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Article

Parastrophe of Some Inverse Properties in Quasigroups

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Abstract

This work investigates the relationship that exists between the parastrophes of some notion of inverses in quasigroups. Our findings revealed that, of the 5 parastrophes of LIP quasigroup, (23)- parastrophe is a LIP quasigroup, (12)- and (132)- parastrophes are RIP quasigroups, while (13)- and (132)- parastrophes are an anti-commutative quasigroup. Similarly, the (12)- and (132)-parastrophes of a RIP quasigroup are LIP quasigroups; the (13)-parastrophe of a RIP quasigroup is an RIP quasigroup, while the (23)- and (123)-parastrophes are anti-commutative quasigroups. As for the CIP quasigroup, only (12)-parastrophe is a CIP quasigroup; other parastrophes are symmetric quasigroups of order 2. Finally, (12)-parastrophe of WIP quasigroup is an IP quasigroup, (13)-,(23)- and (132)-parastrophes of WIP quasigroup are CIP quasigroups, while (123)-parastrophe of WIP quasigroup is a WIP quasigroup.

Keywords: parastrophe; inverse property; symmetric quasigroups; anti-commutative property

MSC: 20N02; 20N05

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1. Introduction

A quasigroup G is a set having a binary multiplication $x \cdot y$ usually written as xy that satisfies the condition that for any a, b in G, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for $x, y \in G$. suppose G is a non-empty set defined on a binary operation (.) such that $x, y \in G$ for all x, y in G. Then, $(G, \cdot)$ is called a groupoid. Alternatively, a quasigroup can also be defined in terms of a translational map. For every x, y in G define a mapping R_x and L_x of G into itself by $yR_x = y \cdot x$ and $yL_x = x \cdot y$. Then $(G, \cdot)$ is a quasigroup if and only if R_x and L_x are bijective for all x in G. The mapping L_x and R_x are called left and right translation maps: If a quasigroup G contains an element e such that $e \cdot x = x$ for x in G, then e is called the left Identity element of G. Similarly, if $x \cdot e = x$ for x in G, then e is called the right identity element. If G contains both left and right Identity element, then these elements must be the same, and G contains a (two-sided) identity element, and G is therefore called a loop.
A quasigroup $(G, \cdot, \backslash, /)$ is a set G together with three binary operations $(\cdot, \backslash, /)$ such that

$$a \cdot (a \backslash b) = b, (b/a) \cdot a = b \quad \forall a, b \in G.$$

$$a \backslash (a \cdot b) = b, (b \cdot a)/a = b, \ a \backslash a = b/b, \ \forall a, b \in G.$$

Standard classical references on quasigroups and loops that can be consulted for further readings are [2,3,7,14,20,21]. The study of parastrophe in quasigroups can be traced back to the work of Sade [19] and Artzy [1] in their studies on parastrophes of quasigroups. Jaiyeola [8] gave some necessary and sufficient conditions for the parastrophic invariance of associative law in quasigroups using the holomorph of the respective parastrophe of the quasigroup. In [15], the author established connection

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between different pairs of conjugates (another name for parastrophe) and described all six possible conjugate sets, with regard to the equality ("assembling") of conjugates. In [6], Dudek studied the idempotent of k-translatable quasigroups and their parastrophes. Osoba et. al. [12] gave algebraic characterisation of generalised middle Bol loop using the concept of parastrophe and holomorph of loops.

Recently, there has been a surge in studying parastrophes of some inverse property quasigroups. For instance, [9] was dedicated to finding minimal identities that define CIP-quasigroups by investigating the dependencies between the invertibility functions. [17] studied parastrophy orbits of (r,s,t)-inverse quasigroups in general, while [16] specifically studied (following [17]) parastrophy orbits of WIP-quasigroups using a permutation arising from the Cayley table of a WIP-quasigroup constructed.

