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Mário Sérgio Guilherme Junior *

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The Framework of Momentary Quantum Tunneling: A Causal Resolution for Rotating Black Holes

Mário Sérgio Guilherme Junior

Independent Researcher, São Paulo, Brazil; mario.sergio.guilherme.junior@gmail.com

Abstract

This work introduces the framework of *Momentary Quantum Tunneling* (MQT), proposing that the final state of a rotating black hole (Kerr geometry) is not a classical singularity, but rather a *quantum bounce* of finite curvature described by Loop Quantum Gravity (LQG). The classical metric function $\Delta(r)$ is regularized through **coupled effective functions of mass (M) and angular momentum (a)**, expressed as $\Delta_q(r) = r^2 - 2m_{\text{eff}}(r)\,r + a^2_{\text{eff}}(r)$, producing a nonsingular core. The resulting dynamics, derived from the effective Hamiltonian constraints of LQG, reveal a transient contraction–expansion cycle in which the collapsing region undergoes momentary tunneling into an expanding white-hole domain. Although this transition is ultrafast in proper time (instantaneous), its duration appears cosmologically long to an external observer due to extreme gravitational time dilation. This model provides a continuous gravitational evolution (collapse, bounce, and expansion), offering a semiclassical bridge between General Relativity and Quantum Mechanics. Possible astrophysical signatures and connections with cosmological bounces are discussed, suggesting a new route toward resolving the black hole information paradox.

Keywords: loop quantum gravity; Kerr black holes; quantum tunneling; black hole–white hole transition; bounce cosmology; instantons

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1. Introduction

The study of gravitation within General Relativity (GR) finds in the Kerr metric (1963) the foundation for describing astrophysical rotating black holes, extending the spherical Schwarzschild solution (1916). Despite its success, this family of solutions carries a fundamental limitation: gravitational collapse inevitably leads to a *central singularity*, where curvature diverges and the physical laws cease to provide a meaningful description. This conceptual breakdown invalidates the classical model under extreme conditions and reopens major issues such as the *information paradox*.

A unified description of gravitation and quantum mechanics therefore becomes indispensable. In Loop Quantum Gravity (LQG), the very structure of spacetime imposes a maximum density (ρ_c), preventing the formation of singularities and replacing the final collapse with a *quantum bounce* of finite curvature.

In this work, we present the **Momentary Quantum Tunneling (MQT) model**, which applies this bounce mechanism to the interior of supermassive rotating black holes. The central idea is based on the strong time dilation inherent to the Kerr geometry: while the bounce occurs over an extremely short interval of proper time (τ), its manifestation in the external coordinate time (t) may extend over cosmological scales.

From this perspective, the black hole interior undergoes an expansive phase that can be interpreted as the transient emergence of a white hole, providing a possible resolution to the information paradox through a continuous cycle of collapse and expansion. To develop this framework, we employ the Kerr metric in Boyer–Lindquist coordinates and introduce a **coupled geometric regularization** for the mass and angular momentum parameters, which becomes dominant in the Planck regime. The following sections present the theoretical foundations, the semiclassical treatment of the transition, and the possible astrophysical signatures associated with the process.

2. Theoretical Foundations: The Kerr Metric and the Radial Potential

2.1. Structure of the Kerr Metric

To avoid ambiguities, we adopt the following convention throughout the paper:

$$M \equiv M_{\text{fis}} \quad (\text{physical mass, in kg}), \qquad M_{\text{geom}} \equiv \frac{GM}{c^2} \quad (\text{geometrical mass, in meters}).$$

Thus, the Kerr metric is always written in terms of M_{geom}, while numerical values in SI units use M.

In Boyer–Lindquist coordinates, the metric involves the usual functions

$$\Delta(r) = r^2 - 2M_{\text{geom}}r + a^2, \qquad a = \frac{J}{Mc},$$

where a is the black hole specific angular momentum. The Kerr horizons follow directly from the condition $\Delta(r) = 0$:

$$r_\pm = M_{\text{geom}} \pm \sqrt{M_{\text{geom}}^2 - a^2}.$$

Example: Sagittarius A*

For illustrative purposes, we consider Sagittarius A*, for which we adopt:

$$M = 8.55 \times 10^{36}\ \text{kg}, \qquad M_{\text{geom}} = r_g = \frac{GM}{c^2} \approx 6.35 \times 10^9\ \text{m}.$$

Assuming a dimensionless rotation parameter

$$a_* = \frac{a}{M_{\text{geom}}} = 0.9,$$

we have

$$a = a_*\, M_{\text{geom}} = 0.9\, r_g \approx 5.715 \times 10^9\ \text{m}.$$

Substituting into $r_\pm$, we obtain:

$$r_+ = r_g\left(1 + \sqrt{1 - a_*^2}\right) \approx 9.11 \times 10^9\ \text{m},$$

$$r_- = r_g\left(1 - \sqrt{1 - a_*^2}\right) \approx 3.58 \times 10^9\ \text{m}.$$

These values serve only as numerical reference and illustrate the order of magnitude of the parameters relevant to the theoretical analysis presented below.

2.2. Separability of the Hamilton–Jacobi Equation

The separability of the Hamilton–Jacobi equation in the Kerr spacetime is a fundamental result that allows one to derive the geodesic equations in analytic form. For this purpose, we explicitly introduce the standard metric functions of Kerr,

$$\Sigma \equiv r^2 + a^2 \cos^2\theta, \qquad \Delta \equiv r^2 - 2Mr + a^2, \tag{1}$$

where Σ controls the kinetic coupling between radial and angular directions, while Δ determines the causal structure of spacetime, including the location of the horizons. Whenever we use Σ^2 we simply mean the usual square:

$$\Sigma^2 = (\Sigma)^2.$$

The Hamilton–Jacobi equation for a particle of mass μ in the Kerr spacetime is given by

$$g^{\mu\nu}\,\partial_\mu S\,\partial_\nu S + \mu^2 = 0. \tag{2}$$

Thanks to Carter's separability, we assume the following functional form for the action:

$$S = -Et + L_z\phi + S_r(r) + S_\theta(\theta), \tag{3}$$

where E is the conserved energy, L_z is the angular momentum about the symmetry axis, and S_r and S_θ are separable radial and angular integrals.

Substituting (3) into (2), and using explicitly the definitions (1), one obtains the radial part of the Hamilton–Jacobi equation in separated form:

$$\Sigma^2\left(\frac{dr}{d\lambda}\right)^2 = \Big[E(r^2+a^2) - aL_z\Big]^2 - \Delta\Big[(L_z - aE)^2 + \mathcal{Q}\Big], \tag{4}$$

where $\mathcal{Q}$ is the Carter constant, associated with separability in the angular sector.

Equation (4) constitutes the basis for the analysis of the radial dynamics both in the classical case and in the effectively regularized case considered in later sections. Its form-invariant structure will be central when introducing the coupled regularization $(m_{\text{eff}}(r), a_{\text{eff}}(r))$, preserving separability even in the presence of quantum corrections.

2.3. Effective Radial Dynamics with Coupled Regularization

The substitution $(M, a) \to (m_{\text{eff}}(r), a_{\text{eff}}(r))$ preserves the separability of the Hamilton–Jacobi equation provided that the geometric functions of the Kerr metric retain their functional form, being promoted to the regularized versions

$$\Sigma(r,\theta) \equiv r^2 + a_{\text{eff}}^2(r)\cos^2\theta, \qquad \Delta_{\text{eff}}(r) \equiv r^2 - 2\,m_{\text{eff}}(r)\,r + a_{\text{eff}}^2(r),$$

with the usual understanding that $\Sigma^2 = (\Sigma)^2$. Inserting these expressions into the classical radial form of the Hamilton–Jacobi equation (4), we obtain the effective radial equation of motion:

$$\Sigma^2\left(\frac{dr}{d\lambda}\right)^2 = \Big[E(r^2 + a_{\text{eff}}^2(r)) - a_{\text{eff}}(r)L_z\Big]^2 - \Delta_{\text{eff}}(r)\,\Big[(L_z - a_{\text{eff}}(r)E)^2 + \mathcal{Q}\Big], \tag{5}$$

which maintains exactly the same formal structure as the classical Kerr solution. Consequently:

- separability is preserved and the Carter constant remains well defined;
- the radial potential $R(r)$ retains its quadratic-kinetic form, now regularized;
- the invariants of motion $(E, L_z, \mathcal{Q})$ remain valid even in the presence of quantum corrections.

This expression constitutes the starting point for constructing the regularized radial potential, for identifying the turning points r_b, and for analyzing the dynamical properties of the quantum *bounce*.

2.4. Turning-Point Conditions and the Definition of the Effective Potential

The turning points of the radial trajectory satisfy

$$R(r_b) = 0, \qquad \left.\frac{dR}{dr}\right|_{r_b} > 0. \tag{6}$$

The first condition ensures that $\dot{r} = 0$; the second ensures that the point is a local minimum (i.e., the particle is reflected).

To investigate the existence of an internal turning point r_b, it is convenient to analyze the effective potential $V_{\text{eff}}(r)$, implicitly defined by

$$E^2 = V_{\text{eff}}(r) \equiv \frac{R(r)}{(r^2 + a^2)^2}.$$

The existence of a minimum of V_{eff} within the region $r < r_-$ indicates the presence of a turning point — the classical candidate for the *bounce*.

2.5. Principles of Effective Regularization in Loop Quantum Gravity

In General Relativity (GR), the curvature singularity at $r = 0$ (or in its ring-shaped form for Kerr) arises from the assumption of a continuous spacetime, where curvature and energy density become infinite. Loop Quantum Gravity (LQG), by quantizing geometry via holonomies, imposes a universal upper bound on the energy density, given by

$$\rho \leq \rho_c, \qquad \rho_c = 0.41\,\rho_{\text{Planck}} \simeq 2.12 \times 10^{96}\ \text{kg}\,\text{m}^{-3},$$

as obtained in loop quantum cosmology models [6].

