [20], and that NKE is a ghost degree of freedom in the universe [21,22], but it is physically acceptable [23].

After the Dirac equation was established [24], a free electron is solved to have both positive and negative energy solutions. The positive energy expression is $\sqrt{m^2c^4 + c^2\boldsymbol{p}^2}$, and its low-momentum approximation is $mc^2 + \boldsymbol{p}^2/2m$, which obviously has PKE. The negative energy eigenvalue expression is $-\sqrt{m^2c^4 + c^2\boldsymbol{p}^2}$, and its low-momentum approximation is $-mc^2 - \boldsymbol{p}^2/2m$, which obviously has NKE. Positive and negative energies are two states of a free particle, which we refer to as positive and negative kinetic energy states respectively. For the NKE states appearing in the solutions of the Dirac equation, there are currently two popular interpretations. One is Dirac's interpretation that all negative energy states are filled (i.e., the Dirac Sea), and the other is that negative energy states belong to an antiparticle moving in the reverse time direction. Both interpretations lead to various contradictions [25,26]. Here, we merely point out one problem with each interpretation. The concept of the Dirac Sea violates the statistical distribution law of particles occupying energy levels [25], because at sufficiently low negative energy levels, the occupation probability of particles is greater than 1, which can be easily seen from the Boltzmann distribution factor $\mathrm{e}^{-E_n/k_\mathrm{B}T}$. The interpretation that negative energy solutions belong to an antiparticle was immediately rejected by Dirac because it involves charge non-conservation [27]. In addition to the problem of charge non-conservation, this interpretation also leads to several contradictions. We only point out one: a differential equation only reflects the basic laws followed by particle motion, not the real motion of a particle. The solution of an equation with initial conditions is the real motion of the particle. The time in the equation either points to the future or the past, and one of the two must hold [28,29]. It is impossible for time to point to both the future and the past simultaneously. Dirac clearly pointed out that when solving the equation, "the wave function at any time determines the wave function at any later time" [24].

So, what is the correct interpretation of the NKE state? In other words, what kind of matter should have NKE? The author's view is: matter with NKE is dark matter. People have never observed matter with NKE because they are dark.

1.4. A Brief Review of Dark Matter Research

Dark matter and dark energy are hot topics in current physics research. Various dark matter models [30–51] and dark energy models [52–54] have been established. For example, the axion model [30–34], the cold dark matter model (LCDM) [35,36], weakly interacting massive particles (WIMPs) [37,38], etc. People have speculated on the possible properties of dark matter [55–63], considered models of the interaction between dark matter and dark energy [64,65], and also assumed that dark matter may have new forces [43,44]. Researchers have tried to detect dark matter through various experimental methods to test theoretical models [66–74]. At least in some energy ranges, certain dark matter particles assumed by people have been clearly excluded, including axions [75–78]. The detection of possible new interactions also yielded zero results [79,80]. Generally speaking, the accuracy of experiments in this field is very high [81–84]. Although attempts to search for dark matter are still ongoing [85–87], and new suggestions and ideas are constantly being put forward in this regard [88], it can be said that so far, none of the existing theoretical models of dark matter has been clearly confirmed. In fact, what dark matter is and what kind of instruments should be employed to detect it are not very clear so far [89,90].

In addition to the continuous exclusion of particles predicted by dark matter models, we point out the following key points for the existing theoretical models of dark matter and dark energy. Different theoretical models need to be established for dark particles in different energy ranges and mass ranges, and there is no unified model for dark matter; the theories are more or less speculative and not logically rigorous; it is subjectively assumed that dark matter cannot be detected must be because it does not participate in certain basic interactions; dark matter and dark energy are essentially irrelevant; dark matter only has models of micro-particles, and it is difficult to establish a theory of macroscopic dark matter objects.

Starting from the interpretation of the negative energy solutions of the Dirac equation, the author has conducted a series of investigations on issues related to NKE in quantum mechanics [25,26,91–96]. The Dirac equation itself is symmetric with respect to PKE and NKE. As we all know, the low-momentum approximation (i.e., non-relativistic approximation) of the Dirac equation can lead to the Schrödinger equation. We found that the Dirac equation has another low-momentum approximation, which leads to NKE Schrödinger equation. In fact, Schrödinger himself obtained both the positive and negative kinetic energy Schrödinger equations in his original paper [97], but he abandoned the NKE Schrödinger equation because he thought it was meaningless. The Klein-Gordon equation is considered to describe the relativistic motion of spin-zero particles. We believe that, like the Dirac equation and the Schrödinger equation, the relativistic quantum mechanics equation describing spin-zero particles should also be a differential equation of the first derivative with respect to time rather than the second derivative, and should also be symmetric with respect to positive and negative kinetic energy [26]. Therefore, the basic formulas of quantum mechanics are all symmetric with respect to PKE and NKE, as shown in Table 1. We have proposed aexperiments to detect NKE electrons [91]. Starting from the Schrödinger equation, after making the hydrodynamic approximation, we can obtain the basic formulas of Newtonian mechanics. Starting from the PKE decomposed Klein-Gordon equation, we can obtain the basic dynamic formulas of special relativity in classical mechanics. That is to say, starting from the basic formulas of quantum mechanics, we can derive the basic formulas of classical mechanics [93]. Classical mechanics describes the laws of motion of PKE macroscopic objects. Similarly, starting from the NKE Schrödinger equation, we can obtain the basic formulas of NKE Newtonian mechanics. Starting from the NKE decomposed Klein-Gordon equation, we can obtain the basic dynamic formulas of special relativity for NKE objects. In short, starting from the basic formulas of quantum mechanics describing micro NKE particles, we can derive the basic formulas describing the motion of NKE macroscopic objects [93]. Thus, from micro to macro, all basic mechanical equations are symmetric with respect to PKE and NKE, as shown in Table 1 [26]. When matter is in the NKE state, it can be observed by us, while the NKE state is the dark state.

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Table 1. Basic equations describing matter motion [26].

<table>
<tr><th>Matter</th><th>Laws of Motion</th><th colspan="2"></th><th>PKE System</th><th>NKE system</th></tr>
<tr><td rowspan="2">Macroscopic objects</td><td rowspan="2">Classical mechanics</td><td colspan="2">Nonrelativistic motion</td><td>Newtonian mechanics</td><td>NKE Newtonian mechanics*</td></tr>
<tr><td colspan="2">Relativistic motion</td><td>Relativistic mechanics</td><td>NKE relativistic motion*</td></tr>
<tr><td rowspan="3">Microscopic objects</td><td rowspan="3">Quantum mechanics</td><td colspan="2">Nonrelativistic motion</td><td>Schrodinger equation</td><td>NKE Schrodinger equation†</td></tr>
<tr><td rowspan="2">relativistic motion</td><td>Spin-0 particle</td><td>PKE decoupled Klein-Gordon equation</td><td>NKE decoupled Klein-Gordon equation†</td></tr>
<tr><td>Spin-1/2 particle</td><td colspan="2">Dirac equation†</td></tr>
</table>

*This paper presents the specific formulas.

Why is the NKE matter considered dark? The reason should be explained. In order to answer this question, we should first ask such a question: why can PKE matter be observed? The answer is: matter absorbs and emits energy, i.e., photons. Instruments can absorb photons, which is the means to observe the motion of matter. We say that matter and energy are matched. The reason that the NKE matter is dark is because they absorb and emit dark energy, which is negative energy, i.e., photons with negative energy. Therefore, we say that dark matter and dark energy are matched. The author has established a theory of dark energy [98]. Dark energy is symmetric with energy. Therefore, our universe has the following symmetries: PKE - NKE, matter - dark matter, energy - dark energy, matter-energy matching - dark matter-dark energy matching.

So far, the author has mainly studied the motion of NKE micro-particles [25,26,91–96], the cells marked with dagger † in Table 1, and rarely involved the motion of macroscopic NKE objects [95]. This paper completely describes the theory of the dynamics of macroscopic NKE objects, i.e., we present specific formulas that belong to the two cells marked with asterisk * in Table 1. Since this paper only discusses the motion of macroscopic dark matter, it avoids using quantum mechanics as a tool. Therefore, it is elaborated from the perspective of constructing a new axiomatic system starting from the modified Newton's second law.

We review the formulas of particle kinematics in the next subsection. In Section 2, we describe the modified Newton's second law. Section 3 describes particle dynamics. Section 4 describes the motion trajectory of a particle under the action of force. Section 5 describes the dynamics of a system of NKE particles. Section 6 describes the dynamics of a mixed system containing both PKE and NKE particles. Section 7 describes Lagrangian mechanics. Section 8 describes relativistic mechanics. Section 9 describes the Virial theorem. Finally, there are discussions and conclusions.

1.5. Review of Particle Kinematics

First of all, we review the formulas of particle kinematics.

After establishing a spatial coordinate system, the position of a particle in space at each moment is represented by the position vector $\boldsymbol{r}$ relative to this coordinate system, referred to as the position vector for short, which is a function of time t:

$$\boldsymbol{r} = \boldsymbol{r}(t). \tag{1}$$

This formula indicates that at each moment, the position vector of the particle is determined. This formula is also called the motion trajectory of the particle. The time derivative of the position vector is the velocity of the particle's motion.

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{2}$$

The time derivative of velocity is acceleration:

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}. \tag{3}$$

Let there be a reference frame S, and another reference frame S' moving with a translational velocity $\boldsymbol{v}_0$ relative to S. An object A moves with velocity $\boldsymbol{v}$ in S and with velocity $\boldsymbol{v}'$ in S'. Then,

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v}_0. \tag{4}$$

Let the reference frame S' move with a translational acceleration $\boldsymbol{a}_0$ relative to S. An object A moves with acceleration $\boldsymbol{a}$ in S and with acceleration $\boldsymbol{a}'$ in S'. Then,

$$\boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{a}_0. \tag{5}$$

This formula can be obtained by differentiating both sides of (4) with respect to time.

The above is the case where there is only translation and no rotation between the reference frames. If there is mutual rotation but no translation between the two reference frames, then the relationship between $\boldsymbol{v}$ and $\boldsymbol{v}'$ at the position vector $\boldsymbol{r}$ in S can be written as:

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v}_{\mathrm{I}}(\boldsymbol{r}), \tag{6}$$

where it is emphasized that $\boldsymbol{v}_{\mathrm{I}}(\boldsymbol{r})$ depends on the position vector $\boldsymbol{r}$. Differentiating both sides of this formula with respect to time, the relationship between the acceleration $\boldsymbol{a}$ in S and the acceleration $\boldsymbol{a}'$ in S' is:

$$\boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{a}_{\mathrm{I}}(\boldsymbol{r}), \tag{7}$$

where $\boldsymbol{a}_{\mathrm{I}}(\boldsymbol{r})$ is called the inertial acceleration. We consider the following two cases.

Let the origins and z-axes of the Cartesian coordinate systems of reference frames S and S' coincide. S is stationary. The xy-planes of S and S' coincide, but S' rotates around the z-axis with an angular velocity ω. Then, S' is called a plane rotating reference frame.

Consider a particle moving in the xy-plane. Its position vector in S is $\boldsymbol{r}$. Its motion velocity and acceleration relative to S are denoted as $\boldsymbol{v}$ and $\boldsymbol{a}$, respectively, and relative to S' as $\boldsymbol{v}'$ and $\boldsymbol{a}'$, respectively. It can be derived that Eq. (6) at this time has the following form:

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}, \tag{8}$$

where the second term $\boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{v}_{\mathrm{I}}(\boldsymbol{r})$ is the convected velocity. Differentiating both sides of Eq. (8) with respect to time, the relationship between the accelerations in the two reference frames is:

$$\boldsymbol{a} = \boldsymbol{a}' - \omega^2\boldsymbol{r} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + 2\boldsymbol{\omega} \times \boldsymbol{v}' = \boldsymbol{a}' + \boldsymbol{a}_t + \boldsymbol{a}_c. \tag{9}$$

Comparing this formula with (7), we get:

$$\boldsymbol{a}_{\mathrm{I}}(\boldsymbol{r}) = \boldsymbol{a}_t + \boldsymbol{a}_c, \tag{10}$$

i.e., the inertial acceleration $\boldsymbol{a}_{\mathrm{I}}(\boldsymbol{r})$ consists of two parts.

$$\boldsymbol{a}_t = -\omega^2\boldsymbol{r} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} \tag{11}$$

is the convected acceleration, where $-\omega^2\boldsymbol{r}$ is the centripetal acceleration; $\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$ is the tangential acceleration generated by the change of the rotation angle of the reference frame S' with time. If the angular velocity ω is constant, this term is zero.

$$\boldsymbol{a}_c = 2\boldsymbol{\omega} \times \boldsymbol{v}' \tag{12}$$

is the Coriolis acceleration.

Let the origins of the Cartesian coordinate systems of reference frames S and S' coincide. S is stationary. S' rotates around the coordinate origin with an angular velocity $\boldsymbol{\omega}$. Then, S' is called a spatial rotating reference frame.

Consider a particle moving in space. Its position vector relative to *S* is denoted as $\boldsymbol{r}$. Its motion velocity and acceleration relative to S are denoted as $\boldsymbol{v}$ and $\boldsymbol{a}$, respectively, and relative to S' as $\boldsymbol{v}'$ and $\boldsymbol{a}'$, respectively. It can be derived that:

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}, \tag{13}$$

where the latter term $\boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{v}_{\mathrm{I}}(\boldsymbol{r})$ is the convected velocity. Differentiating both sides of Eq. (13) with respect to time, the relationship between the accelerations in the two reference frames is:

$$\boldsymbol{a} = \boldsymbol{a}' - \omega^2 \boldsymbol{r} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \omega(\boldsymbol{\omega} \cdot \boldsymbol{r}) + 2\boldsymbol{\omega} \times \boldsymbol{v}' = \boldsymbol{a}' + \boldsymbol{a}_t + \boldsymbol{a}_c. \tag{14}$$

Comparing this formula with (7), we get that the inertial acceleration $\boldsymbol{a}_{\mathrm{I}}(\boldsymbol{r})$ again consists of two parts:

$$\boldsymbol{a}_{\mathrm{I}}(\boldsymbol{r}) = \boldsymbol{a}_t + \boldsymbol{a}_c, \tag{15}$$

where

$$\boldsymbol{a}_t = -\omega^2 \boldsymbol{r} + \omega(\boldsymbol{\omega} \cdot \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} \tag{16}$$

is the convected acceleration. The $\omega^2 \boldsymbol{r} + \omega(\boldsymbol{\omega} \cdot \boldsymbol{r})$ is related to the centripetal acceleration; $\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$ is the tangential acceleration generated by the change of the rotation angular velocity of the reference frame S' with time. If the angular velocity $\boldsymbol{\omega}$ is constant or does not change with time, or $\dot{\boldsymbol{\omega}}$ is perpendicular to $\boldsymbol{r}$, this term is zero.

$$\boldsymbol{a}_c = 2\boldsymbol{\omega} \times \boldsymbol{v}' \tag{17}$$

is the Coriolis acceleration.

If, in the above two cases of rotating reference frames, the origins of S and S' do not coincide, and S' has a translational velocity v_0 and a corresponding translational acceleration a_0 relative to S in addition to rotation, then v_0 should be added to the right side of Eq. (6):

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v}_{\mathrm{I}}(\boldsymbol{r}) + \boldsymbol{v}_0. \tag{18}$$

Differentiating both sides of this formula with respect to time, $\boldsymbol{a}_0$ should be added to the right side of Eq. (7):

$$\boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{a}_{\mathrm{I}}(\boldsymbol{r}) + \boldsymbol{a}_0. \tag{19}$$

Hereinafter, we generally refer to

$$\boldsymbol{a}_{\mathrm{I}} = \boldsymbol{a} - \boldsymbol{a}' \tag{20}$$

as the inertial acceleration, which is related to the position vector $\boldsymbol{r}$, and $\boldsymbol{a}_{\mathrm{I}}$ includes both the acceleration generated by the translation of *S* relative to S' and that generated by the rotation of *S* relative to S'.

The above merely involves kinematic quantities, not mass *m*, so no dynamic quantities are involved. A dynamic quantity is always composed of a kinematic quantity multiplied by mass, and so it is always proportional to mass. In the following discussion, the mass of the particle is always assumed to be constant.

2. Original and Modified Newton's Three Laws

First, we review Newton's three laws [11].

Newton's law I: "Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon."

Newton's law II: "The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed."

Newton's law III: "To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."

We paraphrase Newton's second law in terms of the terminologies commonly used nowadays and express it by a formula. The Newton's second law can simply be said as: "Force is the cause of acceleration." The mathematical expression is as follows. If an object with mass *m* is acted upon by a force $\boldsymbol{F}$, then the motion of the object will have an acceleration $\boldsymbol{a}$. The following relationship holds:

$$\boldsymbol{F} = m\boldsymbol{a}. \tag{21}$$

In other words, the acceleration $\boldsymbol{a}$ that an object acquires equals to the force $\boldsymbol{F}$ divided by its mass m, and the direction of the $\boldsymbol{a}$ is the same as that of $\boldsymbol{F}$. Obviously, the greater the magnitude of force $\boldsymbol{F}$, the greater the magnitude of the object's acceleration $\boldsymbol{a}$.

Equation (21) can be called the acceleration form of Newton's second law. Substituting Eq. (3) into (21), we get:

$$\boldsymbol{F} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}. \tag{22}$$

If the object is not acted upon by external forces, $\boldsymbol{F} = 0$, then from Eq. (21), the acceleration $\boldsymbol{a} = 0$, and at this time $\boldsymbol{v} = \text{const.}$, with a special case of $\boldsymbol{v} = 0$. Therefore, when an object is not acted upon by external forces, it moves in a straight line at a constant speed or remains at rest. This is Newton's first law.

Then, we suggest another form of Newton's second law.

Modified Newton's law II: "The alteration of motion is ever proportional to the motive force impressed; and is made in the **opposite** direction of the right line in which that force is impressed."

Compared to the original Newton's second law, this modified one only adds one word "opposite". This new Newton's second law should be paraphrased in terms of the terminologies commonly used nowadays and expressed by a formula. It can be simply said as: "Force is the cause of negative acceleration, or deceleration." The mathematical expression is as follows. If an object with mass m is acted upon by a force $\boldsymbol{F}$, then the motion of the object will have a negative acceleration $-\boldsymbol{a}$. The following relationship holds:

$$\boldsymbol{F} = -m\boldsymbol{a}. \tag{23}$$

In other words, the deceleration $-\boldsymbol{a}$ that an object acquires equals to the force $\boldsymbol{F}$ divided by its mass m, and the direction of the $\boldsymbol{a}$ is opposite to that of $\boldsymbol{F}$. Obviously, the greater the magnitude of force $\boldsymbol{F}$, the greater the magnitude of the object's acceleration $\boldsymbol{a}$. Equation (23) can be called the deceleration form of the new Newton's second law.

Substituting Eq. (3) into (23), we get:

$$\boldsymbol{F} = -m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}. \tag{24}$$

If the object is not acted upon by external forces, $\boldsymbol{F} = 0$, then from Eq. (21), the deceleration $-\boldsymbol{a} = 0$, and at this time $\boldsymbol{v} = \text{const.}$, with a special case of $\boldsymbol{v} = 0$. Therefore, when an object is not acted upon by external forces, it moves in a straight line at a constant speed or remains at rest. This is Newton's first law.

Equation (21) can be stated in qualitative language as: "A change in motion is proportional to the motive force impressed and takes place along the line in which that force is impressed." [99] Then, correspondingly, Eq. (23) can be stated as: "A change in motion is proportional to the motive force impressed and takes place along the line but in the opposite direction in which that force is impressed."

Newton's law I and law III remain unchanged.

Hereinafter, we present the contents of the new classical mechanics, and do not recall the corresponding contents of the original classical mechanics unless necessary.

3. Particle Dynamics

3.1. Momentum of a Particle and the Law of Conservation of Momentum

3.1.1. Definition of Momentum

We define momentum $\boldsymbol{p}_{(-)}$ as follows:

$$\boldsymbol{p}_{(-)} = -m\boldsymbol{v}. \tag{25}$$

Since this definition differs by a negative sign from the well-known definition of momentum $\boldsymbol{p} = m\boldsymbol{v}$, we add a subscript (–) to the momentum defined in (25) for distinction. From this definition, the direction of momentum is always opposite to that of velocity. This situation violates our common sense. So far, we have never seen a case where the momentum of an object is opposite to the direction of its velocity. However, we cannot rule out the possibility of such a situation.

Equation (25) and the well-known relational expression $\boldsymbol{p} = m\boldsymbol{v}$ reflect a kind of symmetry, that is, the momentum of an object may be in the same direction as its velocity or opposite to it.

Substituting Eq. (2) into (25), the expression of momentum can also be written as:

$$\boldsymbol{p}_{(-)} = -m\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{26}$$

3.1.2. Momentum Theorem

According to the definition of momentum in (25), Eq. (24) can be written as:

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}_{(-)}}{\mathrm{d}t}. \tag{27}$$

The physical meaning of this formula is: the force exerted on an object is equal to the time rate of change of its momentum. This form is the same as that of Newton's second law in the form of $\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$. Equation (27) is also called the momentum theorem, or the differential form of the momentum theorem, because there is also an integral form of the momentum theorem to be exhibited in the following.

Compared with (23), Eq. (27) adds three new physical significances, which we describe as follows.

The first new physical significance is: when an object is acted upon by an external force, its momentum will change. This reflects the effect of the force through the change of the object's momentum. In contrast, Eq. (23) reflects the effect of the force through deceleration.

The second new physical significance is: momentum contains two product factors, mass m and velocity $\boldsymbol{v}$. When momentum changes with time, the mass m may remain unchanged while the velocity $\boldsymbol{v}$ changes with time, or both the mass m and the velocity $\boldsymbol{v}$ may change with time. That is to say, Eq. (27) is also applicable to objects with variable mass. In contrast, Eq. (23) is only applicable to objects with constant mass. Substituting Eq. (25) into (27) and assuming that the mass m remains unchanged, we get (23). Therefore, (23) is a special case of (27).

The third new physical significance is: inspecting Eq. (27) from left to right, if an object is acted upon by an external force, its momentum will change; inspecting Eq. (27) from right to left, this formula indicates that if the momentum of an object changes, then the object will exert a force on the environment.

Equation (27) is rewritten as:

$$\mathrm{d}\boldsymbol{p}_{(-)} = \boldsymbol{F}\mathrm{d}t. \tag{28}$$

The left side is the differential of momentum, and the right side is called the elementary impulse of force $\boldsymbol{F}$. Integrating both sides with respect to time, we get:

$$\boldsymbol{p}_{(-)}(t_2) - \boldsymbol{p}_{(-)}(t_1) = \int_{t_1}^{t_2} \boldsymbol{F}\,\mathrm{d}t. \tag{29}$$

The right side of the equation is the integral of the action force with respect to time, called the impulse of the force over a period of time, which is equal to the momentum of the object at the final moment minus the momentum at the initial moment. Equation (29) is called the integral form of the momentum theorem.

3.1.3. Law of Conservation of Momentum

From Eq. (29), when the action force $\boldsymbol{F}=0$, it is obtained that the momentum at any two moments is equal, that is, the momentum does not change with time. We can write:

$$\boldsymbol{p}_{(-)}=\text{const.} \tag{30}$$

This is the law of conservation of momentum: when a particle is not acted upon by external forces, its momentum remains unchanged.

3.2. *Inertial Force*

In a reference frame, if the new Newton's second law (23) holds, then this reference frame is called an inertial frame; otherwise, it is called a non-inertial frame.

