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Article

The Problem of Heat and Mass Transfer in Melting Snow and Ice Cover Taking into Account the External Heat Flow

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Abstract

Based on the Muskat-Leverett two-phase filtration equations, a model problem of water and air movement in melting snow is considered, taking into account the external heat flow. The maximum principle and the finite-velocity lemma for perturbations are proven for water saturation. An algorithm for numerically studying the self-similar problem is presented. Calculations of the temperature and water saturation distributions by depth are presented. The significant influence of the specified flux at the boundary and the thermal conductivity coefficient on the temperature field in the snow layer is demonstrated, which is important for predicting melting and hydrothermal processes in snow and ice covers. A theorem on the existence of a weak solution to this problem is formulated. A literature review is provided on mathematical models of multiphase filtration in porous media, taking into account phase transitions (melting, sublimation) and external heat fluxes.

Keywords: snow; melting; heat flow; inhomogeneous medium; phase transitions; multiphase filtration; porous medium

1. Introduction

Recently, various snow cover models have become increasingly popular. They are used to solve problems related to avalanche movement, the contribution of snow cover to runoff formation in river basins, and the spread of pollutants in melting snow. The thermal balance and regime of the earth's surface are shaped by a powerful natural factor—solar radiation flux. Accounting for solar radiation flux is important during snowmelt, as it allows for more accurate predictions of the thermal regime of the snow-ice cover and the thermodynamic processes within it. Models typically use surface temperature as a boundary condition, which ignores important physical processes occurring in the uppermost layer of snow. Solar radiation flux significantly influences the thermal balance of the snow and ice surface, where intense processes such as melting, evaporation, and other processes occur in the near-surface layer under the influence of radiation, exerting a decisive influence on the dynamics of the underlying snow cover and the thermal regime. Snow and ice are light-scattering media over a fairly wide range of the solar radiation spectrum, so their optical properties must be taken into account.

One way to account for heat flux in snowmelt modeling is to specify it for the boundary layer on the snow surface. This method for determining heat flux is presented in the work [1], where a condition for heat exchange with the atmosphere was specified at the upper boundary of the soil (or snow cover), while a geothermal heat flux was introduced at the lower boundary. The boundary condition at the snow-soil boundary determined the equality of temperatures and heat fluxes. The work included calculations to assess the influence of various factors on the snow surface temperature and the development of a method for determining the thermal stability of the snow cover and the effective thermal conductivity of snow based on the temperature of the upper soil layer.

Another method is the partial reflection of shortwave radiation from the snow surface and its partial penetration into the snow layer. In many studies, this process is presented in the form of Bouguer's law [2], which takes into account such important optical properties as the scattering coefficient, the attenuation coefficient, and albedo. The albedo for snow, where the attenuation coefficient reaches 30–60 m⁻¹, reaches limit values close to 100 [2]. Thus, freshly fallen snow is characterized by the highest albedo value A>90[2]. A good description of the reflectivity of snow cover is presented in the work [3], which also proposes a model for calculating the albedo based on air temperature and total solar radiation data, which can be used in the absence of observations. In [4], a one-dimensional multiphase model of snow cover was considered, in which the change in heat content for each phase is described by a heat flux equation accounting for convection, thermal conductivity, solar radiation, and phase transitions.

The heat flux in [5] to describe heat and moisture exchange in snow was defined based on P.P. Kuzmin's method. The flux accounted for direct and diffuse shortwave radiation, longwave radiation from the atmosphere, longwave radiation from snow, turbulent heat exchange, heat exchange during evaporation, and heat input with liquid precipitation. To model the hydrothermal regime of snow cover at subzero temperatures, a heat source was used due to the absorption of shortwave radiation flux within the snowpack, based on the Bouguer-Lambert law. Problems of heat and mass transfer in melting snow and ice cover without taking into account an external heat source were considered in works [6–11].

In the paper [12], a heat and moisture exchange model was developed that calculates all components of the heat and water balance on land, as well as state variables (effective landscape surface temperature, soil temperature, soil moisture content, amount of frozen water in the soil, albedo, etc.). Here, the transfer of shortwave radiation within the snowpack is taken into account in the temperature equation, the radiation intensity varies with depth according to the Bouguer-Lambert law, and an empirical relationship is proposed for determining the attenuation coefficient. For the albedo coefficient, the relationship from [13] was used, taking into account the aging of the upper snow layer.

The process of radiative snowmelt is described in detail in the paper [14]. Experimental studies were conducted to examine the influence of the radiation component on the snowmelt process and changes in snow cover albedo. The paper [15] compares two parameterizations for solar radiation transfer in sea ice. The first assumed exponential radiation decay with constant transmittance and attenuation coefficients. The second employed a two-layer scheme for solar radiation penetration into ice, simulating the near-surface transition layer. Numerical experiments were conducted to simulate the seasonal evolution of the snow-ice cover thickness and the vertical temperature profile within it.

2. Statement of the Problem

Melting snow is considered as a three-phase medium consisting of water ($i = 1$), air ($i = 2$), and ice ($i = 3$). The process is described using the mass conservation equation for each phase, the generalized Darcy law, and the energy conservation equation for melting snow [5,16–18]:

$$\frac{\partial \rho_i}{\partial t} + \text{div}(\rho_i \vec{u}_i) = \sum_{j=1}^3 I_{ji}, \quad i = 1, 2, 3, \quad I_{ji} = -I_{ij}, \quad \sum_{i,j=1}^3 I_{ij} = 0; \quad (1)$$

$$s_i \phi (\vec{u}_i - \vec{u}_3) = -K_0 \frac{k_{0i}}{\mu_i} (\nabla p_i + \rho_i^0 \vec{g}), \quad i = 1, 2, \quad p_2 - p_1 = p_c(s_1, \theta), \quad \sum_{i=1}^2 s_i = 1; \quad (2)$$

$$\left(\sum_{i=1}^3 \rho_i^0 c_i \alpha_i \right) \frac{\partial \theta}{\partial t} + \left(\sum_{i=1}^3 \rho_i^0 c_i \vec{v}_i \right) \nabla \theta = \text{div}(\lambda_c \nabla \theta) - \nu I_{31}. \quad (3)$$

Here t is the time; \vec{u}_i is the velocity of the i -th phase; ρ_i - the reduced density related to the true density ρ_i^0 and the volume concentration α_i by the relation $\rho_i = \alpha_i \rho_i^0$ (the condition $\sum_{i=1}^3 \alpha_i = 1$ is a consequence of the definition of ρ_i); I_{ji} is the intensity of the transition of mass from the j -th to the i -th component in unit volume per unit time; $\vec{v}_i = \phi s_i \vec{u}_i$ are the filtration rates of water and air; ϕ is the

porosity of snow; s_1, s_2 – water and air saturation ($\alpha_1 = \phi s_1, \alpha_2 = \phi s_2, \alpha_3 = 1 - \phi$); K_0 – permeability tensor, k_{0i} – relative phase permeabilities ($k_{0i} = k_{0i}(s_i) \geq 0, k_{0i}|_{s_i=0} = 0$), μ_i – dynamic viscosity, p_i – phase pressures, p_c – capillary pressure, \vec{g} is the gravity acceleration vector; θ – temperature of the environment ($\theta_i = \theta, i = 1, 2, 3$); $c_i = \text{const} > 0$ – heat capacity of the i -th phase at constant volume; $\nu = \text{const} > 0$ – specific heat of fusion of ice; λ_c – thermal conductivity of snow ($\lambda_c = a_c + b_c \rho_c^2, \rho_c = \sum_{i=1}^3 \rho_i^0 \alpha_i, a_c = \text{const} > 0, b_c = \text{const} > 0$ [5]).

The system (1)–(3) is supplemented by the hypotheses $\vec{u}_3 = 0, I_{12} = 0, I_{31} = I_{31}(\theta), \rho_i^0 = \rho_i^0(\theta), i = 1, 2, 3$.

The purpose of the work is to construct an exact solution of the system of equations (1)–(3) under natural boundary conditions.

