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Not peer-reviewed version

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Posted Date: 2 December 2025

doi: 10.20944/preprints202512.0286.v1

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Article

On Some Properties of ABC Triples and Radicals of an Integer

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Abstract

In this paper, we show that if a, b, c are numbers such that $c = a + b$, $\gcd(a, b, c) = 1$ and $\text{rad}(abc) < c$, then c can't be square-free, at most one of a and b is square-free, and the square-free factor must be the smallest factor of the triple, we also showed an explicit upper bound for the quality of abc triples. We also discuss some basic properties of the radical function in general.

Keywords: radical of an integer; abc conjecture; Diophantine equations

MSC: 11A25, 11D75, 11J25

1. Introduction

The *radical* of a positive integer n , often denoted as $\text{rad}(n)$, is the product of all prime factors of n , in other words, the largest square-free factor of n .

The *abc conjecture* is a conjecture in number theory first proposed by Joseph Oesterlé and David Masser in 1985 that is related to the radicals of positive integers [1,2]. The statement of the abc conjecture is as follows:

Conjecture 1. *Let a, b, c be positive integers such that $c = a + b$ and $\gcd(a, b, c) = 1$, then for each $\epsilon > 0$, there are only finite triples (a, b, c) such that $\text{rad}(abc)^{1+\epsilon} < c$.*

So far there is no widely accepted proof for the abc conjecture and the conjecture is still regarded as an open problem.

So far, a proven upper bound for c is

$$\log c < K \cdot \text{rad}(abc)^{\frac{1}{3}} (\log \text{rad}(abc))^3 \quad (1)$$

with K being a constant independent of a, b and c . This upper bound was proven by Stewart and Yu in 2001 [3].

Besides, Stewart and Tijdeman proved that the following lower bound holds for infinitely many abc triples in 1986 [4]:

$$\log c > \log \text{rad}(abc) + k \frac{\sqrt{\log c}}{\log \log c} \quad (2)$$

for all $k < 4$, and van Frankenhuysen improved k to 6.068 in 2000 [5].

Moreover, Granville and Tucker ([6]) has proposed the following conjecture:

Conjecture 2. *Let a, b, c be positive integers such that $c = a + b$ and $\gcd(a, b, c) = 1$, then $c < \text{rad}(abc)^2$.*

Besides, there's a stronger conjecture proposed by Baker in 2004 [7], stating that

Conjecture 3. *Let a, b, c be positive integers such that $c = a + b$ and $\gcd(a, b, c) = 1$, then $c < \frac{6}{5} \text{rad}(abc) \frac{(\log \text{rad}(abc))^{\omega(abc)}}{(\omega(abc))!}$*

where $\omega(abc)$ is the number of distinct prime factors of abc .

Since it has been shown by Laishram and Shorey(2011, [8]) that $\frac{6}{5} \frac{(\log rad(abc))^{\omega(abc)}}{(\omega(abc))!} < rad(abc)^{\frac{3}{4}}$,

Conjecture 3 implies that $c < rad(abc)^{\frac{7}{4}}$.

In this paper, we intend to discuss some properties of the radical of integers and the triples involved in the abc conjecture, improving the upper bound for the c in an abc triple, and apply our result to the Nagell-Ljunggren equation as an example.

Unless otherwise specified, $rad(n) = \prod_{i=1}^{\omega(n)} p_i$ indicates the radical of a positive integer $n = \prod_{i=1}^{\omega(n)} p_i^{a_i}$ with $p_1, \dots, p_{\omega(n)}$ being prime numbers and $1 \leq a_1, \dots, a_{\omega(n)}$ being positive integers, and $\log n$ indicates the natural logarithm of n .

2. Properties of Radicals

In this section, we will discuss the general properties of the radical of a positive integer.

Definition 1. $R(n) = \frac{rad(n)}{n}$

Theorem 1. If $\gcd(a, b) = 1$, then $R(ab) = R(a)R(b)$

Proof. Since $rad(n)$ is multiplicative, we have $R(ab) = \frac{rad(ab)}{ab} = \frac{rad(a) rad(b)}{a b} = R(a)R(b)$ \square

Theorem 2. $R(n) = 1$ if and only if n is square-free, $R(n) \leq \frac{1}{2}$ if $R(n)$ is not square-free.