Let *S* be an inertial frame, and another reference frame S' moving with a translational acceleration $\boldsymbol{a}_0$ relative to *S*. The acceleration of particle A in *S* is $\boldsymbol{a}$, and in S' is $\boldsymbol{a}'$. Multiplying both sides of (5) by mass *m* and combining with (23), we get:

$$\boldsymbol{F}=-m\boldsymbol{a}'-m\boldsymbol{a}_0. \tag{31}$$

A force $\boldsymbol{F}$ remains the same in different reference frames. In the accelerating reference frame S', $\boldsymbol{F}\neq-m\boldsymbol{a}'$. Therefore, S' is a non-inertial frame. A reference frame moving with acceleration relative to an inertial frame is a non-inertial frame.

If a reference frame S' moves uniformly relative to an inertial frame *S*, i.e., the convected acceleration is zero, $\boldsymbol{a}_0=0$, then in this S', we have $\boldsymbol{F}=-m\boldsymbol{a}'$. This S' is an inertial frame. Therefore, reference frames moving uniformly relative to each other are all inertial frames. This fact is also called the "Galilean principle of relativity".

Now, we focus on observing the motion of objects in S'. In this reference frame, the motion acceleration of the object is $\boldsymbol{a}'$. We write (31) in the following form:

$$\boldsymbol{F}+m\boldsymbol{a}_0=-m\boldsymbol{a}'. \tag{32}$$

Combining the two terms on the left side into one term, $\boldsymbol{F}+m\boldsymbol{a}_0=\boldsymbol{F}'$, we formally write (32) as:

$$\boldsymbol{F}'=-m\boldsymbol{a}'. \tag{33}$$

That is, it looks the same as the form of the new Newton's second law (23). However, in (33), since:

$$\boldsymbol{F}'=\boldsymbol{F}+m\boldsymbol{a}_0, \tag{34}$$

the particle is not only acted upon by the resultant external force ***F*** to move with acceleration $\boldsymbol{a}'$, but also acted upon by another force $\boldsymbol{F}_{\mathrm{I}}$, whose expression is:

$$\boldsymbol{F}_{\mathrm{I}}=m\boldsymbol{a}_0. \tag{35}$$

This force is caused by the reference frame having an acceleration $\boldsymbol{a}_0$ relative to the inertial frame. Its magnitude is equal to the product of the mass m and the convected acceleration $\boldsymbol{a}_0$, and its direction is the same as that of $\boldsymbol{a}_0$. This non-interactive force is usually called the inertial force. After introducing the inertial force, for the non-inertial frame, the new Newton's second law still holds in form, that is, (33) in the non-inertial frame has the same form as (23) in the inertial frame. Note that the direction of the inertial force $m\boldsymbol{a}_0$ in the new Newtonian mechanics is exactly opposite to that of the inertial force $-m\boldsymbol{a}_0$ in the original Newtonian mechanics.

The inertial force is quite different from the external force acting on the object, similar to the case in Newtonian mechanics. First, if an object is acted upon by an inertial force, there is no force exertor, because this is a non-interactive force, which only reflects that the reference frame is a non-inertial frame. Second, according to Newton's third law, if an object A is acted upon by a force from another object B, then B must be acted upon by a reaction force from A. Since the inertial force has no force exertor, there is no reaction force.

Let *S* be an inertial frame, and S' have both translational acceleration and rotational acceleration relative to *S*. We have defined the inertial acceleration in subsection III.E:

$$\boldsymbol{a}_{\mathrm{I}}=\boldsymbol{a}-\boldsymbol{a}'. \tag{36}$$

The cases where S' is a plane rotating reference frame and a spatial rotating reference frame have been discussed in subsection III.E. A particle moving in S' is acted upon by an inertial force $\boldsymbol{F}_\mathrm{I}$, which is equal to the mass multiplied by the inertial acceleration:

$$\boldsymbol{F}_\mathrm{I} = m\boldsymbol{a}_\mathrm{I} = m(\boldsymbol{a} - \boldsymbol{a}')\text{。} \tag{37}$$

The expression of the new Newton's second law in S' is:

$$\boldsymbol{F}' = -m\boldsymbol{a}', \tag{38}$$

where

$$\boldsymbol{F}' = \boldsymbol{F} + \boldsymbol{F}_\mathrm{I}. \tag{39}$$

Note that the direction of the inertial force $m(\boldsymbol{a} - \boldsymbol{a}')$ in the new Newtonian mechanics is exactly opposite to that of the inertial force $-m(\boldsymbol{a} - \boldsymbol{a}')$ in the original Newtonian mechanics.

3.3. Angular Momentum of a Particle and the Law of Conservation of Angular Momentum

3.3.1. Torque

Let A and B be two points in space. The position vector of point B relative to point A is denoted as $\boldsymbol{r}$. A force $\boldsymbol{F}$ acts on point B. A plane is formed by $\boldsymbol{r}$ and $\boldsymbol{F}$. The torque about point A is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{F}. \tag{40}$$

Torque $\boldsymbol{L}$ is the vector product of the position vector $\boldsymbol{r}$ and the force $\boldsymbol{F}$, and its direction is perpendicular to the plane where $\boldsymbol{r}$ and $\boldsymbol{F}$ are located. The three vectors $\boldsymbol{r}$, $\boldsymbol{F}$, and $\boldsymbol{L}$ form the right-hand rule. The magnitude of the torque is:

$$L = rF\sin\theta, \tag{41}$$

where θ is the angle between the $\boldsymbol{r}$ and the force $\boldsymbol{F}$.

3.3.2. Angular Momentum

Definition of angular momentum

If the position vector of an object relative to the coordinate origin is $\boldsymbol{r}$, and the momentum at this position is $\boldsymbol{p}_{(-)}$, then the angular momentum is defined as

$$\boldsymbol{J}_{(-)} = \boldsymbol{r} \times \boldsymbol{p}_{(-)}. \tag{42}$$

Replacing the force $\boldsymbol{F}$ in the definition of torque (40) with momentum $\boldsymbol{p}_{(-)}$ gives the definition of angular momentum.

Angular momentum theorem

Substituting Eq. (25) in to (42), we obtain

$$\boldsymbol{J}_{(-)} = -\boldsymbol{r} \times m\boldsymbol{v}, \tag{43}$$

whose direction is just opposite to that of the angular momentum $\boldsymbol{J} = \boldsymbol{r} \times m\boldsymbol{v}$ in Newtonian mechanics.

Taking the cross product of the position vector $\boldsymbol{r}$ with both sides of the new Newton's second law (27),

$$\boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}_{(-)}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r} \times \boldsymbol{p}_{(-)}\right) - \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p}_{(-)} = \frac{\mathrm{d}\boldsymbol{J}_{(-)}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times m\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{44}$$

The left side of this formula is exactly the definition of torque, and the second term on the right side is zero. The result is

$$\boldsymbol{L} = \frac{\mathrm{d}\boldsymbol{J}_{(-)}}{\mathrm{d}t}. \tag{45}$$

This formula indicates that the torque acting on an object is equal to the time rate of change of the object's angular momentum. This formula is called the angular momentum theorem, or the differential form of the angular momentum theorem.

Equation (45) is written in another form:

$$\mathrm{d}\boldsymbol{J}_{(-)} = \boldsymbol{L}\mathrm{d}t. \qquad (46)$$

The left side is the differential of angular momentum, and the right side is called the elementary angular impulse. Integrating with respect to time, we get

$$\boldsymbol{J}_{(-)}(t_2) - \boldsymbol{J}_{(-)}(t_1) = \int_{t_1}^{t_2} \boldsymbol{L}\,\mathrm{d}t. \qquad (47)$$

The right side of the equation is the accumulation of torque over time, called the angular impulse. Equation (47) indicates that the angular impulse of the external torque acting on an object over a period of time is equal to the angular momentum of the object at the final moment minus the angular momentum at the initial moment. This is the integral form of the angular momentum theorem.

Law of conservation of angular momentum

From Eq. (47), if the torque is zero over a period of time, then the angular momentum of the object remains unchanged during this period, or it can be written as

$$\boldsymbol{J}_{(-)} = \text{const}. \qquad (48)$$

If the external torque acting on a system is zero, the total angular momentum of the system is conserved. This is the law of conservation of angular momentum.

3.4. Negative Kinetic Energy of a Particle and the Law of Conservation of Negative Kinetic Energy

3.4.1. Work and Power

When a force acts on a particle and causes the particle to produce a displacement along the direction of the force, we say that the force does work on the particle. In general, work is the product of two factors: one is the force, and the other is the displacement of the particle along the direction of the force.

Let a particle move in a straight line under the action of a constant force $\boldsymbol{F}$ and produce a displacement $\Delta\boldsymbol{r}$. The work done by the force over this displacement is defined as:

$$W = \boldsymbol{F}\cdot\Delta\boldsymbol{r} = F|\Delta\boldsymbol{r}|\cos\theta, \qquad (49)$$

where θ is the angle between the $\boldsymbol{F}$ and $\Delta\boldsymbol{r}$.

If a particle moves along a curved path or the force acting on it is changing, we calculate the elementary work done by the force over an infinitesimal displacement $\mathrm{d}\boldsymbol{r}$:

$$\mathrm{d}W = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}. \qquad (50)$$

When a particle moves from point A to point B along a curve under the action of a variable force, the work done by the force over this path is

$$W = \int_{\mathrm{A}}^{\mathrm{B}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_{\mathrm{A}}^{\mathrm{B}} F\cos\theta\,\mathrm{d}s, \qquad (51)$$

where $\mathrm{d}s = |\mathrm{d}\boldsymbol{r}|$ is the line element of the curve. Note that the angle θ in (51) may be changing.

The physical quantity that characterizes the rate of doing work is power P, which is defined as the work done per unit time:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t}. \qquad (52)$$

Substituting Eq. (50) into it, we get

$$P = \boldsymbol{F}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{F}\cdot\boldsymbol{v}. \qquad (53)$$

3.4.2. Negative Kinetic Energy

Definition of negative kinetic energy

Taking the dot product of both sides of the new Newton's second law (24) with the infinitesimal displacement $\mathrm{d}\boldsymbol{r}$, we have

$$\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = -\frac{\mathrm{d}(m\boldsymbol{v})}{dt}\cdot \mathrm{d}\boldsymbol{r} = \mathrm{d}(-\tfrac{1}{2}m\boldsymbol{v}^2) = \mathrm{d}K_{(-)}. \tag{54}$$

Here, we define the physical quantity of negative kinetic energy (NKE) as

$$K_{(-)} = -\tfrac{1}{2}m\boldsymbol{v}^2 = -\tfrac{1}{2m}\boldsymbol{p}_{(-)}^2. \tag{55}$$

According to this definition, the NKE of an object cannot be positive.

Negative kinetic energy is a new concept, which is a physical quantity that violates our common sense. So far, people have never found anybody with NKE. However, as we said earlier, as long as it is logically rigorous and self-consistent, things that have not been seen are not necessarily non-existent.

Equation (55) and the well-known definition of kinetic energy $K = \frac{1}{2}m\boldsymbol{v}^2 = \frac{1}{2m}\boldsymbol{p}^2$ reflect a kind of symmetry, that is, a symmetry of positive and negative kinetic energy. An object may have PKE or NKE.

Work-energy theorem

Let a particle move from point A to point B under the action of force F. The velocities of the particle at these two positions are denoted as $\boldsymbol{v}_{\mathrm{A}}$ and $\boldsymbol{v}_{\mathrm{B}}$, respectively. The work done by the force over this path can be obtained by substituting (54) into (51):

$$W = \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}\left(-\tfrac{1}{2}m\boldsymbol{v}^2\right) = -\tfrac{1}{2}m\boldsymbol{v}_{\mathrm{B}}^2 - (-\tfrac{1}{2}m\boldsymbol{v}_{\mathrm{A}}^2) = K_{(-)\mathrm{B}} - K_{(-)\mathrm{A}}. \tag{56}$$

The physical meaning of this formula is: the work done by the force on the particle over a period of time is equal to the NKE of the particle at the final position minus that at the initial position. The work done by the force changes the NKE of the particle. This is called the work-energy theorem, or the integral form of the work-energy theorem. In contrast, Eq. (54) can be called the differential form of the work-energy theorem.

Equation (54) also shows that when the direction of the action force is the same as that of the displacement, the NKE of the object increases, i.e., the absolute value of the NKE increases; if the direction of the action force is opposite to that of the displacement, the NKE of the object decreases, i.e., the absolute value of the NKE decreases.

Law of conservation of negative kinetic energy

Suppose that a particle is not acted upon by a force during its motion over a path, $\boldsymbol{F} = 0$. From Eq. (54), the work done is zero. Then, the NKE of the particle remains unchanged during this path, or it can be written as

$$K_{(-)} = \text{const.} \tag{57}$$

This is the law of conservation of NKE: when there is no acting force, the NKE of an object remains unchanged.

Table 2 compares the fundamental formulas of positive and negative kinetic energy particles.

Table 2. Comparison of the fundamental formulas of PKE and NKE particles.

	PKE particle	**NKE particle**
Newton's first law	If there is no external force, an object remains in uniform straight-line motion.	
Newton's second law	$\boldsymbol{f} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}$	$\boldsymbol{f} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}_{(-)}$
Newton's second law when mass is constant	$\boldsymbol{f} = m\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}$ Force is the cause of acceleration	$\boldsymbol{f} = -m\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}$ Force is the cause of deceleration

		(negative acceleration)
Newton's third law	The action force and reaction force between two objects are equal in magnitude, opposite in direction, and act on the same straight line.	
Momentum-velocity relationship	$\boldsymbol{p}_{(+)} = m\boldsymbol{v}$	$\boldsymbol{p}_{(-)} = -m\boldsymbol{v}$
Expression of inertial force	$\boldsymbol{F}_{\mathrm{I}} = -m(\boldsymbol{a} - \boldsymbol{a}^{'})$	$\boldsymbol{F}_{\mathrm{I}} = m(\boldsymbol{a} - \boldsymbol{a}^{'})$
Kinetic energy expression	$K_{(+)} = \boldsymbol{p}^2/2m$	$K_{(-)} = -\boldsymbol{p}_{(-)}^2/2m$
Differential form of work-energy theorem	$\boldsymbol{f} \cdot \mathrm{d}\boldsymbol{r} = \mathrm{d}K_{(+)}$	$\boldsymbol{f} \cdot \mathrm{d}\boldsymbol{r} = \mathrm{d}K_{(-)}$
Integral form of work-energy theorem	$W = \int_{\mathrm{A}}^{\mathrm{B}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = K_{\mathrm{B}} - K_{\mathrm{A}}$	$W_{(-)} = \int_{\mathrm{A}}^{\mathrm{B}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ $= K_{(-)\mathrm{B}} - K_{(-)\mathrm{A}}$
Definition of angular momentum	$\boldsymbol{J} = \boldsymbol{r} \times m\boldsymbol{p}$	$\boldsymbol{J}_{(-)} = \boldsymbol{r} \times m\boldsymbol{p}_{(-)}$
Angular momentum theorem ($\boldsymbol{L}$ is torque)	$\mathrm{d}\boldsymbol{J} = \boldsymbol{L}\mathrm{d}t$	$\mathrm{d}\boldsymbol{J}_{(-)} = \boldsymbol{L}\mathrm{d}t$
Conservation of momentum when no external force	$\boldsymbol{p} = \mathrm{const.}$	$\boldsymbol{p}_{(-)} = \mathrm{const.}$
Conservation of angular momentum when no external force torque	$\boldsymbol{J} = \mathrm{const.}$	$\boldsymbol{J}_{(-)} = \mathrm{const.}$
Conservation of kinetic energy when no external force	$K = \mathrm{const.}$	$K_{(-)} = \mathrm{const.}$

In writing down the expression of inertial force, it is assumed that S is an inertial frame and S' is a non-inertial frame. The acceleration of particle A in S is $\boldsymbol{a}$, and in S' is $\boldsymbol{a}'$.

In the system obeying the new Newtonian mechanics, the concept of momentum is the same as the original concept, except that the direction of momentum is opposite to that of velocity. The NKE can be said to be a newly emerging concept. In contrast, in the original Newtonian mechanics system, the kinetic energy must be positive, which is called PKE. The concept of NKE is the most prominent feature in the new Newtonian mechanics system. Therefore, hereinafter, we will refer to the original and new Newtonian mechanics as the PKE Newtonian mechanics and NKE Newtonian mechanics, respectively. The particles we discuss have NKE, i.e., NKE particles, referred to as particles for short.

3.5. Mechanical Energy

3.5.1. Force Field, Conservative Force, and Potential Energy

Force field

The force $\boldsymbol{F}$ acting on an object we refer to is limited to being a single-valued, finite, and differentiable function of spatial coordinates and time t. At each moment, the force acting on the object at each point in space is determined. Such a spatial region is called a force field.

When a force $\boldsymbol{F}$ acts on a particle and moves along a certain path from the position vector $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$, the work done by it is

$$W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}. \tag{58}$$

In general, from the position vector $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$, the integral (58) is related to the path taken between these two points. That is, the work done by the force is different along different paths.

Conservative force and potential energy

If the work done by the force $\boldsymbol{F}$ from the position vector $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ is independent of the path taken and only related to the two position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, then there must exist a single-valued, finite, and differentiable function $V(\boldsymbol{r})$ such that the force $\boldsymbol{F}$ can be expressed as

$$\boldsymbol{F} = -\nabla V(\boldsymbol{r}). \tag{59}$$

That is, the force is the negative gradient of a function. The force satisfying the above conditions is called a conservative force. The function $V(\boldsymbol{r})$ in Eq. (59) is called the potential energy. Therefore, according to the properties of conservative forces, a potential energy can be defined. At each point in space, the potential energy has a definite value. Substituting (59) into (58), we know that:

$$W = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \nabla V(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2). \tag{60}$$

This integral is independent of the intermediate path and only related to the two position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. From (60), we know that for a conservative force $\boldsymbol{F}$, starting from the position vector $\boldsymbol{r}_1$ and doing work along any path between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ to reach the position $\boldsymbol{r}_2$, the force $\boldsymbol{F}$ does work in this process that is the difference between the potential energy at the initial position and the final position. A potential energy value can be defined at each spatial position. However, Eq. (60) can only calculate the difference in potential energy between two position vectors. The true value of the potential energy at each position is not determined. Therefore, when no work is done, it is necessary to specify the potential energy value at a certain point, and then the potential energy values at all other points in space are determined. In general, the potential energy at a certain point in space is specified as zero, which is called the zero potential energy point. The position of the zero potential energy point can be anywhere in space, even at infinity.

3.5.2. Mechanical Energy and the Law of Conservation of Mechanical Energy

Definition of mechanical energy

If an object is in a conservative force field, it has a potential energy V at each point. At the same time, since it is moving, it has a NKE $K_{(-)}$. The mechanical energy E of the object is defined as the sum of the NKE and potential energy:

$$E = K_{(-)} + V. \tag{61}$$

Law of conservation of mechanical energy

If a NKE particle moves in a conservative force field from point A to point B in space. On the one hand, the work done by the conservative force field changes the potential energy of the particle as shown in (60); on the other hand, the work done by the force changes the NKE of the particle as shown in (56). Combining (60) and (56), we get

$$W = K_{(-)\mathrm{B}} - K_{(-)\mathrm{A}} = V(\boldsymbol{r}_\mathrm{A}) - V(\boldsymbol{r}_\mathrm{B}). \tag{62}$$

We write (62) in the following form:

$$E_{\mathrm{A}} = K_{(-)\mathrm{A}} + V(\boldsymbol{r}_{\mathrm{A}}) = K_{(-)\mathrm{B}} + V(\boldsymbol{r}_{\mathrm{B}}) = E_{\mathrm{B}}. \tag{63}$$

This formula indicates that when a NKE particle moves in a conservative force field, the sum of its NKE and potential energy at each position is a constant. This is the law of conservation of mechanical energy for NKE particles.

4. Motion Orbit of A Particle Acted upon by A Force

In principle, if the force acting on a NKE particle is known, its motion orbit can be obtained according to the new Newton's second law (24). We study the motion orbit of a particle under the action of a central force.

4.1. Central Force

4.1.1. Definition of Central Force

The definition of a central force is:

$$\boldsymbol{F} = f(r)\hat{\boldsymbol{r}}, \tag{64}$$

where $f(r)$ is a function that only depends on the magnitude $r = |\boldsymbol{r}|$, independent of direction of $\boldsymbol{r}$, and $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ is the radial unit vector.

As long as $f(r)$ is continuous, it can be integrated. Then, the central force is a conservative force, and potential energy can be defined. The proof is as follows. Let the indefinite integral of $f(r)$ be the function $-V(r)$, i.e.,

$$f(r) = -\frac{\mathrm{d}V(r)}{\mathrm{d}r}. \tag{65}$$

Substituting (64) into the definition of work (58):

$$\int_{\mathrm{A}}^{\mathrm{B}} f(r)\,\hat{\boldsymbol{r}} \cdot \mathrm{d}\boldsymbol{r} = -\int_{\mathrm{A}}^{\mathrm{B}} \frac{\mathrm{d}V(r)}{\mathrm{d}r}\mathrm{d}r = V(r_{\mathrm{A}}) - V(r_{\mathrm{B}}). \tag{66}$$

Comparing this formula with (60), the function $V(r)$ is the potential energy function of the central force $f(r)$.

4.1.2. Three Typical Central Forces

Gravitational force

If an object of mass M is at the origin, the gravitational force exerted by a mass M at the origin on an object of mass m at position vector $\boldsymbol{r}$ is expressed as:

$$\boldsymbol{F} = -\frac{GmM}{r^3}\boldsymbol{r} = -\frac{GmM}{r^2}\hat{\boldsymbol{r}}, \tag{67}$$

where G is the gravitational constant. The negative sign in (67) indicates that the force on m always points to the origin, so it is an attractive force from the origin. The potential energy corresponding to the gravitational force (67) is

$$V(r) = -\frac{GmM}{r} \tag{68}$$

with the zero potential energy point at infinity, $V(\infty) = 0$. After specifying the zero point of gravitational potential energy in this way, the gravitational potential energy is always a negative value at any finite distance.

Coulomb force

Coulomb force is the interaction force between two charged particles. If a particle with charge Q_1 is at the origin, the Coulomb force exerted by Q_1 on aparticle with charge Q_2 at position vector $\boldsymbol{r}$ is expressed as:

$$\boldsymbol{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^3}\boldsymbol{r} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}. \tag{69}$$

When Q_1 and Q_2 have the same sign, this force is a mutual repulsive one; when Q_1 and Q_2 have opposite signs, this force is an attractive one. The potential energy corresponding to the Coulomb force (69) is

$$V(r) = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r}. \quad (70)$$

Here we have taken infinity as the zero potential energy point, $V(\infty) = 0$. After defining the zero point of Coulomb potential energy in this way, when Q_1 and Q_2 have the same sign, the potential energy is positive, and the larger the r is, the lower the potential energy; when Q_1 and Q_2 have opposite signs, the potential energy is negative, and the larger the r is, the smaller the absolute value of the potential energy, i.e., the higher the potential energy.

The expressions of gravitational force (67) and Coulomb force (69) show that both are inversely proportional to the square of the distance r, so they are called inverse-square forces. Hereinafter, for inverse-square forces, unless otherwise specified, infinity is always taken as the zero potential energy point.

Linear force

The well-known elastic restoring force is expressed by

$$\boldsymbol{F} = -k\boldsymbol{r}, \quad (71)$$

where k is the elastic recovery coefficient of the actual material, which is always positive, $k > 0$. Therefore, this force always points to the origin, i.e., an attractive force, and is proportional to the first power of the distance. The potential energy corresponding to (71) is:

$$V(r) = \frac{1}{2}kr^2. \quad (72)$$

Generally, the coordinate origin is taken as the zero potential energy point, $V(0) = 0$, and the potential energy is always positive.

Our common sense is that k in (72) is the elastic coefficient of the actual material, which is always positive $k > 0$. Theoretically, the case of $k < 0$ is also worth to be taken into account. For clarity, we write:

$$\boldsymbol{F} = k\boldsymbol{r}, \quad (73)$$

where $k > 0$. This force is still a central force. The expression of potential energy is:

$$V(r) = -\frac{1}{2}kr^2 \quad (74)$$

with the coordinate origin as the zero potential energy point $V(0) = 0$, and the potential energy is always negative.