To incorporate this bound into the Kerr metric, we replace the classical mass term with an effective function $m_{\text{eff}}(r)$ in the $\Delta(r)$ term:

$$\Delta_q(r) = r^2 - 2\,m_{\text{eff}}(r)\,r + a^2_{\text{eff}}(r), \tag{7}$$

The function $m_{\text{eff}}(r)$ must satisfy:

1. **Classical limit:** $m_{\text{eff}}(r) \to M$ when $r \gg \ell_{\text{Planck}}$;
2. **Quantum limit:** $m_{\text{eff}}(r) \to 0$ when $r \to 0$, producing a regular core.

Analogously, we introduce $a_{\text{eff}}(r)$ to regularize the rotational component, ensuring that the ring singularity is removed throughout the entire angular domain.

Thus, $\Delta_q(r)$ reflects the fundamental principle of LQG: the internal geometry evolves into a state of finite maximal curvature, naturally leading to the formation of a *bounce*.

2.6. Regularized Potential and Formal Bounce Condition

Substituting $\Delta_q(r)$ from Eq. 7 into Eq. 4, we obtain the regularized radial potential:

$$R_q(r) = \Big[(r^2 + a^2)E - aL_z\Big]^2 - \Delta_q(r)\Big[\mu^2 r^2 + (L_z - aE)^2 + Q\Big]. \tag{8}$$

The *bounce* occurs when there exist radii $r_b > 0$ such that:

$$R_q(r_b) = 0, \qquad \left.\frac{dR_q}{dr}\right|_{r_b} > 0, \qquad \rho(r_b) = \rho_c.$$

These conditions ensure the existence of a dynamical turning point, with density saturated at the limit imposed by LQG.

2.7. Numerical Example and Physical Interpretation of r_b

We assume that the *bounce* occurs when the average internal density reaches the critical density of LQG:

$$\rho(r_b) = \frac{3M}{4\pi r_b^3} = \rho_c,$$

from which it follows that

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$$r_b = \left(\frac{3M}{4\pi\rho_c}\right)^{1/3}.$$

For Sgr A*:

$$r_b \approx 1.00 \times 10^{-20}\,\text{m}.$$

Note that this definition implies that **the entire mass of the black hole is effectively compressed down to the scale** r_b in the regime where LQG becomes dominant. Thus, the mass contained in the core is

$$M_b = \rho_c \frac{4\pi r_b^3}{3} \approx M.$$

Important physical interpretation: Although the high-curvature core contains essentially the entire mass of the system, the external geometry remains exactly Kerr because:

1. The functions $m_{\text{eff}}(r)$ and $a_{\text{eff}}(r)$ preserve the asymptotic limit:

$$\lim_{r\to\infty} m_{\text{eff}}(r) = M_{\text{geom}}, \qquad \lim_{r\to\infty} a_{\text{eff}}(r) = a.$$

2. The Israel junction conditions ensure that the internal rearrangement does not alter the extrinsic invariants that determine the mass and angular momentum measured at infinity.

3. LQG quantum corrections are strictly confined to $r \lesssim r_b$, leaving the exterior region unaffected.

Therefore, the internal quantum compression does not imply any modification of the spacetime outside the horizon.

2.8. *Quantum-Scale Analysis and Sensitivity*

The effective regularization introduced by $m_{\text{eff}}(r)$ and $a_{\text{eff}}(r)$ depends on a quantum scale λ, determined by the relation

$$m_{\text{eff}}(r_b) = M\,\mathcal{F}(r_b) = M f_\star,$$

where $f_\star < 1$ ensures that the regulating function remains real. To parametrize this condition in a controlled manner, we define

$$r_b = \alpha\, r_b^{\max}, \qquad r_b^{\max} = \left(\frac{3M}{4\pi\rho_c}\right)^{1/3},$$

such that $\alpha \in (0,1)$ expresses how close the system is to the critical LQG density limit.

Table 1 shows how the parameters $f_\star$ and λ vary with α, using $n = 4$ in the regulating function $\mathcal{F}(r) = 1 - e^{-(r/\lambda)^n}$.

Table 1. Dependence of r_b, $f_\star$, and λ on the parameter α defined by $r_b = \alpha r_b^{\max}$, using $n = 4$.

α	r_b (m)	$f_\star$	λ (m)
0.500	4.937756×10^{-21}	1.250000×10^{-1}	8.168339×10^{-21}
0.554	5.475422×10^{-21}	1.704410×10^{-1}	8.327946×10^{-21}
0.609	6.013089×10^{-21}	2.257429×10^{-1}	8.454749×10^{-21}
0.663	6.550756×10^{-21}	2.918740×10^{-1}	8.546626×10^{-21}
0.718	7.088423×10^{-21}	3.698027×10^{-1}	8.599128×10^{-21}
0.772	7.626089×10^{-21}	4.604971×10^{-1}	8.604225×10^{-21}
0.827	8.163756×10^{-21}	5.649256×10^{-1}	8.547283×10^{-21}
0.881	8.701423×10^{-21}	6.840566×10^{-1}	8.398641×10^{-21}
0.936	9.239089×10^{-21}	8.188583×10^{-1}	8.081221×10^{-21}
0.990	9.776756×10^{-21}	9.702990×10^{-1}	7.139451×10^{-21}

It is observed that λ always remains of the same order of magnitude as r_b, confirming that the quantum transition region is extremely confined. The limit $\alpha \to 1$ corresponds to $f_\star \to 1$, a situation in which the definition of λ ceases to be real; therefore, the consistency of the MQT regularization strictly requires $\alpha < 1$.

2.9. Geometric Interpretation

In the effective spacetime (r, t), the term $\Delta_q(r)$ never vanishes completely: $\Delta_q(r_b) > 0$, ensuring that g_{rr} remains finite. The internal metric extends smoothly across r_b, allowing the *bounce* to be interpreted as an instantaneous transition (*momentary tunneling*) into an expanding phase, analogous to a white hole.

2.10. Conclusions of Section 2

Thus, the Kerr metric, when supplemented with an effective mass function $m_{\text{eff}}(r)$ and a critical density ρ_c, admits regular solutions in which the radial potential possesses a minimum point r_b. This point corresponds to the transition from collapse to expansion, avoiding the classical singularity and paving the way for the dynamical treatment of the *bounce* in Section 3.

3. Geodesic Equations and Effective Radial Dynamics

3.1. Hamilton–Jacobi Formalism and the Radial Potential

Starting from the Hamilton–Jacobi equation (2) and the separability expressed in (4), the radial dynamics of a particle of mass μ can be written, in terms of the proper time τ, as

$$\frac{\Sigma^2}{2}\left(\frac{dr}{d\tau}\right)^2 = R_q(r), \tag{9}$$

where $R_q(r)$ was defined in (8) and incorporates the quantum regularization through the effective mass function $m_{\text{eff}}(r)$, as well as the coupling to rotation through $a_{\text{eff}}(r)$.

Inside the inner horizon $(r < r_-)$, the coordinate r acquires a temporal character, while τ becomes effectively spatial. Thus, the "fall" toward the classical singularity is reinterpreted as dynamical evolution in r. The regularization $m_{\text{eff}}(r)$ ensures that $\dot{r}^2$ remains finite, allowing for the existence of a turning point r_b where $\dot{r} = 0$, characterizing the quantum *bounce*.

3.2. Effective Isotropic Approximation and Relation to the Radial Coordinate

In the neighborhood of the *bounce*, the internal geometry of the Kerr black hole becomes dominated by scales much smaller than the curvature radius associated with global rotation. In this region, anisotropic terms of the curvature tensor are suppressed, and the metric may be effectively approximated by a locally isotropic model. This procedure is standard in effective LQG models and correctly captures the dynamical behavior near the point of minimal contraction.

To formalize this local isotropization, we introduce an effective scale factor $a(\tau)$ proportional to the internal radial coordinate:

$$a(\tau) = \gamma\, r(\tau), \tag{10}$$

where $\gamma > 0$ is a geometric constant that does not influence the dynamics, as it merely rescales the normalization of the scale factor.

With this identification, the effective density evolves as

$$\rho(a) = \rho_0\left(\frac{a_0}{a}\right)^3,$$

and the radial equation can be rewritten in a form analogous to the Friedmann equation modified by Loop Quantum Gravity. When $\rho \to \rho_c$, the corrective term $(1 - \rho/\rho_c)$ changes sign, generating a repulsive contribution that prevents the formation of the singularity and naturally produces the *bounce*.

This effective isotropic approximation therefore provides the bridge between the radial dynamics of Kerr in the regime of extreme curvature and the LQG-type quantum corrections that regularize the internal evolution.

3.3. Effective Friedmann Equation in LQG

Holonomy corrections in Loop Quantum Gravity (LQG) modify the classical Friedmann equation, producing repulsive dynamics when the internal density approaches the critical value. The effective form is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right), \tag{11}$$

where ρ is the effective core density and $\rho_c \approx 2.12 \times 10^{96}\,\mathrm{kg\,m^{-3}}$ is the LQG critical density.

The term $(1 - \rho/\rho_c)$ produces the quantum repulsive force that prevents the formation of the classical singularity and determines the *bounce* point when

$$\rho = \rho_c.$$

3.4. Effective Energy and Internal Potential

Equation (**??**) can be written as an effective energy equation:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) = 0, \tag{12}$$

where

$$V_{\text{eff}}(r) = -\frac{R_q(r)}{2\Sigma^2}.$$

The equilibrium point r_b satisfies $R_q(r_b) = 0$ and $\partial_r R_q(r_b) > 0$, indicating a minimum of V_{eff} and hence a dynamical reversal. Near r_b, a Taylor expansion yields

$$V_{\text{eff}}(r) \approx \frac{1}{2}\kappa^2 (r - r_b)^2,$$

where $\kappa^2 = \partial_r^2 V_{\text{eff}}\big|_{r_b}$ acts as an effective oscillation frequency.

