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Keywords: generalized uncertainty principle; spacetimes with horizons; quantum gravity

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On Quantum Gravity of Space-Time with Horizon

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Abstract

In the present work the high-energy deformation at Planck's scale for space-time with horizon is suggested on condition that the Generalized Uncertainty Principle is fulfilled. The deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition. In the beginning, this deformation is considered for black holes with Schwarzschild's metric. Then the obtained results are generalized for the De Sitter space and black holes with Schwarzschild-De Sitter Metrics. In the concluding section, the problems for further studies are put forward and the applications of the results are shown.

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1. Introduction

The quantum gravity problem still remains basic in fundamental physics, despite the availability of such powerful approaches to its solution as superstring theory [1,2], loop quantum gravity [3],causal dynamical triangulations [4] and some others. Very important for this problem is quantization of the particular definite classical metric (or geometry) ds^2 understood as finding of its high-energy deformation ds_q^2 (valid at Planck's scale) that on going to low energies represents ds^2 to a high accuracy:

$$ds_q^2 \xrightarrow{low\,energies} ds^2. \tag{1}$$

In this work the author suggests an approach to obtain such a deformation, referred to as **q-deformation**, for horizon spaces, specifically for Schwarzschild's black holes, De Sitter space and black holes with the De Sitter-Schwarzschild metric, on condition that the Generalized Uncertainty Principle (**GUP**) [5]–[8]is fulfilled. The work is based on the results in [8]. It is demonstrated that, with the use of these results interpreted differently in [9],one can construct a deformation of the Schwarzschild metric at Planck's scale and study its properties. Next, the obtained results are generalized for the De Sitter space and black holes with Schwarzschild-De Sitter Metrics.

2. Preliminary Information

2.1. The Schwarzschild Metric and GUP

First, let us consider the metric of an arbitrary static, spherically symmetric space-time (formulae (6.1.5) in [10])

$$ds_{sym}^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2}$$

and the adequate general solution of the vacuum Einstein equation – Schwarzschild's black hole (BH) with the metric [10,11]

$$ds_M^2 = -\left(1 - \frac{2MG}{c^2 r}\right)c^2dt^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

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where $d\theta^2 + \sin^2\theta d\phi^2 = d\Omega^2$ and M is a black hole mass.
For $c = 1$, the Schwarzschild radius is as follows:

$$r_M = 2MG/c^2 = 2MG. \tag{4}$$

Then (3) has the form

$$ds_M^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

Specifically, for the energies on the order of Plank energies (quantum gravity scales) $E \simeq E_p$, the Heisenberg Uncertainty Principle [12]

$$(\delta X)(\delta P) \geq \frac{\hbar}{2}, \tag{6}$$

may be replaced by the Generalized Uncertainty Principle (**GUP**) [8]

$$[X, P] = i\hbar \exp\left(\frac{\alpha^2 l_p^2}{\hbar^2}p^2\right), \quad (\delta X)(\delta P) \geq \frac{\hbar}{2}\left\langle \exp\left(\frac{\alpha^2 l_p^2}{\hbar^2}p^2\right)\right\rangle. \tag{7}$$

Then for saturate (**GUP***), $(\delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2$ and in virtue of formula (8) in [8]

$$(\delta X)(\delta P) = \frac{\hbar}{2}\exp\left(\frac{\alpha^2 L_{\rm Pl}^2}{\hbar^2}\left((\delta P)^2 + \langle P \rangle^2\right)\right). \tag{8}$$

It is further assumed that the holds true (**GUP***) i.e., (8). Then there is a Planck Schwarzschild black (further referred to as "minimal") with the minimal mass M_0 and the minimal radius r_{min} (formula (20) in [8]) that is a theoretical minimal length $r_{min} = r_{M_0} \doteq r_0$:

$$r_0 = l_{min} = (\delta X)_0 = \sqrt{\frac{e}{2}}\alpha l_p \doteq \ell, \quad M_0 = \frac{\alpha\sqrt{e}}{2\sqrt{2}}m_p, \tag{9}$$