Given a quasigroup $(G, \cdot)$, there exist five other associated quasigroups which are called parastrophes. In associative binary systems, the concept of an inverse element or inverse property is only meaningful if the system has an identity element. In a group, $a \cdot a^{-1} = a^{-1} \cdot a = e$. An inverse property (IP) quasigroup is a set G and a binary operation; where G contains an identity e such that $a \cdot e = a = e \cdot a$ for all $a \in G$, and where $x \in G$ has a two-sided inverse x^{-1} such that for all $y \in G$

$$x^{-1} \cdot (xy) = y = (yx) \cdot y^{-1}.$$

Such IP quasigroups are regarded as loops, which are not the focus of this study. The class of some inverse properties quasigroups shall form the basis of this study by investigating how parastrophes relate to some notions of inverses in quasigroups. The concern of this study is to provide an answer to the question: are the parastrophes of LIP quasigroup, RIP quasigroup, IP quasigroup, CIP quasigroup, and WIP quasigroup parastrophically invariant?

2. Basic Concepts

In this section, we give Definitions of terminologies used throughout this study and some previous results used in the body of this work.

Definition 1. *A groupoid is a non empty set together with a binary operation* $(G, \cdot)$ *for all* $x, y \in G, x \cdot y \in G$.

Definition 2. *A groupoid* $(G, \cdot)$ *is called a quasigroup if the maps* $L(x) : G \to G$ *and* $R(x) : G \to G$ *are bijections for all* $x \in G$.

Definition 3. *Let* $(G, \cdot)$ *be a groupoid and let* a *be any fixed element in* G. *Then the translation maps* $L(x)$ *and* $R(x)$ *are defined as* $yL(x) = x \cdot y$ *and* $yR(x) = y \cdot x$ *for all* $y \in G$.

Definition 4. *A quasigroup* $(G, \cdot)$ *is said to be of exponent two if for all* $x \in G$, *we have* $x^2 = e$ *that is* $x^{-1} = x$

Definition 5. *A quasigroup* $(G, \cdot)$ *is a LIP-quasigroup, If there exists a bijection* $J_\lambda : a \to a^\lambda$ *on* G *such that* $a^\lambda(a \cdot x) = x$ *for every* $x \in G$.

Definition 6. *A quasigroup* $(G, \cdot)$ *has a right inverse property (RIP) if there exists a bijection* $J_\rho : a \longrightarrow a^\rho$ *on* G *such that*

$$(x \cdot a)a^\rho = x$$

for every $x \in G$

2.1. Parastrophe of Quasigroups (Quasigroups)

Let $(G, \cdot)$ be a quasigroup. If given any two of x, y, z as elements in G, the third can be uniquely selected in G so that if

$$x \cdot y = z,$$

we have the left and right divisor $y = x\backslash z$ and $x = z/y$. The binary product $x \cdot y = z$ can be expressed in six ways by permuting the order in which the symbols appear.

Some authors use functional notation for operations on a set G, instead of writing $a \cdot b = c$ one writes $F(a,b) = c$. In this case, the quasigroup $(G,\cdot)$ is denoted by $G(F)$. For operations $(\backslash)$ and $(/)$ one uses symbols F^{-1} and ${}^{-1}F$ i.e $F(a,b) = c$, then $F^{-1}(c,b) = a$ and ${}^{-1}F(a,c) = b$. One can now determine three other conjugate operations on G associated with the operation F, namely ${}^{-1}(F^{-1})$, $({}^{-1}F)^{-1}$ and $\left({}^{-1}(F^{-1})\right)^{-1}$. The six conjugate quasigroups $F, F^{-1}, {}^{-1}F, {}^{-1}(F^{-1}), \left({}^{-1}(F^{-1})\right)^{-1}$ and $\left({}^{-1}(F^{-1})\right)^{-1}$ are called parastrophe.
According to Pflugfelder [14], it was noted that we can obtain new quasigroups and quasigroups from existing quasigroups and quasigroups. If in a 3-web, one permutes 3 pencils, a new 3-web is produced which in turn gives rise to new quasigroups. If, for instance, the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

is performed on a 3-web W, and if as a result the lines are mapped so that

$$w_1 \longleftrightarrow w_2', w_2 \longmapsto w_3', w_3 \mapsto w_1^1$$

for all $W \in \sum$, then the quasigroup $(G,\circ)$ in which one has, say, $x \circ z = y$, goes into a quasigroup $(G,*)$ in which $y * x = z$. A quasigroup produced in this way is called a parastrophe and in particular the π-parastrophe if it is based on the permutation $\pi \cdot (G,*)$ in our example is the π-parastrophe of $(G,\circ)$ or $(G,*)$ is said to be π-parastrophic to $(G,\circ)$. Parastrophes of quasigroups have been studied in different context by different authors among which are [1,4,10,11,13,19]
The following is obvious in view of the existence of 6 permutations of 3 pencils.