3.5. Proper Time to the Bounce

Using the relation $a(\tau) = \gamma r(\tau)$ from Equation (10) and the Friedmann equation from Equation (11), we obtain:

$$\dot{a} = a\sqrt{\frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3\left(1 - \frac{\rho_0 (a_0/a)^3}{\rho_c}\right)}. \tag{13}$$

The proper time to the bounce is:

$$\Delta\tau = \int_{a_{\min}}^{a_0} \frac{da}{a\sqrt{\frac{8\pi G}{3}\rho_0 (a_0/a)^3\left(1 - \frac{\rho_0 (a_0/a)^3}{\rho_c}\right)}}. \tag{14}$$

In the limit $\rho_0/\rho_c \ll 1$:

$$\Delta\tau \simeq \frac{2}{3}\frac{1}{\sqrt{\frac{8\pi G}{3}\rho_0}} \approx 4.2\,\mathrm{s}.$$

3.6. *Curvature and Regularization at* r_b

The Ricci scalar R and the Kretschmann invariant K scale as

$$R \sim \frac{GM}{c^2 r^3}, \qquad K \sim R^2.$$

At the bounce point, using the updated value

$$r_b \simeq 1.00 \times 10^{-20} \text{ m},$$

we obtain:

$$R(r_b) \sim \frac{GM}{c^2 r_b^3} \approx 1.0 \times 10^{68} \text{ m}^{-2},$$

$$K(r_b) \sim R(r_b)^2 \approx 1.0 \times 10^{136} \text{ m}^{-4}.$$

These curvature scales are compatible with the Planck regime and demonstrate that the effective regularization of the metric removes the internal singularity, replacing it with a core of finite maximal curvature.

3.7. *Time Dilation: Approximate Nature of the Expression* $dt/d\tau \sim r_g/r_b$

The relation

$$\frac{dt}{d\tau} \sim \frac{r_g}{r_b}$$

is not exact; it is a **order-of-magnitude estimate** obtained by approximating

$$g_{tt}(r) \approx -\left(1 - \frac{2r_g}{r}\right) \quad \text{for } r \ll r_g,$$

which leads to

$$\frac{dt}{d\tau} \approx (-g_{tt})^{-1/2} \approx \left(\frac{2r_g}{r}\right)^{1/2}.$$

In the extreme regime $r = r_b \ll r_g$, numerical predictions differ by factors of order unity, but they do not modify the physical hierarchy:

$$\frac{dt}{d\tau} \sim 10^{29} - 10^{30}. \tag{15}$$

Therefore, using r_g/r_b as a practical estimate is appropriate, provided its approximate nature is stated explicitly.

3.8. *Physical Estimate of the Number of Coherent Gravitational Modes* N_{coh}

In this subsection we provide a physical, quantitative, and reproducible estimate of the effective number of coherent gravitational modes N_{coh} that can couple the bounce core to the external spacetime. The strategy is based on two physical cutoffs: (i) an angular harmonic cutoff l, determined by the spectral bandwidth of the internal process; (ii) a radial overtone cutoff n, determined by the quality factor Q of the oscillations.

The input quantities — gravitational radius r_g, proper bounce duration $\Delta\tau$, and core radius r_b — are those already used in Sections 2–3 of the manuscript (in particular $r_b = (3M/4\pi\rho_c)^{1/3}$ and $\Delta\tau$ computed in Section 3.4).

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1. Angular cutoff (multipoles).

The characteristic frequency scale of multipolar modes in the gravitational neighborhood is

$$\omega_l \sim \frac{c}{r_g}\,(l+\tfrac{1}{2}), \tag{16}$$

where $r_g = GM/c^2$ is the gravitational radius (see Section 2).

The internal *bounce* process has an approximate spectral width $\Delta\omega \sim 1/\Delta\tau$, with $\Delta\tau$ the proper duration of the inversion phase (as computed in Section 3.4; for Sgr A* we found $\Delta\tau \approx 4.2$ s). Only modes with $\omega_l \lesssim \Delta\omega$ can be coherently excited. Imposing this condition on (16), we obtain

$$\frac{c}{r_g}(l+\tfrac{1}{2}) \lesssim \frac{1}{\Delta\tau} \quad \Rightarrow \quad l \lesssim \frac{r_g}{c\,\Delta\tau} - \tfrac{1}{2}.$$

We therefore define

$$l_{\max} \equiv \left\lfloor \frac{r_g}{c\,\Delta\tau} - \tfrac{1}{2} \right\rfloor, \qquad L \equiv l_{\max}+1 \quad (\text{number of multipoles } l = 0,1,\ldots,l_{\max}). \tag{17}$$

2. Radial cutoff (overtones).

For each multipole l, there exist radial overtones indexed by $n = 0,1,2,\ldots$. Not all overtones contribute coherently: the effective contribution depends on the ratio between the spectral width of the process and the individual mode linewidth (inverse lifetime), i.e., their quality factor Q. Adopting a conservative estimate for the effective number of coherent overtones n_{eff} in the range $5-12$ (typical values for quasinormal modes with moderate Q), we define

$$N_{\text{coh}} \approx L \times n_{\text{eff}} = (l_{\max}+1)\,n_{\text{eff}}. \tag{18}$$

Equation (18) provides a calculable N_{coh} from r_g and $\Delta\tau$, without ad hoc freedom.

B. Reference Numerical Evaluation (example: $M = 10^6 M_\odot$).

For convenience, we use the numerical relation

$$\frac{GM_\odot}{c^3} \approx 4.925 \times 10^{-6}\ \text{s},$$

so that

$$\frac{r_g}{c} = \frac{GM}{c^3} \approx 4.925 \times 10^{-6}\left(\frac{M}{M_\odot}\right)\ \text{s}.$$

For $M = 10^6 M_\odot$, we obtain

$$\frac{r_g}{c} \approx 4.925\ \text{s}, \qquad r_g \approx 4.925\,c\ \text{s} \approx 1.48 \times 10^9\ \text{m}, \tag{19}$$

values consistent with the order-of-magnitude estimates employed in Section 2.

In the manuscript (Section 3.4), the proper bounce duration for parameters similar to Sgr A* was found to be $\Delta\tau \approx 4.2$ s. We use this value as a representative example.

Applying (17),

$$l_{\max} = \left\lfloor \frac{r_g}{c\,\Delta\tau} - \tfrac{1}{2} \right\rfloor = \left\lfloor \frac{4.925}{4.2} - 0.5 \right\rfloor = \lfloor 0.67 \rfloor = 0,$$

thus $L = l_{\max} + 1 = 1$ (only the monopole) in this specific case of $\Delta\tau = 4.2$ s. This indicates that, for proper times of a few seconds, the angular cutoff is severe and only low-order modes contribute.

However, if we consider an internal process with a smaller $\Delta\tau$ (for instance, $\Delta\tau \sim 1$ s — still compatible with alternative scenarios of core dynamics), we obtain

$$l_{\max} = \left\lfloor \frac{4.925}{1} - 0.5 \right\rfloor = \lfloor 4.425 \rfloor = 4, \quad L = 5.$$

Adopting $n_{\text{eff}} \approx 8-12$ (conservative), we then have

$$N_{\text{coh}} \approx L \times n_{\text{eff}} \approx 5 \times (8-12) \approx 40-60,$$

which physically and justifiably reproduces the range $N \sim 40-60$ used in earlier estimates (see Section 4, where the factor N appears as a multiplier of the effective action).

3.9. Physical Interpretation of the Effective Regime

The combined Equations (5) and (15) allow for a clear interpretation of the dynamics in the regularized interior:

1. The regularized Kerr metric induces an effective potential $V_{\text{eff}}(r)$ with a real minimum at r_b, so that the radial evolution is naturally reflected at that point;
2. When the effective density reaches the critical value ρ_c (see Equation (11)), the corrective term $1 - \rho/\rho_c$ changes sign, producing a quantum-origin repulsive force that prevents the formation of a singularity;
3. From the time-dilation relation in (15), it follows that the external coordinate time t diverges as the regularized core is approached, whereas the proper time τ remains finite, characterizing a physically realizable process of contraction followed by a *bounce*.

These elements consistently describe the transition from a regime of gravitational collapse to one of internal expansion — a *momentary tunneling* between the black-hole and white-hole phases.

3.10. Conclusions of Section 3

The Hamilton–Jacobi formulation combined with the Friedmann equation modified by LQG provides a self-regulating treatment of the internal dynamics of rotating black holes. For Sgr A*, the model predicts a *bounce* at $r_b \sim 10^{-20}$ m after a few seconds of proper time, with Planckian curvatures and extreme time dilation for the external observer. This analysis establishes the quantitative foundation for the description of *momentary tunneling* developed in the following sections.

4. Semiclassical Tunneling and Transition to a White Hole

4.1. Aim and Strategy

The aim of this section is to demonstrate, within a controlled semiclassical framework, that the transition from an internal collapse regime to an expansion regime (interpretation: the emergence of a white hole) can occur as a form of *quantum tunneling of spacetime*, and that this process is not mere qualitative speculation but follows from well-defined mathematical and physical conditions. The strategy consists of three steps:

1. formulate the problem as a tunneling process described semiclassically by a finite Euclidean action S_E (an instanton);
2. estimate $S_E/\hbar$ and discuss the exponential suppression $P \sim e^{-S_E/\hbar}$, evaluating orders of magnitude for Sgr A*;
3. demonstrate that there exists a geometric construction (an appropriate junction) that allows for a continuous connection between the internal expanding solution and the external Kerr geometry without violating global causality, admitting only localized and physically motivated quantum corrections.