where α - model-dependent parameters on the order of 1, e - base of natural logarithms, and $r_0 \propto l_p, M_0 \propto m_p$.
For a black hole of the mass M, the formula giving temperature $T_M = 1/(8\pi GM)$, with regard to the quantum-gravitational corrections within the scope of (**GUP**),(**GUP***), takes the form [8]

$$T_{M,q} = \frac{1}{8\pi MG}\exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{M_0}{M}\right)^2\right)\right) =$$
$$= \frac{1}{8\pi MG}\left(1 + \frac{1}{2e}\left(\frac{M_0}{M}\right)^2 + \frac{5}{8e^2}\left(\frac{M_0}{M}\right)^4 + \frac{49}{48e^3}\left(\frac{M_0}{M}\right)^6 + \ldots\right). \tag{10}$$

Here
a)the Lambert W-function $W(u)$ satisfying the equation (formulae (1.5) in [13] and (9) in [8])

$$W(u)e^{W(u)} = u. \tag{11}$$

$W(u)$ is the multifunction for complex variable $u = x + yi$. However, for real $u = x, -1/e \leq u < 0, W(u)$ is the single-valued continuous function having two branches denoted by $W_0(u)$ and $W_{-1}(u)$, and for real $u = x, u \geq 0$ there is only one branch $W_0(u)$ [13].
a) From (11) obviously:
$W(0) = 0$; if $(u \approx 0) \mapsto W(0) \approx 0, e^{W(u)} \approx 1$

$(u > 0) \equiv W(u) > 0; (u < 0) \equiv W(u) < 0.$

b)For large (classical BH) $M, r_M = \aleph r_0, \aleph \gg 1, \exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_{r_M}\right)\right) \approx 1$
For small (Planck BH or quantum BH) $M, r_M = \wp r_0, \wp \propto 1,$
$\exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_{r_M}\right)\right) < 1$
c)We have the following expression (formula (25) in [8]):

$$\exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{M_0}{M}\right)^2\right)\right) = \exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{r_0}{r_M}\right)^2\right)\right) =$$
$$\exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\alpha_{r_M}\right)\right) = \left(1 + \frac{1}{2e}\alpha_{r_M} + \frac{5}{8e^2}\alpha^2_{r_M} + \frac{49}{48e^3}\alpha^3_{r_M} + \dots\right) \tag{12}$$

As noted in [9], (10) may be interpreted in the following way:

$$T_{M,q} = \frac{1}{8\pi M_q G} \tag{13}$$

where M_q, r_{M_q} BH mass M and its event horizon radius r_M taking into account quantum-gravity corrections

$$M_q = M \exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_{r_M}\right)\right);$$
$$r_{M_q} = 2M_q G = r_M \exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_{r_M}\right)\right), \tag{14}$$

where [14,15]

$$\alpha_{r_M} \doteq \frac{l^2_{min}}{r^2_M} = \frac{M_0^2}{M^2} == \frac{r_0^2}{r_M^2}. \tag{15}$$

The quantities M_q, r_{M_q} are called the "quantum deformations" (or **q-deformation**) of mass and radius M, r_M.

3. The q-Deformed Schwarzschild Metric

3.1 Returning to the Schwarzschild metric (3),(5), due to the formula (14), we introduce its (**q-deformation**) $ds^2_{M_q}$:

$$r \mapsto r_q = r \exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_r\right)\right) \geq r_0$$
$$ds^2_M \mapsto ds^2_{M_q} = -\left(1 - \frac{r_{M_q}}{r_q}\right)dt^2 + \left(1 - \frac{r_{M_q}}{r_q}\right)^{-1} dr_q^2 + r_q^2 d\Omega^2, \tag{16}$$

where r is the radial coordinate in formula (5) and $\alpha_r = r_0^2/r^2$.
It is clear that $ds^2_{M_q}$ is also the Schwarzschild metric with the radial coordinate r_q and the Schwarzschild radius r_{M_q}. On the condition (low-energy domain)

$$r \geq r_M > r_{M_q} \gg r_0, \tag{17}$$

because of point b) in Section 2, we have

$$r_q \approx r, r_{M_q} \approx r_M. \tag{18}$$

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In this case $ds^2_{M_q}$ is coincident with ds^2_M (3) to a high accuracy. But we are interested in their high-energy domain too:

$$r \geq r_{M_q} \cong r_0, \tag{19}$$

where fomula (18) is invalid.