Proof. By definition of $rad(n)$, we have $rad(n) \leq n$, and if n is square-free, then $rad(n) = n$ and vice versa.

If n is not square-free, then there's a prime factor $p \mid n$ such that $p^2 \mid n$, which implies that $p \cdot rad(n) \leq n$.

Therefore we have

$$R(n) = \frac{rad(n)}{n} \leq \frac{rad(n)}{p \cdot rad(n)} = \frac{1}{p} \leq \frac{1}{2}. \quad (3)$$

And this completes the proof. \square

Remark 1. Let $\mu(n)$ be the Möbius function of n , and $\omega(n)$ be the number of distinct prime factors of n , then we have $\mu(n) = (-1)^{\omega(n)} \lfloor R(n) \rfloor$.

Theorem 3. If $\gcd(a, b) = 1$, then $R(a) + R(b) \leq R(ab) + 1$

Proof. Since $R(a) \leq 1$ and $R(b) \leq 1$, we have $R(a) - 1 \leq 0 \wedge R(b) - 1 \leq 0$. Therefore we have

$$\begin{aligned} 0 &\leq (R(a) - 1)(R(b) - 1) \\ &\implies 0 \leq R(ab) - (R(a) + R(b)) + 1 \\ &\implies R(a) + R(b) \leq R(ab) + 1. \end{aligned} \quad (4)$$

And this completes the proof. \square

3. Properties of ABC Triples

In this section, we will discuss the properties of abc triples i.e. positive integers a, b, c such that $a + b = c$, $\gcd(a, b, c) = 1$ and $rad(abc) < c$.

Definition 2. A set of three positive integers (a, b, c) is called an **abc triple** if $c = a + b$, $\gcd(a, b, c) = 1$ and $rad(abc) < c$.

Theorem 4. If (a, b, c) is an abc triple, then c is not square-free

Proof. If c is square-free, then we have $c = \text{rad}(c)$, therefore, we have

$$c = \text{rad}(c) \leq \text{rad}(a)\text{rad}(b)\text{rad}(c) = \text{rad}(abc) \quad (5)$$

□

Theorem 5. If (a, b, c) is an abc triple, then at most one of a or b is square-free.

Proof. First, by Theorem 3, we have

$$R(a) + R(b) + R(c) - 2 = (R(a) + R(b) - 1) + R(c) - 1 \leq R(ab) + R(c) - 1 \leq R(abc). \quad (6)$$

Since (a, b, c) is an abc triple, we have

$$R(a) + R(b) + R(c) \leq 2 + R(abc) = 2 + \frac{\text{rad}(abc)}{abc} < 2 + \frac{c}{abc} = 2 + \frac{1}{ab}. \quad (7)$$

If both of a and b are square-free, then we have

$$\begin{aligned} 2 + R(c) &= R(a) + R(b) + R(c) \leq 2 + \frac{1}{ab} \\ \implies \text{rad}(c) &\leq \frac{c}{ab} = \frac{a+b}{ab} < 2. \end{aligned} \quad (8)$$

But this leads to a contradiction.

Therefore, if (a, b, c) is an abc triple, at most one of a or b is square-free. □

Theorem 6. If (a, b, c) is an abc triple with a being square-free, then $a < b < c$

Proof. $b < c$ follows from the fact that $c = a + b$

Assume that a is square-free and $a > b$, then we have

$$a \cdot \text{rad}(b)\text{rad}(c) = \text{rad}(a)\text{rad}(b)\text{rad}(c) = \text{rad}(abc) < c = a + b. \quad (9)$$

Since b and c are not square-free by Theorem 5, we have $4 \leq b < c$, which implies that $2 \leq \text{rad}(b)$ and $2 \leq \text{rad}(c)$, thus we have

$$4 = 2 \times 2 \leq \text{rad}(b)\text{rad}(c) < 1 + \frac{b}{a} < 2. \quad (10)$$

Which is a contradiction.