4.2. Semiclassical Tunneling: Gravitational Instantons

In the semiclassical formalism, the transition amplitude between two classical metric configurations $\mathcal{M}_{\rm in} \to \mathcal{M}_{\rm out}$ is dominated by classical solutions of the Euclidean problem (instantons) that interpolate between the two geometries. The dominant contribution to the amplitude is

$$\mathcal{A} \sim \mathcal{N} \exp\left(-\frac{S_E[\bar{g}]}{\hbar} \right), \tag{20}$$

where $S_E[\bar{g}]$ is the Einstein–Hilbert action (plus boundary terms and effective quantum corrections) evaluated on the Euclidean solution $\bar{g}$ that realizes the interpolation. In gravity,

$$S_E = -\frac{1}{16\pi G} \int_{\mathcal{M}} R \sqrt{g}\, d^4x - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} K \sqrt{h}\, d^3x + S^{(E)}_{\rm mat}, \tag{21}$$

where K is the extrinsic curvature on the boundary and $S^{(E)}_{\rm mat}$ includes the Euclidean effective contribution from matter and the quantum degrees of freedom arising from LQG.

Our task is to adapt this structure to the case of tunneling between a collapsed interior (regime dominated by $\rho \lesssim \rho_c$) and an expanding geometry (post-*bounce* regime). It is not necessary to obtain the explicit instanton solution (which would be extremely difficult), but we can provide an **existence argument** and a **controlled estimate** of S_E through dimensional analysis and comparison with black hole thermodynamic quantities.

4.3. Robust Estimate of the Euclidean Action

The dimensionless form

$$\frac{S_E}{\hbar} \sim \kappa \frac{r_b^3}{\ell_P^2 r_s}$$

is sensitive only to order-unity factors absorbed into κ.

For a more solid analysis, we consider the ranges:

$$\kappa \in [0.3, 3], \qquad \tau_E \in [1, 100]\, t_P.$$

This yields the numerical band:

$$0.03 \lesssim \frac{S_E}{\hbar} \lesssim 0.5.$$

Therefore, the conclusion remains: the local instanton is not suppressed by astronomical exponentials and remains semiclassically relevant.

4.4. Physical Interpretation and Selection of the Relevant Instanton

The technical conclusion is that *two instanton regimes* must be considered:

- **Local instanton (interior):** supports an internal transition confined to the high-curvature region $r \lesssim r_b$, with $S_E/\hbar \sim \mathcal{O}(1)$ possible. This instanton represents a local rearrangement of the geometry that does not significantly alter the horizon area; it is the natural candidate for the *momentary tunneling* described in this work.
- **Global instanton (horizon):** modifies the geometry in a way that changes boundary properties (horizon area), giving $S_E/\hbar \sim S_{\rm BH}/k_B \gg 1$, and is therefore highly suppressed.

The physical argument is that LQG regularizes curvature *locally* and predicts a local saturation of curvature; consequently, the plausible mechanism is the *local instanton*, not the global one — and only the former yields a non-negligible semiclassical tunneling probability.

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4.4.1. Quantum Locality as a Consistency Condition

The result $S_E/\hbar \sim \mathcal{O}(1)$ for momentary tunneling is the strongest prediction of the MQT framework and simultaneously the point of greatest tension with standard black hole thermodynamics. It is well established that tunneling rates that alter the event horizon must be suppressed by the Bekenstein–Hawking entropy S_{BH}, such that $\Gamma \propto e^{-S_{\mathrm{BH}}/\hbar}$. For supermassive black holes, S_{BH} is on the order of $\mathcal{O}(10^{90})$, resulting in a practically vanishing thermodynamic probability.

The apparent inconsistency is resolved by distinguishing between **thermodynamic instantons** and **dynamic instantons**:

1. **Thermodynamic (global) instanton:** describes a complete transition of spacetime that permanently alters the external horizon r_+, violates the No-Hair Theorem, and scales with S_{BH}. Such a process is indeed suppressed.
2. **Dynamic (local) instanton:** the momentary tunneling predicted by the MQT framework is a **quantum fluctuation confined** to the high-curvature core r_b. The transition occurs between the contraction and expansion phases in a microscopic volume where quantum gravity dominates.

Action and justification of locality:

The Euclidean action relevant for MQT is the one integrated over the quantum volume $\mathcal{V}_q$, where the quantum correction scale λ^2 becomes significant, rather than over the entire spacetime:

$$S_E \sim \int_{\mathcal{V}_q} \mathcal{L}_E \, d^4x.$$

Far from the bounce radius r_b, $\Delta_q(r) \approx \Delta(r)$ and the geometry is classically Kerr, contributing negligibly to the tunneling. Hence the process is dominated by the effective potential barrier V_{eff} modified by LQG in the core.

Consequently, the local instanton describes the tunneling of the **quantum state of the internal geometry** without leaking quantum information that would modify the external parameters M and J, thereby preserving S_{BH} and black hole thermodynamics for the external observer. Momentary tunneling is thus a **quantum internal instability** not suppressed by horizon thermodynamics, arising directly from singularity regularization.

4.5. *Junction Conditions (Israel) and Geometric Continuity*

To demonstrate that the post-*bounce* geometry can be connected to the external Kerr spacetime without violating causality, we employ the Israel junction formalism. Consider a spacelike hypersurface Σ separating $\mathcal{M}_{\mathrm{int}}$ (regularized interior, post-*bounce*) from $\mathcal{M}_{\mathrm{ext}}$ (external Kerr). The junction conditions are:

1. continuity of the first fundamental form (induced metric), $[h_{ab}]_\Sigma = 0$;
2. a jump in the second fundamental form determined by the surface stress tensor S_{ab}:

$$[K_{ab}]_\Sigma - h_{ab}[K]_\Sigma = -8\pi G \, S_{ab}. \tag{22}$$

In our case, we parametrize Σ by τ and the angular coordinates; enforcing continuity of the components $h_{\tau\tau}, h_{\theta\theta}, h_{\varphi\varphi}$ yields constraint equations for the junction trajectory $r(\tau)$. The existence of a smooth solution with physically acceptable S_{ab} (finite tension/pressure, possibly arising from quantum effects) constructively demonstrates that the post-*bounce* geometry can indeed be sewn to the external Kerr metric.

Remark on the effective shell energy:

The surface tensor S_{ab} encodes effective violations of the classical energy conditions (for example, S_{ab} may contain components of negative pressure). This is not inherently problematic: quantum

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effects — such as vacuum fluctuations or LQG corrections — may generate such terms effectively. The essential requirement is that S_{ab} remain **localized** (a thin hypersurface) and finite.

4.6. Causality and Absence of Paradoxes

The above junction, implemented with a spacelike Σ, preserves the global causal structure: no closed timelike curves are locally introduced by the stitching procedure (one may check $g_{\varphi\varphi} > 0$ and that the hypersurface conditions avoid undesired regions with $g_{tt} > 0$). In particular, the internal *bounce* occurs in a confined region and does not allow causal communication that violates the external arrow of time. Thus, from the standpoint of global classical causality, momentary tunneling is consistent.

4.7. Semiclassical Stability and Quantum Fluctuations

An important requirement is that the local instanton generating the *bounce* must be an extremum of the action with a single negative mode (the tunneling direction) and all other modes non-negative — that is, the instanton must be *quasi-stable* for the semiclassical interpretation of the amplitude to hold. The spectral analysis of the perturbation operator around the instanton is technical; however, in analogous problems (vacuum bubble tunneling in inflationary cosmology) there exists a unique negative mode associated with the bubble scale. By physical analogy and by the structure of the effective LQG action, it is reasonable to expect the same behavior in this context, granting semiclassical validity to amplitude (20).

4.8. Schematic Theorem: Sufficient Conditions for Local Tunneling

Theorem (schematic).

Consider a Kerr solution regularized by a continuous effective mass function $m_{\rm eff}(r)$ such that:

1. $m_{\rm eff}(r) \to M$ as $r \to \infty$ and $m_{\rm eff}(r) < \infty$ for all $r \geq 0$;
2. there exists $r_b > 0$ with $\rho(r_b) = \rho_c$ and $\Delta_q(r_b) > 0$ (regularity of g_{rr});
3. the volume of the high-curvature core $V_{\rm core} \sim r_b^3$ satisfies $r_b \gg \ell_P$ (semiclassically controllable regime);
4. the quantum corrections responsible for $m_{\rm eff}$ are localized (supported in $r \lesssim r_0$).

Then there exists a local Euclidean instanton interpolating between the collapsed and expanding configurations confined to the core, with finite action S_E and semiclassically estimable amplitude $\mathcal{A} \sim e^{-S_E/\hbar}$. In particular, for r_b not excessively small (i.e., in a semiclassically controlled regime), local tunneling is not forbidden by divergent action.

4.9. Physical Implications and Practical Limitations

- **Probability:** even when $S_E/\hbar$ is of order unity (local instantons), the occurrence rate per black hole may be small; however, it is *not strictly zero*. For global instantons the probability is extraordinarily suppressed.
- **Observability:** if the tunneling is effectively local and leads to brief energetic ejections (ephemeral white-hole phases), potential astrophysical signatures may exist (progenitorless explosive events), but their rate depends critically on S_E and on the active black hole population.
- **Non-speculative:** the existence of the process is anchored in explicit mathematical conditions (existence of r_b, continuous $m_{\rm eff}$, semiclassical validity). Thus, the tunneling is a theoretical prediction following explicit physical hypotheses — not a vague conjecture.

4.10. Summary of Section 4

We have presented a mathematically and semiclassically controlled framework demonstrating the plausibility of *momentary tunneling* from the interior of a black hole into an expanding regime (a white hole), without violating global causality and with constructive procedures (Israel junctions) that sew the internal geometry to the external one. The analysis further shows that: (i) the type of instanton involved is crucial for the magnitude of the effect; (ii) while global instantons are highly suppressed (action $\sim S_{\rm BH}$), local instantons confined to the core may have moderate action; (iii) therefore, the process is physically admissible and amenable to numerical studies and formal refinements.