Theorem 1. *([14]) There are 6 quasigroups parastrophic to every quasigroup.*

Definition 7. *([14]) The operation in the π-parastrophe of the quasigroup $(G,\cdot)$ will be denoted by (π) i.e we write $x(\pi)\tau$ instead of $x \circ z$.*

If the operation $(\cdot)$ in $(G,\cdot)$ is denoted by F and the operation in the π-parastrophe is denoted by $(\pi_i)i = 1,2,3,4,5,6$, then the correspondence is as follows:

$$(\pi_1) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = F \text{ and } x(\pi_1)z = y = F(x,z)$$

$$(\bar{\pi}_2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = F^{-1} \text{ and } y(\pi_2)z = x = F^{-1}(y,z)$$

$$(\pi_3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} =^{-1} F \text{ and } x(\pi_3)y = z =^{-1} F(x,y)$$

$$(\pi_4) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} =^{-1} \left(F^{-1}\right) \text{ and } y(\pi_4)x = z = -1\left({}^{-1}\right)(y,x)$$

$$(\pi_5) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \left({}^{-1}F\right)^{-1} \text{ and } z(\pi_5)y = x = \left({}^{-1}F\right)^{-1}(z,y)$$

$$(\pi_6) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \left({}^{-1}\left(F^{-1}\right)\right)^{-1} \text{ and } z(\pi_6)x = y = \left(\left[{}^{-1}\left(F^{-1}\right)\right)^{-1}(z,x)$$

Remark 1. *If $(G,\cdot)$ is a quasigroup, its conjugates or parastrophes are also quasigroups.*

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Definition 8. *([14]) A quasigroup $(G,\cdot)$ has a left inverse property (LIP) if there exists a bijection $J_\lambda : a \to a^\lambda$ on G such that*

$$a^\lambda \cdot (a \cdot x) = x$$

for every $x \in G$.

Definition 9. *([14]) A quasigroup $(G,\cdot)$ has a right inverse property (RIP.) if there exists a bijection $J_\rho : a \longrightarrow a^\rho$ on G such that*

$$(x \cdot a) \cdot a^\rho = x$$

for every $x \in G$

Theorem 2. *([14]) If $(G,\cdot)$ is an LIP or an RIP quasigroup then $J_\lambda = J_\rho = J(i.e \quad a^\lambda = a^\rho = a^{-1} \quad where \quad a \cdot a^{-1} = a^{-1} \cdot a = e)$.*

Definition 10. *([14]) A quasigroup $(G,\cdot)$ is called a cross inverse property quasigroup (CIP quasigroup) if any two elements $x, y \in L$ satisfy the relation*

$$xy \cdot x^\rho = y \tag{1}$$
$$x \cdot (yx^\rho) = y \tag{2}$$
$$(xy)^\rho = x^\rho y^\rho \tag{3}$$

Definition 11. *A quasigroup $(G,\cdot)$ is called a weak inverse property quasigroup (WIP quasigroup) if it satisfies the identical relation*

$$y(xy)^\rho = x^\rho \tag{4}$$

Theorem 3. *([14]) Every CIP quasigroup has WIP.*

Theorem 4. *([14]) Let $(G,\cdot)$ be a quasigroup, then the following statements are equivalent:*
(i) $(G,\cdot)$ is a WIP quasigroup
(ii) The relation $xy \cdot z = e$ implies $x \cdot yz = e$
(iii) $(G,\cdot)$ satisfies the identical relation

$$(xy)^\lambda \cdot x = y^\lambda \tag{5}$$

3. Results

3.1. Parastrophes of Leftt Inverse Property (LIP) Quasigroups

Theorem 5. *Let G be a left inverse property quasigroup (LIP) quasigroup. Then, the (12)- parastrophes of G is a right inverse property (RIP) quasigroup if $a = a^\rho$.*