Therefore, if (a, b, c) is an abc triple with a being square-free, then $a < b < c$. □

Theorem 7. If $c = a + b$ and $\gcd(a, b, c) = 1$, then $c < \text{rad}(abc)^2$

Proof. W.l.o.g. assume that $a < b$. First, we cannot have $\text{rad}(abc)^2 = c$, because if $\text{rad}(a)^2\text{rad}(b)^2\text{rad}(c)^2 = \text{rad}(abc)^2 = c$, then we have $\text{rad}(a)\text{rad}(b) \mid c \implies \gcd(a, b, c) > 1$, which is a contradiction, thus we either have $\text{rad}(abc)^2 > c$ or $\text{rad}(abc)^2 < c$.

Claim: Assume that $z = \frac{1}{4\text{rad}(abc)}$ and $\text{rad}(abc)^2 < c$, then for all nonnegative integers $0 \leq m$, $(1 + \frac{z}{2b})^m \text{rad}(abc)^2 < c$ implies that $(1 + \frac{z}{2b})^{m+1} \text{rad}(abc)^2 < c$.

Proof of the claim:

We prove the claim by mathematical induction.

First, since $\text{rad}(abc)^2 < c$ and since $c = a + b < 2b$, we have $\text{rad}(abc)^2 < c < 2b$

Therefore, we have

$$R(ab) = \frac{\text{rad}(ab)}{ab} < \frac{\text{rad}(ab)}{a \left(\frac{\text{rad}(abc)^2}{2}\right)} = \frac{2\text{rad}(ab)}{a \cdot \text{rad}(abc)^2} \quad (11)$$

and

$$R(c) = \frac{\text{rad}(c)}{c} < \frac{\text{rad}(c)}{\text{rad}(abc)^2} \quad (12)$$

This implies that $R(ab) - \frac{2\text{rad}(ab)}{a \cdot \text{rad}(abc)^2} < 0$ and $R(c) - \frac{\text{rad}(c)}{\text{rad}(abc)^2} < 0$

Therefore we have

$$0 < \left(R(ab) - \frac{2\text{rad}(ab)}{a \cdot \text{rad}(abc)^2}\right) \left(R(c) - \frac{\text{rad}(c)}{\text{rad}(abc)^2}\right) \quad (13)$$

Which implies that

$$\frac{2\text{rad}(ab)}{a \cdot \text{rad}(abc)^2} R(c) + \frac{\text{rad}(c)}{\text{rad}(abc)^2} R(ab) < R(abc) + \left(\frac{2}{a \cdot \text{rad}(abc)^3}\right) \quad (14)$$

Multiplying a on both sides, we have

$$\begin{aligned} & \frac{2}{c \cdot \text{rad}(abc)} + \frac{1}{b \cdot \text{rad}(abc)} \\ & < a \cdot R(abc) + \left(\frac{2}{\text{rad}(abc)^3}\right) = a \cdot \frac{\text{rad}(abc)}{abc} + \left(\frac{2}{\text{rad}(abc)^3}\right). \end{aligned} \quad (15)$$

Since $\text{rad}(abc)^2 < c$, (14) implies that

$$\frac{2}{c \cdot \text{rad}(abc)} + \frac{1}{b \cdot \text{rad}(abc)} < \frac{1}{b\sqrt{c}} + \frac{2}{\text{rad}(abc)^3} \quad (16)$$

Let $\rho = \frac{1}{\text{rad}(abc)}$, then by multiplying bc on both sides, we have

$$(2b + c)\rho < \sqrt{c} + 2\rho^3 bc \quad (17)$$

Which implies that

$$\frac{2b + c - \frac{\sqrt{c}}{\rho}}{2bc} < \rho^2 \implies \text{rad}(abc)^2 < \frac{2b}{2b + c - \frac{\sqrt{c}}{\rho}} c \quad (18)$$

If $2b + c - \frac{\sqrt{c}}{\rho} \leq 2b + z$, then we have

$$\begin{aligned} c - \frac{\sqrt{c}}{\rho} - z &\leq 0 \implies \sqrt{c} \leq \frac{1}{2} \left(\frac{1}{\rho} + \sqrt{\frac{1}{\rho^2} + 4z}\right) \\ \implies c &\leq \frac{1}{4} \left(\frac{2}{\rho^2} + \frac{2}{\rho} \sqrt{\frac{1}{\rho^2} + 4z}\right) \leq \frac{1}{\rho^2} + z \end{aligned} \quad (19)$$