3. Self-Similar Solution to the Problem

We introduce the final values of temperature θ^-, θ_1 and θ^+ (ice melting point). Let $0 < \theta^- < \theta_1 < \theta^+$. We assume that for all $\theta \in (0, \infty)$ the following relations hold: $\alpha_3(\theta) = 0$ when $\theta \geq \theta^+$, $\alpha_3(\theta) = 1 - \phi^- - \phi_1(\theta - \theta_1)$ when $\theta_1 \leq \theta \leq \theta^+$, $\alpha_3(\theta) = 1 - \phi^-$ for $\theta \leq \theta_1$. Here, $\phi^- = \phi(\theta^-) \in (0, 1), \phi_1 = (1 - \phi^-)/(\theta^+ - \theta_1)$ are the given parameters. In addition, it is assumed that the porous medium is homogeneous ($K_0 = \text{const} > 0$); $\rho_i^0 = \text{const} > 0$ (these conditions do not affect the generality of the results), in the coordinate system xyz the vector $\vec{g} = (0, 0, -g)$, the functions included in the system (1) – (3) depend on z, t . Eliminating in (1) I_{31} , we obtain the system of equations

$$\frac{\partial}{\partial t}(\phi s_1 \rho_1^0 + \rho_3^0(1 - \phi)) + \frac{\partial}{\partial z}(\rho_1^0 v_1) = 0; \quad (4)$$

$$\frac{\partial}{\partial t}(\phi s_2 \rho_2^0) + \frac{\partial}{\partial z}(\rho_2^0 v_2) = 0; \quad (5)$$

$$v_i = -K_0 \frac{k_{0i}}{\mu_i} \left(\frac{\partial p_i}{\partial z} - \rho_i^0 g \right), \quad i = 1, 2, \quad p_2 - p_1 = p_c(s_1, \theta), \quad s_1 + s_2 = 1; \quad (6)$$

$$\left(\sum_{i=1}^3 \rho_i^0 c_i \alpha_i \right) \frac{\partial \theta}{\partial t} + \left(\sum_{i=1}^2 \rho_i^0 c_i v_i \right) \frac{\partial \theta}{\partial z} = \frac{\partial}{\partial z} \left(\lambda_c \frac{\partial \theta}{\partial z} \right) - \nu \rho_3^0 \frac{\partial \phi}{\partial t}. \quad (7)$$

For the system (4) - (7) consider the following problem: snow occupies the area of $(-\infty, ct), t > 0$. At $z = -\infty$, there is no water ($s_1 = 0, v_1 = 0$), the air is stationary ($v_2 = 0$) and the temperature is set to $\theta = \theta^-$ (below the melting point ice); at $z = ct$ the velocities of water and air are known ($v_1 = v_1^+, v_2 = v_2^+$), the air pressure is ($p_2 = p^+$) and the boundary condition is set $\lambda_c \frac{d\theta}{dz} = Q - \alpha(\theta - \theta^0)$ (α is the heat transfer coefficient, θ^0 is the atmospheric air temperature, Q is the set heat flow). Assuming that all the sought functions depend only on the variable $\xi = z - ct$ (c is an unknown constant) from (4) - (7) we obtain

$$-c \frac{d}{d\xi}(\phi s_1 \rho_1^0 + \rho_3^0(1 - \phi)) + \frac{d}{d\xi}(\rho_1^0 v_1) = 0; \quad (8)$$

$$-c \frac{d}{d\xi}(\phi s_2 \rho_2^0) + \frac{d}{d\xi}(\rho_2^0 v_2) = 0; \quad (9)$$

$$v_i = -K_0 \frac{k_{0i}}{\mu_i} \left(\frac{dp_i}{d\xi} - \rho_i^0 g \right), \quad p_2 - p_1 = p_c(s_1, \theta), \quad s_1 + s_2 = 1; \quad (10)$$

$$\left(\sum_{i=1}^3 \rho_i^0 c_i (v_i - c \alpha_i) \right) \frac{d\theta}{d\xi} - c \nu \rho_3^0 \frac{d\phi}{d\xi} = \frac{d}{d\xi} \left(\lambda_c \frac{d\theta}{d\xi} \right); \quad (11)$$

$$s_1|_{\xi \rightarrow -\infty} = 0, \quad \theta|_{\xi \rightarrow -\infty} = \theta^-, \quad \frac{d\theta}{d\xi}|_{\xi \rightarrow -\infty} = 0, \quad v_i|_{\xi \rightarrow -\infty} = 0; \quad (12)$$

$$p_2(0) = p^+, \quad (\lambda_c \frac{d\theta}{d\xi} + \alpha(\theta - \theta^0) - Q)|_{\xi=0} = 0, \quad v_i(0) = v_i^+, \quad i = 1, 2. \quad (13)$$

The sought functions are $s_1(\xi)$, $v_i(\xi)$, $p_i(\xi)$, and the constant c . The solution to problem (8) - (13) is constructed as follows. We find the constant c and obtain representations for the filtration rates and temperature by integrating equations (8), (9) and (10). Using these representations and (10), we arrive at an equation for the saturation of $s_1(\xi)$. Investigation of the solvability of the problem for $s_1(\xi)$ completes the construction of the solution to the problem (8)-(13). In the following, we use the notation $s_1(\xi) \equiv s(\xi)$.

Definition 1. A weak solution to the problem (8)-(13) is the function $\theta(\xi)$, $s(\xi)$, $v_i(\xi)$, $p_i(\xi)$ and a fixed parameter c if:

1) the function $\theta(\xi)$ has a continuous derivative and satisfies the equation

$$\lambda_c \frac{d\theta}{d\xi} = c\rho_3^0(c_1 - c_3)M(\theta) + (A_1c_1 + A_2c_2)(\theta - \theta^-) - \nu\rho_3^0c(\phi - \phi^-) \equiv f_1(\theta),$$

$$M(\theta) \equiv \int_{\theta^-}^{\theta} (1 - \phi(\zeta))d\zeta = \begin{cases} M_1 = (1 - \phi^-)(\frac{1}{2}\theta^+ - \theta^- + \frac{1}{2}\theta_1), & \theta \geq \theta^+, \\ (1 - \phi^-)(\theta - \theta^-) - \frac{\phi_1}{2}(\theta - \theta_1)^2, & \theta_1 \leq \theta \leq \theta^+, \\ (1 - \phi^-)(\theta - \theta^-), & \theta \leq \theta_1. \end{cases}$$

and the conditions $(\lambda_c \frac{d\theta}{d\xi} + \alpha(\theta - \theta^0) - Q)|_{\xi=0} = 0$, $\theta|_{\xi \rightarrow -\infty} = \theta^-$, $\frac{d\theta}{d\xi}|_{\xi \rightarrow -\infty} = 0$;

2) the function $s(\xi)$ has a continuous derivative with weight $a_0(s)$ ($a_0(0) = a_0(1) = 0$), which satisfies the equation

$$a_0(s) \frac{ds}{d\xi} = \varphi_1\varphi_2\gamma \frac{p'_0 f_1}{p_0 \lambda_c} + \frac{1}{p_0} \bar{g}\varphi_1\varphi_2 + \frac{1}{p_0} |c|\phi As - \frac{1}{p_0} |c|(\phi - \phi^-)B \equiv f_2(s, \theta),$$

and the conditions $s(0) = s^+$, $s|_{\xi \rightarrow -\infty} = 0$;

3) the functions $v_i(\xi)$ satisfy the equalities

$$v_1 = c\phi s + c \frac{\rho_3^0}{\rho_1^0} (\phi^- - \phi), \quad v_2 = c\phi(1 - s) - c\phi^-,$$

and the conditions $v_i(0) = v_i^+$, $v_i|_{\xi \rightarrow -\infty} = 0$;

4) the functions $p_i(\xi)$ satisfy the equalities

$$p_2(\xi) = p(\xi) + p_0(\theta)b(s), \quad p_1(\xi) = p_2(\xi) - p_c(s(\xi), \theta(\xi)),$$

$$p(\xi) = p^+ - p_0(\theta^+)b(s^+) - \int_{\xi}^0 f_3(s(\zeta), \theta(\zeta))d\zeta,$$

and the condition $p_2(0) = p^+$.

Determination of filtration rates

From (8) and (9) it follows that

$$\rho_1^0 v_1 - c(\phi s_1 \rho_1^0 + (1 - \phi) \rho_3^0) = A_1 = const, \quad (14)$$

$$\rho_2^0 v_2 - c\phi s_2 \rho_2^0 = A_2 = const. \quad (15)$$

From (14) and (12) we have $A_1 = -c\rho_3^0(1 - \phi^-)$, $\phi^- \equiv \phi(\theta^-)$. From (15) and (12) we find $A_2 = -c\rho_2^0\phi^-$.

