Concept Paper

A Theta–Regularized Identity for SL_2 and a Fejér–Windowed Strip Bridge for Log|ξ|

Parker Emmerson *

Posted Date: 1 December 2025

doi: 10.20944/preprints202512.0047.v1

Keywords: Riemann zeta function; completed zeta function; Eisenstein series; theta kernel; Rallis inner product formula; Poisson kernel on strips; Fejér kernel; spectral analysis

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 December 2025 doi:10.20944/preprints202512.0047.v1

Concept Paper

A Theta–Regularized Identity for SL_2 and a Fejér–Windowed Strip Bridge for $\log|\xi|$

Parker Emmerson (Yaohushuaon)

Independent Researcher, USA; parkeremmerson@icloud.com

Abstract

A theta–regularized inner product identity in rank one is established, linking a mixed theta–weighted Eisenstein pairing on $\Gamma\backslash\mathbb{H}$ to the σ–derivative of $\log|\xi(s)|$, up to explicit Euler factor correction terms arising from the $G \times G$ doubling formalism. More precisely, for $s = \frac{1}{2} + \sigma + \mathrm{i}t$ it is shown that $\frac{\partial}{\partial\sigma} \log\left|\langle \Theta(\cdot)E(\cdot, s),\, \Theta(\cdot)E(\cdot, 1-\bar{s}) \rangle_{\mathrm{reg}}\right| = 2\,\mathrm{Re}\,\frac{\xi'(s)}{\xi(s)} - 2\,\mathrm{Re}\,\frac{\zeta'(2s)}{\zeta(2s)} + 2\,\mathrm{Re}\,\frac{\zeta'(2-2\bar{s})}{\zeta(2-2\bar{s})}$, as an identity of tempered distributions in t. On the critical line $\sigma = 0$ the Euler corrections cancel and a particularly simple formula is obtained: $\frac{\partial}{\partial\sigma} \log \langle \Theta(\cdot)E(\cdot, \frac{1}{2} + \sigma + \mathrm{i}t),\, \Theta(\cdot)E(\cdot, \frac{1}{2} - \sigma + \mathrm{i}t) \rangle_{\mathrm{reg}}\Big|_{\sigma=0} = 2\,\frac{\partial}{\partial\sigma} \log\left|\xi\left(\frac{1}{2} + \sigma + \mathrm{i}t\right)\right|\Big|_{\sigma=0}$. Fejér–windowed versions of these identities are then obtained, and a Fejér–windowed "strip bridge" is proved: a harmonic operator identity expressing the short–band component of $\partial_\sigma \log|\xi(1/2 + \sigma + \mathrm{i}t)|$ at an interior latitude via a linear combination of Fejér–smeared edge data, with a power–saving $O(H^{-\eta})$ remainder after short–band freezing, uniformly for $|\sigma^\star| \geq \sigma_0 > 0$. A sharp truncation stability result is also established. After subtracting the finitely many Zagier–Arthur cusp counterterms, the Fejér–smeared σ–derivative of the logarithm of the truncated mixed theta–Eisenstein pairing agrees with its regularized version up to $O(H^{-A})$ for any prescribed $A > 0$, provided the truncation height $Y = H^{B(A)}$ is chosen sufficiently large. A brief discussion is included of numerical checks in a sample region, and a short Fourier–analytic proof note is given for the renormalization estimate that underlies the strip bridge.

Keywords: Riemann zeta function; completed zeta function; Eisenstein series; theta kernel; Rallis inner product formula; Poisson kernel on strips; Fejér kernel; spectral analysis

MSC: 11M26; 11F66; 11F70; 42B10

---

1. Introduction and Main Results

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $X = \Gamma\backslash\mathbb{H}$ be the modular surface, with $z = x + \mathrm{i}y$, $y > 0$, and hyperbolic measure $d\mu(z) = y^{-2}\,dx\,dy$. Let $E(z, s)$ denote the spherical Eisenstein series at the cusp ∞ for $G = \mathrm{SL}_2(\mathbb{R})$, normalized so that its constant term at ∞ is

$$E(z, s) = y^s + \phi(s) y^{1-s} + \ldots, \tag{1}$$

where $\phi(s)$ is the scattering coefficient. For the standard normalization (see, e.g., [20, Chap. 3]),

$$\phi(s) = \frac{\xi(2s-1)}{\xi(2s)} = \pi^{\frac{1}{2}} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}, \tag{2}$$

with $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ the completed Riemann zeta function.

The real–analytic Jacobi theta kernel is

$$\theta(z) = \sum_{m,n\in\mathbb{Z}} \exp\left(-\pi \frac{|mz+n|^2}{y}\right), \qquad z = x + \mathrm{i}y, \tag{3}$$

and we write $\Theta(z) := \theta(z) - 1$. Since Θ is x–periodic and grows like $\sqrt{y}$ in the cusp (see §2.2), one must employ the regularized inner products in the sense of Zagier and Arthur for pairings involving Θ and Eisenstein series.

1.1. *Theta–Weighted Mixed Energy and the Main Identity*

Let $s = \frac{1}{2} + \sigma + \mathrm{i}t$ with $\sigma, t \in \mathbb{R}$. With the standard L^2 pairing $\langle f, g \rangle = \int_X f(z)\,\overline{g(z)}\,d\mu(z)$ and its Zagier–Arthur regularized extension, consider the mixed theta–weighted energy

$$\mathcal{J}_\Theta(s) := \big\langle \Theta(\cdot)\,E(\cdot, s),\ \Theta(\cdot)\,E(\cdot, 1-\bar{s}) \big\rangle_{\mathrm{reg}}. \tag{4}$$

The choice $s' = 1 - \bar{s}$ ensures compatibility with $\xi(1-\bar{s}) = \overline{\xi(s)}$ and produces a nontrivial t–dependence.

Our first main result is an explicit identity for the σ–derivative of $\log|\mathcal{J}_\Theta(s)|$.

Theorem 1 (Theta–regularized identity)**.** *Fix Tamagawa measures and spherical local data. For all $\sigma, t \in \mathbb{R}$, in the sense of tempered distributions in t and pointwise away from zeros and poles, one has*

$$\frac{\partial}{\partial\sigma}\log\Big|\mathcal{J}_\Theta\Big(\tfrac{1}{2}+\sigma+\mathrm{i}t\Big)\Big| = 2\,mathrmRe\,\frac{\xi'(\frac{1}{2}+\sigma+\mathrm{i}t)}{\xi(\frac{1}{2}+\sigma+\mathrm{i}t)} \;-\; 2\,\mathrm{Re}\,\frac{\zeta'(1+2\sigma+2\mathrm{i}t)}{\zeta(1+2\sigma+2\mathrm{i}t)} \;+\; 2\,\mathrm{Re}\,\frac{\zeta'(1-2\sigma+2\mathrm{i}t)}{\zeta(1-2\sigma+2\mathrm{i}t)}. \tag{5}$$

Equivalently,

$$\frac{\partial}{\partial\sigma}\log|\mathcal{J}_\Theta(s)| = 2\,\frac{\partial}{\partial\sigma}\log|\xi(s)| - \frac{\partial}{\partial\sigma}\mathrm{Re}\log\zeta(2s) - \frac{\partial}{\partial\sigma}\mathrm{Re}\log\zeta(2-2\bar{s}),$$

with $s = \frac{1}{2} + \sigma + \mathrm{i}t$.

Remark 1. *On the critical line $\sigma = 0$ one has $2s = 2 - 2\bar{s} = 1 + 2\mathrm{i}t$, so the last two terms in (5) cancel identically and*

$$\frac{\partial}{\partial\sigma}\log\Big|\mathcal{J}_\Theta\Big(\tfrac{1}{2}+\sigma+\mathrm{i}t\Big)\Big|\Big|_{\sigma=0} = 2\,\frac{\partial}{\partial\sigma}\log\Big|\xi\Big(\tfrac{1}{2}+\sigma+\mathrm{i}t\Big)\Big|\Big|_{\sigma=0}.$$

Thus the theta–regularized mixed energy has a particularly transparent logarithmic derivative along the critical line.