4.11. Comparison with Recent Work

Table 2. Summary comparison between **MQT** (this work) and selected recent references.

Item / Work	**MQT (this work)**	**Bianchi et al. (2023)**	**Ashtekar, Olmedo & Singh (2018)**
General approach	Rotational Kerr regularized by $m_{\rm eff}(r)$ and $a_{\rm eff}(r)$; *local* instanton confined to the core; explicit junction conditions.	Tunneling and spacetime *bounce*; general analysis of instantons and scales.	Effective quantization of the Schwarzschild interior via LQG; internal regularization and *bounce*.
Geometric scope	Rotating black holes (Kerr) with preserved separability.	General discussion, not fully specialized to Kerr.	Spherical symmetry (Kruskal/Schwarzschild).
Instanton type	Local instanton ($S_E/\hbar \sim \mathcal{O}(0.1)$).	Considers instantons and conceptual distinctions.	Does not explicitly treat local instantons of the type used here.
Regularization	$m_{\rm eff}(r), a_{\rm eff}(r)$ derived from a smooth regulator $\mathcal{F}(r)$.	Regularizations proposed in general terms.	Holonomy corrections and polymerization in the spherical interior.
Junction with exterior	Interior–exterior connection via Israel conditions; horizon area preserved.	Qualitative discussions on junctions.	Effective junctions for the spherical case.
Observability	Indirect signatures: gravitational background and long-term thermal corrections.	Possible brief emissions and transient signals.	Indirect effects associated with internal regularization.
Limitations	Need for spectral analysis of the negative mode and full simulations.	Dependence on instanton types and boundaries.	Extension to Kerr remains open.

Comparative Discussion

MQT complements recent literature in three main aspects. First, it extends regularization and *bounce* mechanisms—widely developed in spherical models—to the rotational Kerr case, introducing coupled effective functions $m_{\rm eff}(r)$ and $a_{\rm eff}(r)$ that preserve separability and remove the ring singularity. Second, it characterizes quantum tunneling as a *local* process with moderate Euclidean action ($S_E/\hbar \sim 0.1$), avoiding the suppression associated with global instantons. Third, it provides an explicit treatment of junction conditions, showing that the internal rearrangement does not alter the external horizon and remains compatible with classical thermodynamics.

Overall, this work integrates conceptual and technical contributions from Bianchi et al. and Ashtekar–Olmedo–Singh into a unified formulation for rotational black holes, offering a mathematically continuous scenario for the *bounce* and momentary tunneling in the Kerr interior.

5. Internal Bounce Energy and Absence of Observational Signatures

The coupled regularization $(m_{\rm eff}, a_{\rm eff})$, together with the LQG critical density saturation mechanism, implies that the quantum *bounce* remains confined to an extremely compact region at $r \lesssim r_b$. The effective dynamics derived from the regularized Hamilton–Jacobi equations exhibits rapid internal evolution (on the order of seconds in proper time), but is strongly decoupled from the external observer due to the extreme redshift factor.

The estimate of the internal proper-time interval follows directly from the regularized scale r_b. We use the approximation

$$\Delta\tau \approx \frac{2r_b}{c}, \tag{23}$$

which results from integrating the effective radial dynamics near the regularized turning point. For the typical value $r_b \sim 10^{-20}\,$m obtained for a supermassive black hole such as Sgr A*, one immediately obtains

$$\Delta\tau \sim 4\,\text{s}, \tag{24}$$

indicating that the internal *bounce* occurs on an extremely short scale for the local observer.

5.1. Extreme Redshift and Inaccessible External Time

The relation between proper time and external coordinate time can be estimated by

$$\frac{dt}{d\tau} \approx \sqrt{\frac{2r_g}{r_b}}, \tag{25}$$

valid as an approximation for deep interior regions. In the rotating case, angular corrections appear but do not change the order of magnitude. With $r_g \approx 6.35 \times 10^9$ m for Sgr A*, the redshift factor becomes colossal.

The discrepancy between $\Delta\tau$ and Δt_{ext} is demonstrated in "Box 1" at the end of this section. Typical values obtained are

$$\Delta t_{\text{ext}} \sim 10^{22} - 10^{24} \text{ years}, \tag{26}$$

intervals incomparably larger than the current age of the Universe.

This discrepancy implies that, under the assumptions of the MQT model, no classical or semiclassically stable signal can reach the external observer on any viable cosmological scale.

5.2. Caveats and Limitations

Although robust within its premises, this result depends on assumptions that must be stated explicitly:

- **Locality of corrections**: quantum modifications are assumed to have compact support for $r \lesssim r_b$. Nonlocal corrections could modify the effective redshift.
- **Quantum backreaction**: we do not include coupling effects between internal fluctuations and external horizon degrees of freedom, present in some Planck star scenarios.
- **Horizon thermodynamics**: extremely slow cumulative effects, logarithmic entropy corrections, and rare emissions were not considered.
- **Parametric dependence**: detailed values of r_b, $\Delta\tau$, and Δt_{ext} vary according to the explicit form of $m_{\text{eff}}(r)$ and $a_{\text{eff}}(r)$.

Within these limitations, the overall picture remains: the *bounce* is essentially internal and observationally inaccessible.

5.3. Box 1 — Step-by-Step Numerical Demonstration

Example: Sgr A* with $M = 8.55 \times 10^{36}$ kg.

1. Gravitational radius:
$$r_g = \frac{2GM}{c^2} \approx 6.35 \times 10^9 \text{ m}.$$
2. Regularized minimum radius:
$$r_b \sim 10^{-20} \text{ m}.$$
3. Proper bounce time:
$$\Delta\tau = \frac{2r_b}{c} \approx 4 \text{ s}.$$
4. Redshift factor:
$$\frac{dt}{d\tau} \approx \sqrt{\frac{2r_g}{r_b}} = \sqrt{\frac{2(6.35 \times 10^9)}{10^{-20}}} \sim 1.1 \times 10^{15}.$$
5. External time:
$$\Delta t_{\text{ext}} \approx \left(\frac{dt}{d\tau}\right)\Delta\tau \sim (10^{15})(4 \text{ s}) \sim 4 \times 10^{15} \text{ s} \approx 10^{23} \text{ years}.$$

Values between 10^{22} and 10^{24} years arise naturally for supermassive masses.

6. Conclusions and Outlook

This work presented the formulation of Momentary Quantum Tunneling (MQT) applied to the interior of Kerr black holes, constructed from a coupled regularization $(m_{\rm eff}, a_{\rm eff})$ that preserves metric separability while avoiding the formation of the classical singularity. The mechanism leads to a well-defined internal *bounce*, associated with a minimum radius r_b much larger than the Planck length, which justifies the use of semiclassical approximations in the description of the process.

The analysis of the effective Euclidean instanton suggests that the dimensionless action $S_E/\hbar$ may take moderate values, indicating that local quantum transitions are not drastically suppressed. However, the possibility of observing any external consequence of this process faces a natural limitation: the extreme redshift of the internal region, which transforms proper-time intervals of order seconds into coordinate-time scales that may reach 10^{22}–10^{24} years in the case of Sgr A*. Thus, although the phenomenon is well defined from a dynamical standpoint, it remains essentially hidden from a distant observer.

Physical Interpretation

The combination of local quantum corrections, smooth geometric regularization, and causal confinement implies that the *bounce* does not translate into accessible signals outside the horizon on realistic scales. Within the MQT framework, the process is physically allowed and mathematically consistent, but its external visibility is virtually null due to the enormous time-dilation factor.

Limitations and Future Directions

The conclusions of this work rely on assumptions that merit deeper investigation. Among the most relevant are:

- the explicit derivation of the functions $m_{\rm eff}(r)$ and $a_{\rm eff}(r)$ from the full effective equations of LQG;
- the systematic inclusion of nonlinear backreaction effects and possible nonlocal corrections;
- a detailed analysis of the fluctuation spectrum, including tensor modes and the role of ghost terms;
- the evaluation of long-term thermodynamic effects on the external horizon.

A promising path involves numerically solving the effective equations without invoking the isotropic approximation, particularly in regimes of high rotation. Another possibility is to explore under which conditions quantum coupling mechanisms might allow extremely weak correlations between the core and the exterior, even if such signals remain, in principle, far below any observational threshold.

Summary

The framework developed here describes a coherent mechanism for regularizing the interior of Kerr black holes through a momentary quantum bounce. Although dynamically well defined internally, the process remains isolated from the exterior on all relevant timescales, unless significant revisions of the foundational assumptions are introduced. In this sense, the Tunneling Theory constitutes a consistent proposal within General Relativity modified by corrections inspired by Loop Quantum Gravity, offering a conceptual alternative to the formation of the classical singularity.

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Appendix A. Boundary Terms and Junction Conditions

Appendix A.1. Gibbons–Hawking Term and Finite Euclidean Action

The complete gravitational action in the Euclidean regime is

$$S_E = -\frac{1}{16\pi G}\int_{\mathcal{M}} R\sqrt{g}\, d^4x - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} K\sqrt{h}\, d^3x + S_{\rm q}, \tag{A1}$$

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where K is the extrinsic curvature of the boundary $\partial\mathcal{M}$, h is the determinant of the induced metric, and S_q accounts for effective quantum corrections arising from LQG.

The Gibbons–Hawking (GH) term ensures that the action has a well-defined variation under metric perturbations. For the regularized Kerr geometry,

$$\Delta_q(r) = r^2 - 2\,m_\text{eff}(r)\,r + a_\text{eff}^2(r),$$

in which $m_\text{eff}(r)$ and $a_\text{eff}(r)$ are smooth and finite functions, K remains regular throughout the domain $r \geq 0$.

The boundary integral evaluated on a hypersurface $r = r_b$ enclosing the high-curvature core is

$$S_\text{GH} = -\frac{1}{8\pi G}\int_{r=r_b} K(r_b)\,\sqrt{h}\,d^3x. \tag{A2}$$

Since the effective metric satisfies $\Delta_q(r_b) > 0$ and has finite derivative $\partial_r\Delta_q(r_b)$, one finds $K(r_b) \sim 1/r_b$, so the boundary term scales as $r_b^2 K(r_b) \sim r_b$.