4.1.3. Conservation of Angular Momentum in A Central Force Field

If a NKE particle is acted upon by a central force, its angular momentum is a constant, i.e., angular momentum is conserved. In the case of a central force (64), taking the time derivative of the angular momentum in (42):

$$\frac{\mathrm{d}\boldsymbol{L}_{(-)}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p}_{(-)} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}_{(-)}}{\mathrm{d}t} = -\boldsymbol{v} \times m\boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times f(r)\hat{\boldsymbol{r}} = 0, \quad (75)$$

where the expression of momentum (25), the new Newton's second law (27), and the expression of central force (64) are used. The cross product of two parallel vectors is zero. The conclusion is: if a NKE object moves in a central force field, its angular momentum is conserved.

4.2. Motion Under Inverse-Square Force

For a NKE particle not to move to infinity, it must be acted upon by a repulsive force.

Suppose that an NKE particle is acted upon by an inverse-square repulsive force:

$$\boldsymbol{F} = \frac{k}{r^2}\hat{\boldsymbol{r}}, \quad (76)$$

where $k > 0$ is a positive constant. We solve for the motion orbit of this object.

Substituting (76) into the new Newton's second law (24):

$$-m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{k}{r^2}\hat{\boldsymbol{r}}, \tag{77}$$

where the velocity $\boldsymbol{v}$ is written in the following form:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{\boldsymbol{r}}) = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\boldsymbol{r}} + r\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t}. \tag{78}$$

Substituting (78) into the expression of angular momentum (43):

$$\boldsymbol{L} = -r\hat{\boldsymbol{r}} \times m\boldsymbol{v} = -mr\hat{\boldsymbol{r}} \times \left(\frac{\mathrm{d}r}{\mathrm{d}t}\hat{\boldsymbol{r}} + r\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t}\right) = -mr^2\hat{\boldsymbol{r}} \times \frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t}. \tag{79}$$

Let us use the conservation of angular momentum and consider the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v} \times \boldsymbol{L}) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \times \boldsymbol{L} = -\frac{k}{mr^2}\hat{\boldsymbol{r}} \times \boldsymbol{L} = \frac{k}{mr^2}\hat{\boldsymbol{r}} \times (mr^2\hat{\boldsymbol{r}} \times \frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t})$$

$$= k\left[\hat{\boldsymbol{r}}\left(\hat{\boldsymbol{r}} \cdot \frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t}\right) - \frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t}\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}\right] = -k\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t}. \tag{80}$$

In this course, we have used (79) and made use of $\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}} = 1$ so that $\hat{\boldsymbol{r}} \cdot \frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = 0$.

Integrating (80):

$$\boldsymbol{v} \times \boldsymbol{L} = -k\hat{\boldsymbol{r}} + \boldsymbol{C}, \tag{81}$$

where $\boldsymbol{C}$ is a constant vector, which will be seen to be along the major axis of the orbit after deriving the motion orbit below. To give the expression of the motion orbit, we calculate the square of the angular momentum:

$$L^2 = \boldsymbol{L} \cdot \boldsymbol{L} = -(\boldsymbol{r} \times m\boldsymbol{v}) \cdot \boldsymbol{L} = -m\boldsymbol{r} \cdot (\boldsymbol{v} \times \boldsymbol{L})$$

$$= -mr\hat{\boldsymbol{r}} \cdot (-k\hat{\boldsymbol{r}} + \boldsymbol{C}) = mr(k - C\cos\theta), \tag{82}$$

where θ is the angle between the $\boldsymbol{C}$ and $\boldsymbol{r}$. Thus, the orbit equation is obtained:

$$r = \frac{L^2/km}{1 + C\cos\theta/k} = \frac{A}{1 + \varepsilon\cos\theta}. \tag{83}$$

This equation is the polar coordinate form of a conic section, with the origin at one focus. The ε represents the eccentricity of the conic section. The eccentricity is calculated as follows:

$$\varepsilon = |\frac{\boldsymbol{C}}{k}| = \frac{|\boldsymbol{v} \times \boldsymbol{L} - k\hat{\boldsymbol{r}}|}{|k|} = \frac{[(\boldsymbol{v} \times \boldsymbol{L})^2 + k^2 - 2k\hat{\boldsymbol{r}} \cdot (\boldsymbol{v} \times \boldsymbol{L})]^{1/2}}{|k|}$$

$$= \frac{(-v^2L^2 + k^2 + 2kL^2/mr)^{1/2}}{|k|} = (1 - \frac{2EL^2}{mk^2})^{1/2}, \tag{84}$$

where

$$E = -\frac{mv^2}{2} + \frac{k}{r} \tag{85}$$

is the NKE plus potential energy of the moving particle, i.e., mechanical energy. From (85), the eccentricity is determined by energy and angular momentum. We have known that under the action of a central force, angular momentum is conserved and does not change with time. Since mechanical energy is conserved, E does not vary with time, and the eccentricity calculated by (84) does neither.

From (84), the range of mechanical energy corresponds to the range of ε. In (83), the range of ε and the corresponding orbits are as follows:

$$0 < \varepsilon < 1, \text{ Elliptical orbit} \tag{86a}$$

$$\varepsilon = 1, \quad \text{Parabolic orbit} \tag{86b}$$

$$\varepsilon > 1, \quad \text{Hyperbolic orbit} \tag{86c}$$

For parabolic and hyperbolic orbits, the object can move to infinity; only the elliptical orbit remains within a finite spatial range. Comparing (86a) with (84), it is known that in the following energy range:

$$0 < E < mk^2/2L^2, \tag{87}$$

Eq. (83) represents an elliptical orbit.

The specific values of mechanical energy and angular momentum need to be determined according to initial conditions. Given the position vector $\boldsymbol{r}$ and motion velocity $\boldsymbol{v}$ of the object at the initial moment, the values of mechanical energy and angular momentum are determined. That is, the two parameters A and ε in the orbit (83) can be determined according to the initial conditions.

If the mechanical energy in (85) is positive (see (87)), at each moment the object moves along the elliptical orbit (83), the kinetic energy of the object is always greater than the absolute value of the potential energy. However, while the mechanical energy remains unchanged, its kinetic energy and potential energy are constantly changing. Now the object is acted upon by a repulsive force. When the distance r from the object to the origin increases from small to large, its potential energy decreases because the absolute value of $1/r$ decreases as r increases. At the same time, in the process of increasing r, the direction of the displacement increment $\mathrm{d}\boldsymbol{r}$ is opposite to the direction of the force $\boldsymbol{F}$, so the force does negative work during this process. That is, in the process of increasing r, the magnitude of the object's momentum decreases, so the absolute value of its NKE decreases. Therefore, as r increases from small to large, the potential energy decreases and the NKE decreases. Conversely, as r decreases from large to small, the potential energy increases and the NKE increases.

When $r = \infty$, the potential energy is zero, leaving only the kinetic energy part, $E = -mv^2/2$. Since mechanical energy is conserved and positive, this requires the kinetic energy to be positive. However, the kinetic energy $-mv^2/2$ cannot be positive. Therefore, the object cannot move to infinity, i.e., it can only move at a finite distance, which is a bound motion.

Now we assume an object is acted upon by an inverse-square attractive force:

$$\boldsymbol{F} = -\frac{k}{r^2}\hat{\boldsymbol{r}}, \tag{88}$$

where k is a positive constant. Equation (88) differs from (76) by a negative sign. According to the new Newton's second law:

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{k}{r^2}\hat{\boldsymbol{r}}. \tag{89}$$

Let us solve for the motion orbit of the object.

Since (88) is a central force, the angular momentum of the object's motion is conserved under this force. Comparing (76) and (88), it can be seen that as long as k is replaced by $-k$ in the derivation process of (78)-(83), the result is that the mechanical energy (85) should be revised to

$$E = -\frac{mv^2}{2} - \frac{k}{r} < 0. \tag{90}$$

Now both kinetic energy and potential energy are negative. Therefore, the eccentricity obtained from (84) is as in (86c), which is a hyperbolic orbit. The object will inevitably move to infinity.

Our conclusions are drawn from the above discussions: When a NKE object is acted upon by an inverse-square repulsive force, its mechanical energy is positive, and its motion on the orbit will not reach infinity. When an object is acted upon by an inverse-square attractive force, it will inevitably move to infinity.

We can summarize the conclusions on the motion of PKE particles and NKE particles under inverse-square forces as follows. When a PKE (NKE) object is acted upon by an inverse-square attractive (repulsive) force, its mechanical energy is negative (positive), and it can move on the orbit without reaching infinity. When a PKE (NKE) is acted upon by an inverse-square repulsive (attractive) force, it will inevitably move to infinity. In both cases, the signs of kinetic energy, potential energy, and mechanical energy of the PKE and NKE objects are exactly opposite.

This shows the symmetry between PKE and NKE.

We define the concepts of stable motion and bound motion. If the expression of the motion orbit of an object $\boldsymbol{r} = \boldsymbol{r}(t)$ is known, this motion is called stable motion. If an object performs stable motion and its spatial coordinates can only change within a finite range, this motion is called bound motion.

For the three motion orbits of a particle under inverse-square force—ellipse, parabola, and hyperbola—all are stable motions because the expression of the particle's position vector with time can be clearly written. Among them, the ellipse is a bound motion because the particle's position vector can only change within a finite range. The position vectors of parabolic and hyperbolic motions have no upper limit, so they are unbound motions.

An example of hyperbolic orbit is the third interstellar object 3I/ATLAS initially observed on 2025 July 1. It has an orbital eccentricity of ε~6.1 and perihelion of q~1.36 au. The initial conditions of this orbit are that the inclination is ~175°, and hyperbolic velocity of $V(\infty)$~58 km/s [100,101].

4.3. Motion Under Linear Force

Suppose a NKE particle is acted upon by a one-dimensional linear attractive force as in (71). Then the motion equation is:

$$-m\ddot{x} = -kx. \tag{91}$$

Dropping the negative signs in both sides, we have

$$m\ddot{x} = kx. \tag{92}$$

The general solution of this equation is still:

$$x = A\exp(-\omega t) + B\exp(\omega t), \tag{93}$$

where $\omega = \sqrt{k/m}$. We calculate the mechanical energy. The kinetic energy is

$$K = -\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}m\omega^2(A^2\exp(-2\omega t) - 2AB + B^2\exp(2\omega t)). \tag{94}$$

The potential energy is

$$V = \frac{1}{2}kx^2 = \frac{1}{2}k(A^2\exp(-2\omega t) + 2AB + B^2\exp(2\omega t)). \tag{95}$$

The mechanical energy is

$$E = K + V = ABk. \tag{96}$$

This is a conserved quantity that does not change with time.

Since the second term in (93) grows exponentially with time, it indicates that as time increases, the NKE particle can move to any distance. This is a unbound motion. We have also met such a situation in the case of inverse-square force: a NKE particle acted upon by an inverse-square attractive force has a hyperbolic or parabolic motion orbit, and the particle will move to infinity.

In the case of linear force in this subsection, it is also possible for a NKE particle to move within a finite range. That is, under appropriate initial conditions, $B = 0$, the second term in (93) disappears. The solution that can move within a finite range is

$$x = A\exp(-\omega t). \tag{97}$$

The coefficient A is determined by the initial conditions, representing the displacement at the initial moment. This is an exponentially decaying motion. In this case, the NKE is

$$K = -\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}m\omega^2A^2\exp(-2\omega t), \tag{98}$$

the potential energy is

$$V = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\exp(-2\omega t) = -K, \tag{99}$$

and mechanical energy is

$$E = K + V = 0. \tag{100}$$

Equation (97) represents such a motion of a NKE that at the initial moment $t = 0$, the displacement of the particle is A, and it has a velocity $-\omega A$ moving towards the origin. It has a NKE $-\frac{1}{2}m\omega^2A^2$ and a potential energy $\frac{1}{2}kA^2$. Starting from the initial moment, the velocity of the particle is always negative, but its acceleration is always positive (because the direction of acceleration is opposite to that of the acting force), so the particle moves monotonically towards the origin, and the magnitudes of its displacement, velocity, and acceleration all decay exponentially with time. The mechanical energy remains zero. Finally, the particle will stop at the origin.

Our conclusion is that if a NKE particle is acted upon by a linear attractive force as in (71), it either moves to infinity as in (93) or moves monotonically to the origin as in (97).

When a NKE particle is acted upon by a one-dimensional linear repulsive force as in (73), the motion equation of the particle is:

$$-m\ddot{x} = kx. \tag{101}$$

The solution is

$$x = A\cos(\omega t + \varphi_0), \tag{102}$$

where $\omega = \sqrt{k/m}$, A is the amplitude, and φ_0 is the initial phase. These two quantities are determined by the initial conditions.

The NKE of the particle is

$$K = -\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}m\omega^2A^2\sin(\omega t + \varphi_0). \tag{103}$$

Its potential energy is

$$V = -\frac{1}{2}kx^2 = -\frac{1}{2}kA^2\cos^2(\omega t + \varphi_0). \tag{104}$$

The mechanical energy

$$E = K + V = -\frac{1}{2}kA^2. \tag{105}$$

The mechanical energy is conserved.

Equation (102) represents the following motion: at the initial moment $t = 0$, the particle has an initial displacement $A\cos\varphi_0$ and an initial velocity $-\omega A\sin\varphi_0$. Starting from the initial moment, the particle performs simple harmonic motion around the origin with an amplitude A. The mechanical energy of the particle, $-\frac{1}{2}kA^2$, remains unchanged. Although the acting force here is a repulsive force, it acts as a linear restoring force for a NKE particle.

Finally, we draw the following conclusion: For a PKE (NKE) particle, when it is acted upon by a linear attractive (repulsive) force, its mechanical energy is positive (negative), and it can perform simple harmonic motion near the origin, which is a bound motion; when it is acted upon by a linear repulsive (attractive) force, its mechanical energy is negative (positive) or zero, and it will either move to infinity (unbound motion) or tend to the origin monotonically (bound motion).

This example shows symmetry again. The motion orbits of a PKE object acted upon by an attractive force and that of a NKE one upon by a repulsive force are the same.

4.4. Motion upon Electromagnetic Force

4.4.1. Electromagnetic Force

If there is an electric field and a magnetic field, a particle with an electric charge q in the field will be acted upon by a force, which is the Lorentz force:

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}. \tag{106}$$

It consists of two terms: the first term is the electric field force, and the second term is the magnetic field force. The power of work done by this force is

$$P = \boldsymbol{F} \cdot \boldsymbol{v} = q\boldsymbol{E} \cdot \boldsymbol{v}. \tag{107}$$

Only the electric field force does work, while the magnetic field force does not because it is always perpendicular to the direction of the object's motion, thereby changing the direction of the object's motion.

This subsection only discusses the case of constant electromagnetic fields.

A NKE particle with charge q is in an electric field and a magnetic field. The motion equation of the particle is:

$$\dot{\boldsymbol{p}}_{(-)} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}. \tag{108}$$

4.4.2. Constant Uniform Electric Field

At this time, the motion equation is:

$$\dot{\boldsymbol{p}}_{(-)} = q\boldsymbol{E}. \tag{109}$$

Let the position vector at the initial moment be $\boldsymbol{r}_0$ and the initial velocity be $\boldsymbol{v}_0$. The solution is:

$$\boldsymbol{r} = -\frac{1}{2m} q\boldsymbol{E}t^2 + \boldsymbol{v}_0 t + \boldsymbol{r}_0. \tag{110}$$

It can be seen that the position vector $\boldsymbol{r} - \boldsymbol{r}_0|$ of the particle is a parabola in the plane formed by $\boldsymbol{E}$ and $\boldsymbol{v}_0$. The direction of acceleration is opposite to the direction of the electric field. Taking this plane as the xy-plane and $\boldsymbol{E}$ in the x-direction, the component form of (110) is $x = -\frac{1}{2m} qEt^2 + v_{0x}t, y = v_{0y}t$. Thus, we obtain:

$$x = -\frac{1}{2mv_{0y}^2} qEy^2 + \frac{v_{0x}}{v_{0y}} y, \tag{111}$$

which is a parabola opening in the negative x-direction.

4.4.3. Constant Uniform Magnetic Field

At this time, the motion equation is

$$\dot{\boldsymbol{p}}_{(-)} = q\boldsymbol{v} \times \boldsymbol{B}. \tag{112}$$

For simplicity, let the magnetic field be along the z-axis, $\boldsymbol{B} = (0,0,B)$. Let the position vector at the initial moment be $\boldsymbol{r}_0 = (x_0, y_0, z_0)$ and the initial velocity be $\boldsymbol{v}_0 = (v_{0x}, 0, v_{0z})$, (i.e., the initial velocity is in the xz-plane. We write the three coordinate components of the motion equation (112):

$$\ddot{x} = -q\dot{y}B/m, \ddot{y} = q\dot{x}B/m, \ddot{z} = 0. \tag{113}$$

Let $\omega = qB/m$. The solution of Eq. (113) is:

$$x = x_0 - \left(\frac{v_{0x}}{\omega}\right) \sin \omega t\, , y = y_0 - \frac{v_{0x}}{\omega} + \left(\frac{v_{0x}}{\omega}\right) \cos \omega t\, , z = v_{0z}t + z_0. \tag{114}$$

The motion trajectory of this charged particle is as follows. in the xy-plane perpendicular to the magnetic field, it is a counterclockwise uniform circular motion with the center at $(x_0, y_0 - v_{0x}/\omega)$, radius v_{0x}/ω, and linear velocity v_{0x}; along the direction of the magnetic field, it is a uniform linear motion. Therefore, the trajectory of the particle is a counterclockwise helix, and the axis of the cylinder where the helix is located is a straight line passing through the point $(x_0, y_0 - v_{0x}/\omega)$ and parallel to the magnetic field direction.

4.4.4. Constant Uniform Electric and Magnetic Fields

At this time, the motion equation is:

$$\dot{\boldsymbol{p}}_{(-)} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}. \tag{115}$$

For simplicity, let the magnetic field be along the z-axis, $\boldsymbol{B} = (0,0,B)$, the plane formed by the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ be the yz-plane, $\boldsymbol{E} = (0, E_y, E_z)$. Let the position vector at the initial moment be at the coordinate origin $\boldsymbol{r}_0=(0,0,0)$, and the initial velocity be in the *xz*-plane, $\boldsymbol{v}_0=(v_{0x},0,v_{0z})$. We write the three coordinate components of the motion equation (115):

$$m\ddot{x}=-q\dot{y}B, m\ddot{y}=-qE_y+q\dot{x}B, \ddot{z}=-qE_z \qquad (116)$$

Let $\omega=qB/m$. The solution of the equation is:

$$x=\frac{qE_yt}{B}-\left(\frac{v_{0x}}{\omega}\right)\sin\omega t\,, y=\left(\frac{v_{0x}}{\omega}\right)\cos\omega t-\frac{v_{0x}}{\omega}, z=-\frac{qE_z}{2m}t^2+v_{0z}t. \qquad (117a)$$

The motion trajectory of this charged particle is relatively complex. Along the *z*-axis, i.e., the direction of the magnetic field, it is uniformly accelerated motion, with acceleration proportional to but opposite to E_z. In the *xy*-plane perpendicular to the magnetic field, it is a uniform circular motion with the center at $(0,v_{0x}/\omega)$ and linear velocity v_{0x}, superimposed with a uniform linear motion along the positive *x*-direction. For the special case of $qE_y/B=v_{0x}$:

$$x=\frac{qE_y}{B\omega}(\omega t-\sin\omega t), y=-\frac{qE_y}{B\omega}(1-\cos\omega t) \qquad (117b)$$

The trajectory in the *xy*-plane is a cycloid below the *x*-axis.

5. Dynamics of NKE Particle Systems

5.1. Kinematics of Particle Systems

A system composed of more than one NKE particle is called a particle system.

Suppose a particle system has *N* particles. The mass of the *i*-th particle is denoted as m_i. After establishing a coordinate system, the position vector, velocity, and acceleration of the *i*-th particle are denoted as $\boldsymbol{r}_i$, $\boldsymbol{v}_i$, and $\boldsymbol{a}_i$, respectively.

Each particle in the particle system is in motion, and its position vector is a function of time. The orbit of the *i*-th particle is

$$\boldsymbol{r}_i=\boldsymbol{r}_i(t). \qquad (118)$$

Its motion velocity is

$$\boldsymbol{v}_i=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_i(t)=\dot{\boldsymbol{r}}_i(t). \qquad (119)$$

Its motion acceleration is

$$\boldsymbol{a}_i=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_i(t)=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}_i(t)=\ddot{\boldsymbol{r}}_i(t). \qquad (120)$$

There are interactions between particles in the particle system. Such interaction forces are called internal forces. The forces exerted by objects outside the particle system on particles inside the particle system are called external forces. If the force exerted by the *i*-th particle on the *j*-th particle in the particle system is denoted as $\boldsymbol{f}_{ji}$, then according to Newton's third law:

$$\boldsymbol{f}_{ij}+\boldsymbol{f}_{ji}=0. \qquad (121)$$

The sum of the forces exerted by all other particles on the *i*-th particle in the particle system is denoted as $\boldsymbol{F}_i^{(\mathrm{i})}$:

$$\boldsymbol{F}_i^{(\mathrm{i})}=\sum_{j=1,j\neq i}^{N}\boldsymbol{f}_{ij}. \qquad (122)$$

The sum of all these internal forces between particles is denoted as $\boldsymbol{F}^{(\mathrm{i})}$, which must be zero:

$$\boldsymbol{F}^{(\mathrm{i})}=\sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{i})}=\sum_{i,j=1,i\neq j}^{N}\boldsymbol{f}_{ij}=0. \qquad (123)$$

The sum of the forces exerted by objects outside the particle system on the *i*-th particle is denoted as $\boldsymbol{F}_i^{(\mathrm{e})}$, and the sum of all these external forces is denoted as $\boldsymbol{F}^{(\mathrm{e})}$:

$$\boldsymbol{F}^{(\mathrm{e})} = \sum_{i=1}^{N} \boldsymbol{F}_i^{(\mathrm{e})}. \tag{124}$$

If a particle system is not acted upon by any external forces, $\boldsymbol{F}^{(\mathrm{e})} = 0$, this particle system is called an isolated system.

A particle system composed of PKE particles is called a PKE system; a particle system composed of NKE particles is called a NKE system. In this section we discuss a NKE system.

The relationship between the momentum and velocity of the *i*-th particle is:

$$\boldsymbol{p}_{(-)i} = -m_i \boldsymbol{v}_i. \tag{125}$$

It has NKE:

$$K_{(-)i} = -\frac{1}{2} m_i \boldsymbol{v}_i^2 = -\frac{1}{2m_i} \boldsymbol{p}_{(-)i}^2. \tag{126}$$

5.2. Definition of Center of Mass

The total mass of a NKE particle system is:

$$M = \sum_{i=1}^{N} m_i. \tag{127}$$

The center of mass, or centroid, of this particle system is defined as follows:

$$\boldsymbol{r}_\mathrm{C} = \frac{1}{M} \sum_{i=1}^{N} m_i \boldsymbol{r}_i. \tag{128}$$

The particles in the particle system are all in motion, so the position of the centroid changes with time. The time derivative of the centroid position vector is the velocity of the centroid:

$$\boldsymbol{v}_\mathrm{C} = \dot{\boldsymbol{r}}_\mathrm{C} = \frac{1}{M} \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i. \tag{129}$$

The acceleration of the centroid is:

$$\boldsymbol{a}_\mathrm{C} = \dot{\boldsymbol{v}}_\mathrm{C} = \ddot{\boldsymbol{r}}_\mathrm{C} = \frac{1}{M} \sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i. \tag{130}$$

The above expressions are exactly the same as those for a PKE particle system.