From (14), (15) we obtain representations for the filtration velocities:

$$v_1 = c\phi s + c \frac{\rho_3^0}{\rho_1^0} (\phi), \quad v_2 = c\phi(1 - s) - c\phi^-. \quad (16)$$

Representation for Temperature

Using (16) and conditions (12), we obtain

$$\lambda_c \frac{d\theta}{d\xi} = c\rho_3^0(c_1 - c_3)M(\theta) + (A_1c_1 + A_2c_2)(\theta - \theta^-) - \nu\rho_3^0c(\phi - \phi^-) \equiv f_1(\theta), \quad (17)$$

where

$$M(\theta) \equiv \int_{\theta^-}^{\theta} (1 - \phi(\zeta))d\zeta = \begin{cases} M_1 = (1 - \phi^-)(\frac{1}{2}\theta^+ - \theta^- + \frac{1}{2}\theta_1), & \theta \geq \theta^+, \\ (1 - \phi^-)(\theta - \theta^-) - \frac{\phi_1}{2}(\theta - \theta_1)^2, & \theta_1 \leq \theta \leq \theta^+, \\ (1 - \phi^-)(\theta - \theta^-), & \theta \leq \theta_1. \end{cases}$$

Considering (14), (15) under the conditions (13), for unknown parameters $c, s^+ = s_1(0), (s_1 \equiv s, s_2 \equiv 1 - s), \theta(0)$ we obtain the following nonlinear system of equations

$$v_1^+ = c(\phi(\theta(0))s^+ + \frac{\rho_3^0}{\rho_1^0}(\phi^- - \phi(\theta(0)))), \quad (18)$$

$$v_2^+ = c(\phi(\theta(0))(1 - s^+) - \phi^-), \quad (19)$$

$$(\lambda_c \frac{d\theta}{d\xi} + \alpha(\theta - \theta^0) - Q)|_{\xi=0} = 0. \quad (20)$$

From the equation for temperature: $\lambda_c \frac{d\theta}{d\xi} \equiv f_1(\theta)$, then the system can be rewritten as:

$$\begin{cases} v_1^+ = c[\phi(\theta(0))s(0) + \frac{\rho_3^0}{\rho_1^0}(\phi^- - \phi(\theta(0)))], \\ v_2^+ = c[\phi(\theta(0))(1 - s(0)) - \phi^-], \\ (f_1(\theta) + \alpha(\theta - \theta^0) - Q)|_{\xi=0} = 0, \end{cases}$$

here $f_1(\theta) = c\{\rho_3^0(c_1 - c_3)M(\theta) - (\theta - \theta^-)(\rho_3^0c_1(1 - \phi^-) + \rho_2^0c_2\phi^-) - \nu\rho_3^0(\phi - \phi^-)\}$.

The solution to the problem (12), (13), (17) can be represented as

$$I(\theta) \equiv \int_{\theta}^{\theta^+} \frac{d\zeta}{f_1(\zeta)} = \int_{\xi}^0 \frac{d\zeta}{\lambda_c(\zeta)} \equiv \psi(\lambda_c(\xi)), \quad \theta(\xi) = I^{-1}(\psi(\lambda_c(\xi))). \quad (21)$$

For $\theta \in [\theta_1, \theta^+]$ from (21) we obtain

$$I(\theta) = \frac{1}{b_1} \int_{\theta}^{\theta^+} \frac{d\zeta}{(\zeta - \theta_1 + \alpha)^2 - \beta^2} = \psi(\lambda_c(\xi)), \quad \theta \in [\theta_1, \theta^+]. \quad (22)$$

Here $d_1 = c\rho_3^0(c_1 - c_3)(1 - \phi^- + A_1c_1 + A_2c_2) - \nu c\rho_3^0\phi_1, b_1 = -c\phi_1\rho_3^0(c_1 - c_3)/2, \alpha = d_1/(2b_1), \beta^2 = (d_1^2 - 4a_1b_1)/(4b_1^2)$.

For $\theta \in [\theta^-, \theta_1]$ from (21) we obtain

$$\theta(\xi) = \theta^- + (\theta_1 - \theta^-) \exp(-b_2 \int_{\xi}^{\xi_1} \frac{d\zeta}{\lambda_c(\zeta)}). \quad (23)$$

Here $b_2 = c\rho_3^0(c_1 - c_3)(1 - \phi^- + A_1c_1 + A_2c_2)$.

The solvability of this problem is proved in detail in [6]. Thus, for a given function $\lambda_c(s, \theta)$, the representation (21) and its special cases (22), (23) determine the temperature for all $\xi \in (-\infty, 0)$.

Algorithm for solving the system (18-20)

The system solution for unknown parameters $\theta(0), c < 0, s^+$ is divided into three stages:

1. Let $\phi(\theta(0))(1 - s(0)) - \phi^- = 0$, then $v_2^+ = 0$, $v_1^+ < 0$ and $s(0) \neq 1$.

$$s(0) = 1 - \frac{\phi^-}{\phi(\theta(0))}, c = \frac{v_1^+}{(\phi(\theta(0)) - \phi^-)(1 - \rho_3^0/\rho_1^0)} < 0, 0 \leq s(0) < 1.$$

To find $\theta(0)$, we consider the options and use the flow condition on the boundary.

If $\theta \geq \theta^+$, then $M(\theta^+) = M_1 = \text{const}$, $\phi(\theta^+) = 1$ - the solution is nonphysical.

If $\theta \leq \theta_1$, then $M(\theta^-) = 0$ and $c = 0$ - the solution is nonphysical.

For the case $\theta_1 \leq \theta(0) \leq \theta^+$, we substitute c into f_1 and get:

$$F_1 \equiv v_1^+ \left(\rho_3^0(c_1 - c_3) \left((1 - \phi^-)(\theta(0) - \theta^-) - \frac{\phi_1}{2}(\theta(0) - \theta_1)^2 \right) - (\theta(0) - \theta^-)(\rho_3^0 c_1(1 - \phi^-) + \rho_2^0 c_2 \phi^-) - \nu \rho_3^0 \phi_1 (\theta(0) - \theta_1) \right) - \phi_1 (\theta(0) + \theta_1) (1 - \rho_3^0/\rho_1^0) (\alpha(\theta(0) - \theta^0) - Q) = 0.$$

This is the quadratic equation for $\theta(0)$:

$$A_1 \theta^2 + B_1 \theta + C_1 = 0, \text{ where}$$

$$A_1 = \alpha \phi_1 (1 - \rho_3^0/\rho_1^0) - v_1^+ \rho_3^0 \frac{\phi_1}{2} (c_1 - c_3),$$

$$B_1 = v_1^+ \left(\rho_3^0(c_1 - c_3) (1 - \phi^- + \phi_1 \theta_1) - (\rho_3^0 c_1 (1 - \phi^-) + \rho_2^0 c_2 \phi^-) - \nu \rho_3^0 \phi_1 \right) - \phi_1 (1 - \rho_3^0/\rho_1^0) (\alpha(\theta^0 + \theta_1) + Q),$$

$$C_1 = v_1^+ \left(\rho_3^0(c_1 - c_3) \left(\frac{\phi_1}{2} \theta_1^2 - \theta^- (1 - \phi^-) \right) + \theta^- (\rho_3^0 c_1 (1 - \phi^-) + \rho_2^0 c_2 \phi^-) + \nu \rho_3^0 \phi_1 \theta_1 \right) + \phi_1 \theta_1 (1 - \rho_3^0/\rho_1^0) (\alpha \theta^0 + Q). \text{ Here } \rho_1^0, \rho_2^0, \rho_3^0, c_1, c_2, c_3, \theta_1, \theta^-, \nu, \phi_1, \phi^- \text{ are given parameters, and the parameters } v_1^+, \alpha, \theta^0, Q \text{ must be subject to conditions so that the following relations are satisfied:}$$

- If $(\frac{B_1}{2A_1})^2 - C_1 < 0$, then there are no real roots.
- If $(\frac{B_1}{2A_1})^2 - C_1 = 0$ and $B_1/A_1 > 0$, then there are no solutions.
- If $(\frac{B_1}{2A_1})^2 - C_1 = 0$ and $B_1/A_1 < 0$, then the solution is:
 $\theta(0) = -B_1/A_1$, here $\theta(0)$ must satisfy the condition $\theta_1 \leq \theta(0) \leq \theta^+$.
- If $(\frac{B_1}{2A_1})^2 - C_1 > 0$, then there are two roots:

$$\theta_1(0) = -B_1/A_1 + ((\frac{B_1}{2A_1})^2 - C_1)^{1/2}, \text{ which}$$

- for $B_1/A_1 < 0$ must satisfy the condition $\theta_1 \leq \theta_1(0) \leq \theta^+$;

- for $B_1/A_1 > 0$ must satisfy the inequalities

$$((\frac{B_1}{2A_1})^2 - C_1)^{1/2} > -B_1/A_1 \text{ and } \theta_1 \leq \theta_1(0) \leq \theta^+.$$

$$\theta_2(0) = -B_1/A_1 - ((\frac{B_1}{2A_1})^2 - C_1)^{1/2}, \text{ which}$$

- for $B_1/A_1 > 0$ is not a solution;

- for $B_1/A_1 < 0$ must satisfy the inequalities

$$((\frac{B_1}{2A_1})^2 - C_1)^{1/2} < -B_1/A_1 \text{ and } \theta_1 \leq \theta_2(0) \leq \theta^+.$$

2. Let $\phi(\theta(0))s(0) + \frac{\rho_3^0}{\rho_1^0}(\phi^- - \phi(\theta(0))) = 0$, then $v_1^+ = 0$, $v_2^+ < 0$.

$$s(0) = \frac{\rho_3^0/\rho_1^0(\phi(\theta(0)) - \phi^-)}{\phi(\theta(0))}, c = \frac{v_2^+}{(\phi(\theta(0)) - \phi^-)(1 - \rho_3^0/\rho_1^0)} < 0, 0 \leq s(0) \leq 1.$$

To find $\theta(0)$, we consider the options and use the flow condition on the boundary.