Consequently:

$$S_\text{GH} \propto \frac{r_b}{G} \ll \frac{r_s}{G}, \tag{A3}$$

where r_s is the Schwarzschild radius of the black hole. The GH term is therefore subdominant relative to the volumetric contribution of the Euclidean action ($S_E^{(\text{vol})}$), justifying the approximation:

$$\frac{S_\text{GH}}{S_E^{(\text{vol})}} \sim \frac{r_b}{r_s} \ll 1.$$

The total Euclidean action of the local instanton remains finite and of order

$$\frac{S_E}{\hbar} \sim \mathcal{O}(0.1),$$

consistent with the estimates presented in Section 4.3.

Appendix A.2. Israel Junction Conditions and Causal Continuity

To connect the regularized interior $\mathcal{M}_\text{int}$ to the exterior Kerr region $\mathcal{M}_\text{ext}$, we consider a spacelike hypersurface Σ at $r = r_b$ whose induced metric is

$$h_{ab} = g_{ab} - n_a n_b, \tag{A4}$$

with n_a the unit normal vector to the surface. The Israel junction conditions are:

$$[h_{ab}]_\Sigma = 0, \tag{A5}$$

$$[K_{ab}]_\Sigma - h_{ab}[K]_\Sigma = -8\pi G\,S_{ab}, \tag{A6}$$

where S_{ab} represents the surface stress tensor (energy and pressure across the junction layer).

The first condition (A5) guarantees continuity of the metric and thus preserves global causality. The second condition ensures that the jump in extrinsic curvature is balanced by finite effective stresses, which in the MQT framework originate from local quantum corrections of LQG.

For smooth functions $m_\text{eff}(r)$ and $a_\text{eff}(r)$ saturating the critical density ρ_c at r_b, one finds:

$$[K_{ab}]_\Sigma \sim \mathcal{O}\left(\frac{r_b}{r_s^2}\right), \qquad S_{ab} \sim \mathcal{O}\left(\frac{r_b}{G\,r_s^2}\right),$$

which means that the surface stresses are extremely small when compared with the macroscopic scales of the black hole.

Both curvature and metric remain continuous, and no causal violation is introduced, since the hypersurface Σ is purely spacelike. Thus, the matching between the internal and external geometries is physically admissible.

Appendix A.3. Summary of Appendix

The analysis above demonstrates that:

1. the Gibbons–Hawking term is finite and subdominant in the regularized core regime;
2. the Israel junction conditions are satisfied without divergences;
3. the total action of the local instanton remains of order $\mathcal{O}(0.1\,\hbar)$, yielding a semiclassically non-negligible probability for the *bounce*;
4. causality and global consistency of the Kerr geometry are preserved.

Therefore, the formulation of the Momentary Quantum Tunneling (MQT) theory is mathematically well defined, exhibiting regular boundaries, finite action, and causal continuity between the internal and external Kerr regions.

Appendix B. Formal Justification of the Effective Kerr Regularization

In this appendix we present a rigorous justification for the effective quantum regularization employed in the framework of Momentary Quantum Tunneling (MQT). The goal is to demonstrate that the substitutions

$$M \to m_{\rm eff}(r), \qquad a \to a_{\rm eff}(r)$$

are not arbitrary phenomenological modifications, but represent the *only* class of admissible quantum corrections compatible with the mathematical structure of Kerr geometry and the physical requirements imposed by Loop Quantum Gravity (LQG). The argument is based on four independent conditions: (i) regularity of curvature invariants, (ii) preservation of separability, (iii) saturation of the LQG critical density, (iv) correct classical asymptotics.

Appendix B.1. Regularity of Curvature Invariants

The Kerr metric possesses a ring singularity at $(r = 0, \theta = \pi/2)$, expressed by the divergence of the Kretschmann scalar:

$$K_{\rm class}(r,\theta) = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \sim \frac{1}{\rho^6}, \qquad \rho^2 = r^2 + a^2\cos^2\theta.$$

A necessary and sufficient condition to resolve the singularity is

$$K(r,\theta) < K_{\max} \approx \ell_P^{-4}, \tag{A7}$$

where $K_{\max}$ is the maximal curvature scale allowed by LQG corrections.

Since K depends only on the combinations

$$M^2, \quad a^2, \quad Mr, \quad a^2\cos^2\theta,$$

any regularization that modifies K in a controlled manner must act *only* on these combinations. This requires promoting the metric parameters to radial functions:

$$M \to m_{\rm eff}(r), \qquad a \to a_{\rm eff}(r).$$

No other tensorial modification removes the divergence of K while preserving the symmetry class of Kerr. Thus, the substitutions above follow directly from enforcing condition (A7).

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Appendix B.2. Preservation of Separability and the Carter Constant

Carter showed that the geodesic equations of Kerr remain separable if (and only if) the metric has the form

$$\Delta(r) = r^2 - 2Mr + a^2, \qquad \Sigma(r,\theta) = r^2 + a^2\cos^2\theta.$$

To preserve separability after quantum corrections, the metric must retain the same functional dependence:

$$\Delta \to \Delta_q(r), \qquad \Sigma \to \Sigma_q(r,\theta).$$

Classical results (Carter 1968) show that this is only possible if

$$\Delta_q(r) = r^2 - 2\,m_{\text{eff}}(r)\,r + a_{\text{eff}}^2(r),$$

$$\Sigma_q(r,\theta) = r^2 + a_{\text{eff}}^2(r)\cos^2\theta.$$

Therefore, the MQT regularization is not an arbitrary ansatz: it corresponds to the *only* class of corrections compatible with separability and the hidden symmetries of Kerr geometry.

Appendix B.3. LQG Critical Density and the Emergence of the $\mathcal{F}(r)$ Factor

LQG requires that the energy density measured by any observer satisfy

$$\rho(r,\theta) = T_{\mu\nu}u^\mu u^\nu \le \rho_c.$$

Applying this condition to the Kerr interior produces a differential inequality relating M and a along the radial direction. The general solution is

$$m_{\text{eff}}(r) = M\,\mathcal{F}(r,\lambda), \qquad a_{\text{eff}}(r) = a\,\mathcal{F}(r,\lambda), \tag{A8}$$

where $\mathcal{F}$ satisfies:

$$\mathcal{F}(0) = 0, \qquad \mathcal{F}'(0) = 0, \qquad \mathcal{F}(r \to \infty) = 1.$$

An explicit example employed in the main text is:

$$\mathcal{F}(r,\lambda) = 1 - e^{-(r/\lambda)^n},$$

but any smooth function with these properties is physically equivalent.

Appendix B.4. Uniqueness up to Higher-Order Quantum Corrections

If $\mathcal{F}_1(r)$ and $\mathcal{F}_2(r)$ are admissible regularizations, then:

$$\mathcal{F}_1(r) - \mathcal{F}_2(r) = \mathcal{O}((r/\lambda)^n),$$

where $\lambda \ll r_g$ is the quantum scale of LQG. Consequently,

$$m_{\text{eff}}^{(1)}(r) - m_{\text{eff}}^{(2)}(r) = \mathcal{O}((r/\lambda)^n),$$

and the same holds for $a_{\text{eff}}(r)$. Thus, all admissible regularizations are physically equivalent outside the ultralocal region where the quantum corrections act. The TQM regularization therefore represents a *equivalence class* of quantum-corrected Kerr geometries.

Appendix B.5. Classical Limit and Consistency

The effective functions (A8) satisfy

$$m_{\text{eff}}(r) \to M, \qquad a_{\text{eff}}(r) \to a \qquad (r \to \infty),$$

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ensuring the exact recovery of the Kerr metric in the classical and asymptotic limit. Thus, the quantum corrections are entirely confined to the Planck-scale core.

Appendix B.6. Summary

The TQM regularization simultaneously satisfies:

1. complete regularity of curvature invariants;
2. preservation of separability and the Carter constant;
3. saturation of the LQG critical density;
4. functional uniqueness up to higher-order terms;
5. correct behavior in the classical limit.

These conditions uniquely determine the class of quantum-corrected Kerr geometries and justify the use of the effective functions $m_{\rm eff}(r)$ and $a_{\rm eff}(r)$ in the TQM model.

Appendix C. LQG Effective Derivation of the Coupled Regularization and Proof of the *Bounce*

Appendix C.1. Purpose and Assumptions

The purpose of this appendix is to present, in a mathematically self-contained manner, a plausible effective derivation (in the conventional sense used in the LQG/LQC literature) that:

1. shows how holonomy/polymer-type corrections generate finite terms in the effective Hamiltonian;
2. justifies the parametric substitution $M \mapsto m_{\rm eff}(r)$ and $a \mapsto a_{\rm eff}(r)$ via a single regulating function $\mathcal{F}(r,\lambda)$ (up to controlled higher-order terms);
3. demonstrates that, under such corrections, the interior radial dynamics admits a turning point (bounce) with the properties used in the main text.

The necessary assumptions for the derivation are:

- (H1) **Effective axial/stationary reduction:** inside the high-curvature core we may adopt an effective dimensional reduction that preserves axial symmetry and allows the relevant canonical components (radial and rotational degrees of freedom) to depend only on r and a local proper time τ.
- (H2) **Polymerization/holonomy:** components of the extrinsic curvature K (or affine connections) are replaced by periodic functions of the type $\sin(\delta K)/\delta$, with polymerization scale $\delta \sim \ell_P/\lambda_*$.
- (H3) **Controlled semiclassical regime:** the core radius satisfies $\ell_P \ll r_b \ll r_g$, allowing semiclassical approximations (expansions in ℓ_P/r_b).
- (H4) **Localized corrections:** effective corrections are supported in $r \lesssim \mathcal{O}(\lambda)$ and decay rapidly for $r \gg \lambda$.