By Bernoulli's inequality, we have $\sqrt{\frac{1}{\rho^2} + 4z} \leq \frac{1}{\rho}(1 + 2z\rho^2) = \frac{1}{\rho} + 2z\rho$, therefore, we have

$$c \leq \frac{1}{4} \left(\frac{2}{\rho^2} + \frac{2}{\rho} \sqrt{\frac{1}{\rho^2} + 4z}\right) \leq \frac{1}{\rho^2} + z = \text{rad}(abc)^2 + \frac{1}{4\text{rad}(abc)} \quad (20)$$

but since $\text{rad}(abc)$ and c are positive integers with $6 \leq \text{rad}(abc)$, we have $\frac{1}{4\text{rad}(abc)} < 1$, this implies that $c \leq \text{rad}(abc)^2$, which contradicts with our assumption.

Therefore we have $2b + z < 2b + c - \frac{\sqrt{c}}{\rho}$, and by (18) we have

$$\text{rad}(abc)^2 < \frac{2b}{2b + c - \frac{\sqrt{c}}{\rho}} c < \frac{2b}{2b + z} c \implies \left(1 + \frac{z}{2b}\right) \text{rad}(abc)^2 < c \quad (21)$$

Thus the claim holds for $m = 0$.

Now assume that the proposition holds for all positive integers $1 \leq m \leq k - 1$, then for the case $m = k$, since $(1 + \frac{z}{2b})^k \text{rad}(abc)^2 < c$ and since $c = a + b < 2b$, we have $(1 + \frac{z}{2b})^k \text{rad}(abc)^2 < c < 2b$. For the sake of brevity, we may have $\zeta = (1 + \frac{z}{2b})^k$ in the following paragraphs.

Therefore, we have

$$R(ab) = \frac{\text{rad}(ab)}{ab} < \frac{\text{rad}(ab)}{a(\zeta \frac{\text{rad}(abc)^2}{2})} = \frac{2\text{rad}(ab)}{\zeta a \cdot \text{rad}(abc)^2} \quad (22)$$

and

$$R(c) = \frac{\text{rad}(c)}{c} < \frac{\text{rad}(c)}{\zeta \text{rad}(abc)^2} \quad (23)$$

This implies that $R(ab) - \frac{2\text{rad}(ab)}{\zeta a \cdot \text{rad}(abc)^2} < 0$ and $R(c) - \frac{\text{rad}(c)}{\zeta \text{rad}(abc)^2} < 0$

Therefore we have

$$0 < \left(R(ab) - \frac{2\text{rad}(ab)}{\zeta a \cdot \text{rad}(abc)^2}\right) \left(R(c) - \frac{\text{rad}(c)}{\zeta \text{rad}(abc)^2}\right) \quad (24)$$

Which implies that

$$\frac{2\text{rad}(ab)}{\zeta a \cdot \text{rad}(abc)^2} R(c) + \frac{\text{rad}(c)}{\zeta \text{rad}(abc)^2} R(ab) < R(abc) + \left(\frac{2}{\zeta^2 a \cdot \text{rad}(abc)^3}\right) \quad (25)$$

Since $\zeta \text{rad}(abc)^2 < c$, (25) implies that

$$\begin{aligned} \frac{2\text{rad}(abc)}{\zeta a c \cdot \text{rad}(abc)^2} + \frac{\text{rad}(abc)}{\zeta b \cdot \text{rad}(abc)^2} &= \frac{2\text{rad}(ab)}{\zeta a \cdot \text{rad}(abc)^2} R(c) + \frac{\text{rad}(c)}{\zeta \text{rad}(abc)^2} R(ab) \\ &< R(abc) + \left(\frac{2}{\zeta^2 a \cdot \text{rad}(abc)^3}\right). \end{aligned} \quad (26)$$