Remark 3.1

3.1.1 The black holes meeting the condition (19) are called *the quantum black holes* (**qbhs**) in accordance with the terminology in [16]–[18] .

3.1.2 Note that, since we have $\exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_r\right)\right) < 1$, by virtue of the first line of (16) $r > r_q, r_M > r_{M_q}$.

For the metric in (16) we can easily find the **q-deformed** Christoffel symbols

$$\Gamma^{\sigma}_{\mu\lambda,q} = \frac{1}{2} g^{\nu\sigma}_q \left(\frac{\partial g_{\mu\nu,q}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu,q}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda,q}}{\partial x^\nu}\right), \tag{20}$$

where $g_{\beta\gamma,q}, g_q^{\beta\gamma}$ are the metric's components $ds^2_{M_q}$ (16).

In particular, nonzero components of the Christoffel symbols **q-deformed**in Schwarzschild's Metric $ds^2_{M_q}$ (16) (for $G = 1$)are of the form:

$$\begin{aligned}
\Gamma^r_{tt,q} = \frac{r_{M_q}(r_q - r_{M_q})}{2r_q},\ \Gamma^r_{rr,q} = -\frac{r_{M_q}}{2r_q(r_q - r_{M_q})},\ \Gamma^r_{\theta\theta,q} = -(r_q - r_{M_q}); \\
\Gamma^r_{\phi\phi,q} = -(r_q - r_{M_q})\sin^2\theta,\ \Gamma^t_{rt,q} = \frac{r_{M_q}}{2r_q(r_q - r_{M_q})},\ \Gamma^\theta_{r\theta,q} = r_q^{-1}; \\
\Gamma^\theta_{\phi\phi,q} = -\sin\theta\cos\theta,\ \Gamma^\phi_{r\phi,q} = r_q^{-1},\ \Gamma^\phi_{\theta\phi,q} = \frac{\cos\theta}{\sin\theta}.
\end{aligned} \tag{21}$$

From the last formula we can derive the **q-deformed** Riemann tensor

$$R^{\mu}{}_{\nu\alpha\beta,q} \equiv \partial_\alpha \Gamma^\mu_{\nu\beta,q} - \partial_\beta \Gamma^\mu_{\nu\alpha,q} + \Gamma^\mu_{\gamma\alpha,q}\Gamma^\gamma_{\nu\beta,q} - \Gamma^\mu_{\gamma\beta,q}\Gamma^\gamma_{\nu\alpha,q}; \tag{22}$$

the **q-deformed** Ricci tensor:

$$R_{\mu\nu,q} \equiv R^{\alpha}{}_{\mu\alpha\nu,q};$$

and the **q-deformed** Ricci scalar:

$$R_q \equiv R_{\mu\nu,q}\, g^{\mu\nu,q}.$$

For these quantities we can have in the explicit form the **q-deformation** of Einstein equations in the **q-deformed** Schwarzschild Metric(16)in the vacuum:

$$R_{\mu\nu,q} - \frac{1}{2} R_q\, g_{\mu\nu,q} = 0. \tag{23}$$

It is obvious that, on the condition (17) (and hence (refCond-Exter1)), the last equation agrees with Einstein equations for the Schwarzschild Metric (3),(5) in the vacuum to a high accuracy

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0. \tag{24}$$

Remark 3.2

3.2.1 As noted in ([9] **Remark 2.3**), knowing the expansion $\exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\alpha_r\right)\right)$ in terms of the small parameter α_r, in formula (12) one can, by the method of undetermined Coefficients, easily obtain the small-parameter expansion of the inverse number $\exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_r\right)\right)$, and also of all the integer powers