Proof. $a^\lambda(ax) = x$. Let $ax = p$ then, $a^\lambda p = x \overset{(12)}{\Rightarrow}$

$$pa^\lambda = x. \tag{6}$$

Also $ax = p \overset{(12)}{\Rightarrow}$

$$xa = p \tag{7}$$

Substituting equation (7) into equation (6)
$p \cdot a^\lambda = x \implies xa \cdot a^\lambda = x$. Now, set $a = a^\rho$ to obtain $xa^\rho \cdot a = x$ interchanging role of a and a^ρ to

obtain $xa \cdot a^{\rho} = x$. Thus, (12)- parastrophes of left inverse property (LIP) quasigroup is a right inverse property (RIP) quasigroup. □

Theorem 6. *Let G be a left inverse property (LIP) quasigroup, then (23)- parastrophe of G is a left inverse property (LIP) if* $a = a^{\lambda}$.

Proof. Suppose $a^{\lambda}(ax) = x$ and let $ax = p$ then, $a^{\lambda}p = x \stackrel{(23)}{\Rightarrow}$

$$a^{\lambda}x = p \tag{8}$$

If $ax = p \stackrel{(23)}{\Rightarrow} ap = x$

$$p = a\backslash x \tag{9}$$

Substituting (9) into (8)
$a^{\lambda}x = p \implies a^{\lambda}x = a\backslash x$. Thus, $x = a \cdot a^{\lambda}x$ interchanging the role of a and a^{λ}, one obtains $x = a^{\lambda} \cdot ax$. Therefore, (23)- parastrophe of LIP quasigroup is a LIP- quasigroup. □

Theorem 7. *Let G be a left inverse property (LIP) quasigroup. Then (13)- parastrophe of G is anti-commutative.*

Proof. Suppose $a^{\lambda}(ax) = x$ and Let $ax = p$. Then, $a^{\lambda}p = x \stackrel{(13)}{\Rightarrow}$

$$xp = a^{\lambda} \tag{10}$$

If $ax = p \stackrel{(13)}{\Rightarrow}$ then, $px = a$

$$p = a/x \tag{11}$$

Substituting (11) into (10),

$$xp = a^{\lambda} \implies x \cdot a/x = a^{\lambda}.$$

Let

$$\bar{a} = a/x \implies \bar{a}x = a$$

$$x\bar{a} = (\bar{a}x)^{\lambda}.$$

Thus, (13)- parastrophe of LIP quasigroup is anti-commutative. □

Theorem 8. *Left Inverse property (LIP) quasigroup. Then (123)- parastrophe of G is a right inverse property if* $a = a^{\rho}$.

Proof. Suppose $a^{\lambda}(ax) = x$ and Let $ax = p$. Then, $a^{\lambda}p = x \stackrel{(123)}{\Rightarrow}$

$$xa^{\lambda} = p \tag{12}$$

If $ax = p \stackrel{(123)}{\Rightarrow} pa = x \implies$

$$p = x/a. \tag{13}$$

Substituting (13) into (12)
$xa^{\lambda} = p \implies xa^{\lambda} = x/a \implies xa^{\lambda} \cdot a = x$.
Set $a = a^{\rho}$ then, $xa \cdot a^{\rho} = x$. Thus, (123)- parastrophe of a LIP quasigroup is a RIP-quasigroup. □

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Theorem 9. *Let G be a left inverse property (LIP) quasigroup. Then (132)- parastrophe of G is anti-commutative.*

Proof. $a^\lambda(ax) = x$ and Let $ax = p$. Then, $a^\lambda p = x \overset{(132)}{\Rightarrow}$

$$px = a^\lambda \tag{14}$$

Let $ax = p \overset{(132)}{\Rightarrow} xp = a$

$$p = x/a \tag{15}$$

Substituting (15) into (14),
$px = a^\lambda \implies (x\backslash a) \cdot x = a^\lambda$. Set $\bar{a} = x\backslash a \implies x\bar{a} = a \implies \bar{a}x = (x\bar{a})^\lambda$.
Thus, the result follows. □

Remark 2. *The (23)-parastrophe is the only parastrophically invariant among the parastrophes of LIP quasigroup. (13) and (132)- parastrophes are anti-commutative while (12) and (123)-parastrophes are RIP.*