If $\theta \geq \theta^+$, then $M(\theta^+) = M_1 = \text{const}$, $\phi(\theta^+) = 1$ - the solution is nonphysical.

If $\theta \leq \theta_1$, then $M(\theta^-) = 0$ and $c = 0$ - the solution is nonphysical.

For the case $\theta_1 \leq \theta(0) \leq \theta^+$, we substitute c into f_1 and get:

$$F_2 \equiv v_2^+ \left(\rho_3^0(c_1 - c_3) \left((1 - \phi^-)(\theta(0) - \theta^-) - \frac{\phi_1}{2}(\theta(0) - \theta_1)^2 \right) - (\theta(0) - \theta^-)(\rho_3^0 c_1(1 - \phi^-) + \rho_2^0 c_2 \phi^-) - \nu \rho_3^0 \phi_1 (\theta(0) - \theta_1) \right) - \phi_1 (\theta(0) + \theta_1) (1 - \rho_3^0/\rho_1^0) (\alpha(\theta(0) - \theta^0) - Q) = 0.$$

This is the quadratic equation for $\theta(0)$:

$$A_2 \theta^2 + B_2 \theta + C_2 = 0, \text{ where}$$

$$A_2 = \alpha \phi_1 (1 - \rho_3^0/\rho_1^0) - v_2^+ \rho_3^0 \frac{\phi_1}{2} (c_1 - c_3),$$

$$B_2 = v_2^+ \left(\rho_3^0 (c_1 - c_3) (1 - \phi^- + \phi_1 \theta_1) - (\rho_3^0 c_1 (1 - \phi^-) + \rho_2^0 c_2 \phi^-) - \nu \rho_3^0 \phi_1 \right) - \phi_1 (1 - \rho_3^0 / \rho_1^0) (\alpha (\theta^0 + \theta_1) + Q),$$

$$C_2 = v_2^+ \left(\rho_3^0 (c_1 - c_3) \left(\frac{\phi_1}{2} \theta_1^2 - \theta^- (1 - \phi^-) \right) + \theta^- (\rho_3^0 c_1 (1 - \phi^-) + \rho_2^0 c_2 \phi^-) + \nu \rho_3^0 \phi_1 \theta_1 \right) + \phi_1 \theta_1 (1 - \rho_3^0 / \rho_1^0) (\alpha \theta^0 + Q).$$

Here $\rho_1^0, \rho_2^0, \rho_3^0, c_1, c_2, c_3, \theta_1, \theta^-, \nu, \phi_1, \phi^-$ are given parameters, and the parameters $v_2^+, \alpha, \theta^0, Q$ must be subject to conditions so that the following relations are satisfied:

- If $(\frac{B_2}{2A_2})^2 - C_2 < 0$, then there are no real roots.
- If $(\frac{B_2}{2A_2})^2 - C_2 = 0$ and $B_2/A_2 > 0$, then there are no solutions.
- If $(\frac{B_2}{2A_2})^2 - C_2 = 0$ and $B_2/A_2 < 0$, then the solution is:
 $\theta(0) = -B_2/A_2$, here $\theta(0)$ must satisfy the condition $\theta_1 \leq \theta(0) \leq \theta^+$.
- If $(\frac{B_2}{2A_2})^2 - C_2 > 0$, then there are two roots:
 $\theta_1(0) = -B_2/A_2 + ((\frac{B_2}{2A_2})^2 - C_2)^{1/2}$, which

– for $B_2/A_2 < 0$ must satisfy the condition $\theta_1 \leq \theta_1(0) \leq \theta^+$;

– for $B_2/A_2 > 0$ must satisfy the inequalities

$$((\frac{B_2}{2A_2})^2 - C_2)^{1/2} > -B_2/A_2 \text{ and } \theta_1 \leq \theta_1(0) \leq \theta^+.$$

$$\theta_2(0) = -B_2/A_2 - ((\frac{B_2}{2A_2})^2 - C_2)^{1/2}, \text{ which}$$

– for $B_2/A_2 > 0$ is not a solution;

– for $B_2/A_2 < 0$ must satisfy the inequalities

$$((\frac{B_2}{2A_2})^2 - C_2)^{1/2} < -B_2/A_2 \text{ and } \theta_1 \leq \theta_2(0) \leq \theta^+.$$

3. Let $\phi(\theta(0))(1 - s(0)) - \phi^- \neq 0$ and $\phi(\theta(0))s(0) + \frac{\rho_3^0}{\rho_1^0}(\phi^- - \phi(\theta(0))) \neq 0$, then $v_1^+ < 0, v_2^+ < 0$.

$$s(0) = \frac{(\phi(\theta(0)) - \phi^-)(v_1^+ + \frac{\rho_3^0}{\rho_1^0}v_2^+)}{\phi(\theta(0))(v_1^+ + v_2^+)}, c = \frac{v_1^+ + v_2^+}{(\phi(\theta(0)) - \phi^-)(1 - \rho_3^0/\rho_1^0)} < 0, 0 \leq s(0) \leq 1.$$

To find $\theta(0)$, we consider the options and use the flow condition on the boundary.

If $\theta \geq \theta^+$, then $M(\theta^+) = M_1 = \text{const}$, $\phi(\theta^+) = 1$ - the solution is nonphysical.

If $\theta \leq \theta_1$, then $M(\theta^-) = 0$ and $c = 0$ - the solution is nonphysical.

For the case $\theta_1 \leq \theta(0) \leq \theta^+$, we substitute c into f_1 and get:

$$F_3 \equiv (v_1^+ + v_2^+) \left(\rho_3^0 (c_1 - c_3) ((1 - \phi^-)(\theta(0) - \theta^-) - \frac{\phi_1}{2}(\theta(0) - \theta_1)^2) - (\theta(0) - \theta^-)(\rho_3^0 c_1 (1 - \phi^-) + \rho_2^0 c_2 \phi^-) - \nu \rho_3^0 \phi_1 (\theta(0) - \theta_1) \right) - \phi_1 (\theta(0) + \theta_1) (1 - \rho_3^0 / \rho_1^0) (\alpha (\theta(0) - \theta^0) - Q) = 0.$$

This is the quadratic equation for $\theta(0)$:

$$A_3 \theta^2 + B_3 \theta + C_3 = 0, \text{ where}$$

$$A_3 = \alpha \phi_1 (1 - \rho_3^0 / \rho_1^0) - \rho_3^0 \frac{\phi_1}{2} (c_1 - c_3) (v_1^+ + v_2^+),$$

$$B_3 = (v_1^+ + v_2^+) \left(\rho_3^0 (c_1 - c_3) (1 - \phi^- + \phi_1 \theta_1) - (\rho_3^0 c_1 (1 - \phi^-) + \rho_2^0 c_2 \phi^-) - \nu \rho_3^0 \phi_1 \right) - \phi_1 (1 - \rho_3^0 / \rho_1^0) (\alpha (\theta^0 + \theta_1) + Q),$$

$$C_3 = (v_1^+ + v_2^+) \left(\rho_3^0 (c_1 - c_3) \left(\frac{\phi_1}{2} \theta_1^2 - \theta^- (1 - \phi^-) \right) + \theta^- (\rho_3^0 c_1 (1 - \phi^-) + \rho_2^0 c_2 \phi^-) + \nu \rho_3^0 \phi_1 \theta_1 \right) + \phi_1 \theta_1 (1 - \rho_3^0 / \rho_1^0) (\alpha \theta^0 + Q).$$

Here $\rho_1^0, \rho_2^0, \rho_3^0, c_1, c_2, c_3, \theta_1, \theta^-, \nu, \phi_1, \phi^-$ are given parameters, and the parameters $v_1^+, v_2^+, \alpha, \theta^0, Q$ must be subject to conditions so that the following relations are satisfied:

- If $(\frac{B_3}{2A_3})^2 - C_3 < 0$, then there are no real roots.
- If $(\frac{B_3}{2A_3})^2 - C_3 = 0$ and $B_3/A_3 > 0$, then there are no solutions.
- If $(\frac{B_3}{2A_3})^2 - C_3 = 0$ and $B_3/A_3 < 0$, then there is a solution:
 $\theta(0) = -B_3/A_3$, where $\theta(0)$ must satisfy the condition $\theta_1 \leq \theta(0) \leq \theta^+$.
- If $(\frac{B_3}{2A_3})^2 - C_3 > 0$, then there are two roots:
 $\theta_1(0) = -B_3/A_3 + ((\frac{B_3}{2A_3})^2 - C_3)^{1/2}$, which

- for $B_3/A_3 < 0$ must satisfy the condition $\theta_1 \leq \theta_1(0) \leq \theta^+$;
- for $B_3/A_3 > 0$ must satisfy the inequalities $((\frac{B_3}{2A_3})^2 - C_3)^{1/2} > -B_3/A_3$ and $\theta_1 \leq \theta_1(0) \leq \theta^+$.