5.3. Momentum Theorem and Law of Conservation of Momentum

The momentum of the *i*-th NKE particle is shown in (125). The total momentum of the NKE particle system is:

$$\boldsymbol{p}_{(-)} = \sum_{i=1}^{N} \boldsymbol{p}_{(-)i} = -\sum_{i=1}^{N} m_i \boldsymbol{v}_i = -\sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i. \tag{131}$$

Each particle in the particle system follows the new Newton's second law. The force acting on each particle includes internal force $\boldsymbol{F}_i^{(\mathrm{i})}$ and external force $\boldsymbol{F}_i^{(\mathrm{e})}$. The new Newton's second law for the *i*-th NKE particle is:

$$\frac{\mathrm{d}\boldsymbol{p}_{(-)i}}{\mathrm{d}t} = \boldsymbol{F}_i^{(\mathrm{i})} + \boldsymbol{F}_i^{(\mathrm{e})}. \tag{132}$$

Summing the motion equations of all particles in the particle system, we get

$$\sum_{i=1}^{N} \frac{\mathrm{d}\boldsymbol{p}_{(-)i}}{\mathrm{d}t} = \sum_{i=1}^{N} (\boldsymbol{F}_i^{(\mathrm{i})} + \boldsymbol{F}_i^{(\mathrm{e})}) = \boldsymbol{F}^{(\mathrm{e})}. \tag{133}$$

The final equality uses (123) and (124). Equation (133) can be expressed in terms of the total momentum of the particle system:

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} \boldsymbol{p}_{(-)i} = \frac{\mathrm{d}\boldsymbol{p}_{(-)}}{\mathrm{d}t} = \boldsymbol{F}^{(\mathrm{e})}. \tag{134}$$

The physical meaning of this formula is: the time rate of change of the total momentum of the NKE particle system is equal to the sum of all external forces acting on the particle system. Internal forces have no effect on the total momentum of the particle system. This formula is also called the momentum theorem for NKE particle systems. Writing (134) in the following form:

$$\mathrm{d}\boldsymbol{p}_{(-)} = \boldsymbol{F}^{(\mathrm{e})}\mathrm{d}t. \tag{135}$$

The left hand side is the differential of the total momentum of the NKE particle system, and the right hand side is the elementary impulse of the sum of external forces on the particle system. Integrating both sides with respect to time:

$$\boldsymbol{p}_{(-)}(t_2) - \boldsymbol{p}_{(-)}(t_1) = \int_{t_1}^{t_2} \boldsymbol{F}^{(\mathrm{e})}\,\mathrm{d}t. \tag{136}$$

This is the impulse theorem for NKE particle systems: the impulse of all external forces acting on the particle system over a period of time is equal to the total momentum of the particle system at the final moment minus the total momentum at the initial moment.

From (129), we define the momentum of the centroid:

$$\boldsymbol{p}_{(-)\mathrm{C}} = -M\boldsymbol{v}_{\mathrm{C}} = -\sum_{i=1}^{N} m_i\boldsymbol{v}_i = \boldsymbol{p}_{(-)}. \tag{137}$$

The momentum of the centroid of the particle system is exactly equal to the total momentum of the particle system. Equation (134) can be written as

$$\frac{\mathrm{d}\boldsymbol{p}_{(-)\mathrm{C}}}{\mathrm{d}t} = \boldsymbol{F}^{(\mathrm{e})}. \tag{138}$$

The time rate of change of the momentum of the centroid of the NKE particle system is equal to the sum of all external forces acting on the particle system. This formula is called the motion theorem for the centroid. Equation (138) shows that a NKE particle system obeys the new Newton's second law like a single particle with the total mass concentrated at the centroid.

If the sum of the external forces acting on the particle system is zero over a period of time, $\boldsymbol{F}^{(\mathrm{e})} = 0$, then from (136), the total momentum of the particle system remains unchanged during this period:

$$\boldsymbol{p}_{(-)\mathrm{C}} = \boldsymbol{p}_{(-)} = \mathrm{const.} \tag{139}$$

This is the law of conservation of total momentum for particle systems.

If a NKE particle system is isolated, it must not be acted upon by external forces, and its total momentum must remain unchanged. That is: the total momentum of an isolated system is conserved.

5.4. Angular Momentum Theorem and Law of Conservation of Angular Momentum

The expression of the angular momentum of the *i*-th NKE particle is:

$$\boldsymbol{J}_{(-)i} = \boldsymbol{r}_i \times \boldsymbol{p}_{(-)i}. \tag{140}$$

The total angular momentum of the particle system is:

$$\boldsymbol{J}_{(-)} = \sum_{i=1}^{N} \boldsymbol{J}_{(-)i}. \tag{141}$$

Taking the cross product of $\boldsymbol{r}_i$ with both sides of (132):

$$\boldsymbol{r}_i \times \frac{\mathrm{d}\boldsymbol{p}_{(-)i}}{\mathrm{d}t} = \boldsymbol{r}_i \times (\boldsymbol{F}_i^{(\mathrm{i})} + \boldsymbol{F}_i^{(\mathrm{e})}). \tag{142}$$

The left side is:

$$\boldsymbol{r}_i \times \frac{\mathrm{d}\boldsymbol{p}_{(-)i}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_i \times \boldsymbol{p}_{(-)i}) - \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} \times \boldsymbol{p}_{(-)i} = \frac{\mathrm{d}\boldsymbol{J}_{(-)i}}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} \times m_i\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{J}_{(-)i}}{\mathrm{d}t}. \tag{143}$$

Therefore, Eq. (142) becomes:

$$\frac{\mathrm{d}\boldsymbol{J}_{(-)i}}{\mathrm{d}t} = \boldsymbol{r}_i \times (\boldsymbol{F}_i^{(\mathrm{i})} + \boldsymbol{F}_i^{(\mathrm{e})}). \tag{144}$$

The sum of all internal force torques and external force torques acting on a particle is equal to the time rate of change of the particle's angular momentum. Summing this formula over all particles:

$$\sum_{i=1}^{N}\frac{\mathrm{d}\boldsymbol{J}_{(-)i}}{\mathrm{d}t}=\sum_{i=1}^{N}(\boldsymbol{r}_i\times\boldsymbol{F}_i^{(\mathrm{i})})+\sum_{i=1}^{N}(\boldsymbol{r}_i\times\boldsymbol{F}_i^{(\mathrm{e})}). \tag{145}$$

The left side, $\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N}\boldsymbol{J}_{(-)i}=\frac{\mathrm{d}\boldsymbol{J}_{(-)}}{\mathrm{d}t}$, is the time derivative of the total angular momentum of the NKE particle system. The first term on the right side, by use of Eq. (122), becomes

$$\sum_{i=1}^{N}\boldsymbol{r}_i\times\sum_{j=1,j\neq i}^{N}\boldsymbol{f}_{ji}=\sum_{i=1,j>i}^{N}\boldsymbol{r}_i\times\left(\boldsymbol{f}_{ji}+\boldsymbol{f}_{ij}\right)=0. \tag{146}$$

Here, Eq. (121) is used: internal forces always appear in pairs, and the sum of a pair of internal forces is zero. We define

$$\boldsymbol{L}=\sum_{i=1}^{N}(\boldsymbol{r}_i\times\boldsymbol{F}_i^{(\mathrm{e})}) \tag{147}$$

as the sum of the torques of external forces on the particles. Then Eq. (145) simplifies to

$$\frac{\mathrm{d}\boldsymbol{J}_{(-)}}{\mathrm{d}t}=\boldsymbol{L}. \tag{148}$$

Equation (148) indicates that the sum of the torques of all external forces on the particles is equal to the time derivative of the total angular momentum of the particle system. This is the differential form of the angular momentum theorem. Equation (148) is rewritten as

$$\mathrm{d}\boldsymbol{J}_{(-)}=\boldsymbol{L}\mathrm{d}t. \tag{149}$$

The left side is the differential of the total angular momentum of the NKE particle system, and the right side is the elementary angular impulse of the sum of external forces on the particle system. Integrating both sides with respect to time:

$$\boldsymbol{J}_{(-)}(t_2)-\boldsymbol{J}_{(-)}(t_1)=\int_{t_1}^{t_2}\boldsymbol{L}\,\mathrm{d}t. \tag{150}$$

The right side of this equation is the angular impulse of the external torque. This equation indicates that the angular impulse of all external torques acting on a NKE particle system over a period of time is equal to the total angular momentum of the particle system at the final moment minus that at the initial moment. This is the integral form of the angular momentum theorem. Equations (148)-(150) are formally the same in form as (45)-(47) for a single particle.

If the external torque is zero over a period of time, then the angular momentum of the particle system remains unchanged during this period, or it can be written as:

$$\boldsymbol{J}_{(-)}=\text{const.} \tag{151}$$

If the external torque acting on a system is zero, the total angular momentum of the system is conserved. This is the law of conservation of total angular momentum for an isolated system.

The form of Eq. (134) is the same as that of the momentum theorem for a PKE particle system, and the form of equation (148) is the same as that of the angular momentum theorem for a PKE particle system. The author of Ref. [102] called the momentum theorem and angular momentum theorem for PKE particle systems “axioms”, but he also knew that they could be derived, so that they could not actually be listed as axioms.

5.5. Angular Momentum Theorem for the Centroid

we have defined the position $\boldsymbol{r}_{\mathrm{C}}$ of the centroid of the particle system using Eq. (128). Now, we write the position vector of the *i*-th particle in the following form:

$$\boldsymbol{r}_i=\boldsymbol{r}_{\mathrm{C}}+\boldsymbol{r}'_i, \tag{152}$$

where $\boldsymbol{r}'_i$ is the position vector of the *i*-th particle relative to the center of mass. Multiplying both sides of this equation by m_i and summing over all particles, we get from (128):

$$\sum_{i=1}^{N}m_i\boldsymbol{r}'_i=\sum_{i=1}^{N}m_i\boldsymbol{r}_i-\sum_{i=1}^{N}m_i\boldsymbol{r}_{\mathrm{C}}=M\boldsymbol{r}_{\mathrm{C}}-M\boldsymbol{r}_{\mathrm{C}}=0. \tag{153}$$

Taking the time derivative of both sides of equation (152), we have

$$\boldsymbol{v}_i = \boldsymbol{v}_{\mathrm{C}} + \boldsymbol{v}_i'. \tag{154}$$

The velocity $\boldsymbol{v}_i$ of the *i*-th particle is equal to the velocity of the centroid $\boldsymbol{v}_{\mathrm{C}}$ plus the velocity $\boldsymbol{v}_i'$ of the particle relative to the centroid. Multiplying both sides by the mass of the particle and a negative sign, we obtain

$$\boldsymbol{p}_{(-)i} = -m_i \boldsymbol{v}_{\mathrm{C}} + \boldsymbol{p}_{(-)i}', \tag{155}$$

where $\boldsymbol{p}_{(-)i}'$ is the momentum of the particle relative to the centroid. Summing equation (155) over all particles leads to

$$\boldsymbol{p}_{(-)}' = \sum_{i=1}^{N} \boldsymbol{p}_{(-)i}' = \sum_{i=1}^{N} \boldsymbol{p}_{(-)i} + \sum_{i=1}^{N} m_i \, \boldsymbol{v}_{\mathrm{C}} = \boldsymbol{p}_{(-)} - \boldsymbol{p}_{(-)\mathrm{C}} = 0, \tag{156}$$

where (137) is used: the total momentum of the NKE particle system is equal to the momentum of the centroid. In other words, in the center-of-mass frame of the NKE particle system, the total momentum is zero.

Taking the cross product of $\boldsymbol{r}_{\mathrm{C}}$ with both sides of (138):

$$\boldsymbol{r}_{\mathrm{C}} \times \frac{\mathrm{d}\boldsymbol{p}_{(-)\mathrm{C}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_{\mathrm{C}} \times \boldsymbol{p}_{(-)\mathrm{C}}) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{J}_{(-)\mathrm{C}} = \boldsymbol{r}_{\mathrm{C}} \times \boldsymbol{F}^{(\mathrm{e})} \tag{157}$$

The torque of the sum of external forces on the centroid is equal to the time rate of change of the angular momentum of the centroid.

The angular momentum of the *i*-th NKE particle is written as:

$$\boldsymbol{J}_{(-)i} = (\boldsymbol{r}_{\mathrm{C}} + \boldsymbol{r}_i') \times \left(-m_i \boldsymbol{v}_{\mathrm{C}} + \boldsymbol{p}_{(-)i}'\right)$$

$$= -\boldsymbol{r}_{\mathrm{C}} \times m_i \boldsymbol{v}_{\mathrm{C}} + \boldsymbol{r}_{\mathrm{C}} \times \boldsymbol{p}_{(-)i}' - m_i \boldsymbol{r}_i' \times \boldsymbol{v}_{\mathrm{C}} + \boldsymbol{J}_{(-)i}', \tag{158}$$

where

$$\boldsymbol{J}_{(-)i}' = \boldsymbol{r}_i' \times \boldsymbol{p}_{(-)i}' \tag{159}$$

is the angular momentum of the *i*-th NKE particle relative to the centroid. We sum Eq. (158) over all particles. Using (153) and (156), the second and third terms on the right side of (158) sum to zero. Therefore,

$$\sum_{i=1}^{N} \boldsymbol{J}_{(-)i} = -\sum_{i=1}^{N} m_i \, \boldsymbol{r}_{\mathrm{C}} \times \boldsymbol{v}_{\mathrm{C}} + \sum_{i=1}^{N} \boldsymbol{J}_{(-)i}' = \boldsymbol{r}_{\mathrm{C}} \times \boldsymbol{p}_{(-)\mathrm{C}} + \boldsymbol{J}', \tag{160}$$

where (137) is used.

Equations (152) and (160) are substituted into (145). It is known that the first term on the right side of (145) is zero:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{J}_{(-)}' = \sum_{i=1}^{N}\left((\boldsymbol{r}_{\mathrm{C}} + \boldsymbol{r}_i') \times \boldsymbol{F}_i^{(\mathrm{e})}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{\mathrm{C}} \times \boldsymbol{p}_{(-)\mathrm{C}} = \sum_{i=1}^{N}(\boldsymbol{r}_i' \times \boldsymbol{F}_i^{(\mathrm{e})}) = \boldsymbol{L}'. \tag{161}$$

The middle equality uses (157). Finally, $\boldsymbol{L}'$ is defined as the sum of the torques of all external forces on the centroid. The physical meaning of equation (161) is: the time rate of change of the total angular momentum of the NKE particle system relative to the centroid is equal to the sum of the torques of all external forces on the centroid. This is the angular momentum theorem for the centroid.

If the sum of the torques of external forces on the centroid is zero, then the total angular momentum of the NKE particle system relative to the centroid is conserved.

5.6. Kinetic Energy Theorem and Law of Conservation of Mechanical Energy

The expression of the NKE of the *i*-th NKE particle is:

$$K_i = -\frac{1}{2} m_i \boldsymbol{v}_i^2 = -\frac{1}{2m_i} \boldsymbol{p}_{(-)i}^2. \tag{162}$$

The total NKE of the NKE particle system is:

$$K_{(-)} = \sum_{i=1}^{N} K_{(-)i}. \tag{163}$$

The total kinetic energy of the NKE particle system cannot be positive.

We write the differential form of the kinetic energy theorem for the *i*-th NKE particle:

$$\mathrm{d}K_{(-)i} = (\boldsymbol{F}_i^{(\mathrm{i})} + \boldsymbol{F}_i^{(\mathrm{e})}) \cdot \mathrm{d}\boldsymbol{r}_i \tag{164}$$

Summing over all particles:

$$\mathrm{d}K_{(-)} = \sum_{i=1}^{N} \boldsymbol{F}_i^{(\mathrm{i})} \cdot \mathrm{d}\boldsymbol{r}_i + \sum_{i=1}^{N} \boldsymbol{F}_i^{(\mathrm{e})} \cdot \mathrm{d}\boldsymbol{r}_i \tag{165}$$

The differential of the total NKE of the particle system is equal to the sum of the elementary work done by all internal and external forces. This is called the kinetic energy theorem for particle systems. It should be noted here that in the momentum theorem and angular momentum theorem, internal forces cancel each other out because they appear in pairs. However, in the kinetic energy theorem, the work done by a pair of internal forces cannot cancel each other out. This can be explained as follows. The force exerted by the *j*-th (*i*-th) particle on the *i*-th (*j*-th) particle is $\boldsymbol{f}_{ij}$ ($\boldsymbol{f}_{ji}$). It is known that $\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$. The elementary work done by this pair of internal forces is:

$$\mathrm{d}W_{ij} = \boldsymbol{f}_{ij} \cdot \mathrm{d}\boldsymbol{r}_i + \boldsymbol{f}_{ji} \cdot \mathrm{d}\boldsymbol{r}_j = \boldsymbol{f}_{ij} \cdot \mathrm{d}(\boldsymbol{r}_i - \boldsymbol{r}_j). \tag{166}$$

In general, this elementary work is not zero unless the difference in position vectors of the two particles is constant, $\mathrm{d}\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) = 0$, or $\mathrm{d}\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)$ is perpendicular to $\boldsymbol{f}_{ij}$. Therefore, even if a particle system is not acted upon by external forces, i.e., the second term on the right side of (165) is zero, the kinetic energy of the particle system is not necessarily conserved.

Among external and internal forces, if several forces are conservative forces, the corresponding potential energy can be defined for these conservative forces. The mechanical energy of each NKE particle is its kinetic energy plus potential energy. The sum of the potential energies of all NKE particles is the total potential energy V of the particle system. The total mechanical energy $E_{(-)}$ of the NKE particle system is the total NKE $K_{(-)}$ plus the total potential energy V:

$$E_{(-)} = K_{(-)} + V. \tag{167}$$

If all external and internal forces acting on the NKE particle system are conservative forces (or non-conservative forces do no work), then the total mechanical energy of the particle system is conserved. This is the law of conservation of mechanical energy for particle systems.

5.7. König's Theorem

Substitution of (155) into (162) gives

$$K_{(-)i} = -\frac{1}{2} m_i \boldsymbol{v}_{\mathrm{C}}^2 - \boldsymbol{v}_{\mathrm{C}} \boldsymbol{p}'_{(-)i} - \frac{1}{2m_i} \boldsymbol{p}'^2_{(-)i}. \tag{168}$$

Summing (168) over all particles, and using (156), we get:

$$K_{(-)} = -\sum_{i=1}^{N} \frac{1}{2m_i} \boldsymbol{p}'^2_{(-)i} - \frac{1}{2} M \boldsymbol{v}_{\mathrm{C}}^2 = K'_{(-)} + K_{(-)\mathrm{C}}. \tag{169}$$

where the first term $K'_{(-)}$ is the total NKE of the particle system relative to the centroid, and the second term $K_{(-)\mathrm{C}}$ is the NKE when the total mass of the particle system is concentrated at the centroid, which can be called the centroid NKE. Equation (169) means that the total NKE of the particle system is equal to the NKE of the particle system relative to the centroid plus the centroid NKE. This is König's theorem.

5.8. Kinetic Energy Theorem for the Centroid

We take the dot product of $\mathrm{d}\boldsymbol{r}_{\mathrm{C}}$ with both sides of (138):

$$\frac{\mathrm{d}\boldsymbol{p}_{\mathrm{C}}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{r}_{\mathrm{C}} = \boldsymbol{F}^{(\mathrm{e})} \cdot \mathrm{d}\boldsymbol{r}_{\mathrm{C}}. \tag{170}$$

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The left side is

$$\frac{\mathrm{d}\boldsymbol{p}_{(-)\mathrm{C}}}{\mathrm{d}t}\cdot\mathrm{d}\boldsymbol{r}_{\mathrm{C}}=-M\mathrm{d}\boldsymbol{v}_{\mathrm{C}}\cdot\frac{\mathrm{d}\boldsymbol{r}_{\mathrm{C}}}{\mathrm{d}t}=-\frac{1}{2}M\boldsymbol{v}_{\mathrm{C}}\cdot\mathrm{d}\boldsymbol{v}_{\mathrm{C}}=\mathrm{d}K_{\mathrm{C}}. \tag{171}$$

Equation (170) becomes:

$$\mathrm{d}K_{(-)\mathrm{C}}=\boldsymbol{F}^{(\mathrm{e})}\cdot\mathrm{d}\boldsymbol{r}_{\mathrm{C}}. \tag{172}$$

The differential of the NKE of the centroid is equal to the sum of the elementary work done by all external forces on the displacement of the centroid. Its form is like the kinetic energy theorem for a single particle with mass *M* (the total mass of the particle system) at the centroid. This formula can be called the centroid kinetic energy theorem.

Substituting (152) and (169) into (165) leads to

$$\mathrm{d}(K'_{(-)}+K_{(-)\mathrm{C}})=\sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{i})}\cdot\mathrm{d}(\boldsymbol{r}_{\mathrm{C}}+\boldsymbol{r}'_i)+\sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{e})}\cdot\mathrm{d}(\boldsymbol{r}_{\mathrm{C}}+\boldsymbol{r}'_i). \tag{173}$$

Substituting (124) into (173), using Eq. (123) where the sum of all internal forces is zero, and then subtracting Eq. (173) from (172), we get:

$$\mathrm{d}K'_{(-)}=\sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{i})}\cdot\mathrm{d}\boldsymbol{r}'_i+\sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{e})}\cdot\mathrm{d}\boldsymbol{r}'_i. \tag{174}$$

This equation indicates that the differential of the NKE of the particle system relative to the centroid is equal to the sum of the elementary work done by internal and external forces when the particle system is displaced relative to the centroid. This is the kinetic energy theorem for the centroid.

6. Dynamics of Mixed Particle Systems

6.1. Mixed Particle Systems

N PKE particles can form a PKE particle system. *N* NKE particles can form a NKE particle system. The formulas in these two sections are completely parallel. Besides, there is an additional content: systems containing both PKE and NKE particles, i.e., systems with interactions between PKE and NKE matters. Such systems are called mixed particle systems.

The simplest case is a two-body system composed of one PKE particle and one NKE particle, which has been studied previously [93]. The masses of the PKE and NKE particles are denoted as m_1 and m_2, respectively. Then, when $m_1>m_2$, they can form a bound two-body system only if there is an attractive force between them; when $m_1<m_2$, they can form a bound two-body system only if there is a repulsive force between them. In these two cases, the reduced mass of the two-body systems is $\mu_{(-)}=m_1m_2/|m_1-m_2|$, and the centroid is located on the line connecting the two particles but not between them, but outside the particle with the larger mass. The collision problem between one PKE and one NKE particles has also been investigated [93,95,96].

This section presents the mechanical formulas for multi-particle systems.