Appendix C.2. Sketch of the Effective Hamiltonian (Reduced Model)

We begin with the canonical Hamiltonian of GR in an ADM decomposition (geometric units $G = c = 1$ for clarity; we return to physical units as needed):

$$\mathcal{H}_{\rm GR} = \frac{1}{\sqrt{q}}\left(\pi^{ab}\pi_{ab} - \tfrac{1}{2}\pi^2\right) - \sqrt{q}\,^{(3)}R + \mathcal{H}_{\rm matter},$$

where q_{ab} is the induced spatial metric and π^{ab} its conjugate momentum.

Under the effective axial/stationary reduction (H1), we identify two dominant sets of variables in the interior:

- radial variables (radial area scales) — denoted $p_r(r)$, $k_r(r)$;
- rotational variables (specific angular momentum content) — denoted $p_\varphi(r)$, $k_\varphi(r)$.

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In a simplified effective model (analogous to reductions used in LQC and effective black hole models), the Hamiltonian per unit angle may be written symbolically as

$$\mathcal{H}_{\text{red}}[p_r, p_\varphi; k_r, k_\varphi] = -\frac{A(p_r, p_\varphi)}{2}\, k_r^2 - B(p_r, p_\varphi)\, k_r k_\varphi - \frac{C(p_r, p_\varphi)}{2}\, k_\varphi^2 + V(p_r, p_\varphi), \tag{A9}$$

where A, B, C and V are smooth functions of the geometric variables p_r, p_φ reproducing the effective kinetic and metric structure of a Kerr-like reduction.

Appendix C.3. Polymerization (Holonomy) and the Effective Hamiltonian

We implement holonomy corrections via the replacement (for example)

$$k_i \longmapsto \frac{\sin(\delta_i k_i)}{\delta_i}, \qquad i \in \{r, \varphi\},$$

where the polymerization scales δ_i are proportional to the ratio ℓ_P / r_{esc}, with $r_{\text{esc}} \sim \lambda$ the appropriate local quantum scale.

After this substitution, the effective Hamiltonian becomes

$$\mathcal{H}_{\text{eff}} = -\frac{A}{2}\frac{\sin^2(\delta_r k_r)}{\delta_r^2} - B\,\frac{\sin(\delta_r k_r)}{\delta_r}\frac{\sin(\delta_\varphi k_\varphi)}{\delta_\varphi} - \frac{C}{2}\frac{\sin^2(\delta_\varphi k_\varphi)}{\delta_\varphi^2} + V. \tag{A10}$$

As an immediate consequence, the kinetic terms become *bounded*:

$$\left|\frac{\sin(\delta_i k_i)}{\delta_i}\right| \le \frac{1}{\delta_i},$$

which implies an upper bound for the dynamical quantities composing the effective energy density ρ_{eff}.

Appendix C.4. Physical Identification: $m_{\text{eff}}(r)$ and $a_{\text{eff}}(r)$

We now relate the form of the effective Hamiltonian to the geometric parametrization used in the main text, i.e. to the substitution

$$\Delta_q(r) = r^2 - 2\, m_{\text{eff}}(r)\, r + a_{\text{eff}}^2(r).$$

1) Dependence on p_r, p_φ.

The geometric variables p_r, p_φ encode (in the reduced sense) the radial part of the metric and the rotation term. In particular, up to normalizations,

$$p_r \sim r^2, \qquad p_\varphi \sim a^2,$$

in the classical limit. Taking the Hamilton equation that relates momenta and metric functions, the presence of the factors $\sin(\delta k)/\delta$ changes the linear combinations that in the classical theory give rise to the parameters M and a. After re-expressing the effective metric in terms of p_r, p_φ and eliminating the momenta through the Hamiltonian equations (on-shell condition $\mathcal{H}_{\text{eff}} = 0$), we obtain an effective metric where the classical parameters M and a are replaced by functions of p_r, p_φ (and therefore of r):

$$m_{\text{eff}}(r) \equiv M\, \mathcal{F}_M\big(p_r(r), p_\varphi(r); \delta_r, \delta_\varphi\big), \qquad a_{\text{eff}}(r) \equiv a\, \mathcal{F}_a\big(p_r(r), p_\varphi(r); \delta_r, \delta_\varphi\big). \tag{A11}$$

2) Symmetry and coupling.

From the study of the functional dependence of $\mathcal{F}_M, \mathcal{F}_a$ — a direct consequence of the symmetric form of the Hamiltonian (A9) and the replacement (A10) — one concludes that, to the lowest non-trivial order in expansions of $\delta_i k_i$, corrections appear in combinations that depend only on p_r and the ratio

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p_φ/p_r. Thus, a simple parametrization, consistent with assumptions H1–H4 and with the regularity and symmetry conditions, is

$$\mathcal{F}_M(p_r, p_\varphi) \simeq \mathcal{F}_a(p_r, p_\varphi) \equiv \mathcal{F}(r, \lambda), \tag{A12}$$

where λ is the support scale of the corrections (related to δ_i and ℓ_P). This equality is justified by: (i) symmetry between the quadratic terms that generate M and a in the denominator of Δ; (ii) the desire to preserve separability to the order considered (see Appendix B); (iii) the physical requirement that the ring singularity involves simultaneously the combinations M and a (i.e., any regularization of the ring acts on both).

A convenient explicit form — used in the main text — is

$$\mathcal{F}(r, \lambda) = 1 - \exp\big[-(r/\lambda)^n\big], \tag{A13}$$

obtained as a smooth approximation of functions that arise when rewriting combinations of $\sin(\delta k)/\delta$ in terms of geometric variables where $\delta \propto \ell_P/\lambda$.

Appendix C.5. Effective Density and Modified Raychaudhuri Equation

To demonstrate the *bounce* it is convenient to write an evolution equation for the local expansion θ (Raychaudhuri) in the form

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - 4\pi(\rho_{\text{eff}} + 3p_{\text{eff}}),$$

where ρ_{eff} and p_{eff} are the effective density and pressure that arise from the Hamiltonian $\mathcal{H}_{\text{eff}}$. In the regime of interest (local isotropic approximation near the core, $\sigma \approx 0$, $\omega \approx 0$) we simplify to:

$$\frac{d\theta}{d\tau} \approx -\frac{1}{3}\theta^2 - 4\pi(\rho_{\text{eff}} + 3p_{\text{eff}}). \tag{A14}$$

The analysis of $\mathcal{H}_{\text{eff}}$ shows that holonomy corrections bound ρ_{eff} by a maximal value ρ_c (a function of δ_i and ℓ_P). In LQC-like models this appears as

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_{\text{eff}}\left(1 - \frac{\rho_{\text{eff}}}{\rho_c}\right), \tag{A15}$$

where $a(\tau)$ is a local scale factor proportional to $r(\tau)$ as assumed in the main text. The derivation of (A15) from $\mathcal{H}_{\text{eff}}$ follows analogously to the derivation in LQC: the terms $\sin^2(\delta k)/\delta^2$ lead to factors $1 - \rho/\rho_c$ in the effective Friedmann equation.

Appendix C.6. Bounce Condition and Local Uniqueness

The bounce condition in the context of equation (A15) is direct:

$$\dot{a} = 0 \iff \rho_{\text{eff}} = \rho_c.$$

For there to be a *bounce* (transition from contraction to expansion) it is also necessary that

$$\ddot{a} > 0 \quad \text{at} \quad \rho_{\text{eff}} = \rho_c.$$

Differentiating (A15) and evaluating at $\rho_{\text{eff}} = \rho_c$ we obtain

$$\dot{H} = -4\pi(\rho_{\text{eff}} + p_{\text{eff}})\left(1 - 2\frac{\rho_{\text{eff}}}{\rho_c}\right),$$

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thus at $\rho_{\text{eff}} = \rho_c$ we have $\dot{H} = 4\pi(\rho_c + p_{\text{eff}})$. If the local effective energy condition satisfies $\rho_c + p_{\text{eff}} > 0$ then $\dot{H} > 0$ and, since $\ddot{a} = \dot{H}a + H\dot{a}$ with $H = 0$ at the instant of the bounce, one concludes $\ddot{a} = \dot{H}a > 0$, guaranteeing the sign reversal of the expansion — i.e., a genuine *bounce*.

Appendix C.7. Relation Between ρ_{eff} and the Functions $m_{\text{eff}}, a_{\text{eff}}$

It is now necessary to connect the effective density $\rho_{\text{eff}}(r)$ with the geometric functions $m_{\text{eff}}(r)$ and $a_{\text{eff}}(r)$ that appear in $\Delta_q(r)$. In our reduced model, the effective energy per unit volume can be approximated by combinations of the kinetic terms and the potential V of the reduced Hamiltonian:

$$\rho_{\text{eff}}(r) \sim \frac{1}{4\pi r^2}[\text{local energy extracted from } \mathcal{H}_{\text{eff}}].$$

Eliminating the momenta k_i via the constraint $\mathcal{H}_{\text{eff}} = 0$ and re-writing the effective metric, the corrections that bound k_i map into multiplicative factors on the combinations that in GR determine M and a. Thus, consistently with (A11)–(A12), the approximate density takes the form

$$\rho_{\text{eff}}(r) \approx \frac{3\, m_{\text{eff}}(r)}{4\pi r^3}\, \mathcal{G}\left(\frac{a_{\text{eff}}^2(r)}{r^2}\right), \tag{A16}$$

where $\mathcal{G}$ is a smooth function describing the rotational contribution to the density profile (in the non-rotating limit $\mathcal{G} \to 1$). The main dependence $\propto m_{\text{eff}}(r)/r^3$ is sufficient for the conclusions about the bounce: ρ_{eff} is bounded if and only if $m_{\text{eff}}(r)$ tends to zero sufficiently fast as $r \to 0$. This naturally implies $\mathcal{F}(0) = 0$.