Remark 2 (Growth of logarithmic derivatives and temperedness)**.** *Standard estimates (see, for example, [14, Ch. 3–5] or [41, Ch. 8]) show that for any fixed compact interval $\sigma \in [\sigma_1, \sigma_2]$,*

$$\frac{\xi'\big(\frac{1}{2}+\sigma+\mathrm{i}t\big)}{\xi\big(\frac{1}{2}+\sigma+\mathrm{i}t\big)},\ \frac{\zeta'(1+2\sigma+2\mathrm{i}t)}{\zeta(1+2\sigma+2\mathrm{i}t)},\ \frac{\zeta'(1-2\sigma+2\mathrm{i}t)}{\zeta(1-2\sigma+2\mathrm{i}t)} \;=\; O\big(\log(|t|+2)\big)$$

as $|t| \to \infty$. In particular, each term on the right–hand side of (5) defines a tempered distribution in t, justifying the distributional interpretation of Theorem 1.

1.2. *Fejér–Windowed Version*

The Fourier transform convention $\widehat{f}(\lambda) = \int_{\mathbb{R}} f(u)\,\mathrm{e}^{-\mathrm{i}\lambda u}\,du$ is used. Let K_H denote the Fejér kernel

$$K_H(u) := \frac{H}{2\pi}\left(\frac{\sin(Hu/2)}{Hu/2}\right)^2, \qquad \widehat{K}_H(\lambda) = \left(1 - \frac{|\lambda|}{H}\right)_+. \tag{6}$$

Then $K_H \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} K_H(u)\,du = 1$, and $\widehat{K}_H$ is supported in $[-H, H]$.

Theorem 2 (Fejér–windowed theta identity). *Let $H \geq 1$ and K_H be as in (6). For any $\sigma^\star, t_0 \in \mathbb{R}$,*

$$\frac{\partial}{\partial\sigma}\int_{\mathbb{R}} K_H(t-t_0)\,\log\big|\mathcal{J}_\Theta\big(\tfrac{1}{2}+\sigma+\mathrm{i}t\big)\big|\,dt\Big|_{\sigma=\sigma^\star} = 2\int_{\mathbb{R}} K_H(t-t_0)\,\mathrm{Re}\,\frac{\xi'(\frac{1}{2}+\sigma^\star+\mathrm{i}t)}{\xi(\frac{1}{2}+\sigma^\star+\mathrm{i}t)}\,dt$$
$$-2\int_{\mathbb{R}} K_H(t-t_0)\,\mathrm{Re}\,\frac{\zeta'(1+2\sigma^\star+2\mathrm{i}t)}{\zeta(1+2\sigma^\star+2\mathrm{i}t)}\,dt$$
$$+2\int_{\mathbb{R}} K_H(t-t_0)\,\mathrm{Re}\,\frac{\zeta'(1-2\sigma^\star+2\mathrm{i}t)}{\zeta(1-2\sigma^\star+2\mathrm{i}t)}\,dt.$$

All integrals are to be interpreted as pairings with tempered distributions and are valid pointwise away from zeros and poles.

1.3. Fejér–Windowed Strip Bridge for $\log|\xi|$

An independent harmonic analysis argument is used to prove a "strip bridge" for $u(\sigma,t) = \log|\xi(1/2+\sigma+\mathrm{i}t)|$, expressing the short–band component of the interior normal derivative $\partial_\sigma u$ in terms of boundary data.

Fix $h \in (0,1/2]$ and set

$$u(\sigma,t) := \log\Big|\xi\Big(\tfrac{1}{2}+\sigma+\mathrm{i}t\Big)\Big|, \qquad F_\pm(t) := \partial_\sigma u(\pm h,t).$$

Write $F_{\mathrm{ev}} = \frac{1}{2}(F_+ + F_-)$ and $F_{\mathrm{odd}} = \frac{1}{2}(F_+ - F_-)$. Fix a short Fourier band

$$B_{\lambda_0,\Delta} = \{\lambda\in\mathbb{R} : \,|\lambda\pm\lambda_0| \leq 2\Delta\}, \qquad \lambda_0 \asymp T, \quad \Delta = T^{-1-\varepsilon},$$

with small fixed $\varepsilon > 0$, and let P_Δ be the corresponding frequency projection.

Theorem 3 (Fejér–windowed strip bridge). *Fix $\sigma_0 \in (0,h)$ and let $H \asymp T$ be chosen so that $|\lambda_0| \leq (1-c)H$ for some fixed $c \in (0,1)$, hence $B_{\lambda_0,\Delta} \subset [-H,H]$. For any fixed $\sigma^\star \in [-h,h]$ with $|\sigma^\star| \geq \sigma_0$ and any $t_0 \in \mathbb{R}$ with $|t_0| \leq CH$, one has*

$$\int_{\mathbb{R}} K_H(t-t_0)\,(P_\Delta\partial_\sigma u)(\sigma^\star,t)\,dt = \widetilde{m}_{\mathrm{ev}}(\lambda_0;\sigma^\star)\,(K_H * P_\Delta F_{\mathrm{ev}})(t_0)$$
$$+\,\widetilde{m}_{\mathrm{odd}}(\lambda_0;\sigma^\star)\,(K_H * P_\Delta F_{\mathrm{odd}})(t_0) \;+\; O(H^{-\eta}), \tag{7}$$

uniformly for $|t_0| \leq CH$, where the frozen strip Poisson multipliers are

$$\widetilde{m}_{\mathrm{ev}}(\lambda_0;\sigma^\star) = \frac{\cosh(\lambda_0\,\sigma^\star)}{\cosh(\lambda_0 h)}, \qquad \widetilde{m}_{\mathrm{odd}}(\lambda_0;\sigma^\star) = \frac{\sinh(\lambda_0\,\sigma^\star)}{\sinh(\lambda_0 h)},$$

and $\eta > 0$ depends only on ε. The error $O(H^{-\eta})$ arises from short–band freezing and from local renormalization of nearby poles and zeros, after Fejér smoothing and application of P_Δ, uniformly for $|\sigma^\star| \geq \sigma_0$.

Remark 3 (Dependence on the horizontal zero geometry). *In the proof of Theorem 3, the renormalization error arising from $R^{(1)}_{t_0}$ admits the explicit bound*

$$\int_{\mathbb{R}} K_H(t-t_0)\,(P_\Delta R^{(1)}_{t_0})(\sigma^\star,t)\,dt \;\ll\; \sum_{\rho:\,|\Im\rho-t_0|\leq 2H} m_\rho\,\mathrm{e}^{-c\,H\,|\sigma^\star-(\Re\rho-\frac{1}{2})|},$$

where the sum runs over the zeros and the pole of ξ, counted with multiplicity m_ρ, and $c > 0$ depends only on h. Combining this with $N(T) \ll T\log T$ and the band width $\Delta = T^{-1-\varepsilon}$ yields the uniform bound $O(H^{-\eta})$, for some $\eta > 0$ depending only on ε. Under the Riemann Hypothesis, $\Re\rho = \frac{1}{2}$ for all zeros, so $|\sigma^\star - (\Re\rho - \frac{1}{2})| = |\sigma^\star| \geq \sigma_0$, and the sum is $\ll \mathrm{e}^{-c'\sigma_0 H}$; the error is then exponentially small in H (hence $O(H^{-A})$ for every $A > 0$).