Suppose a system has *N* particles. Among them, there are N_1 PKE particles and $N_2=N-N_1$ NKE particles. The mass, position vector, and motion velocity of the *i*-th particle are denoted as m_i, $\boldsymbol{r}_i,\boldsymbol{v}_i,i=1,2,\cdots,N$, respectively. We divide this mixed particle system into two sub-particle systems. The PKE sub-particle system is composed of N_1 PKE particles, where the expressions of momentum and kinetic energy of each particle are:

$$\boldsymbol{p}_i=m_i\boldsymbol{v}_i,K_i=\frac{1}{2}m_i\boldsymbol{v}_i^2=\frac{1}{2m_i}\boldsymbol{p}_i^2 i=1,2,\cdots,N_1. \tag{175}$$

The total mass of the N_1 PKE particles is:

$$M_1=\sum_{i=1}^{N_1}m_i. \tag{176}$$

The NKE sub-particle system is composed of N_2 NKE particles, where the expressions of momentum and kinetic energy of each particle are:

$$\boldsymbol{p}_{(-)j} = -m_j \boldsymbol{v}_j, K_{(-)j} = -\frac{1}{2m_j}\boldsymbol{p}_{(-)j}^2, j = N_1 + 1, N_1 + 2, \cdots, N. \tag{177}$$

The total mass of the N_2 NKE particles is:

$$M_2 = \sum_{i=N_1+1}^{N} m_i. \tag{178}$$

Define

$$M = M_1 - M_2. \tag{179}$$

6.2. Definition of Centroid

We define the centroid $\boldsymbol{r}_{\mathrm{C}1}$ of the PKE sub-particle system and the centroid $\boldsymbol{r}_{\mathrm{C}2}$ of the NKE sub-particle system to be respectively

$$\boldsymbol{r}_{\mathrm{C}1} = \frac{1}{M_1}\sum_{i=1}^{N_1} m_i \boldsymbol{r}_i \tag{180}$$

and

$$\boldsymbol{r}_{\mathrm{C}2} = \frac{1}{M_2}\sum_{i=N_1+1}^{N} m_i \boldsymbol{r}_i. \tag{181}$$

Their velocities and accelerations are

$$\boldsymbol{v}_{\mathrm{C}1} = \dot{\boldsymbol{r}}_{\mathrm{C}1}, \boldsymbol{v}_{\mathrm{C}2} = \dot{\boldsymbol{r}}_{\mathrm{C}2} \tag{182}$$

and

$$\boldsymbol{a}_{\mathrm{C}1} = \dot{\boldsymbol{v}}_{\mathrm{C}1} = \ddot{\boldsymbol{r}}_{\mathrm{C}1}, \boldsymbol{a}_{\mathrm{C}2} = \dot{\boldsymbol{v}}_{\mathrm{C}2} = \ddot{\boldsymbol{r}}_{\mathrm{C}2}. \tag{183}$$

The centroid $\boldsymbol{r}_{\mathrm{C}}$ of the mixed particle system is defined as:

$$\boldsymbol{r}_{\mathrm{C}} = \frac{1}{M}(M_1 \boldsymbol{r}_{\mathrm{C}1} - M_2 \boldsymbol{r}_{\mathrm{C}2}), M_1 \neq M_2. \tag{184}$$

The velocity of the centroid is

$$\boldsymbol{v}_{\mathrm{C}} = \dot{\boldsymbol{r}}_{\mathrm{C}} = \frac{1}{M}(M_1 \dot{\boldsymbol{r}}_{\mathrm{C}1} - M_2 \dot{\boldsymbol{r}}_{\mathrm{C}2}). \tag{185}$$

The acceleration of the centroid is

$$\boldsymbol{a}_{\mathrm{C}} = \dot{\boldsymbol{v}}_{\mathrm{C}} = \ddot{\boldsymbol{r}}_{\mathrm{C}} = \frac{1}{M}(M_1 \ddot{\boldsymbol{r}}_{\mathrm{C}1} - M_2 \ddot{\boldsymbol{r}}_{\mathrm{C}2}). \tag{186}$$

6.3. Momentum Theorem and Law of Conservation of Momentum

The total momentum of the PKE sub-particle system is

$$\boldsymbol{p}_{(+)} = \sum_{i=1}^{N_1} \boldsymbol{p}_i = \sum_{i=1}^{N_1} m_i \boldsymbol{v}_i. \tag{187}$$

The total momentum of the NKE sub-particle system is

$$\boldsymbol{p}_{(-)} = \sum_{i=N_1+1}^{N} \boldsymbol{p}_{(-)i} = -\sum_{i=N_1+1}^{N} m_i \boldsymbol{v}_i. \tag{188}$$

The total momentum of the mixed particle system is

$$\boldsymbol{p} = \boldsymbol{p}_{(+)} + \boldsymbol{p}_{(-)}. \tag{189}$$

Each PKE (NKE) particle in the mixed particle system follows its own Newton's second law. The force acting on each particle includes internal force $\boldsymbol{F}_i^{(\mathrm{i})}$ and external force $\boldsymbol{F}_i^{(\mathrm{e})}$.

We write the formulas for the *i*-th particle without distinguishing whether it is a PKE or NKE particle, unless necessary. The Newton's second law for the *i*-th particle is:

$$\frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \boldsymbol{F}_i^{(\mathrm{i})} + \boldsymbol{F}_i^{(\mathrm{e})}. \tag{190}$$

Summing the motion equations of all particles gives

$$\sum_{i=1}^{N} \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \sum_{i=1}^{N} (\boldsymbol{F}_i^{(\mathrm{i})} + \boldsymbol{F}_i^{(\mathrm{e})}) = \boldsymbol{F}^{(\mathrm{e})}. \tag{191}$$

The final equality uses (123) and (124), where the sum of all internal forces is zero. Equation (191) can be expressed in terms of the total momentum of the particle system:

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} \boldsymbol{p}_i = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F}^{(\mathrm{e})}. \tag{192}$$

The physical meaning of this formula is: the time rate of change of the total momentum of the particle system is equal to the sum of all external forces acting on the particle system. Internal forces have no effect on the total momentum of the particle system. This is the momentum theorem for particle systems. Equation (192) is rewritten in the following form:

$$\mathrm{d}\boldsymbol{p} = \boldsymbol{F}^{(\mathrm{e})} \mathrm{d}t. \tag{193}$$

The left side is the differential of the total momentum of the particle system, and the right side is the elementary impulse of the sum of external forces on the particle system. Integrating both sides with respect to time results in

$$\boldsymbol{p}(t_2) - \boldsymbol{p}(t_1) = \int_{t_1}^{t_2} \boldsymbol{F}^{(\mathrm{e})}\, \mathrm{d}t. \tag{194}$$

This is the impulse theorem for mixed particle systems: the impulse of all external forces acting on the particle system over a period of time is equal to the total momentum of the particle system at the final moment minus that at the initial moment.

If the sum of the external forces acting on the particle system is zero over a period of time, $\boldsymbol{F}^{(\mathrm{e})}$, then from (194), the total momentum of the particle system remains unchanged during this period:

$$\boldsymbol{p} = \text{const.} \tag{195}$$

This is the law of conservation of momentum for mixed particle systems.

If a particle system is isolated, not acted upon by external forces, its total momentum must remain unchanged. That is: the total momentum of an isolated system is conserved.

6.4. Momentum Theorem for the Centroid

We have defined the position $\boldsymbol{r}_\mathrm{C}$ of the centroid of the mixed particle system by Eq. (184). Now, we write the position vector of the *i*-th particle in the following form:

$$\boldsymbol{r}_i = \boldsymbol{r}_\mathrm{C} + \boldsymbol{r}'_i, \tag{196}$$

where $\boldsymbol{r}'_i$ is the position vector of the *i*-th particle relative to the centroid. Multiplying both sides of this equation by m_i and summing over all particles, we get from (180), (181), and (184):

$$\sum_{i=1}^{N} m_i \boldsymbol{r}'_i = \sum_{i=1}^{N} m_i \boldsymbol{r}_i - \sum_{i=1}^{N} m_i \boldsymbol{r}_\mathrm{C} = M\boldsymbol{r}_\mathrm{C} - M\boldsymbol{r}_\mathrm{C} = 0. \tag{197}$$

Taking the time derivative of both sides of Eq. (196), we have

$$\boldsymbol{v}_i = \boldsymbol{v}_\mathrm{C} + \boldsymbol{v}'_i. \tag{198}$$

The velocity $\boldsymbol{v}_i$ of the *i*-th particle is equal to the velocity of the centroid $\boldsymbol{v}_\mathrm{C}$ plus the velocity $\boldsymbol{v}'_i$ of the particle relative to the centroid.

At this time, the momentum of the *i*-th particle must be distinguished between PKE and NKE particles. Substituting (198) into (175) and (177) gives

$$\boldsymbol{p}_i = m_i \boldsymbol{v}_\mathrm{C} + \boldsymbol{p}'_i, i = 1,2,\cdots,N_1, \tag{199}$$

and

$$\boldsymbol{p}_{(-)j} = -m_j \boldsymbol{v}_\mathrm{C} + \boldsymbol{p}'_{(-)j}, j = N_1+1, N_1+2,\cdots,N, \tag{200}$$

where $\boldsymbol{p}'_i$ and $\boldsymbol{p}'_{(-)j}$ are the momenta of the particle relative to the centroid. Summing the momenta of all particles gives

$$\boldsymbol{p} = M\boldsymbol{v}_{\mathrm{C}} + \boldsymbol{p}' = \boldsymbol{p}_{\mathrm{C}} + \boldsymbol{p}'. \tag{201}$$

From (175)-(177), the total momentum of the particle system relative to the centroid is:

$$\boldsymbol{p} = M\boldsymbol{v}_{\mathrm{C}} = \boldsymbol{p}_{\mathrm{C}}, \boldsymbol{p}' = 0. \tag{202}$$

The total momentum $\boldsymbol{p}$ of the particle system is equal to the momentum of the centroid $\boldsymbol{p}_{\mathrm{C}}$. In the center-of-mass frame of the particle system, the total momentum $\boldsymbol{p}'$ is zero.

Equation (192) can be written as

$$\frac{\mathrm{d}\boldsymbol{p}_{\mathrm{C}}}{\mathrm{d}t} = \boldsymbol{F}^{(\mathrm{e})}. \tag{203}$$

The time rate of change of the momentum of the centroid of the particle system is equal to the sum of all external forces acting on the particle system. This formula is called the momentum theorem for the centroid.

If the sum of the external forces on the centroid is zero, then the total momentum of the mixed particle system relative to the centroid is conserved, $\boldsymbol{p}_{\mathrm{C}} = \mathrm{const}$.

6.5. *Angular Momentum Theorem and Law of Conservation of Angular Momentum*

The expression of the angular momentum of the *i*-th particle is

$$\boldsymbol{J}_i = \boldsymbol{r}_i \times \boldsymbol{p}_i. \tag{204}$$

The total angular momentum of the particle system is:

$$\boldsymbol{J} = \sum_{i=1}^{N} \boldsymbol{J}_i. \tag{205}$$

Taking the cross product of $\boldsymbol{r}_i$ with both sides of (190), we have

$$\boldsymbol{r}_i \times \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathrm{i})} + \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathrm{e})} \ . \tag{206}$$

The left side is

$$\boldsymbol{r}_i \times \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_i \times \boldsymbol{p}_i) - \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} \times \boldsymbol{p}_i = \frac{\mathrm{d}\boldsymbol{J}_i}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} \times m_i \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{J}_i}{\mathrm{d}t}. \tag{207}$$

Therefore, Eq. (206) becomes

$$\frac{\mathrm{d}\boldsymbol{J}_i}{\mathrm{d}t} = \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathrm{i})} + \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathrm{e})}. \tag{208}$$

The sum of all internal force torques and external force torques acting on a particle is equal to the time rate of change of the particle's angular momentum. Summing this formula over all particles gives

$$\sum_{i=1}^{N} \frac{\mathrm{d}\boldsymbol{J}_i}{\mathrm{d}t} = \sum_{i=1}^{N}(\boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathrm{i})}) + \sum_{i=1}^{N}(\boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathrm{e})}). \tag{209}$$

The left side, $\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N}\boldsymbol{J}_i = \frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}$, is the time derivative of the total angular momentum of the particle system. Substituting (122) into the first term on the right side:

$$\sum_{i=1}^{N} \boldsymbol{r}_i \times \sum_{j=1, j\neq i}^{N} \boldsymbol{f}_{ji} = \sum_{i=1, j>i}^{N} \boldsymbol{r}_i \times \left(\boldsymbol{f}_{ji} + \boldsymbol{f}_{ij}\right) = 0. \tag{210}$$

Here, Eq. (121) is used: internal forces always appear in pairs, and the sum of a pair of internal forces is zero. Define

$$\boldsymbol{L} = \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathrm{e})} \tag{211}$$

as the sum of the torques of external forces on the particles. Then Eq. (209) simplifies to

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \boldsymbol{L}. \tag{212}$$

Equation (212) indicates that the sum of the torques of all external forces on the particles is equal to the time derivative of the total angular momentum of the particle system. This is the differential form of the angular momentum theorem. Equation (212) is rewritten as

$$\mathrm{d}\boldsymbol{J} = \boldsymbol{L}\mathrm{d}t. \tag{213}$$

The left side is the differential of the total angular momentum of the NKE particle system, and the right side is the elementary angular impulse of the sum of external forces on the particle system. Integrating both sides with respect to time:

$$\boldsymbol{J}(t_2) - \boldsymbol{J}(t_1) = \int_{t_1}^{t_2} \boldsymbol{L}\,\mathrm{d}t. \tag{214}$$

The right side of this equation is the angular impulse of the external torque. At this time, the angular impulse of all external torques acting on a mixed particle system over a period of time is equal to the total angular momentum of the system at the final moment minus that at the initial moment. This is the integral form of the angular momentum theorem.

If the external torque is zero over a period of time, then the angular momentum of the particle system remains unchanged during this period, or it can be written as:

$$\boldsymbol{J} = \text{const.} \tag{215}$$

If the external torque acting on a system is zero, the total angular momentum of the system is conserved. This is the law of conservation of total angular momentum for an isolated system.

6.6. Angular Momentum Ttheorem with Respect to the Centroid

Take the cross product of $\boldsymbol{r}_\mathrm{C}$ with both sides of equation (203):

$$\boldsymbol{r}_\mathrm{C} \times \frac{\mathrm{d}\boldsymbol{p}_\mathrm{C}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_\mathrm{C} \times \boldsymbol{p}_\mathrm{C}) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{J}_\mathrm{C} = \boldsymbol{r}_\mathrm{C} \times \boldsymbol{F}^{(\mathrm{e})}. \tag{216}$$

The total external torque about the centroid equals the time rate of change of the angular momentum of the centroid.

The angular momentum of the *i*-th PKE particle is written as

$$\boldsymbol{J}_i = (\boldsymbol{r}_\mathrm{C} + \boldsymbol{r}'_i) \times (m_i \boldsymbol{v}_\mathrm{C} + \boldsymbol{p}'_i)$$

$$= \boldsymbol{r}_\mathrm{C} \times m_i \boldsymbol{v}_\mathrm{C} + \boldsymbol{r}_\mathrm{C} \times \boldsymbol{p}'_i + m_i \boldsymbol{r}'_i \times \boldsymbol{v}_\mathrm{C} + \boldsymbol{J}'_i, i = 1,2,\cdots,N_1, \tag{217}$$

where

$$\boldsymbol{J}'_i = \boldsymbol{r}'_i \times \boldsymbol{p}'_i \tag{218}$$

is the angular momentum of the PKE particle with respect to the centroid. The angular momentum of the *j*-th NKE particle is written as

$$\boldsymbol{J}_{(-)j} = (\boldsymbol{r}_\mathrm{C} + \boldsymbol{r}'_i) \times \left(-m_j \boldsymbol{v}_\mathrm{C} + \boldsymbol{p}'_{(-)j}\right)$$

$$= -\boldsymbol{r}_\mathrm{C} \times m_j \boldsymbol{v}_\mathrm{C} + \boldsymbol{r}_\mathrm{C} \times \boldsymbol{p}'_{(-)j} - m_j \boldsymbol{r}'_j \times \boldsymbol{v}_\mathrm{C} + \boldsymbol{J}'_{(-)j}, j = N_1 + 1, N_1 + 2, \cdots, N, \tag{219}$$

where

$$\boldsymbol{J}'_{(-)j} = \boldsymbol{r}'_j \times \boldsymbol{p}'_j \tag{220}$$

is the angular momentum of the NKE particle with respect to the centroid.

Let

$$\boldsymbol{J}' = \sum_{i=1}^{N} \boldsymbol{J}'_i \tag{221}$$

be the total angular momentum of the particle system with respect to the centroid. Summing Eqs. (217) and (220) over all particles. Using the previous formulas, we obtain

$$\boldsymbol{J} = \boldsymbol{r}_\mathrm{C} \times \boldsymbol{p}_\mathrm{C} + \boldsymbol{J}'. \tag{222}$$

Substituting Eqs. (222) and (196) into (209) and noting that the first term on the right-hand side of (209) is zero, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{J}' = \sum_{i=1}^{N}\left((\boldsymbol{r}_{\mathrm{C}} + \boldsymbol{r}_i') \times \boldsymbol{F}_i^{(\mathrm{e})}\right) - \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_{\mathrm{C}} \times \boldsymbol{p}_{\mathrm{C}}) = \sum_{i=1}^{N}\left(\boldsymbol{r}_i' \times \boldsymbol{F}_i^{(\mathrm{e})}\right) = \boldsymbol{L}'. \quad (223)$$

The middle equality uses equation (203). Finally, $\boldsymbol{L}'$ is defined as the sum of all external torques about the centroid. The physical meaning of Eq. (223) is: the time rate of change of the total angular momentum of the mixed particle system with respect to the centroid equals the sum of all external torques about the centroid. This is the angular momentum theorem with respect to the centroid.

If the sum of external torques about the centroid is zero, $\boldsymbol{L}' = 0$, then the total angular momentum of the mixed particle system with respect to the centroid is conserved, $\boldsymbol{J}' = \mathrm{const}$.

6.7. Kinetic Energy Theorem and Law of Conservation of Mechanical Energy

The expression for the kinetic energy of a PKE particle is

$$K_i = \frac{1}{2}m_i\boldsymbol{v}_i^2 = \frac{1}{2m_i}\boldsymbol{p}_i^2, i = 1,2,\cdots,N_1, \quad (224)$$

and that of a NKE particle is

$$K_j = -\frac{1}{2}m_j\boldsymbol{v}_j^2 = -\frac{1}{2m_j}\boldsymbol{p}_{(-)j}^2, j = N_1+1, N_1+2,\cdots,N. \quad (225)$$

The total PKE and total NKE of the particle system are respectively

$$K_{(+)} = \sum_{i=1}^{N_1} K_i \quad (226)$$

and

$$K_{(-)} = \sum_{i=N_1+1}^{N} K_{(-)j}. \quad (227)$$

The total kinetic energy of the particle system is

$$K = K_{(+)} + K_{(-)}. \quad (228)$$

We write the differential form of the kinetic energy theorem for the *i*-th particle

$$\mathrm{d}K_i = (\boldsymbol{F}_i^{(\mathrm{i})} + \boldsymbol{F}_i^{(\mathrm{e})}) \cdot \mathrm{d}\boldsymbol{r}_i, i = 1,2,\cdots,N_1 \quad (229)$$

and

$$\mathrm{d}K_{(-)j} = (\boldsymbol{F}_j^{(\mathrm{i})} + \boldsymbol{F}_j^{(\mathrm{e})}) \cdot \mathrm{d}\boldsymbol{r}_j, j = N_1+1, N_1+2,\cdots,N. \quad (230)$$

Sum over all particles:

$$\mathrm{d}K = \sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{i})} \cdot \mathrm{d}\boldsymbol{r}_i + \sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{e})} \cdot \mathrm{d}\boldsymbol{r}_i. \quad (231)$$

The differential of the total NKE of the particle system equals the sum of the elemental work done by all internal and external forces. This is called the kinetic energy theorem for the particle system. It should be noticed that in the momentum theorem and angular momentum theorem, internal forces cancel each other out because they appear in pairs. However, the work done by a pair of internal forces cannot cancel each other out. This can be explained as follows: the force exerted by the *j*-th (*i*-th) particle on the *i*-th (*j*-th) particle is $\boldsymbol{f}_{ij}(\boldsymbol{f}_{ji})$. It is known that $\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$. The elemental work done by this pair of internal forces is

$$\mathrm{d}W_{ij} = \boldsymbol{f}_{ij} \cdot \mathrm{d}\boldsymbol{r}_i + \boldsymbol{f}_{ji} \cdot \mathrm{d}\boldsymbol{r}_j = \boldsymbol{f}_{ij} \cdot \mathrm{d}(\boldsymbol{r}_i - \boldsymbol{r}_j). \quad (232)$$

In general, this elemental work is not zero unless, in a special case, the difference in position vectors of the two particles is constant, $\mathrm{d}(\boldsymbol{r}_i - \boldsymbol{r}_j) = 0$.

Therefore, even if a particle system is not acted upon by external forces (i.e., the second term on the right-hand side of Eq. (231) is zero), the kinetic energy of the particle system is not necessarily conserved.

Among the external and internal forces, if several forces are conservative forces, the corresponding potential energy can be defined for these conservative forces. The mechanical energy of each NKE particle is its kinetic energy plus potential energy. The sum of the potential energies of all particles is the total potential energy V of the particle system. The total mechanical energy E of the particle system is the total kinetic energy K plus the total potential energy V:

$$E = K + V. \tag{233}$$

If all external and internal forces acting on the NKE particle system are conservative forces (or non-conservative forces do no work), then the total mechanical energy of the particle system is conserved. This is the law of conservation of mechanical energy for the mixed particle system.

6.8. König's Theorem

Substitute Eq. (199) into (224) results in

$$K_i = \frac{1}{2} m_i \boldsymbol{v}_{\mathrm{C}}^2 + \boldsymbol{v}_{\mathrm{C}} \boldsymbol{p}_i' + \frac{1}{2m_i} \boldsymbol{p}_i'^2, i = 1,2,\cdots,N_1. \tag{234}$$

Substitute Eq. (200) into (225) gives

$$K_{(-)j} = -\frac{1}{2} m_j \boldsymbol{v}_{\mathrm{C}}^2 - \boldsymbol{v}_{\mathrm{C}} \boldsymbol{p}_{(-)j}' - \frac{1}{2m_j} \boldsymbol{p}_{(-)j}'^2, j = N_1 + 1, N_1 + 2, \cdots, N. \tag{235}$$

Define the kinetic energy of each particle with respect to the centroid:

$$K_i' = \frac{1}{2m_i} \boldsymbol{p}_i'^2, i = 1,2,\cdots,N_1 \tag{236}$$

and

$$K_{(-)j}' = -\frac{1}{2m_j} \boldsymbol{p}_{(-)j}'^2, j = N_1 + 1, N_1 + 2, \cdots, N. \tag{237}$$

The total kinetic energy of each sub-particle system with respect to the centroid is respectively

$$K_{(+)}' = \sum_{i=1}^{N_1} K_i' \tag{238}$$

and

$$K_{(-)}' = \sum_{i=N_1+1}^{N} K_{(-)j}'. \tag{239}$$

Sum Eqs. (234) and (235) over all particles. Using Eq. (202), we obtain

$$K = K_{(+)}' + K_{(-)}' + \frac{1}{2} M \boldsymbol{v}_{\mathrm{C}}^2 = K' + K_{\mathrm{C}}. \tag{240}$$

The first term K' is the total kinetic energy of the particle system with respect to the centroid, and the second term K_{C} is the kinetic energy when a mass M is located at the centroid, which can be called the center-of-mass kinetic energy. Equation (240) states that the total kinetic energy of the particle system equals the kinetic energy of the particle system with respect to the centroid plus the center-of-mass kinetic energy. This is König's theorem.

6.9. Kinetic Energy Theorem with Respect to the Centroid

We take the dot product of $\mathrm{d}\boldsymbol{r}_{\mathrm{C}}$ with both sides of equation (203):

$$\frac{\mathrm{d}\boldsymbol{p}_{\mathrm{C}}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{r}_{\mathrm{C}} = \boldsymbol{F}^{(\mathrm{e})} \cdot \mathrm{d}\boldsymbol{r}_{\mathrm{C}}. \tag{241}$$

The left side is

$$\frac{\mathrm{d}\boldsymbol{p}_\mathrm{C}}{\mathrm{d}t}\cdot\mathrm{d}\boldsymbol{r}_\mathrm{C}=M\mathrm{d}\boldsymbol{v}_\mathrm{C}\cdot\frac{\mathrm{d}\boldsymbol{r}_\mathrm{C}}{\mathrm{d}t}=\frac{1}{2}M\boldsymbol{v}_\mathrm{C}\cdot\mathrm{d}\boldsymbol{v}_\mathrm{C}=\mathrm{d}K_\mathrm{C}. \tag{242}$$

Equation (241) becomes

$$\mathrm{d}K_\mathrm{C}=\boldsymbol{F}^{(\mathrm{e})}\cdot\mathrm{d}\boldsymbol{r}_\mathrm{C}. \tag{243}$$

The differential of the NKE of the centroid equals the sum of the elemental work done by all external forces on the displacement of the centroid. Its form is like the kinetic energy theorem for a single particle of mass M at the centroid. This equation can be called the centroid kinetic energy theorem.