$\theta_2(0) = -B_3/A_3 - ((\frac{B_3}{2A_3})^2 - C_3)^{1/2}$, which

- for $B_3/A_3 > 0$ is not a solution;
- for $B_3/A_3 < 0$ must satisfy the inequalities $((\frac{B_3}{2A_3})^2 - C_3)^{1/2} < -B_3/A_3$ and $\theta_1 \leq \theta_2(0) \leq \theta^+$.

Determination of saturation and pressure

From (10) and (16) we have

$$v_1 = c\phi s - c\frac{\rho_3^0}{\rho_1^0}(\phi - \phi^-) = -K_0\frac{k_{01}}{\mu_1}\left(\frac{dp_1}{d\zeta} - \rho_1^0 g\right),$$

$$v_2 = c\phi(1-s) - c\phi^- = -K_0\frac{k_{02}}{\mu_2}\left(\frac{dp_2}{d\zeta} - \rho_2^0 g\right).$$

Eliminating p_1 and p_2 from these relations using the second equation in (10), we obtain

$$-k_{01}k_{02}K_0\frac{dp_c}{d\zeta} = -\mu_1 v_1 k_{02} + \mu_2 v_2 k_{01} + K_0 k_{01} k_{02} g(\rho_1^0 - \rho_2^0).$$

Following [6], we set: $p_c(s, \theta) = p_0(\theta)\gamma(s)$, $\frac{d\gamma}{ds} \equiv \gamma' < 0$. For $s \in (-\infty, +\infty)$, the relative phase permeabilities k_{0i} are defined as follows: $k_{0i} = 0$ for $s_i \leq 1$, $k_{0i} = \bar{k}_{0i}(s_i, \theta)s_i^{n_i}$ for $0 \leq s_i \leq 1$, $k_{0i} = \bar{k}_{0i}(1, \theta)$ for $s_i \geq 1$. Here we have the constants $n_i > 1$.

The function $a_0(s) = -\gamma'k_{01}k_{02} > 0$ for $s \in (0, 1)$ and $a_0(s) = 0$ for $s \leq 0$ and $s \geq 1$ is assumed. Using the equation for temperature (17), we represent the equation for saturation in the form

$$\begin{aligned} a_0(s)\frac{ds}{d\zeta} &= \varphi_1\varphi_2\gamma\frac{p'_0}{p_0}\frac{f_1}{\lambda_c} + \frac{1}{p_0}\bar{g}\varphi_1\varphi_2 + \frac{1}{p_0}|c|\phi As - \\ &- \frac{1}{p_0}|c|(\phi - \phi^-)B \equiv f_2(s, \theta), \end{aligned} \quad (24)$$

where $\varphi_i = 0$ for $s_i \leq 0$, $\varphi_i = s_i^{n_i}$ for $0 \leq s_i \leq 1$, $\varphi_i = 1$ for $s_i \geq 1$,

$$a_0 = -\varphi_1\varphi_2\gamma', \quad A = \bar{\mu}_1\varphi_2 + \bar{\mu}_2\varphi_1, \quad B = \bar{\mu}_1\frac{\rho_3^0}{\rho_1^0}\varphi_2 + \bar{\mu}_2\varphi_1, \quad \bar{g} = g(\rho_1^0 - \rho_2^0),$$

$$p'_0 = \frac{dp_0}{d\theta}, \quad \bar{\mu}_i = \frac{\mu_i}{K_0\bar{k}_{0i}}, \quad i = 1, 2.$$

Equation (24) is considered for $\zeta < 0$ and the condition $s(0) = s^+$, that is, for $s(\zeta)$ the Cauchy problem is considered, and the condition $s|_{\zeta=-\infty} = 0$ must be justified.

To find the pressures p_1 and p_2 , we consider the equality following from Darcy's laws (the first equations in (10))

$$\sum_{i=1}^2 (\bar{\mu}_i v_i - g\rho_i^0 \varphi_i) = -(\varphi_1 + \varphi_2) \left(\frac{dp_2}{d\zeta} - \frac{\varphi_1}{\varphi_1 + \varphi_2} \frac{dp_c}{d\zeta} \right).$$

Let's put [6]

$$p(\zeta) \equiv p_2(\zeta) - p_0(\theta)b(s), \quad b(s) = \int_0^s \frac{\varphi_1(\zeta)\gamma'(\zeta)}{\varphi_1(\zeta) + \varphi_2(\zeta)} d\zeta.$$

Then

$$\begin{aligned} \frac{dp}{d\zeta} &= -\frac{1}{(\varphi_1(s) + \varphi_2(s))} [\sum_{i=1}^2 (\bar{\mu}_i v_i(\zeta) - g\rho_i^0 \varphi_i) - p'_0(\theta)\frac{f_1}{\lambda_c}(\varphi_1(s)\gamma(s) + \\ &+ b(s)(\varphi_1(s) + \varphi_2(s)))] \equiv f_3(s, \theta), \end{aligned} \quad (25)$$

$$p(0) = p^+ - p_0(\theta^+)b(s^+).$$

From (25) it follows

$$p(\xi) = p^+ - p_0(\theta^+)b(s^+) - \int_{\xi}^0 f_3(s(\zeta), \theta(\zeta))d\zeta,$$

$$p_2(\xi) = p(\xi) + p_0(\theta)b(s), \quad p_1(\xi) = p_2(\xi) - p_c(s(\xi), \theta(\xi)). \quad (26)$$

If the functions $s(\xi)$ and $\theta(\xi)$ are found, then the filtration rates $v_i(\xi)$ and pressures $p_i(\xi)$ are determined by formulas (16), (26).

Let us assume that the functions $\bar{\mu}_i(s, \theta), \bar{k}_{0i}(s, \theta), p_0(\theta), \gamma(s), a_0(s)$ satisfies the following conditions ($\bar{\Omega}^* = [0, 1] \times [\theta^-, \theta^+]$):

$$a) 0 < v_0^{-1} \leq (\mu_i(s, \theta), \bar{k}_{0i}(s, \theta), p_0(\theta), |\frac{d\gamma(s)}{ds}|) \leq v_0, \quad \frac{a_0(s)}{s} \Big|_{s=0} = 0,$$

$$b) (\|\frac{d\gamma}{ds}\|_{C[0,1]}, \|\frac{dp_0}{d\theta}\|_{C[\theta^-, \theta^+]}, \|\mu_i(s, \theta), \bar{k}_{0i}(s, \theta)\|_{C(\bar{\Omega}^*)}) \leq v_0.$$

Theorem 1. If conditions "a", "b" and condition $s^+ \in (0, 1]$ are satisfied, there exists at least one weak solution of the problem (8)–(13). This solution (in addition to Definition 1) has the following properties:

$0 \leq s(\xi) \leq 1, \quad \theta^- \leq \theta(\xi) \leq \theta^+$ and $c = \frac{v_1^+ + v_2^+}{(\phi(\theta(0)) - \phi^-)(1 - \rho_3^0/\rho_1^0)} < 0$ (found from solving the system (18)–(20)).

The function $\theta(\xi)$ is monotone in ξ and $\theta_1 = \theta(\xi_1), \xi_1 < 0$. There exists a point $\xi_* \in (-\infty, \xi_1]$ such that $s(\xi) = 0$ for all $\xi \leq \xi_*$.

Consider the problem

$$a_0(s) \frac{ds}{d\xi} = f_2(s, \theta), \quad \frac{d\theta}{d\xi} = \frac{f_1(\theta)}{\lambda_c(s, \theta)}, \quad \xi < 0, \quad s(0) = s^+, \quad \theta(0) = \theta^+. \quad (27)$$

The solvability of this problem is proved in detail in [6].