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 December 2025 doi:10.20944/preprints202512.0047.v1

2. Background: Eisenstein Series, Maass–Selberg, Theta Kernel, Weil Representation

2.1. Eisenstein Series and Maass–Selberg

Let $G = \mathrm{SL}_2$, let B denote the upper Borel, and let $I(s) = \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(|\cdot|^s)$. Let $f_s = \otimes_v f_{s,v}$ be the spherical section with $f_{s,\infty}(g_z) = y^s$ on $g_z = n(x)a(y)$, and $f_{s,v}(k_v) = 1$ at finite v. Then

$$E(g,s) := \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} f_s(\gamma g)$$

defines an Eisenstein series convergent for $\Re s > 1$ and meromorphic in s. On X, $E(z,s) = E(g_z,s)$ has the Fourier expansion

$$E(z,s) = y^s + \phi(s)y^{1-s} + \sum_{n \neq 0} a_n(s)\sqrt{y}\,K_{s-\frac{1}{2}}(2\pi|n|y)\,e(nx),$$

with $\phi(s)$ as in (2).

The Maass–Selberg relation (see e.g. [18,20]) gives, for the truncated domain $\mathcal{F}_Y$,

$$\int_{\mathcal{F}_Y} |E(z,\tfrac{1}{2}+\mathrm{i}t)|^2\,d\mu(z) = c_0 \log Y + c_1(t) + R(Y,t), \tag{8}$$

where $c_0 > 0$ is absolute, $c_1(t)$ is bounded in t and expressible in terms of $\phi'(1/2+\mathrm{i}t)/\phi(1/2+\mathrm{i}t)$, and $R(Y,t) = O(Y^{-c})$ uniformly in t for some $c > 0$.

2.2. Theta Kernel and Cusp Asymptotics

For $\theta(z)$ as in (3), Poisson summation in n yields

$$\theta(z) = \sqrt{y} \sum_{m,k \in \mathbb{Z}} e^{-\pi(m^2+k^2)y}\,e^{2\pi\mathrm{i}kmx}.$$

As $y \to \infty$, only $(m,k) = (0,0)$ contributes significantly, hence

$$\theta(z) = \sqrt{y} + O\big(\sqrt{y}\,e^{-\pi y}\big), \qquad \Theta(z) = \sqrt{y} - 1 + O\big(\sqrt{y}\,e^{-\pi y}\big). \tag{9}$$

Thus $\Theta(z)$ grows like $\sqrt{y}$ in the cusp.

2.3. Weil Representation and the Theta Lift

Let V be the split binary quadratic space over $\mathbb{Q}$ with $q(x) = x_1x_2$. In the Schrödinger model of the Weil representation ω for $(G, \mathrm{O}(V))$ on $(V(\mathbb{A}))$, for $\varphi = \otimes_v \varphi_v$ define

$$\theta(g,h;\varphi) := \sum_{x \in V(\mathbb{Q})} \omega(g,h)\varphi(x).$$

With $\varphi_p = \mathbf{1}_{V(\mathbb{Z}_p)}$ for $p < \infty$ and $\varphi_\infty(x) = e^{-\pi(x_1^2+x_2^2)}$, one verifies that $\theta(g_z,1;\varphi) = \theta(z)$ and hence the regularized theta lift (in the sense of Kudla–Rallis and Ichino)

$$\Theta_\varphi(g) := \int^{\mathrm{reg}}_{\mathrm{O}(V)(\mathbb{Q})\backslash\mathrm{O}(V)(\mathbb{A})} \theta(g,h;\varphi)\,dh$$

satisfies $\Theta_\varphi(g_z) = \Theta(z)$.

3. Local Doubling Zeta Integrals and Global Completion

Let f_s be as in §2.1. The local doubling zeta integrals are

$$Z_v(s, \varphi_v, f_{s,v}) := \int_{G(\mathbb{Q}_v)} f_{s,v}(g)\, \langle \omega_v(g)\varphi_v, \varphi_v \rangle\, dg.$$

For $p < \infty$ unramified, $Z_p(s, \varphi_p, f_{s,p}) = \zeta_p(s)$, and for $v = \infty$ there is a (unique up to scalar) spherical choice of $f_{s,\infty}$ for which $Z_\infty(s, \varphi_\infty, f_{s,\infty}) = \pi^{-s/2}\Gamma(s/2)$ (see [56,58,60] and Appendix B). Thus, with Tamagawa measures and compatible local data,

$$\prod_v Z_v(s, \varphi_v, f_{s,v}) = \xi(s).$$

4. Rallis Inner Product Formula and the Theta Identity

For $s, s' \in$ with $\Re s, \Re s'$ sufficiently large, the regularized Siegel–Weil/doubling method yields the Rallis inner product formula (see [56,58,60]):

Theorem 4 (Rallis inner product formula)**.** *With Tamagawa measures and spherical local data, one has*

$$\left\langle \Theta_\varphi E(\cdot, s),\ \Theta_\varphi E(\cdot, s') \right\rangle_{\mathrm{reg}} = C_{\mathrm{glob}} \frac{\zeta(s + s')}{\zeta(2s)\zeta(2s')}\, \xi(s)\, \xi(s'), \tag{10}$$

as a meromorphic identity in $(s, s') \in^2$, where $C_{\mathrm{glob}} \neq 0$ is an absolute constant independent of s, s'.

Remark 4. *All subsequent considerations involve $\partial/\partial\sigma$ of logarithms, so the constant C_{glob} is irrelevant for the identities derived.*

Specializing to $s' = 1 - \bar{s}$ and using $\xi(1 - \bar{s}) = \overline{\xi(s)}$ yields

$$\mathcal{J}_\Theta(s) = \langle \Theta(\cdot)E(\cdot, s),\ \Theta(\cdot)E(\cdot, 1 - \bar{s}) \rangle_{\mathrm{reg}} = C_{\mathrm{glob}} \frac{\zeta(1 + 2it)}{\zeta(2s)\zeta(2 - 2\bar{s})}\, |\xi(s)|^2. \tag{11}$$

Proof of Theorem 1. Taking absolute values in (11) gives

$$|\mathcal{J}_\Theta(s)| = |C_{\mathrm{glob}}| \frac{|\zeta(1 + 2it)|}{|\zeta(2s)|\, |\zeta(2 - 2\bar{s})|}\, |\xi(s)|^2.$$

Taking real logarithms and differentiating in σ yields

$$\frac{\partial}{\partial\sigma} \log |\mathcal{J}_\Theta(s)| = -2\,\mathrm{Re}\frac{\zeta'(2s)}{\zeta(2s)} + 2\,\mathrm{Re}\frac{\zeta'(2 - 2\bar{s})}{\zeta(2 - 2\bar{s})} + 2\,\mathrm{Re}\,\frac{\xi'(s)}{\xi(s)},$$

since both $|C_{\mathrm{glob}}|$ and $\zeta(1 + 2it)$ are independent of σ. Writing $2s = 1 + 2\sigma + 2it$ and $2 - 2\bar{s} = 1 - 2\sigma + 2it$ yields (5). The interpretation as an identity of tempered distributions follows from Remark 2. □

Remark 5. *The factor $\zeta(s + s')/(\zeta(2s)\zeta(2s'))$ in (10) is intrinsic to the doubling construction. After specializing $s' = 1 - \bar{s}$, the numerator becomes $\zeta(1 + 2it)$, independent of σ, whereas the denominators $\zeta(2s)$ and $\zeta(2 - 2\bar{s})$ retain nontrivial σ–dependence and produce the correction terms in (5). These Euler corrections cancel only on the critical line.*

Proof of Theorem 2. Convolution with K_H is continuous on $\mathcal{S}'(\mathbb{R})$ and commutes with ∂_σ. Integrating (5) against $K_H(t - t_0)$ in t and evaluating at $\sigma = \sigma^\star$ gives the asserted identity. □

5. Strip Poisson Calculus and Short–Band Freezing

Let $h \in (0, 1/2]$ be fixed. For the purposes of this section, we regard $u(\sigma,t) = \log|\xi(\frac{1}{2} + \sigma + it)|$ as a tempered distribution in t for each fixed σ, as justified by Remark 2. Away from zeros and the pole of ξ, u is a genuine harmonic function of (σ,t) on vertical strips.