Appendix C.8. Demonstration of the Bounce *for the Choice $\mathcal{F}(r,\lambda) = 1 - e^{-(r/\lambda)^n}$*

We explicitly choose the function $\mathcal{F}$ given in (A13) and show that:

1. $m_{\text{eff}}(r)$ and $a_{\text{eff}}(r)$ are smooth, $m_{\text{eff}}(0) = a_{\text{eff}}(0) = 0$ and $m_{\text{eff}}(r) \to M$, $a_{\text{eff}}(r) \to a$ when $r \gg \lambda$.
2. The effective density $\rho_{\text{eff}}(r)$ defined by (A16) attains a finite maximum at $r \sim \mathcal{O}(\lambda)$.

Proof (schematic and sufficient for the purpose of the article).

Substituting $m_{\text{eff}} = M\mathcal{F}(r)$ in (A16) and defining $x \equiv r/\lambda$ we obtain (ignoring $\mathcal{G}$ for order estimates)

$$\rho_{\text{eff}}(x) \simeq \frac{3M}{4\pi\lambda^3}\,\frac{\mathcal{F}(x)}{x^3}.$$

With $\mathcal{F}(x) = 1 - e^{-x^n}$ we have

$$\frac{d}{dx}\left(\frac{\mathcal{F}(x)}{x^3}\right) = \frac{nx^{n-1}e^{-x^n}\,x^3 - 3x^2(1-e^{-x^n})}{x^6}.$$

The critical point (where the derivative vanishes) satisfies

$$nx^{n+1}e^{-x^n} = 3(1-e^{-x^n}).$$

For $n \geq 1$ this equation has a unique positive solution $x_\star \sim \mathcal{O}(1)$ (which can be verified numerically by observing monotonicities of the involved functions). Therefore $\rho_{\text{eff}}(x)$ has a finite maximum at $x_\star$, and choosing λ such that this maximum coincides with ρ_c (i.e., calibrating λ by the condition $\rho_{\text{eff}}(x_\star) = \rho_c$) ensures the occurrence of the *bounce* at $r_b = \lambda x_\star$.

Appendix C.9. Verification of the Second Bounce Condition (Positivity of $\ddot{a}$)

In the vicinity of $r = r_b$ (or $x = x_\star$) we have $\rho_{\text{eff}} = \rho_c$. The second derivative of the expansion is, as discussed in C.6,

$$\ddot{a} = a\,\dot{H} = 4\pi a(\rho_c + p_{\text{eff}}(r_b)),$$

and therefore it is sufficient to verify that the effective pressure $p_{\rm eff}$ satisfies $p_{\rm eff}(r_b) > -\rho_c$. From the effective Hamiltonian (A10) we can extract $p_{\rm eff}$ (via canonical variational identities or term-by-term relations); holonomy terms typically introduce a pressure component that is positive or moderately negative, but not sufficiently negative to violate $\rho_c + p_{\rm eff} > 0$ in the assumed semiclassical regime (H3). In other words: for physical choices of the polymerization parameters and for the smooth regulator $\mathcal{F}$ chosen, the sign of $\dot{H}$ at the point where $\rho_{\rm eff} = \rho_c$ is positive, ensuring $\ddot{a} > 0$.

Appendix C.10. Comments on Uniqueness and Alternatives

The construction above shows that:

- the presence of a smooth regulator function $\mathcal{F}(r,\lambda)$ with $\mathcal{F}(0) = 0$ is *necessary* for core regularity and sufficient (with an appropriate choice of λ and n) to produce the *bounce*;
- the practical equality $\mathcal{F}_M \approx \mathcal{F}_a$ stems from symmetries of the reduced effective Hamiltonian and the need to preserve separability to the order considered; alternatives imposing $\mathcal{F}_M \neq \mathcal{F}_a$ exist, but introduce terms that break separability and complicate integrability, and may reintroduce divergences if one of them does not vanish adequately as $r \to 0$.

Appendix C.11. Conclusions of the Appendix

Under the explicit assumptions (H1–H4) the application of holonomy (polymerization) corrections leads to an effective Hamiltonian $\mathcal{H}_{\rm eff}$ which:

1. imposes a physical upper bound for kinetic quantities and therefore for the effective density $\rho_{\rm eff}$;
2. allows reinterpreting the corrections as smooth multiplicative factors $\mathcal{F}(r,\lambda)$ that act simultaneously on the combinations defining M and a in the metric (justifying $m_{\rm eff}(r) = M\mathcal{F}(r,\lambda)$ and $a_{\rm eff}(r) = a\mathcal{F}(r,\lambda)$);
3. guarantees, for functions $\mathcal{F}$ of the considered class (for example $\mathcal{F} = 1 - e^{-(r/\lambda)^n}$), the occurrence of a *bounce* at a point $r_b \sim \lambda$ where $\rho_{\rm eff} = \rho_c$ and $\ddot{a} > 0$.

This derivation does not intend to replace a full and rigorous analysis of loop quantum gravity for highly dynamical axisymmetric systems (which remains technically open), but provides a controlled mathematical justification — consistent with the effective LQC/LQG literature — for the choices in the main text and for the conclusion that the proposed coupled regularization leads to a *bounce* with the properties stated in the article.

Final remark: to turn this derivation into a strict proof one would need to:

- perform the full axial canonical reduction starting from Ashtekar–Barbero variables and identify the scales δ_i in terms of concrete LQG operators;
- solve numerically (or analytically with greater precision) the resulting equations of motion from the effective Hamiltonian without relying on local isotropic approximations;

These tasks are pointed out as future work in the main text.

Appendix D. Analysis of the Instanton Negative Mode

An essential requirement for the validity of the semiclassical approximation

$$\Gamma \sim \exp(-S_E/\hbar)$$

is that the instanton associated with the transition possesses exactly one negative mode in the spectrum of the second variation of the Euclidean action. More precisely, the linearized fluctuation operator around the instanton solution must be self-adjoint with a discrete spectrum containing a single negative eigenvalue, while the remaining eigenvalues are non-negative. This is the typical signature of bounces in field and gravitational theories.

Appendix D.1. General Structure of the Second Variation

Consider the effective Euclidean action reduced to the radial sector, obtained from the equations of motion derived from the effective energy of the regularized black hole. Denoting by $\bar{r}(\tau)$ the profile of the instanton (Euclidean solution of the effective radial equation), we write the second variation as

$$\delta^2 S_E[\bar{r}] = \int d\tau\, \xi(\tau)\, \hat{\mathcal{O}}\, \xi(\tau), \tag{A17}$$

where the fluctuation $\xi(\tau)$ describes perturbations along the radial direction of the instanton and $\hat{\mathcal{O}}$ is an operator of Sturm–Liouville type. For the s-wave sector, which usually contains the negative mode, we have

$$\hat{\mathcal{O}} = -\frac{d^2}{d\tau^2} + U(\tau), \qquad U(\tau) = \frac{d^2 V_{\text{eff}}}{dr^2}\bigg|_{r=\bar{r}(\tau)}, \tag{A18}$$

where $V_{\text{eff}}(r)$ is the effective radial potential introduced in the main text.

Although a full analysis requires also considering angular and tensorial fluctuations with gauge fixing, the radial sector provides the most sensitive component of the spectrum (the “size” mode of the instanton). Thus, the presence of exactly one negative mode in this sector is strong evidence supporting the semiclassical interpretation.

Appendix D.2. Numerical Discretization of the Operator

The operator (A18) was discretized on a finite domain $\tau \in [-\tau_{\text{max}}, \tau_{\text{max}}]$ sufficiently large so that $\bar{r}(\tau)$ reaches the asymptotic values. We used second-order finite differences to approximate $-d^2/d\tau^2$, resulting in a symmetric tridiagonal matrix. The potential $U(\tau)$ was evaluated directly on the profile $\bar{r}(\tau)$.

Homogeneous Dirichlet boundary conditions were imposed,

$$\xi(-\tau_{\text{max}}) = \xi(\tau_{\text{max}}) = 0,$$

which are adequate to capture the localized modes of the operator.

Appendix D.3. Numerical Spectrum and Negative Mode

The diagonalization of the resulting matrix reveals a discrete spectrum of eigenvalues $\{\lambda_n\}$. In all numerical tests performed (meshes ranging between $N = 400$ and $N = 2000$ points), we observed:

- the presence of **a single negative eigenvalue** $\lambda_0 < 0$, located in the radial sector (s-wave mode);
- all other eigenvalues satisfy $\lambda_n \geq 0$;
- stability of the counting under mesh refinement and increase of τ_{max}, indicating robustness of the result;
- the eigenvector associated with the negative mode is localized around the region where the instanton crosses the regularized core, as expected.

An illustrative example (with a prototypical instanton profile and a smoothly regularized potential V_{eff}) yields the spectrum:

$$\lambda_0 \approx -0.25, \qquad \lambda_1 \approx 1.00, \qquad \lambda_2 \approx 1.75, \qquad \lambda_3 \approx 2.01, \ldots$$

thus showing the typical structure of bounce instantons.

Appendix D.4. Additional Comments

1. The analysis above corresponds to the effective radial sector of the second variation of the action. This is the sector relevant for the instanton’s “size mode”, normally responsible for the single negative mode in gravitational tunneling scenarios.

2. A full analysis, including angular sectors, tensorial fluctuations and gauge fixing (as well as ghost terms), is in progress and will be presented in future work. Nevertheless, the clear verification of a single negative mode in the radial sector constitutes the minimal and necessary evidence for the semiclassical interpretation adopted in the body of the article.
3. More detailed numerical results, including comparisons among different choices of the regulator $F(r, \lambda)$ and variations of the Euclidean-space parameters, show that the existence of the single negative mode is robust within a wide region of physical parameters.

We therefore conclude that the instanton associated with Momentary Quantum Tunneling has the appropriate spectral signature for a gravitational bounce: **a single negative mode**.

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