Multiplying a on both sides, we have

$$\begin{aligned} \frac{2}{\zeta c \cdot \text{rad}(abc)} + \frac{1}{\zeta b \cdot \text{rad}(abc)} \\ < a \cdot R(abc) + \left(\frac{2}{\zeta^2 \text{rad}(abc)^3}\right) = a \cdot \frac{\text{rad}(abc)}{abc} + \left(\frac{2}{\zeta^2 \text{rad}(abc)^3}\right). \end{aligned} \quad (27)$$

Since $\zeta \text{rad}(abc)^2 < c$, we have

$$\begin{aligned} \frac{2}{\zeta c \cdot \text{rad}(abc)} + \frac{1}{\zeta b \cdot \text{rad}(abc)} &< \frac{\text{rad}(abc)}{bc} + \left(\frac{2}{\zeta^2 \text{rad}(abc)^3}\right) \\ &< \frac{\sqrt{\frac{c}{\zeta}}}{bc} + \frac{2}{\zeta^2 \text{rad}(abc)^3} = \frac{1}{b\sqrt{\zeta c}} + \frac{2}{\zeta^2 \text{rad}(abc)^3} \end{aligned} \quad (28)$$

Let $\rho = \frac{1}{\sqrt{\zeta \text{rad}(abc)}}$, then by multiplying $\sqrt{\zeta}bc$ on both sides, we have

$$(2b + c)\rho < \sqrt{c} + 2\rho^3 bc \quad (29)$$

Which implies that

$$\frac{2b + c - \frac{\sqrt{c}}{\rho}}{2bc} < \rho^2 \implies \zeta rad(abc)^2 < \frac{2b}{2b + c - \frac{\sqrt{c}}{\rho}} c \quad (30)$$

If $2b + c - \frac{\sqrt{c}}{\rho} \leq 2b + z$, then since $\lfloor \zeta rad(abc)^2 \rfloor + 1 \leq c$, $2b + c - \frac{\sqrt{c}}{\rho} \leq 2b + z$ imply that

$$\begin{aligned} c &\leq (\sqrt{\zeta} rad(abc))\sqrt{c} + z \\ \implies \sqrt{c}(\sqrt{c} - \sqrt{\zeta} rad(abc)) &\leq z = \frac{1}{4rad(abc)} \\ \implies \sqrt{c} &\leq \sqrt{\zeta} rad(abc) + \frac{1}{4rad(abc)\sqrt{c}}. \end{aligned} \quad (31)$$

which implies that

$$\begin{aligned} c &\leq \zeta rad(abc)^2 + \frac{2\sqrt{\zeta} rad(abc)}{4rad(abc)\sqrt{c}} + \frac{1}{16c \cdot rad(abc)^2} \\ &\leq \zeta rad(abc)^2 + \frac{2\sqrt{\zeta} rad(abc)}{4rad(abc)\sqrt{\zeta} rad(abc)} + \frac{1}{16c \cdot rad(abc)^2} \\ &= \zeta rad(abc)^2 + \frac{1}{2rad(abc)} + \frac{1}{16c \cdot rad(abc)^2} \\ &\leq \lfloor \zeta rad(abc)^2 \rfloor + 1 + \frac{1}{2rad(abc)} + \frac{1}{16c \cdot rad(abc)^2} \end{aligned} \quad (32)$$

But since $3 \leq c$ is a positive integer and since $\gcd(a, b, c) = 1$ and $a + b = c$, we have $6 \leq rad(abc)$, thus $\frac{1}{2rad(abc)} + \frac{1}{16c \cdot rad(abc)^2} < \frac{1}{12} + \frac{1}{16 \times 6^2} < 1$, therefore, (32) implies that $\lfloor \zeta rad(abc)^2 \rfloor + 1 \leq c \leq \lfloor \zeta rad(abc)^2 \rfloor + 1$, which indicates that $c = \lfloor \zeta rad(abc)^2 \rfloor + 1$.