5.1. Strip Poisson Multipliers

Lemma 1 (Strip Poisson multipliers for the normal derivative). *Let $u(\sigma,t)$ be harmonic on $|\sigma| \le h$. For any $\sigma^\star \in (-h,h)$ and $\lambda \in \mathbb{R}$, with $F_\pm(t) = \partial_\sigma u(\pm h, t)$ and*

$$F_{\mathrm{ev}} = \frac{F_+ + F_-}{2}, \qquad F_{\mathrm{odd}} = \frac{F_+ - F_-}{2},$$

one has

$$\widehat{\partial_\sigma u}(\sigma^\star,\lambda) = \widetilde{m}_{\mathrm{ev}}(\lambda;\sigma^\star)\,\widehat{F_{\mathrm{ev}}}(\lambda) + \widetilde{m}_{\mathrm{odd}}(\lambda;\sigma^\star)\,\widehat{F_{\mathrm{odd}}}(\lambda),$$

where

$$\widetilde{m}_{\mathrm{ev}}(\lambda;\sigma^\star) = \frac{\cosh(|\lambda|\sigma^\star)}{\cosh(|\lambda|h)}, \qquad \widetilde{m}_{\mathrm{odd}}(\lambda;\sigma^\star) = \frac{\sinh(|\lambda|\sigma^\star)}{\sinh(|\lambda|h)}.$$

In particular, as $|\lambda| \to 0$,

$$\widetilde{m}_{\mathrm{ev}}(\lambda;\sigma^\star) \to 1, \qquad \widetilde{m}_{\mathrm{odd}}(\lambda;\sigma^\star) \to \frac{\sigma^\star}{h}.$$

Moreover, for each fixed $\sigma^\star \in (-h,h)$,

$$\left|\partial_\lambda \widetilde{m}_{\mathrm{ev/odd}}(\lambda;\sigma^\star)\right| \ll e^{-(h-|\sigma^\star|)\,|\lambda|}\,(1+|\lambda|), \qquad \lambda \in \mathbb{R},$$

with an implied constant depending only on h.

Proof. Fourier transforming in t gives $(\partial_\sigma^2 - \lambda^2)\widehat{u} = 0$, hence $\widehat{u}(\sigma,\lambda) = Ae^{|\lambda|\sigma} + Be^{-|\lambda|\sigma}$. Matching the normal derivatives at $\sigma = \pm h$ and solving for A, B produces the stated formula. The bounds on $\partial_\lambda \widetilde{m}_{\mathrm{ev/odd}}$ follow by differentiating these explicit expressions and using $\cosh(|\lambda|h), \sinh(|\lambda|h) \asymp e^{|\lambda|h}$ as $|\lambda| \to \infty$. ☐

5.2. Short–Band Freezing

Lemma 2 (Short–band freezing). *Let $B_{\lambda_0,\Delta}$ be the band $\{|\lambda \pm \lambda_0| \le 2\Delta\}$ with $\lambda_0 \asymp T$ and $\Delta = T^{-1-\varepsilon}$, and let P_Δ be the corresponding spectral projector. Then, uniformly for $\sigma^\star$ in compact subsets of $(-h,h)$,*

$$\sup_{\lambda \in B_{\lambda_0,\Delta}} |\widetilde{m}_{\mathrm{ev/odd}}(\lambda;\sigma^\star) - \widetilde{m}_{\mathrm{ev/odd}}(\lambda_0;\sigma^\star)| \ll \Delta,$$

and for any $f \in L^2(\mathbb{R})$ and $t_0 \in \mathbb{R}$,

$$|(\widetilde{m}_{\mathrm{ev/odd}}(D_t;\sigma^\star) - \widetilde{m}_{\mathrm{ev/odd}}(\lambda_0;\sigma^\star))(K_H * P_\Delta f)(t_0)| \ll \Delta^{3/2}\,\|f\|_2 = o(H^{-1})\|f\|_2.$$

Here D_t denotes the Fourier multiplier with symbol λ, and the implied constants may depend on h but are uniform for $|\lambda_0| \asymp T$.

Proof. The mean value theorem and Lemma 1 give

$$|\widetilde{m}_{\mathrm{ev/odd}}(\lambda;\sigma^\star) - \widetilde{m}_{\mathrm{ev/odd}}(\lambda_0;\sigma^\star)| \le |\lambda - \lambda_0| \sup_{\xi \in B_{\lambda_0,\Delta}} |\partial_\lambda \widetilde{m}(\xi;\sigma^\star)| \ll \Delta.$$

The corresponding operator on L^2 is a multiplier of size $\ll \Delta$ supported on a set of width $\ll \Delta$, so its $L^2 \to L^2$ norm is $\ll \Delta$, and Bernstein's inequality for band–limited functions implies the pointwise bound $\ll \Delta^{3/2}\|f\|_2$. With $\Delta = T^{-1-\varepsilon}$ and $H \asymp T$, this is $o(H^{-1})$ as claimed. ☐

5.3. *Local Renormalization for* $\partial_\sigma u$

Lemma 3 (Local renormalization for $\partial_\sigma u$)**.** *Fix* $h \in (0, 1/2]$, $\sigma_0 \in (0, h)$ *and* $\varepsilon > 0$. *For each* $T \geq 1$, $H \asymp T$, $|t_0| \leq CH$, *let* $\mathcal{Z}_{t_0,H}$ *be the multiset consisting of all zeros and the pole* ρ *of* ξ, *counted with multiplicities* $m_\rho \in \mathbb{Z}$, *such that* $|\Im\rho - t_0| \leq 2H$ *and* $|\Re\rho - \frac{1}{2}| \leq h$. *Define*

$$R^{(1)}_{t_0}(\sigma, t) = \sum_{\rho \in \mathcal{Z}_{t_0,H}} m_\rho \operatorname{Re}\left(\frac{1}{\frac{1}{2} + \sigma + \mathrm{i}t - \rho}\right).$$

Then there exists a harmonic function $\widetilde{u}(\sigma, t)$ *on the rectangle* $\{|\sigma| \leq h,\ |t - t_0| \leq 2H\}$ *such that*

$$\widetilde{v}(\sigma, t) := \partial_\sigma \widetilde{u}(\sigma, t) = \partial_\sigma u(\sigma, t) - R^{(1)}_{t_0}(\sigma, t)$$

is harmonic on this rectangle. Moreover, for any short band P_Δ *with* $\Delta = T^{-1-\varepsilon}$ *and any* $\sigma^\star$ *with* $|\sigma^\star| \geq \sigma_0$,

$$\int_{\mathbb{R}} K_H(t - t_0)\,(P_\Delta R^{(1)}_{t_0})(\sigma^\star, t)\,dt \;=\; O(H^{-\eta}),$$

uniformly in $|t_0| \leq CH$, *where* $\eta > 0$ *depends only on* ε.

Proof. Local analysis of $u(s) = \operatorname{Re}\log\xi(s)$ near zeros and the pole shows that in a neighbourhood of each such ρ one has

$$u(s) = m_\rho \log|s - \rho| + h_\rho(s),$$

with h_ρ harmonic. Subtracting these logarithmic terms over $\rho \in \mathcal{Z}_{t_0,H}$ yields

$$\widetilde{u}(\sigma, t) := u(\sigma, t) - \sum_{\rho \in \mathcal{Z}_{t_0,H}} m_\rho \log\left|\tfrac{1}{2} + \sigma + \mathrm{i}t - \rho\right|,$$

which is harmonic on $\{|\sigma| \leq h,\ |t - t_0| \leq 2H\}$ and satisfies $\partial_\sigma \widetilde{u} = \partial_\sigma u - R^{(1)}_{t_0}$.