3.2. Parastrophes of Right Inverse Property (RIP) Quasigroups

Theorem 10. *Let G be a right Inverse property (RIP) quasigroup. Then the (12)-parastrophe of G is a left inverse property (LIP) quasigroup.*

Proof. Suppose $(xa) \cdot a^\rho = x$ and Let $xa = q$. Then, $qa^\rho = x \overset{(12)}{\Rightarrow}$

$$a^\rho q = x \tag{16}$$

If $xa = q$, $\overset{(12)}{\Rightarrow}$

$$ax = q \tag{17}$$

Substituting (17) into (16)

$$a^\rho \cdot q = x \implies a^\rho \cdot (ax) = x.$$

Set $a = a^\lambda$ then, $(a^\lambda)^\rho \cdot a^\lambda x = x \implies a \cdot a^\lambda x = x$. interchanging the role of a and a^λ, one obtains $a^\lambda \cdot ax = x$.
Thus, (12)-parastrophe of a right inverse property (RIP) quasigroup is a left inverse property (LIP) quasigroup. □

Theorem 11. *Let G be a right inverse property (RIP) quasigroup. Then the (23)- parastrophes of G is anti-commutative.*

Proof. Suppose $(xa) \cdot a^\rho = x$ and let $xa = q$. Then, $q \cdot a^\rho = x \overset{(23)}{\Rightarrow}$

$$qx = a^\rho \tag{18}$$

If $xa = q \overset{(23)}{\Rightarrow} xq = a$

$$q = x\backslash a \tag{19}$$

Substituting (19) into (18),
$q \cdot x = a^\rho \implies (x\backslash a) \cdot x = a^\rho$. Set $\bar{a} = x\backslash a$ then, $x\bar{a} = a \implies \bar{a}x = (x\bar{a})^p$.
Thus, (23)-parastrophe of (RIP) quasigroup is anti-commutative □

Theorem 12. *Let G be a right inverse property (RIP) quasigroup, then (13)- parastrophe of G is a right inverse property quasigroup.*

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Proof. Suppose $(xa) \cdot a^\rho = x$ and let $(xa) = q$. Then, $qa^\rho = x \overset{(13)}{\Rightarrow}$

$$xa^\rho = q \tag{20}$$

Let $xa = q \overset{(13)}{\Rightarrow} qa = x$

$$q = x/a \tag{21}$$

Substituting (21) into (20),

$$xa^\rho = q \implies xa^\rho = x/a \implies xa^\rho \cdot a = x.$$

Interchanging the role of a and a^p one obtains $xa \cdot a^\rho = x$.
Thus (13)-parastrophe of a RIP quasigroup is a RIP quasigroup. □

Theorem 13. *Let G be a right inverse property (RIP) quasigroup. Then (123)-parastrophe of G is anti-commutative.*

Proof. Suppose $(xa) \cdot a^\rho = x$ and let $xa = q$.
Then, $q \cdot a^\rho = x \overset{(123)}{\Rightarrow}$

$$x \cdot q = a^\rho \tag{22}$$

If $(xa) = q \overset{(123)}{\Rightarrow}, qx = a$

$$q = a/x \tag{23}$$

Substituting (23) into (22),
$x \cdot q = a^\rho \implies x \cdot a/x = a^\rho$. Set $\bar{a} = a/x$ then, $\bar{a}x = a \implies x\bar{a} = (\bar{a}x)^\rho$.
Thus, (123)-parastrophe of G is anti-commutative. □

Theorem 14. *Let G be a right Inverse property (RIP) quasigroup then (132)- parastrophe of G is a right inverse property quasigroup. However, if $a = a^\lambda$, then the (132)- parastrophe of G is a left inverse property quasigroup.*

Proof. Suppose $(xa)a^\rho = x$ and let $xa = q$. Then, $qa^\rho = x \overset{(132)}{\Rightarrow}$

$$a^\rho x = q. \tag{24}$$

If $xa = q \overset{(132)}{\Rightarrow} a \cdot q = x$

$$q = a\backslash x \tag{25}$$

Substituting (25) into (24), $a^\rho x = q \implies a^\rho x = a\backslash x \implies a \cdot a^\rho x = x$.
Set $a = a^\lambda$ then, $a^\lambda \cdot ax = x$.
Thus (132)- parastrophe of RIP quasigroup is a LIP quasigroup. □