Let $\varepsilon \in (0, 1), a_\varepsilon(s) \equiv a_0(s) + \varepsilon > 0$. For $\xi < 0$, instead of (27), we consider the problem

$$a_\varepsilon(s^\varepsilon) \frac{ds^\varepsilon}{d\xi} = f_2(s^\varepsilon, \theta^\varepsilon), \quad \frac{d\theta^\varepsilon}{d\xi} = \frac{f_1(\theta^\varepsilon)}{\lambda_c(s^\varepsilon, \theta^\varepsilon)}, \quad s^\varepsilon(0) = s^+, \quad \theta^\varepsilon(0) = \theta^+. \quad (28)$$

The local solvability of the problem (28) for each $\varepsilon > 0$ follows from the known results [19], p. 21.

Lemma 1. If $s^\varepsilon(\xi)$ is a solution problem (28) and $s^+ \in [0, 1]$, then $0 \leq s^\varepsilon(\xi) \leq 1$.

Proof of Lemma 1. From (28) we get

$$\frac{ds^\varepsilon}{d\xi} - R s^\varepsilon = -Q,$$

Where

$$R = \frac{1}{p_0 a_\varepsilon} (|c| \phi A + \bar{g} \frac{\varphi_1(s^\varepsilon)}{s^\varepsilon} \varphi_2 + \frac{\varphi_1(s^\varepsilon)}{s^\varepsilon} \varphi_2 \gamma \frac{f_1}{\lambda_c} (p_0')^+),$$

$$Q = \frac{1}{p_0 a_\varepsilon} (|c| (\phi - \phi^-) B + \varphi_1 \varphi_2 \gamma \frac{f_1}{\lambda_c} (p_0')^-),$$

$$(p_0')^+ = \max(0, p_0'), \quad (p_0')^- = -\min(0, p_0'), \quad p_0' = (p_0')^+ - (p_0')^-.$$

Since $A > 0, B > 0$ and $R > 0, Q \geq 0$. Therefore

$$s^\varepsilon(\xi) = (s^+ + \int_{\xi}^0 Q(x)e^{-\int_0^x R(\zeta)d\zeta} dx)e^{\int_0^{\xi} R(\zeta)d\zeta} \geq 0.$$

For the function $(1 - s^\varepsilon(\xi))$ from (28) we find

$$\frac{d(1 - s^\varepsilon)}{d\xi} - R_1(1 - s^\varepsilon) = -Q_1,$$

where

$$R_1 = \frac{1}{p_0 a_\varepsilon} (|c|\phi A + \frac{\varphi_2(s^\varepsilon)}{1 - s^\varepsilon} \varphi_1 \gamma \frac{f_1}{\lambda_c} (p_0')^-) > 0,$$

$$Q_1 = \frac{1}{p_0 a_\varepsilon} (|c|(\phi \bar{\mu}_1 \varphi_2 (1 - \frac{\rho_3^0}{\rho_1^0}) + \phi^- B) + \bar{g} \varphi_1 \varphi_2 + \varphi_1 \varphi_2 \gamma \frac{f_1}{\lambda_c} (p_0')^+) > 0.$$

Then

$$1 - s^\varepsilon(\xi) = (1 - s^+ + \int_{\xi}^0 Q_1(x)e^{-\int_0^x R_1(\zeta)d\zeta} dx)e^{\int_0^{\xi} R_1(\zeta)d\zeta} \geq 0,$$

which completes the proof. \square

The function $s(\xi)$ is continuous on the interval $[\xi_1, 0]$ and, therefore, there exists a value $s^1 \equiv s(\xi_1) \in [0, 1]$. Therefore, we can consider the problem

$$a_0(s) \frac{ds}{d\xi} = f_2(s, \theta), \quad \frac{d\theta}{d\xi} = \frac{f_1(\theta)}{\lambda_c(s, \theta)}, \quad \xi < \xi_1, \quad s(\xi_1) = s^1, \quad \theta(\xi_1) = \theta_1, \quad (29)$$

where

$$f_2 = \varphi_1 \varphi_2 \gamma \frac{p_0'}{p_0} \frac{f_1}{\lambda_c} + \frac{1}{p_0} \bar{g} \varphi_1 \varphi_2 + \frac{1}{p_0} |c| \phi^- A s.$$

Lemma 2. Let $s(\xi)$ be a solution to problem (29) and let the condition $\frac{\varphi_1(s)}{s} \gamma(s) \leq \nu_0, s \in [0, 1]$ be satisfied. Then there exists a point $\xi_* \leq \xi_1$ such that $s(\xi) \equiv 0$ for all $\xi \leq \xi_*$. If $s^1 = 0$ and additionally $\frac{\varphi_1(s)}{s} \gamma(s)|_{s=0} = 0$, then $\xi_* = \xi_1$.

Proof of Lemma 2. From (29) it follows

$$\frac{du}{d\xi} + D_1(s, \theta) = D_2(s, \theta), \quad u = \int_0^s \frac{a_0(\zeta)}{\zeta} d\zeta. \quad (30)$$

Here

$$D_1(s, \theta) = \frac{\varphi_1(s)}{s} \varphi_2 \gamma (p_0')^- \frac{f_1}{p_0 \lambda_c}, \quad f_1 = b_2(\theta - \theta^-) \geq 0,$$

$$D_2(s, \theta) = \frac{1}{p_0} (|c| \phi^- A + \bar{g} \frac{\varphi_1}{s} \varphi_2 + \frac{\varphi_1(s)}{s} \varphi_2 \gamma (p_0')^+ \frac{f_1}{\lambda_c}).$$

We have

$$D_2(s, \theta) \geq |c| \phi^- A \min_{s, \theta} \frac{\bar{\mu}_1}{p_0} \min(1, \alpha, \pi_\mu) \equiv D_2^0 > 0,$$

where $\alpha = \min_{s,\theta} \frac{\bar{\mu}_1}{\bar{\mu}_2} > 0$, $\theta \in [\theta^-, \theta^+]$, $s \in [0, 1]$. Using the representation (26) for $\theta(\xi) \in [\theta^-, \theta_1]$, we obtain

$$|D_1| \leq D_1^0 e^{-\frac{1}{\lambda_c^+}(\xi_1 - \xi)}, \quad \int_{\xi}^{\xi_1} |D_1(\zeta)| d\zeta \leq \frac{1}{\lambda_c^+} D_1^0,$$

where

$$D_1^0 = \frac{1}{\lambda_c^-} b_2(\theta_1 - \theta^-) \max_{s,\theta} \left(\frac{1}{p_0} \frac{\varphi_1(s)}{s} \varphi_2 \gamma(p_0^-) \right) \leq \frac{1}{\lambda_c^-} b_2(\theta_1 - \theta^-) v_0^3,$$

$$\lambda_c^- = a_c + b_c(\rho_2^0 \phi^-)^2, \quad \lambda_c^+ = a_c + b_c(\rho_1^0 \phi^- + \rho_3^0(1 - \phi^-))^2.$$

Integrating equation (30) over ξ from an arbitrary value ξ to ξ_1 , we obtain

$$u(s(\xi_1)) + \frac{1}{\lambda_c^+} D_1^0 \geq D_2^0(\xi_1 - \xi) + u(s(\xi)). \quad (31)$$

Here $u(s(\xi_1)) = 0$ with $s(\xi_1) = 0$ and $u(s(\xi_1)) \leq \int_0^1 \frac{a_0(\xi)}{\xi} d\xi$ for $s(\xi_1) > 0$. The last integral converges by the assumptions of Lemma 2, so $u(s(\xi)) \leq 0$ for all $\xi \leq \xi_*$, where ξ_* satisfies the condition

$$D_2^0 \xi_* = D_2^0 \xi_1 - \frac{1}{\lambda_c^+} D_1^0 - \int_0^1 \frac{a_0(\xi)}{\xi} d\xi.$$

Then it follows from the definition of $u(s)$ that $u(s(\xi)) \equiv 0$ for $\xi \leq \xi_*$. Let $s(\xi_1) = 0$, $\frac{\varphi_1(s)}{s} \gamma(s)|_{s=0} = 0$ (in this case $s(\xi) = 0$ satisfies the first equation in (29)). If $s(\xi)$ is a solution of (29), then, by Lemma 1, the functions $u(\xi), s(\xi)$ are continuous in ξ . Let us consider a small neighborhood of the point ξ_1 , assuming that at the point $\xi = \xi_1 - \delta$, $\delta > 0$ the inequality $s(\xi) > 0$ holds. For $\xi \in [\xi_1 - \delta, \xi_1]$ from equation (30) we obtain

$$\frac{du}{d\xi} \geq D_2^0 - \min_{s,\theta} ((p_0^-) \frac{f_1}{p_0 \lambda_c}) \frac{\varphi_1(s)}{s} \varphi_2 \gamma \geq \frac{1}{2} D_2^0$$

by choosing δ accordingly. Then $0 = u(s(\xi_1)) \geq \frac{1}{2} D_2^0 \delta + u(s(\xi))$, i.e. $u(s(\xi)) < 0$ and, therefore, $s(\xi) = 0$. Repeating this process, at the k -th step we obtain $s(\xi_k) = 0$, $\xi_k = \xi_1 - k\delta$, $k > 1$. When k reaches a value for which the inequality $D_1^0 \leq \lambda_c D_2^0 k\delta$ is satisfied, using (31), we obtain $s(\xi_k) = 0$, $\xi \in (-\infty, \xi_1]$. The lemma is proved. \square