For the bound on the renormalization term, fix $\sigma^\star$ with $|\sigma^\star| \geq \sigma_0$ and write each summand of $R^{(1)}_{t_0}(\sigma^\star, t)$ as

$$f_{\delta,\gamma}(t) = \operatorname{Re}\left(\frac{1}{\delta + \mathrm{i}(t - \gamma)}\right) = \frac{\delta}{\delta^2 + (t - \gamma)^2}, \quad \delta = \sigma^\star - (\Re\rho - \tfrac{1}{2}),\ \gamma = \Im\rho.$$

Its Fourier transform is explicit:

$$\widehat{f}_{\delta,\gamma}(\lambda) = \pi \operatorname{sgn}(\delta)\, \mathrm{e}^{-\mathrm{i}\lambda\gamma}\, \mathrm{e}^{-|\delta||\lambda|}, \qquad \lambda \in \mathbb{R},$$

so in particular

$$\left|\widehat{f}_{\delta,\gamma}(\lambda)\right| \leq \pi\, \mathrm{e}^{-|\delta||\lambda|}.$$

See, for example, [53, §3.3]. Therefore

$$\widehat{K_H * P_\Delta f_{\delta,\gamma}}(\lambda) = \widehat{K}_H(\lambda)\, \mathbf{1}_{B_{\lambda_0,\Delta}}(\lambda)\, \widehat{f}_{\delta,\gamma}(\lambda),$$

and so

$$\|K_H * P_\Delta f_{\delta,\gamma}\|_{L^\infty} \;\ll\; \int_{B_{\lambda_0,\Delta}} \mathrm{e}^{-|\delta||\lambda|}\, d\lambda \;\ll\; \Delta,$$

uniformly in δ and γ. The number of ρ with $|\Im\rho - t_0| \leq 2H$ and $|\Re\rho - \frac{1}{2}| \leq h$ is $O(H \log T)$ by the zero–counting bound $N(T) \ll T \log T$ (see, for instance, [14, Thm. 9.3]). Hence

$$\int_{\mathbb{R}} K_H(t - t_0)\,(P_\Delta R^{(1)}_{t_0})(\sigma^\star, t)\,dt \;\ll\; H\Delta \log T \asymp T^{-\varepsilon} \log T = O(H^{-\eta})$$

for some $\eta > 0$ depending only on ε. $\square$

Remark 6 (Local harmonicity vs. global subharmonicity)**.** *For* $u(\sigma, t) = \log|\xi(\frac{1}{2} + \sigma + \mathrm{i}t)|$, *the function* u *is subharmonic on the strip* $|\sigma| \le h$ *and harmonic away from the zeros and the pole of* ξ. *Lemma 3 constructs, for each window centered at* t_0, *a locally harmonic function* $\widetilde{u}$ *on the rectangle* $\{|\sigma| \le h,\ |t - t_0| \le 2H\}$ *by subtracting the logarithmic singularities arising from zeros and the pole in that window. It is this locally harmonic function* $\widetilde{u}$ *to which the strip Poisson calculus of Lemma 1 is applied in the proof of Theorem 3.*

Remark 7 (Short Fourier–side proof note)**.** *The key ingredients are: (i) the explicit transform* $\delta/(\delta^2 \widehat{+ (t-\gamma)}^2)(\lambda) \asymp \mathrm{e}^{-|\delta||\lambda|}$, *which exhibits exponential damping in frequency proportional to the horizontal distance* $|\delta|$ *from the zero (or pole); and (ii) the standard zero–counting estimate* $N(T) \ll T \log T$, *which bounds the number of such singularities in a given height window. The band width is* $\asymp \Delta$, *and the Fejér multiplier is bounded on* $B_{\lambda_0,\Delta}$, *leading to the* $O(H\Delta \log T)$ *bound.*

5.4. Proof of the Strip Bridge

Proof of Theorem 3. Write $\partial_\sigma u = \widetilde{v} + R_{t_0}^{(1)}$ as in Lemma 3, and let $\widetilde{u}$ be the harmonic function constructed there so that $\widetilde{v} = \partial_\sigma \widetilde{u}$. Applying Lemma 1 to $\widetilde{u}$ gives, in frequency,

$$\widehat{\widetilde{v}}(\sigma^\star, \lambda) = \widetilde{m}_{\mathrm{ev}}(\lambda; \sigma^\star)\, \widehat{\widetilde{F}_{\mathrm{ev}}}(\lambda) + \widetilde{m}_{\mathrm{odd}}(\lambda; \sigma^\star)\, \widehat{\widetilde{F}_{\mathrm{odd}}}(\lambda),$$

where $\widetilde{F}_\pm(t) = \partial_\sigma \widetilde{u}(\pm h, t) = F_\pm(t) - R_{t_0}^{(1)}(\pm h, t)$, and $\widetilde{F}_{\mathrm{ev/odd}}$ are defined analogously.

Because P_Δ and convolution by K_H commute with Fourier multipliers, and $B_{\lambda_0,\Delta} \subset [-H, H]$,

$$K_H * P_\Delta(\partial_\sigma u)(\sigma^\star, \cdot) = \widetilde{m}_{\mathrm{ev}}(D_t; \sigma^\star)\, K_H * P_\Delta F_{\mathrm{ev}} + \widetilde{m}_{\mathrm{odd}}(D_t; \sigma^\star)\, K_H * P_\Delta F_{\mathrm{odd}} + E,$$

where the error E collects the $K_H * P_\Delta$ transforms of $R_{t_0}^{(1)}$ and of $R_{t_0}^{(1)}(\pm h, \cdot)$ fed through the multipliers. By Lemma 3 and the same Fourier–side argument for the boundary terms, each such contribution is $O(H^{-\eta})$ uniformly for $|\sigma^\star| \ge \sigma_0$.

Finally, Lemma 2 is applied to freeze the multipliers at λ_0, introducing an $o(H^{-1})$ error that can be absorbed into the $O(H^{-\eta})$ term (possibly with a smaller $\eta > 0$). Evaluating at t_0 gives (7). $\square$

6. Fejér Smoothing, Cusp Truncation with Counterterms, and Stability

In this section the regularized inner product is compared to a counterterm–subtracted truncated integral. The correct object for comparison with $\langle \cdot, \cdot \rangle_{\mathrm{reg}}$ is the truncated integral of the original integrand with the Zagier–Arthur cusp counterterms removed.

Let $\mathcal{F}$ be a fixed fundamental domain for X, and for $Y \ge 1$ let $\mathcal{F}_Y$ be its standard truncation at height Y (cf. [15,64]). Define

$$\mathsf{I}(s; Y) := \int_{\mathcal{F}_Y} \Theta(z) E(z, s)\, \overline{\Theta(z) E(z, 1 - \overline{s})}\, d\mu(z).$$

The cusp asymptotics (9) and (1) show that the constant term of $\Theta(z)E(z,s)\,\overline{\Theta(z)E(z, 1-\overline{s})}$ in the cusp consists of a finite linear combination of monomials in y and $\log y$ with s–dependent coefficients. Integrating termwise in y over $[1, Y]$ and in x over $[-1/2, 1/2]$ produces finitely many explicit counterterms $\{\mathsf{CT}_j(s; Y)\}_{j=1}^{J}$, each a polynomial in Y and $\log Y$ with coefficients depending meromorphically on s, such that (cf. [64, §2])

$$\mathcal{J}_\Theta(s) = \lim_{Y \to \infty} \left(\mathsf{I}(s; Y) - \sum_{j=1}^{J} \mathsf{CT}_j(s; Y) \right) \tag{12}$$

and, moreover, the tail

$$\mathsf{R}(s;Y) := \left(\mathsf{I}(s;Y) - \sum_{j=1}^{J} \mathsf{CT}_j(s;Y)\right) - \mathcal{J}_\Theta(s)$$

satisfies

$$\forall M \ge 0, \qquad \mathsf{R}(s;Y) = O_M(Y^{-M}), \tag{13}$$

locally uniformly on vertical strips in s. This follows from the fact that, after removal of the finitely many nondecaying terms in the cusp, the remaining integrand decays exponentially in y and its integral over $y \ge Y$ is $O(e^{-cY})$.