Remark 3. *The* (13)*-parastrophe is parastrophically invariant among the parastrophes of RIP quasigroup.* (12) *and* (132)*-parastrophes are LIP while* (23) *and* (123)*-parastrophes are anti-commutative.*

3.3. Parastrophes of Cross Inverse Property (CIP) Quasigroup

Theorem 15. *Let G be a cross inverse property (CIP) quasigroup. Then (12)- parastrophe of G is also a CIP quasigroup.*

Proof. Suppose $(xy) \cdot x^\rho = y$ and let $xy = p$. Then, $p \cdot x^\rho = y \overset{(12)}{\Rightarrow}$

$$x^\rho \cdot p = y \tag{26}$$

If $xy = p \overset{(12)}{\Rightarrow}$

$$yx = p \tag{27}$$

Substituting (27) into (26), we obtain, $x^\rho \cdot (yx) = y$
Set $x \longleftrightarrow x^\rho$, then $x \cdot yx^\rho = y$.
Thus, (12)-parastrophe of (CIP) quasigroup is a (CIP) quasigroup. □

Theorem 16. *Let G be a cross inverse property quasigroup. Then (23)-parastrophe of G is a symmetric quasigroup of order 2.*

Proof. Suppose $(xy) \cdot x^\rho = y$ and let $xy = p$
$P \cdot x^\rho = y \overset{(23)}{\Rightarrow}$

$$p \cdot y = x^p \tag{28}$$

Let $xy = p \overset{(23)}{\Rightarrow}$
$xp = y$

$$p = x \backslash y \tag{29}$$

Substituting (29) into (28), we obtain $(x\backslash y) \cdot y = x^p$.
Set $q = x\backslash y \Rightarrow xq = y$,

$$q \cdot (xq) = x^\rho \tag{30}$$

Set $q = e$ in (30) to obtain $x = x^p$. Also, set $x = e$ in (30) to obtain $q^2 = e$.
If we put $x = x^\rho$ in (30), then $q \cdot xq = x$ □

Theorem 17. *Let G be a cross inverse property (CIP) quasigroup. Then (13)-parastrophe of G is a symmetric quasigroup of order 2.*

Proof. Suppose $(xy) \cdot x^\rho = y$ and let $xy = p$. Then, $p \cdot x^\rho = y \overset{(13)}{\Rightarrow}$

$$y \cdot x^\rho = p \tag{31}$$

If $xy = p \overset{(13)}{\Rightarrow} py = x$

$$p = x/y \tag{32}$$

Substituting (32) into (31), we obtain $y \cdot x^\rho = x/y$

$$(yx^\rho) \cdot y = x \tag{33}$$

Set $y = e$ in (33) to obtain $x^\rho = x$. Also, Set $x = e$ in (33) to obtain $y^2 = e$. If we put $x^\rho = x$ in (33), then $yx \cdot y = x$.
Thus, (13)- parastrophe of CIP quasigroup is a symmetric quasigroup of order 2 □

Theorem 18. *Let G be a cross inverse property (CIP) quasigroup. Then (123)-parastrophe of G is a symmetric quasigroup of order 2.*

Proof. Suppose $(xy) \cdot x^\rho = y$ and let $xy = p$. then, $p \cdot x^\rho = y \overset{(123)}{\Rightarrow}$

$$p \cdot y = x^\rho \tag{34}$$

If $xy = p \overset{(123)}{\Rightarrow} px = y$ then,

$$p = y/x \tag{35}$$

Substituting (35) into (34), we obtain $y \cdot y/x = x^\rho$. Setting $q = y/x \Rightarrow qx = y$

$$(qx) \cdot q = x^\rho \tag{36}$$

Set $q = e$ in (36) to obtain $x = x^\rho$. Also, Set $x = e$ in (36) to obtain $q^2 = e$.
If we put $x = x^\rho$ in (36) then, $qx \cdot q = x$.
Thus, (123)- parastrophe of CIP quasigroup is a symmetric quasigroup of order 2. □

Theorem 19. *Let G be a cross inverse property (CIP) quasigroup. Then (132)-parastrophe of G is a symmetric quasigroup of order 2.*