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for this exponent. Specifically, with formula (12)we can verify that in this case in $\exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_r\right)\right)$ the factor of α_r is $-1/(2e)$ and of α_r^2 it is, respectively, $-3/(8e^2)$, and so on.
3.2.2 It is clear, that, when the condition (17)is valid, the value of r_q is, to a high accuracy, equal to the leading term of its series expansion in terms of the small parameter α_r that is equal to r. Because of this, we can use the approximation $r_q \approx r$. However, this is not the case for the condition (19). r becomes small but contributions made by the expansion terms in r_q according to α_r, containing α_r^n for the integer $n \geq 1$, are growing. There is no approximation $r_q \approx r$ in this pattern. To calculate r_q, in a series expansion in terms of α_r we should use the terms involving $\alpha_r^n, n \geq 1\, integer$.
Remark 3.2 remains valid on the substitution $r \to r_M$.
Using the substitution $r \mapsto r_q = r\exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_r\right)\right)$ from (16), we can derive a **q-deformation** for the metric of an arbitrary static, spherically symmetric space-time ds^2_{sym} in formulae (2) in a similar way:

$$ds^2_{sym,q} = -f(r_q)dt^2 + h(r_q)dr_q^2 + r_q^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{25}$$

On the validity of (17),(18), to a high accuracy, we have

$$ds^2_{sym,q} = ds^2_{sym}. \tag{26}$$

As follows from formulae (7)–(9), in the metric (16) there is no central singularity $r = 0$ as it "closed"by a minimal sphere with the radius $r_0 = l_{min} = \ell$ from (9). Consequently, the Kretschmann scalar (formula (31.7) in [19]) for the metric (16), as distinct from the metric (5), has a finite maximum $K^{max}_{M_q}$:

$$\begin{aligned} K_{M_q} &= R_{abcd,q}R^{abcd,q} = \frac{48GM_q^2}{r_q^6} = \frac{24r^2_{M_q}}{r_q^6}, \\ \lim_{r_q\to r_0} K_{M_q} &= \lim_{r_q\to r_0}\frac{24r^2_{M_q}}{r_q^6} = \frac{24r^2_{M_q}}{r_0^6} \doteq K^{max}_{M_q}. \end{aligned} \tag{27}$$

In this **q-deformed** pattern let us briefly recall the formulae required for the interior solution in the case of a Schwarzschild black hole with the metric (3), taking due regard for the substitutions $r_M \to r_{M_q}, r \to r_q$ from formula (16).
Then, within this black hole, the matter energy-momentum tensor takes the form corresponding to the perfect fluid

$$T_{ab,q} = \rho_q u_{a,q}u_{b,q} + P_q(g_{ab,q} + u_{a,q}u_{b,q}), \tag{28}$$

where ρ_q and P_q are the density and the pressure, respectively. Note that in the suggested pattern the four-velocity u^a [10] is also **q-deformed**. Indeed, as in the classical consideration we use formula (6.2.2) from [10]

$$u^a = f(r)^{1/2}(dt)^a, \tag{29}$$

in this case we have the following substitution:

$$u^a \to u^a_q = f(r_q)^{1/2}(dt)^a, \tag{30}$$

where $f(r)$ is the function from (2),$f(r_q)$ is a **q-deformation** of $f(r)$ from (25). Similar to the classical case, it is assumed that the index of the four-velocity is rising up and falling down by means of the metric $g_{ab,q}, (g^{ab}_q)$ and the following condition is met:

$$g_{ab,q}u^a_q u^b_q = g_{ab}u^a u^b = 1 \tag{31}$$

Normalization is used here $G = \hbar = c = 1$ as in [10].
For the appropriate $T_{\mu\nu}$ (formulae (6.2.3) and (6.2.4) in [10]) we have the **q-deformed** analogue $T_{\mu\nu,q}$:

$$\begin{aligned}
8\pi T_{00,q} = 8\pi\rho_q = G_{00,q} = R_{00,q} + \frac{1}{2}(R^0_{0,q} + R^1_{1,q} + R^2_{2,q} + R^3_{3,q}) = \\
= (r_q h(r_q)^2)^{-1} h(r_q)' + r_q^{-2}(1 - h(r_q)^{-1}); \\
8\pi T_{11,q} = 8\pi P_q = G_{11,q} = R_{11,q} - \frac{1}{2}(R^0_{0,q} + R^1_{1,q} + R^2_{2,q} + R^3_{3,q}) = \\
= (r_q f(r_q) h(r_q))^{-1} f(r_q)' - r_q^{-2}(1 - h(r_q)^{-1}).
\end{aligned} \tag{32}$$