Set

$$\mathsf{I}_{\mathrm{reg}}(s;Y) := \mathsf{I}(s;Y) - \sum_{j=1}^{J} \mathsf{CT}_j(s;Y).$$

Proposition 1 (Truncation stability under Fejér smoothing with counterterms). *Fix $\sigma^\star$ in a compact subset of $\mathbb{R}$ and $t_0 \in \mathbb{R}$. For every $A > 0$ there exists $B = B(A) > 0$ such that, setting $Y = H^B$,*

$$\begin{aligned} &\int_{\mathbb{R}} K_H(t-t_0)\, \frac{\partial}{\partial\sigma} \log\left|\mathsf{I}_{\mathrm{reg}}\left(\tfrac{1}{2}+\sigma+\mathrm{i}t;Y\right)\right| dt\Big|_{\sigma=\sigma^\star} \\ &\qquad = \int_{\mathbb{R}} K_H(t-t_0)\, \frac{\partial}{\partial\sigma} \log\left|\mathcal{J}_\Theta\left(\tfrac{1}{2}+\sigma+\mathrm{i}t\right)\right| dt\Big|_{\sigma=\sigma^\star} + O(H^{-A}), \end{aligned}$$

uniformly for $|t_0| \le CH$. The implied constant may depend on A and $\sigma^\star$ but is independent of H and t_0.

Proof. By (13),

$$\mathsf{I}_{\mathrm{reg}}(s;Y) = \mathcal{J}_\Theta(s)\,(1+\varepsilon(s;Y)), \qquad \varepsilon(s;Y) = O_M(Y^{-M}),$$

for any chosen $M \ge 1$, locally uniformly on vertical strips. Differentiating in σ and using Cauchy estimates on ε in small discs in s implies $\partial_\sigma \varepsilon(s;Y) = O_M(Y^{-M})$ as well. Hence

$$\frac{\partial}{\partial\sigma} \log\left|\mathsf{I}_{\mathrm{reg}}(s;Y)\right| = \frac{\partial}{\partial\sigma} \log|\mathcal{J}_\Theta(s)| + O_M(Y^{-M}),$$

pointwise away from zeros and poles of $\mathcal{J}_\Theta$ and in the sense of distributions in t in general. Convolution in t against K_H preserves the bound, uniformly for $|t_0| \le CH$. Given $A > 0$, choose M and then $B > 0$ such that $Y^{-M} = H^{-BM} \ll H^{-A}$; for instance $B(A) = 2A/M$ suffices. This yields the claim. □

Remark 8. *Without subtracting the Zagier–Arthur counterterms, $\mathsf{I}(s;Y)$ differs from $\mathcal{J}_\Theta(s)$ by explicit polynomials in Y and $\log Y$ with coefficients depending meromorphically on s. These do not decay with Y and would persist after Fejér smoothing. The counterterms remove precisely these divergent pieces, and the remaining cusp tail is exponentially small in Y, which is the source of the $O(H^{-A})$ stability in Proposition 1.*

7. Numerical Verification in a Sample Region

This brief section records a qualitative description of numerical checks supporting the identity of Theorem 1. It is included only to indicate that the analytic formula admits direct numerical verification in a modest range; no new insights are claimed.

Description of the Numerical Scheme

Let $s = \frac{1}{2} + \sigma + \mathrm{i}t$ with a modest choice, for example $\sigma = 0.1$ and t in the range $5 \le t \le 20$. The following steps may be implemented:

1. *Approximate $\mathcal{J}_\Theta(s)$ by truncation.* Use the Fourier expansion of $E(z,s)$ and the truncated double sum defining $\theta(z)$ to approximate $\Theta(z)$ and $E(z,s)$ on a rectangular grid (x,y) with $|x| \le 1/2$ and $1 \le y \le Y$, for a fixed but large Y (e.g. $Y \in [20,100]$). Integrate $\Theta(z)E(z,s)\,\overline{\Theta(z)E(z,1-\bar{s})}$ over

this domain with the hyperbolic measure, and subtract the explicit Zagier–Arthur counterterms described in §6 to obtain a numerical approximation to $\mathsf{I}_{\mathrm{reg}}(s;Y)$.

2. *Approximate* $\partial_\sigma \log|\mathcal{J}_\Theta(s)|$ *by finite differences.* For a small step $h>0$ (for instance $h=10^{-3}$), form

$$D_h(s) := \frac{\log|\mathsf{I}_{\mathrm{reg}}(s+h;Y)| - \log|\mathsf{I}_{\mathrm{reg}}(s-h;Y)|}{2h}.$$

Proposition 1 shows that, with Y chosen as a sufficiently large power of H (and H comparable to $|t|$), the difference between $D_h(s)$ and $\partial_\sigma \log|\mathcal{J}_\Theta(s)|$ is bounded by a power of H^{-1}, plus the usual finite–difference discretization error.

3. *Evaluate the right–hand side.* The right–hand side of (5),

$$R(s) := 2\,\mathrm{Re}\,\frac{\xi'(s)}{\xi(s)} \;-\; 2\,\mathrm{Re}\,\frac{\zeta'(2s)}{\zeta(2s)} \;+\; 2\,\mathrm{Re}\,\frac{\zeta'(2-2\bar{s})}{\zeta(2-2\bar{s})},$$

can be evaluated using high–precision complex arithmetic and standard routines (or finite differences) for ζ'/ζ and Γ'/Γ.

Preliminary implementations following this scheme, with reasonable truncation parameters for the Eisenstein series and the theta series and with Y chosen sufficiently large, show that the numerical values of $D_h(s)$ and $R(s)$ agree to several digits in the above sample range of s, in line with the analytic error estimates.

Remark 9. *No attempt is made here to optimize the numerical scheme or to investigate regions of large t. The purpose is only to indicate that* (5) *can be checked numerically for moderate s directly from the defining integrals, once the counterterms are correctly taken into account.*

Acknowledgments: The author is grateful to Ryan J. Buchanan for valuable correspondence and discussions related to this work.

Appendix A. Model Toeplitz Curvature for a J–Bessel Ridge Kernel

For completeness, a simplified Toeplitz curvature lemma for a J–Bessel ridge kernel is recorded, independent of Kloosterman sums. The arguments are standard and serve only as a model for the appearance of the H^{-1}–scale curvature in a related, but simpler, setting.

Let $C \asymp T^{1/2}$ and X_0 be a fixed parameter with $1 \ll X_0 \ll C$. Let $K_\Delta(x)$ be the short–band kernel

$$K_\Delta(x) = \frac{\mathrm{i}}{\pi}\int_{\mathbb{R}} H_\Delta(\lambda)\,\frac{\lambda}{\cosh(\pi\lambda)}\,J_{2\mathrm{i}\lambda}(x)\,d\lambda,$$

where H_Δ is supported on $|\lambda \pm \lambda_0| \le 2\Delta$ with $\lambda_0 \asymp T$, $\Delta = T^{-1-\varepsilon}$. Define

$$\Psi_c := K_\Delta(X_0/c), \qquad m_{\mathrm{ridge}}(\theta;C) := \left|\sum_{c\sim C} W\!\left(\frac{c}{C}\right)\Psi_c\,e^{-\mathrm{i}c\theta}\right|^2,$$

for a fixed smooth envelope $W \in C_c^\infty([1,2])$. Debye–Watson asymptotics for $J_{2\mathrm{i}\lambda}(x)$ yield an expansion

$$\Psi_c = A(c)e^{\mathrm{i}\phi(c)} + \overline{A(c)}\,e^{-\mathrm{i}\phi(c)} + R(c),$$

with $|A(c)| \asymp CM_\Delta$, $\phi'(c) = -2\lambda_0/c$, and $R(c)$ smaller. The ridge frequency is $\omega_C = 2\lambda_0/C$.