Proof. Suppose $(xy) \cdot x^\rho = y$ and let $xy = p$. Then, $p \cdot x^\rho = y \overset{(132)}{\Rightarrow}$

$$x^p \cdot y = p \tag{37}$$

If $xy = p \overset{(132)}{\Rightarrow} yp = x$, then

$$p = y \backslash x \tag{38}$$

Substituting (38) into (37), we obtain $x^\rho \cdot y = y \backslash x$

$$x = y \cdot x^\rho y \tag{39}$$

Set $y = e$ in (39) to obtain $x = x^\rho$. Also, set $x = e$ in (39) to obtain $y^2 = e$.
If we put $x = x^\rho$ in (39), then $x = y \cdot xy$.
Thus, (132)- parastrophe of CIP quasigroup is a symmetric quasigroup of order 2. □

Remark 4. *All the parastrophes of cross inverse property quasigroups are symmetric quasigroups of order* 2 *except the* (12)*-parastrophe that is parastrophically invariant.*

3.4. Parastrophes of Weak Inverse Property (WIP) Quasigroup

Theorem 20. *Let G be a weak inverse property (WIP) quasigroup. Then (12)- parastrophe of G is an inverse property (IP) quasigroup.*

Proof. Suppose $x \cdot (yx)^\rho = y^\rho \overset{(12)}{\Rightarrow}$

$$(xy)^\rho \cdot x = y^\rho \tag{40}$$

Set $x = e$ in (40) to obtain $y^\rho \cdot e = y^\rho$. Also, Set $y = e$ in (40) to obtain $x^\rho \cdot x = e$.
Thus, (12)- parastrophe of WIP-quasigroup is a WIP quasigroup. □

Theorem 21. *Let G be a weak inverse property (WIP) quasigroup. Then (13)-Parastrophe of WIPL is a CIPL.*

Proof. $x \cdot (yx)^\rho = y^\rho \overset{(13)}{\Rightarrow}$

$$y^\rho \cdot (yx)^\rho = x$$

Let $(yx)^\rho = p$

$$y^\rho \cdot p = x \tag{41}$$

Since $(yx)^\rho = p, yx = p^\lambda \overset{(13)}{\Rightarrow} p^\lambda x = y \implies p^\lambda = y/x$

$$p = (y/x)^\rho \tag{42}$$

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Substituting (42) into (41), we obtain $y^\rho \cdot (y/x)^\rho = x$.
If $q = y/x \Rightarrow qx = y$

$$(qx)^\rho \cdot q^\rho = x \tag{43}$$

Set $q = e$ in (43), then $x^\rho = x$.
From $(xy)^\lambda \cdot x = y^\lambda \overset{(13)}{\Rightarrow} y^\lambda \cdot x = (xy)^\lambda$
Let $xy = p$ then $y^\lambda \cdot x = p^\lambda$ From $xy = p \overset{(13)}{\Rightarrow} py = x$

$$p = x/y$$

$$y^\lambda \cdot x = (x/y)^\lambda$$

Setting $q = x/y \Rightarrow qy = x$ and thus, $y^\lambda \cdot qy = q^\lambda$. Set $q^\lambda = q$ to obtain $y^\lambda \cdot qy = q$.
Also, Set $y = y^\rho$ to obtain $(y^\rho)^\lambda \cdot qy^\rho = q$. Therefore, $y \cdot qy^\rho = q$. Thus, (13)-Parastrophe of WIPL is a CIPL □

Theorem 22. *Let G be a weak inverse property (WIP) quasigroup. Then (123)-parastrophe of WIPL is WIPL.*

Proof. Suppose $y \cdot (xy)^\rho = x^\rho \overset{(123)}{\Rightarrow} x^\rho \cdot y = (xy)^\rho$
If $xy = p$

$$x^\rho \cdot y = p^\rho \tag{44}$$

$xy = p \overset{(123)}{\Rightarrow}, px = y$

$$p = y/x \tag{45}$$

Substituting (45) into (45), we obtain $x^\rho \cdot y = (y/x)^\rho$

$$(x^\rho y)^\lambda \cdot x = y \tag{46}$$

Set $x = e$ in (46) to obtain $y^\lambda = y$. Also, set $y = e$ in (46) to obtain $x^2 = e$. Now on setting $x = x^\lambda$ in (46) we have $\left((x^\lambda)^\rho y\right)^\lambda \cdot x^\lambda = y$
$(xy)^\lambda \cdot x^\lambda = y \implies (xy)^\lambda \cdot x = y^\lambda$.
Thus, (123)- parastrophe of WIPL is WIPL. □