In (32) a component of the energy-momentum tensor $8\pi T_{22,q} = G_{22,q}$ is not written as it is also equal to $8\pi P_q$, the prime means a derivative with respect to r_q, though in the case that (17),(18)are valid, we should have $T_{\mu\nu,q} = T_{\mu\nu}$ to a high accuracy.
Thus, in the general case we can write a **q-deformation** of Einstein equations for the **q-deformed** Schwarzschild Metric (formula (16))

$$G_{\mu\nu,q} \equiv R_{\mu\nu,q} - \frac{1}{2} R_q \, g_{\mu\nu,q} = 8\pi T_{\mu\nu,q}, \tag{33}$$

where equation (23) is valid for the vacuum solution. With formulae (17),((18)),the equations in (33) are, to a high accuracy, coincident with the canonical Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{34}$$

As it has been noted above, in this case the vacuum solution involves the limiting transition from (23) in (24).
It should be noted that for the condition (19)this is not the case. As in the pattern of (19) the consideration is given to the Planckian scales and energies only, for this pattern there are no equations (34). It is associated with the **q-deformed** Einstein equations (33),which, in going to the low-energy limit (17), result in (34).
It is important that, due to the condition (19) and **Remark 3.2.2**, the left and right sides of (33) are power series expanded in terms of the small parameter α_r, the terms which contain $\alpha_r^n, n \geq 1$, *integer* playing a significant role.
Now consider a massive Schwarzschild black hole of the mass M, with the large event horizon radius $r_M > r_{M_q} \gg r_0$.In this case, for the interior of this black hole, (17) is replaced by the condition

$$r_M > r_{M_q} \geq r_q \geq r_0, r_{M_q} \gg r_0. \tag{35}$$

As follows from the last formula, in this case we should consider only the **q-deformed** analogs of the corresponding formulae in [10]with replacement of $r = 0$ by $r_q = r_0$. Then the following formulae represent the **q-deformed** analog of formulae (6.2.6)–(6.2.8) from [10]:

$$\frac{1}{r_q^2}\frac{d}{dr_q}[r_q(1 - h(r_q)^{-1})] = 8\pi\rho_q \tag{36}$$

$$h(r_q) = [1 - \frac{2m(r_q)}{r_q}]^{-1} \tag{37}$$

and

$$m(r_q) = 4\pi \int_{r_0}^{r_q} \rho(r'_q) r'^2_q dr'_q + a_q, \tag{38}$$

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where a_q is a constant.
In the classical pattern the corresponding formula for $m(r)$ ((6.2.8) in [10])is of the form

$$m(r) = 4\pi \int_0^r \rho(r')r'^2 dr' + a, \tag{39}$$

where a is a constant.
According to the classical consideration, the constant a in (39) is chosen to comply with the requirement for smoothness of the metric at $r = 0$ on the space-like hypersurface Σ orthogonal to the time-like Killing vector field ξ^a (p.126 in [10]). From where it follows that, as $r \to 0$, "the area of spheres approaches 4π times the square of their proper radius (p.126 in [10])". Then we have $a = 0$. At the same time, in the classical pattern the space-time geometry is independent of the scales under study (i.e., of the energies) [10]. But by our approach, according to **GUP**,**GUP*** (7)–(9), this is not the case and the Planck-scales physics, and hence the spatial geometry at these scales, differs from the classical one and is yet unknown. In our consideration the condition $r \to 0$ is replaced by the condition $r_q \to r_0$, whereas the condition "the area of spheres approaches 4π times the square of their proper radius " in the limit $r_q \to r_0$ gives the area of a minimal sphere with the radius r_0 from (9):

$$A_{min} = 4\pi r_0^2. \tag{40}$$

It is clear that the constants a and a_q have the dimensions of mass. $a = 0$ - mass of the As in In the case under study we have $r_q \to r_0$, then $a_q = M_0$. The equation (39) in this case takes the form

$$m(r_q) = 4\pi \int_{r_0}^{r_q} \rho(r'_q)r'^2_q dr'_q + M_0 \doteq \widetilde{m}(r_q) + M_0. \tag{41}$$

This is quite natural because in
3.a formula (9) cuts out a minimal sphere with the radius r_0, whose mass is not included by the first term in the left side (41) and which is added into the second term in the right side of this equation.