4. Numerical Investigation

In the one-dimensional self-similar case, the system (1-3) is written in the form (8-13). Equations (7) for θ and (24) for s , which will be calculated numerically, will take the form:

$$\frac{d\theta}{d\xi} = (c\rho_3^0(c_1 - c_3)M(\theta) + (A_1c_1 + A_2c_2)(\theta - \theta^-) - v\rho_3^0c(\phi - \phi^-))/\lambda_c \equiv F_1(\theta), \quad (32)$$

$$\frac{ds}{d\xi} = (c\varphi_1\varphi_2\gamma\frac{p_0^0}{p_0}\frac{f_1}{\lambda_c} + \frac{1}{p_0}\bar{g}\varphi_1\varphi_2 + \frac{1}{p_0}|c|\phi As - \frac{1}{p_0}|c|(\phi - \phi^-)B)/a_0(s) \equiv F_2(s, \theta). \quad (33)$$

Since the coefficient $a_0(s)$ can turn to zero, then when dividing by it, ε is added and $\max(a_0(s), \varepsilon)$ is taken.

Algorithm for numerical solution of self-similar problem

Let us introduce a grid $\xi = ih$, $i = 0, \dots, N$, h — step along the spatial coordinate.

We approximate the Newton's boundary condition:

$$(\lambda_c \frac{\theta_1 - \theta_0}{h} + \alpha(\theta_0 - \theta^0) - Q) = 0. \quad (34)$$

Equation (32) is approximated by an implicit Runge-Kutta scheme of fourth order accuracy [20]

$$\theta_{i+1} = \theta_i + (h/6)(k_1 + 2k_2 + 2k_3 + k_4), \quad (35)$$

where $k_1 = hF_1(\theta_i)$, $k_2 = hF_1(\theta_i + 0.5k_1)$, $k_3 = hF_1(\theta_i + 0.5k_2)$,
 $k_4 = hF_1(\theta_i + k_3)$.

From here

$$\theta_1 = \theta_0 + (h/6)(k_1 + 2k_2 + 2k_3 + k_4). \quad (36)$$

To find $\theta(0)$ the nonlinear equations (32-33) were solved by successive approximations with an inner cycle over k :

1) assuming $k = 0$ for the first approximation, θ_0 was taken equal to θ^+ , then θ_1 was found from (36);

2) in the next iteration $k = k + 1$, θ_1 was substituted into (34), where θ_0 was found from the boundary condition;

3) steps 1-2 are repeated until the conditions $\|\theta_0^{k+1} - \theta_0^k\| \leq \varepsilon$ and $\|\theta_1^{k+1} - \theta_1^k\| \leq \varepsilon$, where $\varepsilon = 10^{-6}$, are satisfied.

To plot the graphs of s and θ from the first two equations of the system (18-20), we express

$$s(0) = \frac{(\phi(\theta(0)) - \phi^-)(v_1^+ + \frac{\rho_3^0}{\rho_1^0} v_2^+)}{\phi(\theta(0))(v_1^+ + v_2^+)}, c = \frac{v_1^+ + v_2^+}{(\phi(\theta(0)) - \phi^-)(1 - \rho_3^0/\rho_1^0)},$$

that is, case 3 of the solution to the system is considered (18-20) for $v_1^+ < 0$, $v_2^+ < 0$.

The velocity c is substituted in the equation for the temperature (32), which is considered to be the Runge-Kutta method of the 4th order of accuracy.

The numerical algorithm for the temperature equation was verified by computational experiments. To simplify the calculations, $F_1 = \theta$ was taken as a trial temperature function. An exact solution for this equation was analytically found. Runge's rule was applied: three calculations were performed on grids with steps $h_1 = h = 0.1$, $h_2 = h/2 = 0.05$, $h_3 = h/4 = 0.025$.

The equation for water saturation (33) is also approximated by an implicit Runge-Kutta scheme of fourth order accuracy

$$s_{i+1} = s_i + (h/6)(k_1 + 2k_2 + 2k_3 + k_4),$$

where $k_1 = hF_2(s_i, \theta_i)$, $k_2 = hF_2(s_i + 0.5k_1, \theta_i)$, $k_3 = hF_2(s_i + 0.5k_2, \theta_i)$,
 $k_4 = hF_2(s_i + k_3, \theta_i)$.

s_0 is calculated using the formula $s_0 = \frac{(\phi(\theta_0) - \phi^-)(v_1^+ + \frac{\rho_3^0}{\rho_1^0} v_2^+)}{\phi(\theta_0)(v_1^+ + v_2^+)}$, taking into account the found θ_0 .

Figure 1 shows the results of the numerical modeling of temperature and water saturation with depth for the following parameters: $h = 0.01$, $Q = 7000$, $\alpha = 0.2$, $\lambda_c = 40$, $\phi^- = 0.62$, $v_1^+ = v_2^+ = -0.000001$, $\theta^- = 268.15$, $\theta_1 = 270.15$, $\theta^+ = 273.15$, $\theta^0 = 278.15$. The graph of the porosity function is shown in Figure 2.

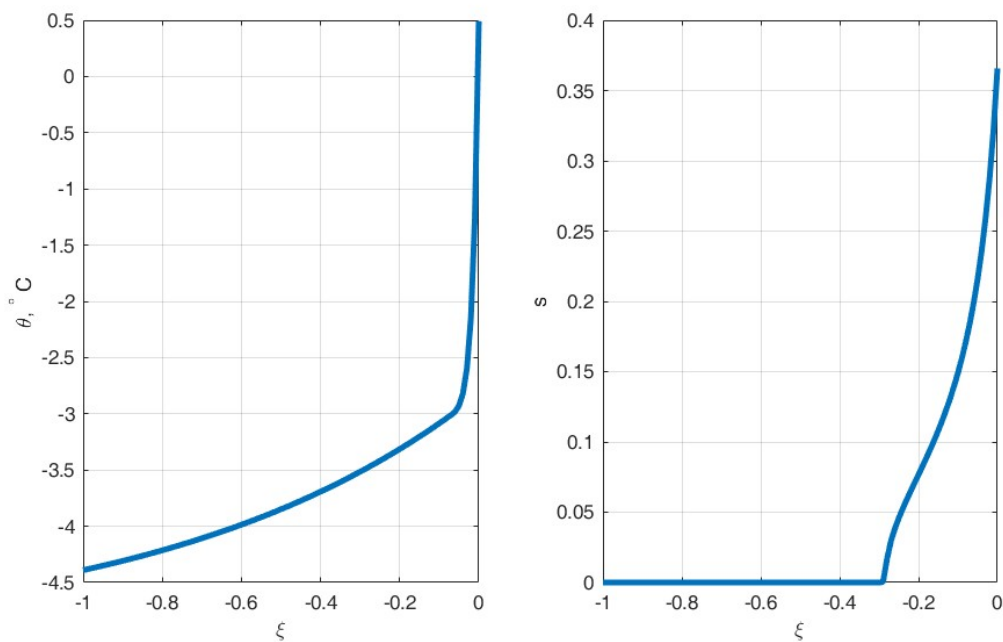


Figure 1. Change in temperature and water saturation

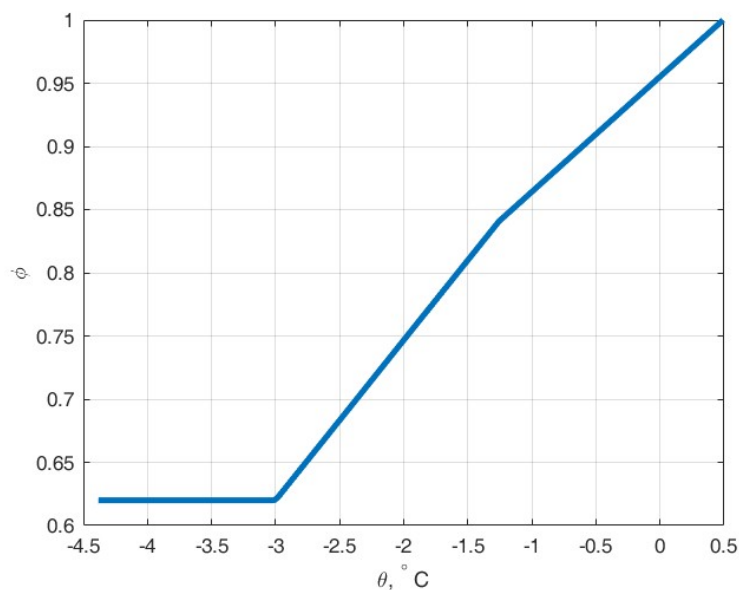


Figure 2. Change in porosity.