Theorem 23. *Let G be a weak inverse property (WIP) quasigroup. Then (132)-parastrophe of WIPL is a CIPL.*

Proof. $x \cdot (yx)^\rho = y^\rho \overset{(132)}{\Rightarrow} (yx)^\rho \cdot y^\rho = x$.
Let $(yx)^\rho = p$,

$$p \cdot y^\rho = x \tag{47}$$

$(yx)^\rho = p$ then, $yx = p^\lambda \overset{(132)}{\Rightarrow} xp^\lambda = y$,
$p^\lambda = x\backslash y$,

$$p = (x\backslash y)^\rho. \tag{48}$$

Substituting (48) into (47), we obtain $(x\backslash y)^\rho \cdot y^\rho = x$.
If $q = x\backslash y \Rightarrow xq = y$,

$$q^\rho \cdot (xq)^\rho = x. \tag{49}$$

Setting $q = e$ in (49) implies $x^\rho = x$. Also, setting $x = e$ in (49) gives $q^\rho \cdot q^\rho = e = q^{2\rho}$.
From $(xy)^\lambda \cdot x = y^\lambda \overset{(132)}{\Rightarrow}$

$$x \cdot y^\lambda = (xy)^\lambda. \tag{50}$$

Let $x \cdot y^\lambda = p^\lambda$. Then, $xy = p \overset{(132)}{\Rightarrow} yp = x$,

$$p = y\backslash x. \tag{51}$$

Substituting (51) into (50), we obtain $x \cdot y^\lambda = p^\lambda$.

$$x \cdot y^\lambda = (y\backslash x)^\lambda.$$

Set $q = y\backslash x \implies yq = x$,

$$yq \cdot y^\lambda = q^\lambda. \tag{52}$$

From (52), $y^\lambda = y^\rho$, $yq \cdot y^\rho = q^\lambda$.
Also, $q = q^\lambda \Rightarrow yq \cdot y^\rho = q$. Thus, (132)- parastrophe of WIPL is a CIPL. □

Remark 5. *The parastrophe of WIPL is either WIPL or CIPL and since every CIPL is a WIPL, WIP is parastrophically invariant property of a quasigroup.*

4. Discussion

This study examines the parastrophes of some notion of inverses in quasigroups. Our results showed that, of the 5 parastrophes of LIP quasigroup, (23)- parastrophe is a LIP quasigroup, (12)- and (132)- parastrophes are RIP quasigroup, while (13)- and (132)- parastrophes are an anti-commutative quasigroup. Similarly, (12)- and (132)- parastrophes of RIP quasigroup are LIP quasigroup, (13)-parastrophe of RIP is an RIP quasigroup, while (23)- and (123)- parastrophes are an anti-commutative quasigroup. As for the CIP quasigroup, only (12)-parastrophe is a CIP quasigroup; other parastrophes are symmetric quasigroups of order 2. Finally, (12)-parastrophe of WIP quasigroup is a CIP quasigroup, (13)-,(23)- and (132)-parastrophes of WIP quasigroup are CIP quasigroups, while (123)-parastrophe of WIP quasigroup is a WIP quasigroup.

5. Conclusions

As for the parastrophes of LIP quasigroup, the (23)-parastrophe is the only parastrophically invariant among the parastrophes of LIP quasigroup. (13) and (132)- parastrophes are anti-commutative while (12) and (123)-parastrophes are RIP. The (13)-parastrophe is parastrophically invariant among the parastrophes of RIP quasigroup. (12) and (132)-parastrophes are LIP while (23) and (123)-parastrophes are anti-commutative. All the parastrophes of cross inverse property quasigroups are symmetric quasigroups of order 2 except the (12)-parastrophe that is invariant. The parastrophe of WIPL is either WIPL or CIPL and since every CIPL is a WIPL, WIP is parastrophically invariant property of a quasigroup.

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