3.b It is obvious that formula (41) presents a **q-deformation** of (39), as in the absence of a nonzero minimal length we have

$$r_0 \to 0, r_q \to r, M_0 \to 0, \widetilde{m}(r_q) \to m(r_q). \tag{42}$$

Then (41) and (39) are equivalent.
In a similar way we can obtain a **q-deformation** for other formulae in [10],in particular a **q-deformed** expression for the total *proper mass*:

$$M_{prop,q} = 4\pi \int_{r_0}^{r_{M_q}} \rho(r_q)r_q^2[1 - \frac{2m(r_q)}{r_q}]^{-1/2} dr_q + M_0. \tag{43}$$

In the limiting transition $r_0 \to 0$ we can obtain

$$\lim_{r_0 \to 0} M_{prop,q} = 4\pi \int_0^{r_M} \rho(r)r^2[1 - \frac{2m(r)}{r}]^{-1/2} dr = M_{prop}, \tag{44}$$

in line with the corresponding formula for the total *proper mass* in the classical pattern (formula (6.2.11) in [10]). Using the notation of [10], we have $M_{prop} \doteq M_p, r_M \doteq R$.
In this way the above formulae are applicable to large (high-mass) black holes ($r_M \approx r_{M_q} \gg r_0$), that is the calculations in ([10],Section 6) may be applied for such black holes with the substitution $r \to r_q, r_M \to r_{M_q}$ and $(r = 0) \mapsto (r = r_0)$. In the limiting transition (42) we can derive the initial formulae of of Section 6 in [10].
Thus, the calculations of this section may be applied to massive black holes with the masses $M \gg M_0$.Such black holes may be formed according to the following two scenarios:

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3.a.1 when massive stars collapse at the end of their life cycle [11];

3.b.1 as primordial black holes in the Early Universe due to the gravitational collapse of the high-density matter [20]–[26].
The quantum black holes (**qbhs**) **Remark 3.1.1** May be formed as a result of the following three different scenarios:

3.a.2 as primordial black holes similarly to point **3.b.1** in the Early Universe but only at the earliest stages. In [9] the author has studied the effect from inclusion of **qgcs** for such black holes generated during the pre-inflationary era, elucidating how this inclusion may result in shifting of the inflation parameters and the parameters of cosmological perturbations.

3.b.2 If **GUP**,**GUP*** (7)–(9)are valid, **qbh** are formed as Planck remnants of Hawking evaporation for massive black holes [8,27,28]. As demonstrated in [28], in this case evaporation of a black hole is suppressed close to the Planck scale, the hole tends to some finite nonzero limit (Planck remnant with the mass $\tilde{m}$) through the absorption of quanta facilitating a balance of the black hole in a thermal bath.

3.c.2 Besides, in [29,30] it has been shown that **qbhs** may be formed due to the process of quantum gravitational tunneling from a hot flat space.
But for **qbhs** (**3.a.2–3.c.2**) we can apply only the Formulas (16)–(23) associated with the vacuum solution, whereas they are inapplicable for the black hole interior because the space geometry at Planck's scale is unknown.

4. Remarks on De Sitter Space and the Schwarzschild-De Sitter Metrics q-deformation

De Sitter space presents a solution for Einstein Equations in the vacuum with the positive cosmological constant Λ:

$$\Lambda = 3/\ell_{\Lambda}^{2}, \tag{45}$$

where $\ell_{\Lambda} = \sqrt{\frac{3}{\Lambda}}$ is its characteristic length or the same curvature radius.
As a rule, de Sitter space metric is given in a static coordinate system (formulae (2.3),(2.4) in [31]):

$$\frac{ds^2}{\ell_{\Lambda}^2} \doteq d\hat{s}_{dS}^2 = -V(\hat{r})\,dt^2 + \frac{1}{V(\hat{r})}d\hat{r}^2 + r^2 d\Omega_2^2,$$
$$V(\hat{r}) = 1 - \hat{r}^2, \tag{46}$$

where r is a radial coordinate and $\hat{r} = r/\ell_{\Lambda}$ is a dimensionless quantity. And in the general case we obviously have $0 \le \hat{r} \le 1$, i.e., $r \le \ell_{\Lambda}$.
In the notation of (39), the Schwarzschild de Sitter (SdS) black holes present black-hole solutions of Einstein equations in the De Sitter space given by the metric [31,32]

$$\frac{ds_{SdS}^2}{\ell_{\Lambda}^2} \doteq d\hat{s}_{SdS}^2 = -V(\hat{r})\,dt^2 + \frac{1}{V(\hat{r})}d\hat{r}^2 + r^2 d\Omega_2^2,$$
$$V(\hat{r}) = 1 - \frac{2\hat{M}}{\hat{r}} - \hat{r}^2. \tag{47}$$