Lemma A1 (Model symbol concavity). *There exists $\kappa_\star > 0$ such that, for $|\theta - \omega_C| \le \kappa_0\Delta/C$,*

$$m_{\mathrm{ridge}}(\theta;C) \le m_{\mathrm{ridge}}(\omega_C;C) - \kappa_\star (CM_\Delta)^2 C^3 (\theta-\omega_C)^2.$$

Sketch. Stationary phase in c shows that the sum $S(\theta) = \sum_c W(c/C)\Psi_c e^{-\mathrm{i}c\theta}$ has a nondegenerate maximum at $\theta = \omega_C$, with second derivative of size $\asymp -(CM_\Delta)C^{3/2}$ at the level of $S(\theta)$, leading to a quadratic upper bound for $|S(\theta)|^2$ near ω_C. $\square$

With a central difference

$$(D_\omega\gamma)(c) := \frac{e^{-\mathrm{i}\omega_C\delta c}\gamma(c+\delta c) - e^{\mathrm{i}\omega_C\delta c}\gamma(c-\delta c)}{2(\delta c/C)},$$

the Fourier multiplier is

$$L_D(\theta) = \mathrm{i}\,\frac{C}{\delta c}\,\sin\big(\delta c(\theta-\omega_C)\big).$$

Theorem A1 (Model curvature at scale H^{-1})**.** *There exists $c > 0$ such that, for all γ supported on $c \sim C$,*

$$\frac{1}{2\pi}\int_{\mathbb{R}}\big(m_{\mathrm{ridge}}(\theta;C) - m_{\mathrm{ridge}}(\omega_C;C)\big)\,|L_D(\theta)|^2\,|\widehat{\gamma}(\theta)|^2\,\widehat{K}_H(\theta-\omega_C)\,d\theta \le -\frac{c}{H}\|\gamma\|_{\ell^2}^2 + o(H^{-1})\|\gamma\|_{\ell^2}^2.$$

Sketch. Near ω_C, $m_{\mathrm{ridge}}(\theta;C) - m_{\mathrm{ridge}}(\omega_C;C) \ll -(CM_\Delta)^2C^3(\theta-\omega_C)^2$, and $|L_D(\theta)|^2 \asymp C^2(\theta-\omega_C)^2$. Their product is negative and of size $\asymp -(CM_\Delta)^2C^5(\theta-\omega_C)^4$. The Fejér weight localizes to $|\theta-\omega_C| \lesssim H^{-1}$, and scaling gives the $-c/H$ contribution, with $o(H^{-1})$ arising from tails and the error term $R(c)$. $\square$

Appendix B. The Archimedean Local Integral $Z_\infty(s,\varphi_\infty,f_{s,\infty})$

The computation of $Z_\infty(s,\varphi_\infty,f_{s,\infty}) = \int_{G(\mathbb{R})} f_{s,\infty}(g)\,\langle\omega_\infty(g)\varphi_\infty,\varphi_\infty\rangle\,dg$ for the split binary quadratic space $V \simeq \mathbb{R}^2$ with $q(x) = x_1x_2$, the standard additive character $\psi_\infty(x) = e^{2\pi\mathrm{i}x}$, and $\varphi_\infty(x) = e^{-\pi(x_1^2+x_2^2)}$ is recalled.

Appendix B.1. Weil Representation and the Spherical Matrix Coefficient

A direct computation shows that for $z = x+\mathrm{i}y$ and $t \ge 0$ related by $d_{\mathbb{H}}(z,\mathrm{i}) = 2t$,

$$\langle\omega_\infty(g_z)\varphi_\infty,\varphi_\infty\rangle = \frac{y^{1/2}}{\sqrt{(y+1)^2+x^2}} = \frac{1}{2\cosh t}.$$

Appendix B.2. Cartan Decomposition and the Spherical Section

With Haar measure normalized so that $dg = \sinh(2t)\,dt\,dk\,dk'$ on $G(\mathbb{R}) = KA^+K$, one may take $f_{s,\infty}(a_t) = c_\infty(s)\,(2\cosh t)^{-s}$.

Appendix B.3. Evaluation of $Z_\infty(s,\varphi_\infty,f_{s,\infty})$

One obtains

$$Z_\infty(s,\varphi_\infty,f_{s,\infty}) = c_\infty(s)\int_0^\infty (2\cosh t)^{-s}\,\frac{1}{2\cosh t}\,\sinh(2t)\,dt = c_\infty(s)\int_0^\infty (2\cosh t)^{-s}\,\sinh t\,dt.$$

With substitutions $u = \tanh t$ and then $v = u^2$,

$$\int_0^\infty (2\cosh t)^{-s}\,\sinh t\,dt = 2^{-s-1}\,\frac{\Gamma\big(\frac{s-1}{2}\big)}{\Gamma\big(\frac{s+1}{2}\big)}.$$

Choosing

$$c_\infty(s) = 2^{s+1}\,\pi^{-s/2}\,\Gamma\Big(\frac{s}{2}\Big)\,\frac{\Gamma\big(\frac{s+1}{2}\big)}{\Gamma\big(\frac{s-1}{2}\big)}$$

gives $Z_\infty(s,\varphi_\infty,f_{s,\infty}) = \pi^{-s/2}\Gamma(s/2)$, matching the archimedean Γ–factor of $\xi(s)$. Together with the unramified finite place identities $Z_p(s,\varphi_p,f_{s,p}) = \zeta_p(s)$, this yields $\prod_v Z_v(s,\varphi_v,f_{s,v}) = \xi(s)$.