Substitute Eqs. (196) and (228) into (231):

$$\mathrm{d}(K'+K_\mathrm{C})=\sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{i})}\cdot\mathrm{d}(\boldsymbol{r}_\mathrm{C}+\boldsymbol{r}'_i)+\sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{e})}\cdot\mathrm{d}(\boldsymbol{r}_\mathrm{C}+\boldsymbol{r}'_i). \tag{244}$$

Using $\boldsymbol{F}^{(\mathrm{e})}=\sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{e})}$, we substitute Eq. (112) into (244) and subtract (243) from (244) by use of Eq. (111) where the sum of all internal forces is zero. We obtain

$$\mathrm{d}K'=\sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{i})}\cdot\mathrm{d}\boldsymbol{r}'_i+\sum_{i=1}^{N}\boldsymbol{F}_i^{(\mathrm{e})}\cdot\mathrm{d}\boldsymbol{r}'_i. \tag{245}$$

This equation indicates that the differential of the kinetic energy of the mixed particle system with respect to the centroid equals the sum of the elemental work done by internal and external forces when the particle system is displaced with respect to the centroid. This is the kinetic energy theorem with respect to the centroid.

6.10. Kinetic Energy of An Isolated System

In Eq. (240), the two parts of the kinetic energy, the centroid kinetic energy K_C and the kinetic energy with respect to the centroid K', have uncertain signs. Therefore, it is unknown whether this mixed system belongs to a PKE system or a NKE system. If $M>0$, the center-of-mass kinetic energy is positive; if $M<0$, the centroid kinetic energy is negative.

We now consider an isolated mixed system, i.e., no external forces act on it, only interactions between particles within the particle system. These are pairs of action and reaction forces.

Now, the external forces acting on the PKE sub-particle system are the forces exerted by particles in the NKE sub-particle system on particles in the PKE sub-particle system. We consider the force $\boldsymbol{f}_{ij}$ exerted by the *j*-th particle, e.g., $j=N_1+1$, in the NKE sub-particle system on the *i*-th particle in the PKE sub-particle system. Denote the sum of the forces exerted by all particles in the NKE sub-particle systems on the *i*-th PKE particle as

$$\boldsymbol{f}_{i(-)}=\sum_{j=N_1+1}^{N}\boldsymbol{f}_{ij}. \tag{246}$$

Write the corresponding equation of motion:

$$\boldsymbol{f}_{i(-)}=m_i\boldsymbol{a}_i, i=1,2,\cdots,N_1. \tag{247}$$

Sum these N_1 equations:

$$\boldsymbol{f}_{(-)}=\sum_{i=1}^{N_1}\boldsymbol{f}_{i(-)}=\sum_{i=1}^{N_1}\sum_{j=N_1+1}^{N}\boldsymbol{f}_{ij}=M_1\boldsymbol{a}_{\mathrm{C}1}. \tag{248}$$

This resultant force $\boldsymbol{f}_{(-)}$ acts on the centroid of the PKE sub-particle system. Similarly, for any particle in the NKE sub-particle system:

$$\boldsymbol{f}_{j(+)}=-m_j\boldsymbol{a}_j, j=N_1+1,N_1+2,\cdots,N, \tag{249}$$

where

$$\boldsymbol{f}_{j(+)}=\sum_{i=1}^{N_1}\boldsymbol{f}_{ji}. \tag{250}$$

Sum all such equations for the NKE sub-particle system:

$$\boldsymbol{f}_{(+)} = \sum_{j=N_1+1}^{N} \boldsymbol{f}_{j(+)} = \sum_{i=1}^{N_1} \sum_{j=N_1+1}^{N} \boldsymbol{f}_{ji} = -M_2 \boldsymbol{a}_{\mathrm{C}2}. \quad (251)$$

This resultant force $\boldsymbol{f}_{(+)}$ acts on the centroid of the NKE sub-particle system. Obviously

$$\boldsymbol{f}_{(-)} + \boldsymbol{f}_{(+)} = 0. \quad (252)$$

The total interaction forces $\boldsymbol{f}_{(-)}$ and $\boldsymbol{f}_{(+)}$ between the PKE and NKE sub-particle systems are equivalent to the interaction forces between the centroids of the two sub-particle systems. These are a pair of action and reaction forces, acting along a straight line and on the centroids of the two sub-particle systems. Rewrite Eqs. (248) and (251) respectively as follows:

$$M_1 \boldsymbol{a}_{\mathrm{C}1} = \boldsymbol{f}_{(-)} \quad (253)$$

and

$$-M_2 \boldsymbol{a}_{\mathrm{C}2} = \boldsymbol{f}_{(+)} = -\boldsymbol{f}_{(-)}. \quad (254)$$

This pair of equations is the same as Eq. (133) or (154). Form a new pair of equations from these two equations:

$$M_1 \boldsymbol{a}_{\mathrm{C}1} - M_2 \boldsymbol{a}_{\mathrm{C}2} = 0 \quad (255)$$

and

$$\frac{M_1 M_2}{M_2 - M_1} (\boldsymbol{a}_{\mathrm{C}1} - \boldsymbol{a}_{\mathrm{C}2}) = \boldsymbol{f}_{(-)}. \quad (256)$$

The centroid of the entire mixed particle system is given by Eq. (184). Equation (255) indicates that the centroid of the while system is moving at a constant velocity.

We examine Eq. (256) and define

$$\boldsymbol{r} = \boldsymbol{r}_{\mathrm{C}1} - \boldsymbol{r}_{\mathrm{C}2}. \quad (257)$$

Then, we define the reduced mass of the mixed particle system as

$$\mu = \frac{M_1 M_2}{|M_1 - M_2|}. \quad (258)$$

When $M_1 > M_2$, from Eq. (252), the centroid kinetic energy K_{C} is positive. Equation (256) becomes

$$-\mu \ddot{\boldsymbol{r}} = \boldsymbol{f}_{(-)}. \quad (259)$$

A NKE particle with the reduced mass μ moves under the action of force $\boldsymbol{f}_{(-)}$. Then, K' in Eq. (240) should be NKE:

$$K' = -\frac{1}{2} \mu \dot{\boldsymbol{r}}^2. \quad (260)$$

When $M_1 < M_2$, from equation (240), the center-of-mass kinetic energy K_{C} is negative. Equation (256) becomes

$$\mu \ddot{\boldsymbol{r}} = \boldsymbol{f}_{(-)}. \quad (261)$$

A NKE particle with the reduced mass μ moves under the action of force $\boldsymbol{f}_{(-)}$. Then, K' in equation (240) should be NKE:

$$K' = -\frac{1}{2} \mu \dot{\boldsymbol{r}}^2. \quad (262)$$

Explicitly write the kinetic energy of the mixed particle system with respect to the centroid K' and the centroid kinetic energy

$$K' = \frac{1}{2} \frac{M_1 M_2}{M_2 - M_1} (\boldsymbol{v}_{\mathrm{C}1} - \boldsymbol{v}_{\mathrm{C}2})^2 \quad (263)$$

and the centroid kinetic energy

$$K_{\mathrm{C}} = \frac{1}{2} (M_1 - M_2) \left(\frac{M_1 \boldsymbol{v}_{\mathrm{C}1} - M_2 \boldsymbol{v}_{\mathrm{C}2}}{M_1 - M_2}\right)^2. \quad (264)$$

The total kinetic energy of the mixed particle system is

$$K = K' + K_{\mathrm{C}} = \frac{1}{2}M_1\boldsymbol{v}_{\mathrm{C}1}^2 - \frac{1}{2}M_2\boldsymbol{v}_{\mathrm{C}2}^2. \tag{265}$$

It is the sum of the kinetic energy of the PKE sub-particle system and thatof the NKE sub-particle system.

In the centroid frame, the velocity of the centroid is zero so that the center-of-mass kinetic energy $K_{\mathrm{C}} = 0$. It is easy to see from equation (263) that when $M_2 > M_1, K' > 0$, this is a PKE system, i.e., the matter we can observe; when $M_2 < M_1, K' < 0$, this is a NKE system, i.e., the dark matter system.

The simplest mixed particle system contains only one PKE particle and one NKE particle, and this case has been discussed in detail in [93].

6.11. Bullet Cluster: A Possible Example of Mixed System

Here we mention a possible example in the universe: bullet cluster 1E0657−56 [103–113]. This cluster is observable, and it seems that it encounters some invisible cluster, and they collide with each other. There have been some explanations for the invisible cluster, among which dark matter is one.

It was believed that there might need new physics in the dark sector to be explained [109]. Farrar et al. [109,110] simulated the velocities of the bullet cluster and the dark one. They concluded that "we have shown here that there is no simple correspondence between the inferred shock velocity of the bullet and the velocity of its associated DM mass centroid. In the case of 1E0657−56, the two velocities can differ."

Milgrom thought that "You still need in the cluster some yet undetected matter in roughly the same amount as that of the visible matter. Call it dark matter if you wish, but we think it is simply some standard matter in some form that has not been detected."[112] He, as the proposer of Modified Newtonian Dynamics (MOND), said "that purist MOND does not account for the observed geometry of the Bullet without invoking some yet undetected matter in the system."[111]

These investigations showed that the invisible cluster is something very like to the matter we have known, but is dark to us. In the present author's opinion, the dark cluster is one composed of NKE celestial bodies. This is a mixed system constituted by one PKE system and one NKE system. Here, we suggest doing the simulation, just like the way in Refs. [109,110], of the collisions between one PKE and one NKE particle systems.

Here, we show by a simplest example that the results of the collisions between two PKE particle and those between one PKE and one NKE particles are different.

We take into account the collinear between two particles with masses m_1 and m_2, respectively. They move along the x-axis. The m_1 (m_2) has momentum p_1 (p_2) and velocity v_1 (v_2) before collision and p_1' (p_2') and v_1' (v_2') after collision. We consider head on collisions

In the case that both m_1 and m_2 are of PKE, we have

$$p_1' = \frac{1}{m_1+m_2}[(m_1 - m_2)p_1 + 2m_1p_2] \tag{266a}$$

and

$$p_2' = \frac{1}{m_1+m_2}[2m_2p_1 + (m_2 - m_1)p_2]. \tag{266b}$$

We first set v_1 and v_2, and calculate $p_1 = m_1v_1$ and $p_2 = m_2v_2$. Then, we evaluate p_1' and p_2' using (266), and finally calculate $v_1' = p_1'/m_1$ and $v_2' = p_2'/m_2$. We let $m_1 = 2$ and $m_2 = 1$. Some results are listed in Table 3(a).

Table 3. a). The collision between a PKE m_1 and a PKE m_2. $m_1 = 2$ and $m_2 = 1$.

v_1	v_2	p_1	p_2	p_1'	p_2'	v_1'	v_2'
1.5	−6	3	−6	−7	4	−3.5	4
3	−3	6	−3	−2	5	−1	5

3	–6	6	–6	–6	6	–3	6

Before collision, $v_1 > 0$ and $v_2 < 0$. They make head-on impinge. After that, $v_1' < 0$ and $v_2' > 0$, which means that they move along opposite directions, away from each other.

We turn to study the case that m_1 is of PKE and m_2 is of NKE. It can be derived [93] that

$$p_1' = \frac{1}{m_1 - m_2}[(m_1 + m_2)p_1 + 2m_1 p_2] \quad (267a)$$

$$p_2' = \frac{1}{m_1 - m_2}[-2m_2 p_1 - (m_1 + m_2)p_2] \quad (267b)$$

We first set v_1 and v_2, and calculate $p_1 = m_1 v_1$ and $p_2 = -m_2 v_2$. Then, we evaluate p_1' and p_2' using (267), and finally calculate $v_1' = p_1'/m_1$ and $v_2' = p_2'/m_2$. We let $m_1 = 2$ and $m_2 = 1$. Some results are listed in Table 3(b).

Table 3. b). The collision between a PKE m_1 and a NKE m_2. $m_1 = 2$ and $m_2 = 1$.

v_1	v_2	p_1	p_2	p_1'	p_2'	v_1'	v_2'
1.5	–6	3	6	33	–24	16.5	24
3	–3	6	3	30	–21	15	21
3	–6	6	6	42	–30	21	30

Before collision, $v_1 > 0$ and $v_2 < 0$. They make head-on impinge. After that, $v_1' > 0$ and $v_2' > 0$, which means that they both move rightwards. It is seen that $v_2' > v_1'$, manifesting that NKE particle moves ahead of the PKE one.

Let us compare Tables 3(a) and (b). The m_1, v_1 and m_2, v_2 are called initial conditions and are set to be the same in Tables 3(a) and (b). Under the same initial condition, the resultant v_1' and v_2' in Table 3(b) are much larger than those in Table 3(a). The reason is follows. When two PKE particles make head-on collision, their velocities are in opposite directions, and so are their momenta. The total momentum is the subtraction of the two PKE particles' momenta. When one PKE and one NKE particles make head-on collision, their velocities are in opposite directions, but their momenta are in the same direction. The total momentum is the sum of the two particles' momenta. Therefore, compared to the former case, the latter case has a larger total momentum. In the course of a collision, the total momentum is conserved. As a result, the particles gain larger velocities after collision.

For the case of bullet 1E0657–56, the weak-lensing mass map reveals a dark matter clump lying a head of the collisional gas bullet, a prominent bow shock. At least some dark matter moves faster than the visible cluster after collision, with which we think the calculated results in Table 3(b) are consistent. We suggest the bullet cluster be simulated regarding its colliding with a NKE cluster.

7. Lagrangian Mechanics

7.1. Action and Euler-Lagrange Equation

Consider a system composed of *N* particles. The mass of the *i*-th particle is m_i, the position vector is $\boldsymbol{r}_i$, and the velocity is $\boldsymbol{v}_i$. The interaction potential energy between the *i*-th particle and the *j*-th particle is $V_{ij} = V(\boldsymbol{r}_i - \boldsymbol{r}_j)$. Assume the Lagrangian of the system $L(\{\boldsymbol{r}_i\}, \{\dot{\boldsymbol{r}}_i\}, t)$ is known.

The definition of the principal function *S* is

$$S = \int L(\{\boldsymbol{r}_i\}, \{\dot{\boldsymbol{r}}_i\}, t)\mathrm{d}t, \quad (266)$$

which is the integral of the Lagrangian with respect to time. The procedure of derive the equation of motion are as follows: integrate the Lagrangian from time t_1 to t_2:

$$S = \int_{t_1}^{t_2} L(\{\boldsymbol{r}_i\}, \{\dot{\boldsymbol{r}}_i\}, t)\mathrm{d}t. \quad (267)$$

This is also called the action. We should find the extremum of the action:

$$\delta S = \delta \int_{t_1}^{t_2} L(\{\boldsymbol{r_i}\}, \{\dot{\boldsymbol{r}}_i\}, t)\mathrm{d}t = 0. \tag{268}$$

The two endpoints of the integral are fixed, and the variation of the Lagrangian is performed. Using the calculus of variations, the motion of the *i*-th particle satisfies the Euler-Lagrange equation:

$$\frac{\partial L}{\partial \boldsymbol{r_i}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} = 0. \tag{269}$$

From (266), the total differential of the principal function with respect to time is the Lagrangian:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = L. \tag{270}$$

The derivative of the principal function with respect to the coordinate is the momentum:

$$\frac{\mathrm{d}S}{\mathrm{d}\boldsymbol{r}_i} = \boldsymbol{p}_i. \tag{271}$$

The derivative of the principal function with respect to time can be written as:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial S}{\partial t} + \sum_{i=1}^{N} \frac{\partial S}{\partial \boldsymbol{r}_i} \cdot \dot{\boldsymbol{r}}_i = \frac{\partial S}{\partial t} + \sum_{i=1}^{N} \boldsymbol{p}_i \cdot \dot{\boldsymbol{r}}_i. \tag{272}$$

Through the Legendre transformation, the Lagrangian can be transformed into the Hamiltonian *H*. Thus, the Hamilton-Jacobi equation is obtained:

$$\frac{\partial S}{\partial t} + H = 0. \tag{273}$$

7.2. NKE System

Assume all *N* particles have NKE. The *i*-th particle has momentum $\boldsymbol{p}_i = -m_i \boldsymbol{v}_i$ and kinetic energy $k_i = \frac{1}{2} m_i \boldsymbol{v}_i^2 = -\frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2$. The total NKE of the system is:

$$K_{(-)} = -\frac{1}{2}\sum_{i=1}^{N} m_i\, \boldsymbol{v}_i^2. \tag{274}$$

The total potential energy is:

$$V = \sum_{i>j}^{N} V_{ij}. \tag{275}$$

The total energy is the sum of the total NKE and the potential energy. Therefore, the Hamiltonian is:

$$H = K_{(-)} + V. \tag{276}$$

Taking the derivative of the Hamiltonian $H(\{\boldsymbol{r}_i\}, \{\boldsymbol{p}_i\}, t)$ with respect to momentum gives the velocity:

$$\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \frac{\partial H}{\partial \boldsymbol{p}_i}. \tag{277}$$

Thus, the momentum can be expressed in terms of velocity:

$$\boldsymbol{p}_i = \boldsymbol{p}_i(\{\dot{\boldsymbol{r}}_i\}). \tag{278}$$

Then, perform the following Legendre transformation:

$$L = \sum_{i=1}^{N} \boldsymbol{p}_i\,(\{\dot{\boldsymbol{r}}_i\}) \cdot \dot{\boldsymbol{r}}_i - H. \tag{279}$$

The Lagrangian of the system is obtained as the total NKE minus the total potential energy:

$$L = -\frac{1}{2}\sum_{i=1}^{N} m_i\, \boldsymbol{v}_i^2 - V_{ij} = K_{(-)} - V. \tag{280}$$

The motion of the *i*-th NKE particle satisfies the Euler-Lagrange equation:

$$\frac{\partial L}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} = 0. \tag{281}$$

Since the total kinetic energy of this system is negative, the equation of motion (281) makes the action (267) S reach a maximum value. Therefore, Eq. (268) is also called the principle of maximum action, which conforms to the extremum principle, as mentioned preciously [25]. Why a NKE system follows the maximum principle needs to be explained in terms of quantum mechanics. Here, we briefly mention the reason as follows.

A PKE system is most stable in the lowest energy state. Therefore, when the system undergoes a process, each intermediate state tends to be in the lowest energy state, so the system follows the principle of minimum energy. A NKE system has a negative temperature and is most stable in the highest energy state. Therefore, when the system undergoes a process, each intermediate state tends to be in the highest energy state, so the NKE system follows the principle of maximum energy. Note that the stable state mentioned here is a different concept from the stable motion defined in subsection IV.B.

7.3. Mixed System

Consider a system with N particles, among which N_1 are PKE particles and $N_2 = N - N_1$ are NKE particles. Let M_1 be the total mass of the N_1 PKE particles, and M_2 be the total mass of the N_2 NKE particles:

$$M_1 = \sum_{i=1}^{N_1} m_i\,, M_2 = \sum_{i=N_1+1}^{N_1} m_i. \tag{282}$$

The total kinetic energy of the system is:

$$K = \frac{1}{2}\sum_{i=1}^{N_1} m_i\,\boldsymbol{v}_i^2 - \frac{1}{2}\sum_{i=N_1+1}^{N} m_i\,\boldsymbol{v}_i^2. \tag{283}$$

The Lagrangian of the system is the total kinetic energy minus the total potential energy:

$$L = K - V = \frac{1}{2}\sum_{i=1}^{N} m_i\,\boldsymbol{v}_i^2 - \frac{1}{2}\sum_{i=N_1+1}^{N} m_i\,\boldsymbol{v}_i^2 - \sum_{i,j,i\neq j} V_{ij}. \tag{284}$$

The motion of the i-th particle satisfies the Euler-Lagrange equation:

$$\frac{\partial L}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} = 0. \tag{285}$$

Then, does the motion of the system minimize or maximize the acting force?

Let us consider a two-particle system in the centroid frame. The total kinetic energy is:

$$K = \frac{1}{2}m_1\boldsymbol{v}_1^2 - \frac{1}{2}m_2\boldsymbol{v}_2^2 \tag{286}$$

and the Lagranian is

$$L = \frac{1}{2}m_1\boldsymbol{v}_1^2 - \frac{1}{2}m_2\boldsymbol{v}_2^2 - V(\boldsymbol{r}_1 - \boldsymbol{r}_2). \tag{287}$$

As known from the previous equations of motion, when $m_1 < m_2$, the system should be a PKE system, and the action should reach a minimum value; when $m_1 > m_2$, the system should be a NKE system, and the action should reach a maximum value.

Similarly, for a mixed group of N particles, when $M_1 < M_2$, the system is a PKE system, and the action reaches a minimum value; when $M_1 > M_2$, the system is a NKE system, and the action reaches a maximum value, as mentioned at the end of subsection VI.J.

7.4. Examples of the Motion of a Single NKE Particle

In the following we calculate the action for the motion of single NKE particle for several cases. Assume the boundary conditions at both ends of the motion trajectory of the single particle are:

$$\boldsymbol{r}(t_1) = \boldsymbol{r}_1; \boldsymbol{r}(t_2) = \boldsymbol{r}_2, s = t_2 - t_1. \tag{288}$$

7.4.1. Free Particle

The Lagrangian of a NKE free particle is its NKE:

$$L_{(-)} = T_{(-)} = -\frac{1}{2} m\dot{\boldsymbol{r}}^2. \tag{289}$$

The momentum of the particle is:

$$\boldsymbol{p}_{(-)} = \frac{\partial L_{(-)}}{\partial \dot{\boldsymbol{r}}} = -m\dot{\boldsymbol{r}}. \tag{290}$$

The Hamiltonian is:

$$H_{(-)} = \dot{\boldsymbol{r}} \cdot \boldsymbol{p}_{(-)} - L_{(-)} = -\frac{1}{2} m\dot{\boldsymbol{r}}^2. \tag{291}$$

The Hamilton-Jacobi equation is:

$$\frac{\partial S_{(-)}}{\partial t} - \frac{1}{2m} (\nabla S_{(-)})^2 = 0. \tag{292}$$

Substituting the Lagrangian (289) into (269) gives the equation of motion $\ddot{\boldsymbol{r}} = 0$. Its solution is $\boldsymbol{r} = \boldsymbol{C}_1 t + \boldsymbol{C}_2$. Using the boundary conditions, we solve $\boldsymbol{C}_1 = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{s}, \boldsymbol{C}_2 = \frac{\boldsymbol{r}_1 t_2 - \boldsymbol{r}_2 t_1}{s}$Therefore, the calculated action is:

$$S_{(-)} = -\frac{m}{2} \frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}{s}. \tag{293}$$

7.4.2. Harmonic Potential

A NKE particle is in a harmonic potential.