By (29) and $c = \lfloor \zeta rad(abc)^2 \rfloor + 1$, we have

$$2b < \frac{\sqrt{c}}{\rho} + (2\rho^2 b - 1)c = \sqrt{\frac{\lfloor \zeta rad(abc)^2 \rfloor + 1}{\zeta rad(abc)^2}} + \left(\frac{2b}{\zeta rad(abc)^2} - 1\right) (\lfloor \zeta rad(abc)^2 \rfloor + 1) \quad (33)$$

Here we must have $\frac{2b}{\zeta rad(abc)^2} - 1 \leq 1$, this is because if $1 < \frac{2b}{\zeta rad(abc)^2} - 1$, then we have $\zeta rad(abc)^2 < b \leq (c - 1)$, which implies that $\zeta rad(abc)^2 + 1 < c \leq \lfloor \zeta rad(abc)^2 \rfloor + 1$, a contradiction.

Also, we must have $b - a = 1$, this is because if $2 \leq b - a$, then we have

$$2 \leq b - a \implies a + 2 \leq b \implies a \leq b - 2 \implies c = a + b \leq 2b - 2 \implies c + 2 \leq 2b \quad (34)$$

Therefore, (33) implies that

$$\begin{aligned} \lfloor \zeta rad(abc)^2 \rfloor + 3 = c + 2 &\leq 2b < \lfloor \zeta rad(abc)^2 \rfloor + 1 + \sqrt{\frac{\lfloor \zeta rad(abc)^2 \rfloor + 1}{\zeta rad(abc)^2}} \\ \implies 4 < \frac{\lfloor \zeta rad(abc)^2 \rfloor + 1}{\zeta rad(abc)^2} &\implies 4\zeta rad(abc)^2 < \lfloor \zeta rad(abc)^2 \rfloor + 1 \end{aligned} \quad (35)$$

However, since $\lfloor \zeta rad(abc)^2 \rfloor \leq \zeta rad(abc)^2$, (35) implies that $4\zeta rad(abc)^2 < \zeta rad(abc)^2 + 1$, which further implies that $3rad(abc)^2 \leq 3\zeta rad(abc)^2 < 1$, a contradiction.

Therefore, we have $b - a = 1$, and by $b - a = 1$ we have $a = b - 1$, which implies that $c = a + b = 2b - 1$.

Again, by (29), we have

$$\begin{aligned}
2b &= c + 1 < \frac{\sqrt{c}}{\rho} + (2\rho^2b - 1)c \\
\implies 2c + 1 &< \frac{\sqrt{c}}{\rho} + \rho^2(c + 1) \\
\implies 2c + 1 &< c + \frac{[\zeta \operatorname{rad}(abc)^2] + 2}{\zeta \operatorname{rad}(abc)^2} \leq c + 1 + \frac{2}{\zeta \operatorname{rad}(abc)^2} \\
\implies c &< \frac{2}{\zeta \operatorname{rad}(abc)^2}
\end{aligned} \tag{36}$$

But since $6 \leq \operatorname{rad}(abc)$ and $1 < \zeta = (1 + \frac{z}{2b})^k$, we have $\frac{2}{\zeta \operatorname{rad}(abc)^2} < 1$, therefore, (36) implies that $c < 1$, which is a contradiction.

Therefore we have $2b + z < 2b + c - \frac{\sqrt{c}}{\rho}$, and by $2b + z < 2b + c - \frac{\sqrt{c}}{\rho}$ and (30), we have

$$\begin{aligned}
(1 + \frac{z}{2b})^k \operatorname{rad}(abc)^2 &< \frac{2b}{2b + c - \frac{\sqrt{c}}{\rho}} c < \frac{2b}{2b + z} c \\
\implies (1 + \frac{z}{2b})^{k+1} \operatorname{rad}(abc)^2 &= (1 + \frac{z}{2b})^k (1 + \frac{z}{2b}) \operatorname{rad}(abc)^2 < c
\end{aligned} \tag{37}$$

Thus the claim holds for $m = k$ as well.

Therefore, by mathematical induction, the claim holds for all $0 \leq m$, which implies that if $\operatorname{rad}(abc)^2 < c$, then $(1 + \frac{z}{2b})^m \operatorname{rad}(abc)^2 < c$ holds for all non-negative integers $0 \leq m$.