References

1. F. V. Atkinson, *The mean-value of the Riemann zeta-function*, Acta Math. 81 (1949), 353–376.
2. J. B. Conrey and A. Ghosh, *Mean values of the Riemann zeta-function*, Invent. Math. 94 (1988), 571–583.
3. H. Davenport, *Multiplicative Number Theory*, 3rd ed., revised by H. L. Montgomery, Springer, 2000.
4. H. M. Edwards, *Riemann's Zeta Function*, Academic Press, 1974; reprint Dover, 2001.
5. L. Euler, *De summis serierum reciprocarum*, Comment. Acad. Sci. Petrop. 7 (1740), 123–134. (Originally presented 1735.) Reprinted in *Opera Omnia* Ser. I, Vol. 14, pp. 217–267.
6. L. Euler, *Variae observationes circa series infinitas*, Comment. Acad. Sci. Petrop. 9 (1744), 160–188. (Originally presented 1737.) Reprinted in *Opera Omnia* Ser. I, Vol. 14.
7. L. Euler, *Introductio in analysin infinitorum*, 2 vols., Lausanne, 1748.
8. G. H. Hardy and J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the distribution of primes*, Acta Math. 41 (1916), 119–196.
9. A. Ivić, *The Riemann zeta-function: Theory and Applications*, Dover, 2003.
10. H. L. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Math. 227, Springer, 1971.
11. H. L. Montgomery and K. Soundararajan, *Beyond pair correlation*, Adv. Math. 217 (2008), 1254–1290.
12. B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsber. Preuss. Akad. Wiss. Berlin (1859), 671–680. (English transl. in [4].)
13. B. Riemann, *Gesammelte Mathematische Werke und Wissenschaftlicher Nachlaß*, Teubner, Leipzig, 1892.
14. E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford Univ. Press, 1986.
15. J. Arthur, *An introduction to the trace formula*, in *Harmonic Analysis, the Trace Formula, and Shimura Varieties*, Clay Math. Proc. 4 (2005), 1–263.
16. R. Bruggeman, *Fourier coefficients of cusp forms*, Invent. Math. 60 (1980), 1–18.
17. R. Bruggeman and R. J. Miatello, *Sum formula for $SL_2(\mathbb{R})$ and Selberg–Kuznetsov trace formula*, Developments in Mathematics 14, Springer, 2006.
18. D. A. Hejhal, *The Selberg Trace Formula for* $\mathrm{PSL}(2,\mathbb{R})$, Vols. I–II, Lecture Notes in Math., Springer, 1983–1984.
19. H. Iwaniec, *Spectral Methods of Automorphic Forms*, 2nd ed., Amer. Math. Soc., GSM 53, 2002.
20. H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc., Colloquium Publ. 53, 2004.
21. R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Math. 544, Springer, 1976.
22. Y. Motohashi, *Spectral Theory of the Riemann Zeta-Function*, Cambridge Univ. Press, 1997.
23. A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. 20 (1956), 47–87.
24. J.-M. Deshouillers and H. Iwaniec, *Kloosterman sums and Fourier coefficients of cusp forms*, Invent. Math. 70 (1982), 219–288.
25. A. Good, *The square mean of Dirichlet series associated with cusp forms*, Math. Z. 157 (1977), 1–9.
26. N. V. Kuznetsov, *Petersson's trace formula and the Hecke eigenvalues for Maass forms*, Math. USSR Sb. 39 (1981), 299–342.
27. N. Watt, *Kloosterman sums and a large sieve inequality*, Math. Proc. Camb. Phil. Soc. 109 (1991), 419–430.
28. S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge Univ. Press, 1991.
29. M. N. Huxley, *Area, Lattice Points, and Exponential Sums*, Oxford Univ. Press, 1996.
30. M. Jutila, *Lectures on a Method in the Theory of Exponential Sums*, Tata Institute, 1987.
31. M. Jutila, *On spectral large sieve inequalities*, Funct. Approx. Comment. Math. 28 (2000), 7–18.
32. J. B. Conrey and H. Iwaniec, *The cubic moment of central values of automorphic L-functions*, Ann. of Math. 151 (2000), 1175–1216.
33. W. Duke, J. Friedlander, and H. Iwaniec, *Bounds for automorphic L-functions*, Invent. Math. 112 (1993), 1–8.
34. E. Kowalski, P. Michel, and J. VanderKam, *Rankin–Selberg L-functions in the level aspect*, Duke Math. J. 114 (2002), 123–191.
35. E. Kowalski, P. Michel, and W. Sawin, *Bilinear forms with Kloosterman sums and applications*, Ann. of Math. 186 (2017), 413–500.
36. P. Michel and A. Venkatesh, *The subconvexity problem for GL_2*, Publ. Math. IHÉS 111 (2010), 171–271.
37. M. P. Young, *The fourth moment of Dirichlet L-functions*, Ann. of Math. 173 (2011), 1–50.
38. V. Blomer and G. Harcos, *The spectral decomposition of shifted convolution sums*, Duke Math. J. 144 (2008), 321–339.

39. V. Blomer and D. Milićević, *The second moment of twisted modular L-functions*, Geom. Funct. Anal. 25 (2015), 453–516.
40. E. Fouvry and P. Michel, *Sur certaines sommes d'exponentielles sur les entiers friables*, Ann. Sci. Éc. Norm. Supér. 31 (1998), 93–126.
41. A. Ivić, *The Riemann zeta-function: Theory and Applications*, Dover, 2003.
42. P. Sarnak, *Letter to N. Katz on the Zeros of L-functions*, 2007.
43. A. Böttcher and S. M. Grudsky, *Spectral Properties of Banded Toeplitz Matrices*, SIAM, 2005.
44. A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, 2nd ed., Springer, 2006.
45. U. Grenander and G. Szegő, *Toeplitz Forms and Their Applications*, California Monographs, 1958.
46. N. K. Nikolski, *Operators, Functions, and Systems: An Easy Reading*, Vols. I–II, Amer. Math. Soc., 2002.
47. F. W. J. Olver et al. (eds.), *NIST Digital Library of Mathematical Functions*, https://dlmf.nist.gov/.
48. F. W. J. Olver, *Asymptotics and Special Functions*, AKP Classics, 1997.
49. N. M. Temme, *Special Functions*, Wiley, 1996.
50. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, 1995.
51. J. B. Garnett, *Bounded Analytic Functions*, Springer, 2007.
52. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.
53. E. M. Stein and R. Shakarchi, *Harmonic Analysis*, Princeton Univ. Press, 2003.
54. D. Bump, *Automorphic Forms and Representations*, Cambridge Univ. Press, 1998.
55. D. Goldfeld, *Automorphic Forms and L-Functions for* $\mathrm{GL}(n)$, Cambridge Univ. Press, 2006.
56. W. T. Gan, Y. Qiu, and S. Takeda, *The regularized Siegel–Weil formula (the second term identity) and the Rallis inner product formula*, Ann. of Math. 197 (2013), 1845–1884.
57. R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. 313 (1989), 539–570.
58. A. Ichino, *On the regularized Siegel–Weil formula*, J. reine angew. Math. 539 (2001), 201–234.
59. A. Ichino, *The Doubling Method and the Rallis Inner Product Formula*, Duke Math. J. 147 (2009), 281–356.
60. S. Kudla and S. Rallis, *A regularized Siegel–Weil formula: the first term identity*, Ann. of Math. 140 (1994), 1–80.
61. I. Piatetski-Shapiro and S. Rallis, *L-functions for the classical groups*, Lecture Notes in Math. 1254, Springer, 1987.
62. F. Shahidi, *On certain L-functions*, Amer. J. Math. 103 (1981), 297–355.
63. A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. 111 (1964), 143–211.
64. D. Zagier, *The Rankin–Selberg method for automorphic functions which are not of rapid decay*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), 415–437.
65. P. Emmerson. *Connectorz: Pro-étale Transfer Actions with Branch Monodromy, Baselines, and Central Fukaya Deformations*. Authorea, October 17, 2025. DOI: `10.22541/au.176071884.42388686/v1`.
66. P. Emmerson. *Eight Symmetry-Aware Rules for Symbolic Integration and Order Selection: A Group-Theoretic and Algorithmic View with Theorem-Example Format*. Authorea, October 03, 2025. DOI: `10.22541/au.175950942.25933217/v1`.
67. R. J. Buchanan and P. Emmerson. *An Application of Modal Logic to Costello–Gwilliam Factorization Algebras*. Authorea, July 15, 2025. DOI: `10.22541/au.175131320.01270086/v2`.
68. P. Emmerson. *Mathematical Formalization of Non-Commutative Tensor Fields and Neural Manifold Evolution*. Authorea, March 24, 2025. DOI: `10.22541/au.174283917.78138091/v1`.
69. P. Emmerson. *A Rigorous Proof of the Riemann Hypothesis*. Authorea, March 13, 2025. DOI: `10.22541/au.174188999.98612005/v1`.
70. P. Emmerson. *Symmetry, Spectral Gaps, and Fractal Geometry in Non-Commutative Optimization*. Authorea, March 13, 2025. DOI: `10.22541/au.174188980.04674251/v1`.
71. P. Emmerson. *Formalizing Mechanical Analysis Using Sweeping Net Methods I*. Authorea, November 06, 2024. DOI: `10.22541/au.173092114.44460197/v1`.
72. P. Emmerson. *Analyzing the Zeros of the Riemann Zeta Function Using Set-Theoretic and Sweeping Net Methods*. Authorea, November 12, 2024. DOI: `10.22541/au.173144157.73514354/v1`.