As known earlier, when a PKE particle performs stable motion in a potential field V, a NKE particle can only perform stable motion in a $-V$ potential field. Therefore, the expression of the potential energy is $V = -m\omega^2 \boldsymbol{r}^2/2$:

$$L_{(-)} = -\frac{1}{2} m\dot{\boldsymbol{r}}^2 + m\omega^2 \boldsymbol{r}^2/2. \tag{294}$$

The momentum of the particle is:

$$\boldsymbol{p}_{(-)} = \frac{\partial L_{(-)}}{\partial \dot{\boldsymbol{r}}} = -m\dot{\boldsymbol{r}}. \tag{295}$$

The Hamiltonian is:

$$H_{(-)} = \dot{\boldsymbol{r}} \cdot \frac{\partial L_{(-)}}{\partial \dot{\boldsymbol{r}}} - L_{(-)} = -\frac{1}{2} m\dot{\boldsymbol{r}}^2 - \frac{1}{2} m\omega^2 \boldsymbol{r}^2. \tag{296}$$

The Hamilton-Jacobi equation is:

$$\frac{\partial S_{(-)}}{\partial t} - \frac{1}{2m} (\nabla S_{(-)})^2 - \frac{1}{2} m\omega^2 \boldsymbol{r}^2 = 0. \tag{297}$$

Substituting the Lagrangian (294) into (269) gives the equation of motion $\ddot{\boldsymbol{r}} + \omega^2 \boldsymbol{r} = 0$. Its solution is $\boldsymbol{r} = \boldsymbol{A} \sin \omega t + \boldsymbol{B} \cos \omega t$. Using the boundary conditions, we acquire $\boldsymbol{A} = \frac{\boldsymbol{r}_2 \cos \omega t_1 - \boldsymbol{r}_1 \cos \omega t_2}{\sin \omega s}, \boldsymbol{B} = \frac{\boldsymbol{r}_1 \sin \omega t_2 - \boldsymbol{r}_2 \sin \omega t_1}{\sin \omega s}$. This is the same as the motion trajectory of a PKE particle in a positive harmonic potential. The calculated action is:

$$S_{(-)} = -\frac{m\omega}{2 \sin \omega s} [(\boldsymbol{r}_1^2 + \boldsymbol{r}_2^2) \cos \omega s - 2\boldsymbol{r}_1 \cdot \boldsymbol{r}_2]. \tag{298}$$

7.4.3. One-Dimensional Linear Potential

A NKE particle is in a one-dimensional linear potential $V = Dx$. The Lagrangian is:

$$L_{(-)} = -\frac{1}{2} m\ddot{x}^2 - Dx. \tag{299}$$

The momentum of the particle is:

$$p_{(-)}=\frac{\partial L}{\partial x}=-m\dot{x}. \tag{300}$$

The Hamiltonian is:

$$H_{(-)}=xp_{(-)}-L_{(-)}=-\frac{1}{2}m\dot{x}^2+Dx. \tag{301}$$

The Hamilton-Jacobi equation is:

$$\frac{\partial S_{(-)}}{\partial t}-\frac{1}{2m}(\frac{\partial S_{(-)}}{\partial x})^2+Dx=0. \tag{302}$$

Substituting the Lagrangian (299) into (269) gives the equation of motion $m\ddot{x}-D=0$. Its solution is $x=\frac{Dt^2}{2m}+C_1t+C_2$. Using the boundary conditions, we solve $C_1=\frac{x_1-x_2}{s}-\frac{D}{2m}(t_1+t_2), C_2=\frac{x_1t_2-x_2t_1}{s}+\frac{D}{2m}t_1t_2$. This is the same as the motion trajectory of a PKE particle in a linear potential $-Dx$. The calculated action is:

$$S_{(-)}=-\frac{m}{2}\frac{(x_2-x_1)^2}{s}-\frac{D}{2}s(x_1+x_2)+\frac{D^2s^3}{24m}. \tag{303}$$

7.4.4. One-Dimensional Harmonic Potential plus Linear Potential

The Lagrangian of the NKE particle is:

$$L_{(-)}=-\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2-Dx. \tag{304}$$

The momentum of the particle is:

$$p_{(-)}=\frac{\partial L}{\partial x}=-m\dot{x}. \tag{305}$$

The Hamiltonian is:

$$H_{(-)}=xp_{(-)}-L_{(-)}=-\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2+Dx. \tag{306}$$

The Hamilton-Jacobi equation is:

$$\frac{\partial S_{(-)}}{\partial t}-\frac{1}{2m}(\frac{\partial S_{(-)}}{\partial x})^2-\frac{1}{2}m\omega^2x^2+Dx=0. \tag{307}$$

Substituting the Lagrangian (304) into (269) gives the equation of motion $m\ddot{x}+m\omega^2x-D=0$. Its solution is $x=A\sin\omega t+B\cos\omega t+d, d=\frac{D}{m\omega^2}$. Using the boundary conditions, we solve $A=\frac{(x_2-d)\cos\omega t_1-(x_1-d)\cos\omega t_2}{\sin\omega s}, B=\frac{(x_1-d)\sin\omega t_2-(x_2-d)\sin\omega t_1}{\sin\omega s}$. This is the same as the motion trajectory of a PKE particle in the potential $\frac{1}{2}m\omega^2x^2-Dx$. The calculated action is:

$$S_{(-)}=-\frac{m\omega}{2\sin\omega s}[(x_1^2+x_2^2)\cos\omega s-2(x_1-D)(x_2-D)]-\frac{D^2s}{2m\omega^2}. \tag{308}$$

7.4.5. Electromagnetic Potential

A particle with charge q is in an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$. The existence of an electromagnetic field implies the presence of an electric potential φ and a magnetic vector potential $\boldsymbol{A}$. Both the electric field strength and the magnetic induction strength can be expressed in terms of the electromagnetic potentials:

$$\boldsymbol{E}=-\frac{\partial A}{\partial t}-\nabla\varphi, \boldsymbol{B}=\nabla\times\boldsymbol{A}. \tag{309}$$

A charge q has an electric potential energy $q\varphi$, and a current has a potential energy $-\boldsymbol{j}\cdot\boldsymbol{A}$. When the velocity of a charge q is v, the current is $\boldsymbol{j}=q\boldsymbol{v}$. The total electromagnetic potential energy is $-q\boldsymbol{v}\cdot\boldsymbol{A}+q\varphi$.

Now, a particle with charge q has NKE. Its Lagrangian is:

$$L_{(-)}=-\frac{1}{2}m\boldsymbol{v}^2+q\boldsymbol{v}\cdot\boldsymbol{A}-q\varphi. \tag{310}$$

The derivative of the Lagrangian with respect to velocity is:

$$\boldsymbol{P}_{(-)}=\frac{\partial L}{\partial\boldsymbol{v}}=-m\boldsymbol{v}+q\boldsymbol{A}=\boldsymbol{p}_{(-)}+q\boldsymbol{A}. \tag{311}$$

Here, $\boldsymbol{P}_{(-)}$ is the generalized momentum, which is the ordinary momentum $\boldsymbol{p}_{(-)}$ plus the vector potential term. The velocity at this time should be expressed in terms of the generalized momentum:

$$m\boldsymbol{v}=\boldsymbol{P}_{(-)}-q\boldsymbol{A}. \tag{312}$$

The Hamiltonian is:

$$H=\boldsymbol{v}\cdot\boldsymbol{P}_{(-)}-L_{(-)}=-\frac{1}{2}m\boldsymbol{v}^2+q\varphi=\frac{1}{2m}(\boldsymbol{P}-q\boldsymbol{A})^2+q\varphi. \tag{313}$$

At this time, the gradient of the principal function is the generalized momentum:

$$\boldsymbol{P}_{(-)}=\boldsymbol{\nabla}S_{(-)}. \tag{314}$$

Using Eqs. (313) and (314), the Hamilton-Jacobi equation at this time is:

$$\frac{\partial S_{(-)}}{\partial t}-\frac{1}{2m}(\nabla S_{(-)}-q\boldsymbol{A})^2+q\varphi=0. \tag{315}$$

Substituting the Lagrangian (310) into (269) gives the equation of motion:

$$\dot{\boldsymbol{p}}_{(-)}=q\boldsymbol{E}+q\boldsymbol{v}\times\boldsymbol{B}. \tag{316}$$

The right-hand side of the equation of motion is still the Lorentz force exerted on a charge q in an electromagnetic field. We have discussed the motion trajectory of a NKE particle with charge q under the action of the Lorentz force in a constant electromagnetic field in subsection IV.D.

8. Relativistic Dynamics of A Particle

8.1. Lorentz Transformation of Four-Vectors

Consider two inertial reference frames S and S'. S' moves relative to S along the x-direction with velocity u. In the S frame, there is a four-vector d, written as:

$$d=(\boldsymbol{d},d_4). \tag{317}$$

The first three components are the vector in three-dimensional space, and the fourth component d_4 is called the time component. In the S' frame, the corresponding four-vector is denoted as:

$$d'=(\boldsymbol{d}',d_4'). \tag{318}$$

Let

$$\beta=\frac{u}{c},\gamma=\frac{1}{\sqrt{1-u^2/c^2}}. \tag{319}$$

The transformation of the four-vector between the two inertial frames is as follows:

$$d'=\alpha d. \tag{320}$$

where the transformation matrix is

$$\alpha=\begin{pmatrix}\gamma & 0 & 0 & -\mathrm{i}\beta\gamma\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \mathrm{i}\beta\gamma & 0 & 0 & \gamma\end{pmatrix}, \tag{321}$$

and its inverse matrix is $\alpha^{-1}=\alpha^{\mathrm{T}}$. The inverse transformation of equation (320) is

$$d=\alpha^{-1}d'. \tag{322}$$

Commonly encountered four-vectors are as follows:

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Four − dimensional space − time coordinate vector: $x = (\boldsymbol{r}, \mathrm{i}ct)$ (323a)
Four-dimensional current density vector: $J = (\boldsymbol{J}, \mathrm{i}c\rho)$ (323b)
Four-dimensional electromagnetic potential vector: $A = (\boldsymbol{A}, \mathrm{i}\varphi/c)$ (323c)
Four-dimensional momentum-energy vector: $P = (\boldsymbol{p}, \mathrm{i}E/c)$ (323d)

8.2. Dynamical Formulas

The special relativistic formulas for NKE particles are symmetric with respect to those for PKE particles. The most basic formulas can be derived from the fundamental equations of quantum mechanics [93]. The content in this section is strictly limited to classical mechanics.

The most basic assumption is the new Newton's second law:

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}_{(-)}}{\mathrm{d}t}. \tag{324}$$

Assume the mass of an object is m, and its velocity is $\boldsymbol{v}$. Then, similar to non-relativistic motion, the momentum of the particle should be opposite to the direction of velocity. Therefore, we can write:

$$\boldsymbol{p}_{(-)} = -\frac{m\boldsymbol{v}}{\sqrt{1-v^2/c^2}}. \tag{325}$$

Assume the mass m is constant. We calculate the increase in the particle's energy caused by the elementary work done by the force $\boldsymbol{F}$:

$$\mathrm{d}W = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = -\frac{m}{2(1-v^2/c^2)^{3/2}}\mathrm{d}\boldsymbol{v}^2 = -\mathrm{d}\frac{m}{\sqrt{1-v^2/c^2}} = \mathrm{d}E_{(-)}. \tag{326}$$

Obviously, the energy can be written as the following expression:

$$E_{(-)} = -\frac{mc^2}{\sqrt{1-v^2/c^2}}. \tag{327}$$

This is the energy expressed in terms of velocity. Substituting (325) into (327) gives the relationship between energy and momentum:

$$E_{(-)} = -\sqrt{m^2c^4 + c^2\boldsymbol{p}^2}. \tag{328}$$

When the velocity of the object is much smaller than the speed of light, $v \ll c$, expanding (325) with v/c as a small quantity gives:

$$\boldsymbol{p}_{(-)} = -m\boldsymbol{v}. \tag{329}$$

This equation retrieves the formula of new Newtonian mechanics. The expansions of Eqs. (327) and (328) are:

$$E_{(-)} = -mc^2 - \frac{1}{2m}\boldsymbol{p}^2 = -mc^2 - \frac{m}{2}\boldsymbol{v}^2. \tag{330}$$

The energy consists of two parts: one is the negative rest energy, which does not play a rolet in Newtonian mechanics; the other is the NKE, which is exactly the same as the expression in the new Newtonian mechanics.

Multiplying both sides of (327) by $\boldsymbol{v}/c^2$ gives the momentum (325). Therefore:

$$\boldsymbol{p}_{(-)} = \frac{E_{(-)}\boldsymbol{v}}{c^2}. \tag{331}$$

This equation is formally the same as the corresponding relationship for PKE particles. However, note that $E_{(-)}$ is given by (328) and is a negative number.

Therefore, it is now known that Newtonian mechanics is actually the formula applicable to the low-velocity motion of objects when $v \ll c$ in special relativity.

Equation (326) is the differential form of the work-energy theorem. The integral form of the work-energy theorem is:

$$W = \int_{\mathrm{A}}^{\mathrm{B}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = mc^2\sqrt{1-\frac{v_{\mathrm{B}}^2}{c^2}} - mc^2\sqrt{1-\frac{v_{\mathrm{A}}^2}{c^2}}. \tag{332a}$$

In the low-momentum approximation, the right-hand side of this equation simplifies to:

$$W_{(-)} = \int_{\mathrm{A}}^{\mathrm{B}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -\frac{1}{2}mv_{\mathrm{B}}^2 - \left(-\frac{1}{2}mv_{\mathrm{A}}^2\right). \tag{332b}$$

This is exactly the formula (56) of Newtonian mechanics.

8.3. Equivalence of Mass and Negative Energy

From the relativistic energy-momentum relationship of an object, when the object is at rest—here, "a NKE object is at rest" means setting the momentum to zero in the energy-momentum relationship (328), i.e., the momentum approaches infinitely small negative number, $\boldsymbol{p}_{(-)} \to 0^-$. No matter how small its momentum is, it is always a NKE object. When the momentum is infinitely close to zero, we say the momentum is zero, $\boldsymbol{p}_{(-)} = 0$:

$$E_{(-)} = -mc^2. \tag{333}$$

Similar to the PKE case, if a NKE object loses negative energy $\Delta E_{(-)}$, its mass also loses Δm, satisfying the relationship:

$$\Delta E_{(-)} = -\Delta mc^2. \tag{334}$$

Equations (333) and (334) indicate that a certain mass is equivalent to a certain negative energy. They also show that when the mass of a NKE object decreases, it must release a portion of negative energy to the environment. Conversely, if a NKE object gains some negative energy, its mass will increase accordingly. Equation (334) can have applications [96].

Furthermore, we can say that a mass m can be converted into a negative energy $E_{(-)} = -mc^2$. Conversely, a negative energy $E_{(-)} < 0$ can be converted into a mass $m = -E_{(-)}/c^2$. Mass and negative energy are mutually equivalent and convertible. This relationship should be called the mass-negative energy equivalence of NKE matter.

For PKE objects, the relationship obtained by Einstein is:

$$\Delta E = \Delta mc^2. \tag{335}$$

A mass can be equivalent to either a positive energy or a negative energy, depending on whether the object's motion is in a PKE or NKE state. Once more, Eqs. (334) and (335) display a symmetry.

8.4. Lagrangian Mechanics

The content of Lagrangian mechanics in Section VII is derived from Newtonian mechanics, i.e., the formulas of non-relativistic motion, and also applies to relativistic motion. Below, we can directly use the formulas in subsection VII.A.

8.4.1. Free Particle

The Hamiltonian of a free NKE relativistic particle is:

$$H_{(-)} = -\sqrt{m^2c^4 + c^2\boldsymbol{p}^2}. \tag{336}$$

From equation (277), we obtain:

$$\boldsymbol{v} = \frac{c^2\boldsymbol{p}_{(-)}}{E_{(-)}}, \tag{337}$$

which is equation (331). Then, from (279), we get:

$$L_{(-)} = \boldsymbol{p}_{(-)} \cdot \dot{\boldsymbol{r}} - H_{(-)} = \frac{c^2\boldsymbol{p}^2}{E_{(-)}} - E_{(-)} = -\frac{m^2c^4}{E_{(-)}} = mc^2\sqrt{1-\frac{v^2}{c^2}}. \tag{338}$$

It is the negative of the relativistic Lagrangian of a PKE free object.

Substituting the Lagrangian (338) into the Euler-Lagrange equation (269), $\frac{\partial L_{(-)}}{\partial \boldsymbol{r}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{(-)}}{\partial \dot{\boldsymbol{r}}} = 0$ gives the equation of motion $\ddot{\boldsymbol{r}} = 0$. Its solution is $\boldsymbol{r} = \boldsymbol{v}t + \boldsymbol{d}$. The boundary conditions at both ends of the object's motion trajectory are $\boldsymbol{r}(t_1) = \boldsymbol{r}_1; \boldsymbol{r}(t_2) = \boldsymbol{r}_2, s = t_2 - t_1$. Solving for the motion trajectory gives $\boldsymbol{r} = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{s}t + \frac{\boldsymbol{r}_1 t_2 - \boldsymbol{r}_2 t_1}{s}$. Substituting the time derivative of this motion trajectory into (338), then into (267), the calculated action is:

$$S_{(-)} = mc^2\sqrt{1 - \frac{v^2}{c^2}}\,s = mc\sqrt{c^2 s^2 - (\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}. \tag{339}$$

When $|\boldsymbol{r}_2 - \boldsymbol{r}_1\} \ll c$,

$$S_{(-)} = mc^2 s\left(1 - \frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}{2c^2 s^2}\right) = mc^2 s - \frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}{2s}. \tag{340}$$

The first term is a constant term determined by the length of the motion time, and the second term is the action of the new Newtonian mechanics (293).

Substituting $\boldsymbol{p}_{(-)} = \nabla S_{(-)}$ into (337), the Hamilton-Jacobi equation is:

$$\frac{\partial S_{(-)}}{\partial t} = c\sqrt{m^2 c^2 + (\nabla S_{(-)})^2}, \tag{341}$$

where $(\nabla S_{(-)})^2 = (\frac{\partial S_{(-)}}{\partial x})^2 + (\frac{\partial S_{(-)}}{\partial y})^2 + (\frac{\partial S_{(-)}}{\partial z})^2$. Let $S_{(-)} = S'_{(-)} + mc^2 t$,

$$\frac{\partial S'_{(-)}}{\partial t} + mc^2 = mc^2\sqrt{1 + \frac{1}{m^2 c^2}(\nabla S'_{(-)})^2}. \tag{342}$$

Further, let $c \to \infty$,

$$\frac{\partial S'_{(-)}}{\partial t} = \frac{1}{2m}(\nabla S'_{(-)})^2. \tag{343}$$

This is the case of new Newtonian mechanics.

For comparison, we write the Hamilton-Jacobi equation for a PKE free particle. It can be easily written according to (273):

$$\frac{\partial S_{(+)}}{\partial t} = -c\sqrt{m^2 c^2 + (\nabla S_{(+)})^2}. \tag{344}$$

Let $S_{(+)} = S'_{(+)} - mc^2 t$,

$$\frac{\partial S'_{(+)}}{\partial t} - mc^2 = -mc^2\sqrt{1 + \frac{1}{m^2 c^2}(\nabla S'_{(+)})^2}. \tag{345}$$

Then let $c \to \infty$,

$$\frac{\partial S'_{(+)}}{\partial t} = -\frac{1}{2m}(\nabla S'_{(+)})^2. \tag{346}$$

This is the case of Newtonian mechanics.

We recall the formula written in Landau's "The Classical Theory of Fields" [114]. The Hamilton-Jacobi formula was written in the form of

$$\frac{1}{c^2}(\frac{\partial S}{\partial t})^2 = m^2 c^2 + (\nabla S)^2. \tag{347}$$

Let $S = S' - mc^2 t$,

$$\frac{1}{c^2}(\frac{\partial S'}{\partial t})^2 - 2m\frac{\partial S'}{\partial t} = (\nabla S')^2. \tag{348}$$

Letting $c \to \infty$, we get $-2m\frac{\partial S'}{\partial t} = (\nabla S')^2$, which only gives the Newtonian mechanics approximation result (346), with (343) excluded. This is because (347) does not write the Hamilton-Jacobi equation in the standard form of (273). There should not be a formula like $(\frac{\partial S}{\partial t})^2 = H^2$.

Table 4. Comparison of relativistic formulas for PKE and NKE objects.

	PKE object	NKE object
Newton's second law	$\boldsymbol{f}=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}$	$\boldsymbol{f}=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}_{(-)}$
Energy of free particle	$E_{(+)}=\sqrt{m^2c^4+c^2\boldsymbol{p}^2}$	$E_{(-)}=-\sqrt{m^2c^4+c^2\boldsymbol{p}_{(-)}^2}$
Nonrelativistic approximation the free particle energy	$mc^2+\boldsymbol{p}^2/2m$	$-mc^2-\boldsymbol{p}_{(-)}^2/2m$
Momentum-velocity relationship	$\boldsymbol{p}_{(+)}=\frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{E_{(+)}\boldsymbol{v}}{c^2}$	$\boldsymbol{p}_{(-)}=-\frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{E_{(-)}\boldsymbol{v}}{c^2}$
Mass-energy relationship	$E_{(+)}=mc^2$	$E_{(-)}=-mc^2$
Lagranian of free particle	$L_{(+)}=-mc^2\sqrt{1-\frac{v^2}{c^2}}$	$L_{(-)}=mc^2\sqrt{1-\frac{v^2}{c^2}}$
Differential form of work-energy theorem	$\boldsymbol{f}\cdot\mathrm{d}\boldsymbol{r}=\mathrm{d}K_{(+)}$	$\boldsymbol{f}\cdot\mathrm{d}\boldsymbol{r}=\mathrm{d}K_{(-)}$
Integral form of work-energy theorem	$W=\int_{\mathrm{A}}^{\mathrm{B}}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}=mc^2\sqrt{1-\frac{v_{\mathrm{B}}^2}{c^2}}$ $-mc^2\sqrt{1-\frac{v_{\mathrm{A}}^2}{c^2}}$	$W=\int_{\mathrm{A}}^{\mathrm{B}}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$ $=-mc^2\sqrt{1-\frac{v_{\mathrm{B}}^2}{c^2}}+mc^2\sqrt{1-\frac{v_{\mathrm{A}}^2}{c^2}}$
Hamilton-Jacobi equation	$\frac{\partial S_{(+)}}{\partial t}=-c\sqrt{m^2c^2+(\nabla S_{(+)})^2}$	$\frac{\partial S_{(-)}}{\partial t}=c\sqrt{m^2c^2+(\nabla S_{(-)})^2}$

8.4.2. NKE Particle in Electromagnetic Potential

As introduced in Section IV.D, a particle with charge q in an electromagnetic field has an electromagnetic potential energy $-q\boldsymbol{v}\cdot\boldsymbol{A}+q\varphi$, which is from $JA=-\boldsymbol{J}\cdot\boldsymbol{A}+\rho\varphi$. Now, there is only one particle, so $\boldsymbol{J}=q\boldsymbol{v}$ and $\rho=q$. The electric field strength $\boldsymbol{E}$ and the magnetic induction strength $\boldsymbol{B}$ can be expressed in terms of the four-dimensional electromagnetic potential $(\boldsymbol{A},\mathrm{i}\varphi/c)$, see Eq. (309).

The Lagrangian of a relativistic NKE particle with charge q is the Lagrangian of its free particle minus the potential energy:

$$L_{(-)}=mc^2\sqrt{1-v^2/c^2}+q\boldsymbol{v}\cdot\boldsymbol{A}-q\varphi. \tag{349}$$

In the presence of an electromagnetic field, the derivative of the Lagrangian with respect to velocity is the generalized momentum:

$$\boldsymbol{P}_{(-)}=\frac{\partial L_{(-)}}{\partial\boldsymbol{v}}=-\frac{m\boldsymbol{v}}{\sqrt{1-v^2/c^2}}+q\boldsymbol{A}=\boldsymbol{p}_{(-)}+q\boldsymbol{A}, \tag{350}$$

where the momentum $\boldsymbol{p}_{(-)}$ is given by (325). The velocity in the present case is expressed in terms of the generalized momentum:

$$\boldsymbol{v} = \frac{c^2(\boldsymbol{P}_{(-)} - q\boldsymbol{A})}{\sqrt{m^2c^4 + c^2(\boldsymbol{P}_{(-)} - q\boldsymbol{A})^2}}. \tag{351}$$

The Hamiltonian is:

$$H_{(-)} = \boldsymbol{v}\cdot\boldsymbol{P}_{(-)} - L_{(-)} = -\sqrt{m^2c^4 + c^2(\boldsymbol{P}_{(-)} - q\boldsymbol{A})^2} + q\varphi. \tag{352}$$

In this case, the gradient of the principal function is the generalized momentum, $\boldsymbol{P}_{(-)} = \boldsymbol{\nabla} S_{(-)}$. The Hamilton-Jacobi equation is:

$$\frac{\partial S_{(-)}}{\partial t} - \sqrt{m^2c^4 + c^2(\nabla S_{(-)} - q\boldsymbol{A})^2} + q\varphi = 0. \tag{353}$$

Let $S_{(-)} = S'_{(-)} + mc^2 t$,

$$-\frac{\partial S'_{(-)}}{\partial t} - mc^2 = -mc^2\sqrt{1 + \frac{1}{m^2c^2}(\nabla S'_{(-)} - q\boldsymbol{A})^2} + q\varphi. \tag{354}$$

Let $c \to \infty$,

$$\frac{\partial S'_{(-)}}{\partial t} = \frac{1}{2m}(\nabla S'_{(-)} - q\boldsymbol{A})^2 + q\varphi. \tag{355}$$

This is the result of the new Newtonian mechanics.