Figure 3 shows the effect of a given heat flux Q on the temperature at the boundary. It is evident that with increasing Q , the temperature of the upper snow layer also increases. Water saturation remains unchanged.

The results of modeling temperature variations with depth as a function of thermal conductivity λ_c are presented in Figure 4. The graph shows that thermal conductivity directly influences snow heating: the higher the value of λ_c , the more intense the snow heating. Minor changes in water saturation are observed.

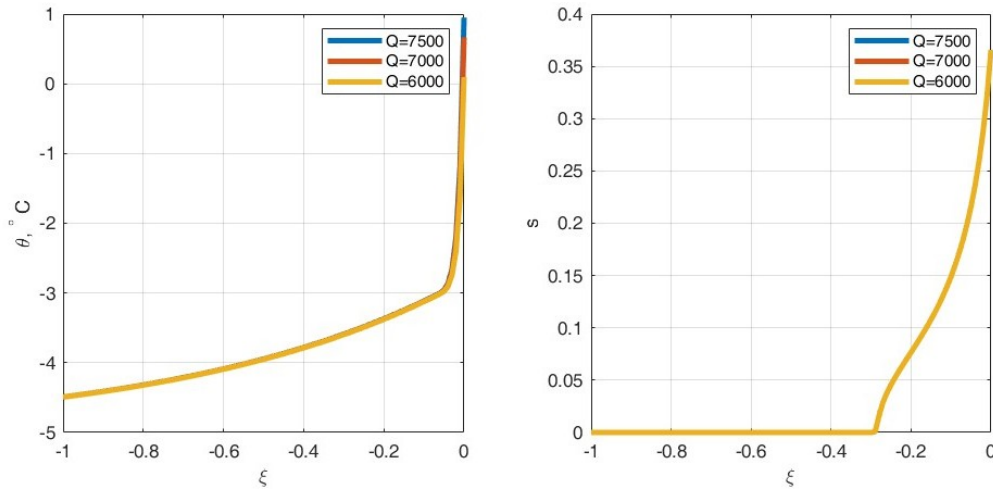


Figure 3. Effect of heat flux Q on boundary temperature at $Q = 7500, Q = 7000, Q = 6000, \lambda_c = 35$.

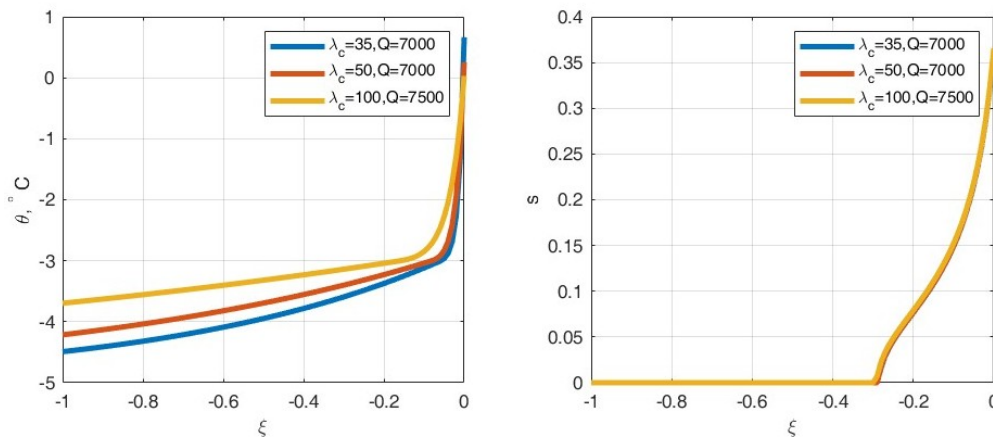


Figure 4. Effect of thermal conductivity coefficient λ_c on temperature at $\lambda_c = 35, \lambda_c = 50, \lambda_c = 100$

The specified porosity ϕ^- influences water saturation at the boundary (Figure 5), as is evident from the formula for $s(0)$. The calculations were performed with the specified parameters $\lambda_c = 35, Q = 7000$, and three different values of ϕ^- . The temperature remains unchanged.

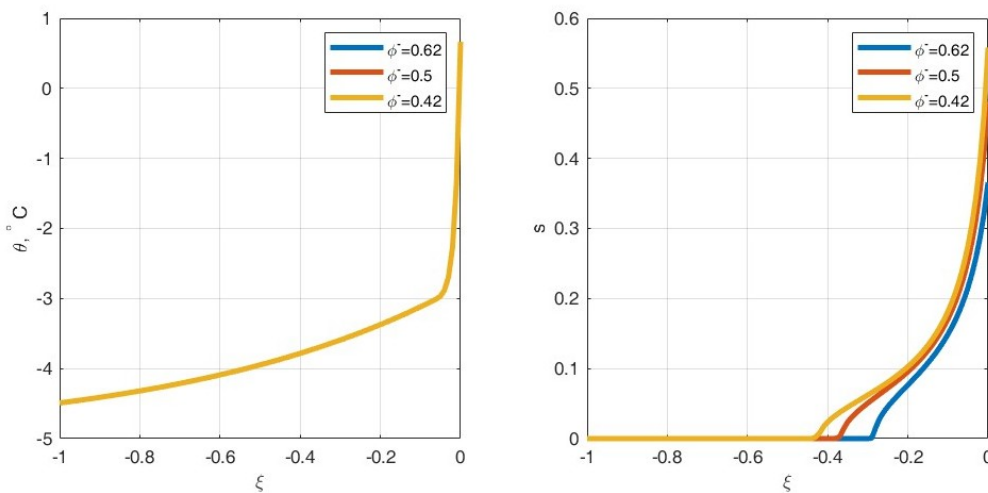


Figure 5. Results of numerical modeling of temperature and water saturation changes for different values of ϕ^-

5. Conclusions

An exact self-similar solution to the problem of heat and mass transfer in snow and ice cover is constructed, taking into account variable porosity and phase transitions. The solution takes into account the influence of solar radiation flux at the boundary. For water saturation, the maximum principle and the lemma on the finite propagation velocity of disturbances are proven. A theorem on the existence of a weak solution to the problem is formulated. A numerical study of the problem of heat and mass transfer in snow is conducted, taking into account the external heat flux. The results of numerical modeling of temperature and water saturation are presented. The influence of a given heat flux at the boundary and the thermal conductivity coefficient on the temperature distribution over depth is demonstrated. The program is written in C++.

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Appendix A

Table A1. Parameter values.

Parameter	Value
true density of water, ρ_1^0	1000 [kg/m ³]
true density of air, ρ_2^0	1.2928 [kg/m ³]
true density of ice, ρ_3^0	925 [kg/m ³]
heat capacity of water at constant pressure, c_1	4187 [m ² /s ² K]
heat capacity of air at constant pressure, c_2	1005 [m ² /s ² K]
heat capacity of ice at constant pressure, c_3	[m ² /s ² K]
specific heat of melting of ice, ν	335000 [m ² /s ²]
boundary value of temperature, θ^-	268.15 [K]
freezing point of water, θ_1	270.15 [K]
melting point of ice, θ^+	273.15 [K]
water filtration rate, v_1	$c\phi s + c \frac{\rho_3^0}{\rho_1^0} (\phi^- - \phi)$
air filtration rate, v_2	$c\phi(1-s) - c\phi^-$
air pressure, p_2	1.0332 [kg/ms ²]
filtration tensor, K_0	10 ⁻⁹ [m ²]
dynamic viscosity coefficient of water, μ_1	0.17865 [kg/ms]
relative phase permeability of water, k_{01}	1
relative phase permeability of air, k_{02}	1
acceleration vector of gravity, \vec{g}	9.8 [m/s ²]
snow porosity at temperature $\theta = \theta^-$, ϕ^-	0.62
snow porosity at temperature $\theta = \theta_1$, ϕ_1	$(1 - \phi^-) / (\theta^+ - \theta_1)$
snow porosity, ϕ	$\phi^- + (\theta - \theta_1) \left(\frac{1 - \phi^-}{\theta^+ - \theta_1} \right)$
a_0	$-\varphi_1 \varphi_2 \gamma'$
thermal conductivity of snow, λ_c	$7 \cdot 10^{-5} + 10^{-3} \rho_c^2$ [kg m/s ³ K]
snow density, ρ_c	$\rho_1^0 \phi s + \rho_2^0 \phi(1-s) + \rho_3^0(1-\phi)$
reduced relative phase permeability of water, φ_1	s^n
reduced relative phase permeability of air, φ_2	1 - s

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