However, since c is finite and $1 < 1 + \frac{z}{2b}$, by the Archimedean property, there is a positive integer τ such that $c < (1 + \frac{z}{2b})^\tau \operatorname{rad}(abc)^2$, therefore, the claim leads to a contradiction, thus $\operatorname{rad}(abc)^2 < c$ is false.

Therefore $c < \operatorname{rad}(abc)^2$. \square

4. Additional Result

In this section, we will apply our result on the Nagell-Ljunggren equation as an example of the application of our results. The *Nagell-Ljunggren equation* is a Diophantine equation of the form $\frac{x^n - 1}{x - 1} = y^q$, with x, y, n, q being positive integers and $3 \leq n, 2 \leq q$. It has been conjectured that the following solutions are all positive solutions of the Nagell-Ljunggren equation:

$$\frac{3^5 - 1}{3 - 1} = 11^2, \frac{7^4 - 1}{7 - 1} = 20^2 \text{ and } \frac{18^3 - 1}{18 - 1} = 7^3 \tag{38}$$

In this section, we will show that the above solutions are all positive solutions of the Nagell-Ljunggren equation by applying Theorem 7.

Theorem 8. *The solutions listed in (38) are all positive solutions of the Nagell-Ljunggren equation.*

Proof. Assume that (x, y, n, q) is a set of positive integers such that $\frac{x^n - 1}{x - 1} = y^q$, then since 29 is the smallest prime factor of n for this solution, we have $29 \leq n$ and $3 \leq q$ ([9])

By moving terms, we get $(x - 1)y^q + 1 = x^n$, then by Theorem 7, we have

$$\begin{aligned}
(x - 1)y^q + 1 &= x^n < \operatorname{rad}(1 \cdot (x - 1)y^q \cdot x^n)^2 \leq x^2(x - 1)^2y^2 < x^4y^2 \\
\implies y^{q-2} &< x^2(x - 1) < x^3 \\
\implies x^n &< x^4y^2 < x^{4 + \frac{6}{q-2}} \implies n < 4 + \frac{6}{q-2}.
\end{aligned} \tag{39}$$

Since $29 \leq n$, we have

$$29 \leq n < 4 + \frac{6}{q-2} \implies 25q - 50 < 6 \implies q < \frac{56}{25} \implies q = 2. \tag{40}$$

But since we have $3 \leq q$ as well, this leads to a contradiction.

Therefore, the solutions listed in (38) are all positive solutions of the Nagell-Ljunggren equation. \square

5. Discussions

In this paper, we showed that if (a, b, c) is an abc triple, then at most one of a and b is square-free, and c can't be a square-free number, which is consistent with a casual observation on computational results like ones from ABC@home [10,11].

Also, note that in general $a + b = c$ does not imply $rad(a) + rad(b) \leq rad(c)$ or $rad(c) \leq rad(a) + rad(b)$, and this is one major barrier in the research of relevant topics. To see this, both of $(a, b, c) = (2, 3^{10} \times 109, 23^5)$ and $(a, b, c) = (1, 2 \times 3^7, 5^4 \times 7)$ satisfy the condition $a + b = c$, [10] but the former one has $rad(c) \leq rad(a) + rad(b)$ and the latter one has $rad(a) + rad(b) \leq rad(c)$. There's no fixed relationship for the logarithm of radicals of integers either, since again $(a, b, c) = (2, 3^{10} \times 109, 23^5)$ gives $\log rad(c) < \log rad(a) + \log rad(b)$ and $(a, b, c) = (1, 2 \times 3^7, 5^4 \times 7)$ gives $\log rad(a) + \log rad(b) < \log rad(c)$.

Regarding the abc conjecture specifically, it has been shown from the data of ABC@Home [11] that there are abc triples with the square roots of its members also form a Pythagorean triple like $(a, b, c) = (2^4 \times 11^2, 3^4 \times 13^2, 5^6)$, and also abc triples like $(a, b, c) = (2^7, 17^3, 71^2)$ with all its members being perfect powers; however, such triples do not seem common among all known pairs. A question is, are there only finitely many abc triples such that both of a , b and c are perfect powers? More specifically, are there only finitely many abc triples that the square roots of its members also form a Pythagorean triple?

Funding: This research received no funding.

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