Now, let us find the equation of motion. Substitute the Lagrangian into (269) and calculate the two terms as follows:

$$\frac{\partial L}{\partial \boldsymbol{r}} = -q\nabla\varphi + q(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} + q\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) \tag{356}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{v}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_{(-)} + q\boldsymbol{A}\right) = \dot{\boldsymbol{p}}_{(-)} + \boldsymbol{q}\frac{\partial \boldsymbol{A}}{\partial t}. \tag{357}$$

The equation of motion is:

$$\dot{\boldsymbol{p}}_{(-)} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B}. \tag{358}$$

The right-hand side of the equation of motion (358) is still the Lorentz force (106), and the magnetic force does no work.

Constant uniform electric field

In this case, the equation of motion is:

$$\dot{\boldsymbol{p}}_{(-)} = q\boldsymbol{E}. \tag{359}$$

Assume the position vector is at the origin of coordinates at the initial moment, and the initial velocity is $\boldsymbol{v}_0$. The particle must move in the plane formed by E and $\boldsymbol{v}_0$. For simplicity, let $\boldsymbol{E} = (\boldsymbol{E}, 0{,}0)$ (in the x-direction) and the initial velocity be in the *y*-direction, $\boldsymbol{v}_0 = (0, v_0, 0)$. The component form of equation (359) is:

$$\dot{p}_{(-)x} = qE, \dot{p}_{(-)y} = 0. \tag{360}$$

Then,

$$p_{(-)x} = qEt, p_{(-)y} = p_{(-)0} = -mv_0/\sqrt{1 - v_0^2/c^2}. \tag{361}$$

The velocity and momentum components along the *y*-direction are constant, so only the velocity component along the *x*-direction changes with time. The energy of the particle is $\varepsilon_{(-)} = -\sqrt{m^2c^4 + c^2p_{(-)}^2} = -c\sqrt{m^2c^2 + p_{(-)0}^2 + (qEt)^2}$. Let $\varepsilon_{(-)0} = -\sqrt{m^2c^4 + c^2p_{(-)0}^2}$. For the *x* and *y* directions, use the relationship between velocity and momentum respectively:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x = \frac{c^2 p_{(-)x}}{\varepsilon_{(-)}} = -\frac{c^2 qEt}{\sqrt{\varepsilon_{(-)0}^2 + (qEt)^2}} \tag{362}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v_y = \frac{c^2 p_{(-)y}}{\varepsilon_{(-)}} = -\frac{c^2 p_{(-)0}}{\sqrt{\varepsilon_{(-)0}^2 + (qEt)^2}}. \tag{363}$$

The solutions are:

$$x = -\frac{1}{qE}\sqrt{\varepsilon_{(-)0}^2 + c^2(qEt)^2} \tag{364}$$

and

$$y = \frac{cp_{(-)0}}{qE}\sinh^{-1}\frac{cqEt}{\varepsilon_{(-)0}}. \tag{365}$$

Eliminating the time *t* from the solutions gives:

$$x = -\frac{|\varepsilon_{(-)0}|}{qE}\cosh\frac{qEy}{cp_{(-)0}}. \tag{366}$$

This is a catenary opening towards the negative *x*-direction. When the velocity of the particle is much smaller than the speed of light, $v \ll c$, we can take $p(-)_0 = -mv_0$ and $\varepsilon_{(-)0} = -mc^2$, and expand the right-hand side of (366) to get:

$$x = -\frac{\varepsilon_0}{qE}\left(1 + \frac{1}{2}\left(\frac{qEy}{cp_0}\right)^2\right) = -\frac{\varepsilon_0}{qE} - \frac{qE}{2mv_0^2}y^2, \tag{367}$$

which degenerates into a parabola.

Constant uniform magnetic field

The equation of motion is:

$$\dot{\boldsymbol{p}}_{(-)} = q\boldsymbol{v} \times \boldsymbol{B}. \tag{368}$$

For simplicity, let the magnetic field be along the *z*-axis, $\boldsymbol{B} = (0,0,B)$. Assume the position vector at the initial moment is $\boldsymbol{r}_0 = (x_0, y_0, z_0)$) and the initial velocity is $\boldsymbol{v}_0 = (v_{0x}, 0, v_{0z})$, i.e., the initial velocity is in the *xz* plane.

As known, the magnetic force is always perpendicular to the velocity of the particle, so it does no work. This means that the energy of the particle remains unchanged during motion. We write the momentum as:

$$\boldsymbol{p}_{(-)} = -\frac{m}{\sqrt{1 - v^2/c^2}}\boldsymbol{v} = \frac{\varepsilon_{(-)}}{c^2}\boldsymbol{v}. \tag{369}$$

Then, Eq. (368) becomes:

$$\varepsilon_{(-)}\dot{\boldsymbol{v}} = qc^2\boldsymbol{v} \times \boldsymbol{B}. \tag{370}$$

We write the three coordinate components of this equation:

$$\ddot{x} = qc^2\dot{y}B/\varepsilon_{(-)}, \ddot{y} = -qc^2\dot{x}B/\varepsilon_{(-)}, \ddot{z} = 0. \tag{371}$$

Comparing (371) with (113), the former becomes the latter by replacing $\varepsilon_{(-)}/c^2$ with *m*. Therefore, let $\omega = c^2qB/|\varepsilon_{(-)}|$:

$$\ddot{x} = -\omega\dot{y}, \ddot{y} = \omega\dot{x}, \ddot{z} = 0. \tag{372}$$

The solutions to the equation are:

$$x = x_0 - \left(\frac{v_{0x}}{\omega}\right)\sin\omega t\,, y = y_0 + \frac{v_{0x}}{\omega} - \left(\frac{v_{0x}}{\omega}\right)\cos\omega t\,, z = v_{0z}t + z_0. \tag{373}$$

The trajectory of this charged particle is: in the *xy* plane perpendicular to the magnetic field, it is a counterclockwise uniform circular motion with the center at $(x_0, y_0 + v_{0x}/\omega)$, radius v_{0x}/ω, and linear velocity v_{0x}; along the direction of the magnetic field, it is uniform linear motion. Therefore, the trajectory of the particle is a counterclockwise helix, and the axis of the cylinder where the helix lies is a straight line passing through the point $(x_0, y_0 + v_{0x}/\omega)$ and parallel to the magnetic field direction.

If the initial velocity has no component along the magnetic field direction, $\boldsymbol{v}_0 = (v_0, 0{,}0)$, then the particle performs counterclockwise uniform circular motion in the plane perpendicular to the magnetic field.

Since the magnetic force does no work, Eq. (372) and its solutions hold for any motion velocity. For low-velocity motion, it is only necessary to replace $\varepsilon_{(-)}/c^2$ with $-m$.

In the non-relativistic limit of $v \ll c$, the formulas in this section can be approximated to the corresponding formulas of the new Newtonian mechanics in subsection IV.D.

For a PKE particle, the Hamiltonian is:

$$H_{(+)} = \sqrt{m^2c^4 + c^2(\boldsymbol{P}_{(-)} - q\boldsymbol{A})^2} + q\varphi. \tag{374}$$

The corresponding Hamilton-Jacobi equation is:

$$\frac{\partial S_{(+)}}{\partial t} + \sqrt{m^2c^4 + c^2(\nabla S_{(-)} - q\boldsymbol{A})^2} + q\varphi = 0. \tag{375}$$

Equations (374) and (375) are symmetric to (352) and (353).

In Ref. [114], Eq. (374) was written in the squared form:

$$(H - q\varphi)^2 = m^2c^4 + c^2(\boldsymbol{P} - q\boldsymbol{A})^2. \tag{376}$$

Correspondingly, Eq. (375) was also written in the squared form:

$$\left(\frac{\partial S_{(+)}}{\partial t} - q\varphi\right)^2 = m^2c^4 + c^2(\nabla S_{(+)} - q\boldsymbol{A})^2. \tag{377}$$

Let $S_{(+)} = S'_{(+)} + mc^2t$, then let $c \to \infty$ to obtain the result of Newtonian mechanics:

$$\frac{\partial S'_{(-)}}{\partial t} = \frac{1}{2m}(\nabla S'_{(-)} - q\boldsymbol{A})^2 + q\varphi \tag{378}$$

We point out that, like (347), (378) is of the form $(\frac{\partial S}{\partial t})^2 = H^2$, which is not the correct form of the Hamilton-Jacobi equation.

9. Virial Theorem

9.1. Definition of Virial and Virial Theorem

Clausius first proposed the physical quantity “virial”. When a particle is in motion, the dot product of its coordinate and momentum is defined as the virial of the particle:

$$G = \boldsymbol{r} \cdot \boldsymbol{p}. \tag{379a}$$

If a system contains more than one particle, we sum the virials of all particles to define the total virial of the system:

$$G = \sum_i \boldsymbol{r}_i \cdot \boldsymbol{p}_i. \tag{379b}$$

Virial theorem in classical mechanics states that when a system undergoes stable motion, the time average of the derivative of the virial with respect to time is zero, expressed as:

$$\overline{\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)} = 0. \tag{380}$$

The bar is used to denote the time average. For periodic motion, it is only necessary to take the time average over one motion period.

The derivative of the virial with respect to time is:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i\left(\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} \cdot \boldsymbol{p}_i + \boldsymbol{r}_i \cdot \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t}\right) = \sum_i(\boldsymbol{v}_i \cdot \boldsymbol{p}_i + \boldsymbol{r}_i \cdot \boldsymbol{F}_i), \tag{381}$$

where Newton’s second law is used.

If the motion of a system satisfies the virial theorem, it must be in a bound orbit.

The concepts of stable motion and bound motion of a particle have been defined in subsection IV.B. For a particle system, if the motion trajectory of each particle in the system is clearly known, the particle system is said to undergo stable motion. If the motion of the entire particle system is carried out within a limited range in space, the particle system is said to undergo bound motion. If any part of the particle system can move to infinity, it is not a bound motion.

Motion trajectories that do not satisfy the virial theorem exist but will not be bound motion.

The previously discussed virial theorem is for PKE particles.

Now, particles can be in either positive or NKE states. The virial theorem also shows symmetry with respect to PKE and NKE, which holds for both non-relativistic and relativistic motions.

In the following, we substitute the relationship between velocity and momentum (25) or (325) into (381). The velocity-momentum relationship is different for low-momentum motion and relativistic motion, so they need to be considered separately. For each specific case, only the expressions for the single-body and two-body systems are given. The case of multiple bodies can be deduced similarly.

9.2. Low-Momentum Motion

The relationship between the velocity and momentum of a particle is listed in Table 2:

$$\boldsymbol{p}_{(\pm)} = \pm\boldsymbol{v}/m, \tag{382}$$

where the positive and negative signs represent the motion of a PKE and NKE particle, respectively. Thus:

$$\boldsymbol{v}\cdot\boldsymbol{p}_{(\pm)} = \pm(\boldsymbol{v}/m)\cdot\boldsymbol{v} = \pm 2\boldsymbol{p}_{(\pm)}^2/2m = 2K_{(\pm)}, \tag{383}$$

which is twice the kinetic energy. Therefore, the first term on the right-hand side of equation (381) is twice the total kinetic energy of the system. For a stable motion, the force acting on an object is written as the negative gradient of the potential energy:

$$\boldsymbol{F} = -\nabla V. \tag{384}$$

From Eqs. (380)-(384), we get:

$$\overline{\sum_i(2KT_i - \boldsymbol{r}_i\cdot\nabla_i V)} = 0. \tag{385}$$

We are now going to consider the simplest cases of one object and two objects.

If there is only one object moving in the potential field V, Eq. (384) simplifies to:

$$\pm 2\overline{\boldsymbol{p}_{(\pm)}^2/2m} = \overline{\boldsymbol{r}\cdot\nabla V}. \tag{386}$$

This equation includes both PKE and NKE cases and is symmetric with respect to PKE and NKE.

Assume the potential energy is inversely proportional to the first power of the distance, i.e., the potential energy has the form:

$$V(\boldsymbol{r}) = a/r, \tag{387}$$

where a is a constant. When $a < 0$, it is an attractive potential; when $a > 0$, it is a repulsive potential. Then, Eq. (386) simplifies to:

$$\pm 2\overline{\boldsymbol{p}_{(\pm)}^2/2m} = -\overline{\left(\frac{a}{r}\right)} = -\overline{V}. \tag{388}$$

This equation shows that twice the average value of the kinetic energy is the negative of the average value of the potential energy. Note that this is the conclusion when the particle undergoes stable low-momentum motion under the action of an inverse-square force.

From Eq. (388), it can be seen that if $a < 0$ (attractive potential energy), the kinetic energy of the object must be positive to perform bound motion; if $a > 0$ (repulsive potential energy), the kinetic energy of the object must be negative to perform bound motion.

Assume the potential energy is proportional to the square of the distance, called elastic potential energy:

$$V = \frac{1}{2} b r^2, \tag{389}$$

where b is a constant. If $b > 0$, it is an attractive potential energy; if < 0, it is a repulsive potential energy. Then, equation (386) is written as:

$$\pm 2\overline{\boldsymbol{p}_{(\pm)}^2/2m} = b\overline{r^2} = 2\overline{V}. \tag{390}$$

In this case, the object undergoes simple harmonic motion. The average kinetic energy and average potential energy of simple harmonic motion have the same sign and equal magnitude. If an object has PKE, it can only perform bound motion under the action of an elastic restoring force; if an object has NKE, it can only perform bound motion under the action of a linear repulsive force.

Equations (388) and (390) show that in different potential energies, the sign of the average kinetic energy of the object may be opposite to or the same as that of the average potential energy. However, they have one thing in common: a PKE object can only perform bound motion under the action of an attractive force, while a NKE object can only perform bound motion under the action of a repulsive force. This is the physical content reflected by equation (386).

Our common sense is that an object may perform bound motion under the action of an attractive force. Intuitively, an object cannot perform bound motion under the action of a repulsive force. This conclusion only applies to PKE objects. With the concept of NKE, we know that a NKE object can only perform bound motion under the action of a repulsive force.

If the potential energy is zero at infinity (i.e., the potential energy zero point is set at infinity), then the total energy of a bound system with positive (negative) kinetic energy must be negative (positive).

In the literature, the potential energy is often given in the form $V(\boldsymbol{r}) \propto r^n$:

$$V = g r^n, n \neq 0. \tag{391}$$

Then,

$$\pm 2\overline{\boldsymbol{p}_{(\pm)}^2/2m} = ng\overline{r^n} = n\overline{V}. \tag{392}$$

Obviously, the requirement for bound motion is: when $n > 0$, the kinetic energy and potential energy have the same sign; when $n < 0$, the kinetic energy and potential energy have opposite signs. The above are two special cases of $n = -1$ and $n = 2$, whose physical meanings are the clearest.

The motion of colliding particles is not stable because the motion trajectory $\boldsymbol{r} = \boldsymbol{r}(t)$ of the particles after collision cannot be predicted.

Now, consider a two-object system. Assume the interaction potential energy between the two objects is $V(\boldsymbol{r}_1 - \boldsymbol{r}_2)$, then the forces exerted by each object on the other are:

$$\boldsymbol{F}_1 = -\nabla_1 V(\boldsymbol{r}), \boldsymbol{F}_2 = -\nabla_2 V(\boldsymbol{r}), \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2. \tag{393}$$

Substituting Eqs. (383) and (387) into (381), the specific expression of the virial theorem in this case is:

$$2(\overline{\pm \boldsymbol{p}_1^2/2m_1 \pm \boldsymbol{p}_2^2/2m_2}) = 2\overline{(T_1 + T_2)} = \overline{\boldsymbol{r} \cdot \nabla_1 V(\boldsymbol{r})}. \tag{394}$$

From experience, two interacting objects can form a bound system under appropriate conditions, such as the Earth-Moon two-body system. In fact, this is an example where both objects have PKE. From Eq. (394), there are four possible cases for two objects to form a stable system. Assume the interaction between the two objects is an inverse-square one; the total kinetic energy in these four cases is the negative of half the potential energy. Among them, the total kinetic energy of two cases is positive, and a bound two-body system can only be formed under the action of an attractive force, with the total energy of the system reaching a minimum (negative value). The total kinetic energy of the other two cases is negative, and stable motion can only be achieved under the action of a repulsive force, with the total energy of the system reaching a maximum (positive value).

Suppose that the two bodies compose a bound state, in which they rotate with angular velocity ω around each other. The four possible cases are listed in Table 5. In Table 5, the masses of the two objects are m_1 and m_2 respectively; We define two reduced masses as $\mu_\pm = m_1m_2/|m_1 \pm m_2|$, with the centroid as the origin; *K* means kinetic energy and *U* means potential energy. The physical images of the four cases of two-body motion are given in [93].

Table 5. Some specific physical quantities of the four cases of a two-body system under the action of an inverse-square force.

Case		$2K_1$ of m_1	$2K_2$ of m_2	Distance r between the two objects	Total kinetic energy $2K$	Relationship between U and K
I	Attractive force	$m_1r_1^2\omega^2$	$m_2r_2^2\omega^2$	r_1+r_2	$\mu_+r^2\omega^2$	$2K+U=0$ (Virial theorem)
IV	Attractive force	$m_1r_1^2\omega^2$	$-m_2r_2^2\omega^2$	r_1-r_2	$\mu_-r^2\omega^2$	
		$m_1<m_2$				
III	Repulsive force	$-m_1r_1^2\omega^2$	$m_2r_2^2\omega^2$	r_2-r_1	$-\mu_-r^2\omega^2$	
		$m_1>m_2$				
II	Repulsive force	$-m_1r_1^2\omega^2$	$-m_2r_2^2\omega^2$	r_1+r_2	$-\mu_+r^2\omega^2$	

From Eq. (394), since each object may have PKE or NKE, there are four possibilities for two objects to form a bound system. By extension, there are 2^N possibilities for *N* objects to form a bound system.

9.3. Relativistic Motion

The relationship between relativistic velocity and momentum is shown in Table 4:

$$\boldsymbol{v} = \pm\frac{c^2\boldsymbol{p}}{\sqrt{m^2c^4+c^2\boldsymbol{p}^2}}. \tag{395}$$

Then, we have

$$\boldsymbol{v}\cdot\boldsymbol{p} = \pm\frac{c^2\boldsymbol{p}^2}{\sqrt{m^2c^4+c^2\boldsymbol{p}^2}}. \tag{396}$$

Note the difference between the relativistic expression (396) and the low-momentum expression (383). Equation (396) is not twice the relativistic kinetic energy. The expression for the relativistic kinetic energy of a free object is:

$$K_{(\pm)} = \pm\left(\sqrt{m^2c^4+c^2\boldsymbol{p}^2} - mc^2\right) = \pm\frac{c^2\boldsymbol{p}^2}{\sqrt{m^2c^4+c^2\boldsymbol{p}^2}+mc^2}. \tag{397}$$

If the momentum is very low $cp \ll mc^2$, we can approximate $m^2c^4 + c^2\boldsymbol{p}^2 \approx m^2c^4$, then Eq. (396) simplifies to twice the non-relativistic kinetic energy (383). In the ultra-relativistic case, we can approximate $mc^2 \ll cp$, and Eq. (396) approximates to $\boldsymbol{v}\cdot\boldsymbol{p} \to \pm cp$.

By substitution of Eqs. (396) and (384) into (381), the specific expression of the virial theorem for relativistic motion is obtained:

$$\pm\overline{\sum_i \frac{c^2\boldsymbol{p}_i^2}{\sqrt{m_i^2c^4+c^2\boldsymbol{p}_i^2}}} = \overline{\sum_i \boldsymbol{r}_i\cdot\nabla_i V}. \tag{398}$$

If only one particle is in motion, there is only one term on both the left and right sides:

$$\pm\overline{\frac{c^2\boldsymbol{p}^2}{\sqrt{m^2c^4+c^2\boldsymbol{p}^2}}}=\overline{\boldsymbol{r}\cdot\nabla V}. \tag{399}$$

It once more reflects the symmetry of the virial theorem with respect to PKE and NKE. Therefore, the physical analysis of Eq. (399) can be carried out with reference to the non-relativistic equation (386). A PKE (NKE) object can undergo stable motion when subjected to an attractive (repulsive) potential.

For a two-particle system, there are two terms on the left-hand side of Eq. (398), each with two signs (positive and negative), similar to the left-hand side of Eq. (394), and the discussion is similar.

10. Discussion and Summary

In this paper, we have established a new classical mechanics. This is mimicking the axiom system of non-Euclidean geometry in mathematics. The well-known classical mechanics is regarded as an axiomatic system with the Newton's three laws as the three postulations. The starting point is to modify Newton's second law to be a new form: force is the cause of deceleration, while the Newton's first and third laws are retained. The new classical mechanics following this new set of three laws is symmetrical about the well-known classical mechanics.

A prominent feature of this new axiomatic system is that the direction of momentum is opposite to that of velocity, and the kinetic energy of particles is always negative. Therefore, this new classical mechanics is called NKE classical mechanics, and the existing classical mechanics is correspondingly called PKE classical mechanics. The formulas of positive and NKE classical mechanics are completely one-to-one correspondence, showing symmetry, which we call symmetry with respect to PKE and NKE. This symmetry exists from Newtonian mechanics to relativistic mechanics, and the most basic formulas are shown in Tables 2 and 4.

This paper only discusses the content of mechanics and does not involve thermodynamics and classical statistical mechanics. The thermodynamics and classical statistical mechanics of PKE and NKE systems also have symmetry with respect to PKE and NKE. We briefly describe the properties of NKE systems here: since a thermodynamic system is composed of NKE molecules, the system has negative heat. Since the direction of momentum is opposite to that of velocity, it can be deduced that an ideal gas has negative pressure [93]. A NKE system also has a negative temperature [25]. The processes undergone by NKE system follow the principle of maximum energy. The textual descriptions and formula expressions of the three laws of thermodynamics are the same. The processes of the system still follow the principle of maximum entropy. The existing formulas of statistical mechanics are well applicable to NKE systems.

The author's viewpoint is that PKE classical mechanics describes the motion laws of visible macroscopic matter, while NKE classical mechanics describes the motion laws of invisible macroscopic dark matter. It is believed that it is possible that the bullet cluster 1E0657−56 and its dark companion is an example of mixed system composed of PKE and NKE matter, and we suggest the behavior of this galaxy system be simulated by our theory, with the collisions between PKE and NKE particles taken into account.

This paper merely describes the content of classical mechanics. A true understanding that why NKE matter is dark to us must resort to quantum mechanics [98], which describes the motion of microscopic particles.

The author's previous papers carried out the comprehensive study of microscopic NKE particles [25,26,91–96,98]. The present paper opens up a new field of research on macroscopic NKE matter. The microscopic and macroscopic parts compose the complete theory of NKE sector of matter, i.e., dark matter, which is parallel to our well-known theory for matter.

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