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For $\hat{r} \ll 1$, the metric ds^2_{SdS} is, to a high accuracy, identical to the Schwarzschild metric (3),(5). Actually, in this case r_M, r, r_0 from (3),(5),(9) is replaced by dimensionless quantities (for $G = 1$ in (5))

$$r_M \mapsto \hat{r}_{\hat{M}} \doteq 2\hat{M}, r \mapsto \hat{r}, r_0 \mapsto \hat{r}_0 > 0. \tag{48}$$

Within the scope of **GUP**,**GUP*** (7)–(9), all the quantities from Formulas (46) and (47) are replaced by the corresponding quantities with the subscript q:

$$\begin{aligned} \hat{r} > \hat{r}_q = \hat{r}\exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_{\hat{r}}\right)\right) \geq \hat{r}_0 > 0, \\ \hat{r}_{\hat{M}} > \hat{r}_{\hat{M},q} = \hat{r}_{\hat{M}}\exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_{\hat{r}_{\hat{M}}}\right)\right) \geq \hat{r}_0 > 0, \end{aligned} \tag{49}$$

where we have $\alpha_{\hat{r}_M} = \hat{r}_0^2/\hat{r}^2_{\hat{M}}, \alpha_{\hat{r}} = \hat{r}_0^2/\hat{r}^2$.
So, in this case **q-deformations** result from Formulas (46) and (47) $d\hat{s}^2_{dS,q}, d\hat{s}^2_{SdS,q}$ in accordance with

$$\begin{aligned} d\hat{s}^2_{dS} \mapsto d\hat{s}^2_{dS,q} = -V(\hat{r}_q)\,dt^2 + \frac{1}{V(\hat{r}_q)}d\hat{r}_q^2 + r_q^2 d\Omega_2^2, \\ V(\hat{r}_q) = 1 - \hat{r}_q^2, \end{aligned} \tag{50}$$

and

$$\begin{aligned} d\hat{s}^2_{SdS} \mapsto d\hat{s}^2_{SdS,q} = -V(\hat{r}_q)\,dt^2 + \frac{1}{V(\hat{r}_q)}d\hat{r}^2 + r_q^2 d\Omega_2^2, \\ V(\hat{r}_q) = 1 - \frac{2\hat{M}_q}{\hat{r}_q} - \hat{r}_q^2. \end{aligned} \tag{51}$$

The last formula for $\hat{r} \ll 1$ is a quantum deformation of Schwarzschild's black holes studied in the previous sections. As we have $\ell_\Lambda \gg r_0$, the general condition

$$[\hat{r} \ll 1] \equiv [\hat{r} \simeq \hat{r}_0] \bigcup [\hat{r}_0 \ll \hat{r}_{\hat{M}} \leq \hat{r} \ll 1] \tag{52}$$

conforms to all the cases considered in Section 3 (both to large-size Schwarzschild black holes and **qbhs**).
Remark 4.1
4.1.1 When **GUP**,**GUP*** (7)-(9) are valid, the **q-deformed** De Sitter metric $d\hat{s}^2_{dS,q}$ (50) has, as distinct from the initial one $d\hat{s}^2_{dS}$ (46), a very important property:
because $\hat{r}_{max} = 1$ and from the first line of formula (49) it follows that

$$\hat{r}_q < 1,$$

is always the case, the **q-deformed** De Sitter metric $d\hat{s}^2_{dS,q}$ has no singularities in a static coordinate system, as distinct from the initial (classical) metric $d\hat{s}^2_{dS}$ having the apparent horizon singularity $\hat{r} = 1$;

4.1.2 For the **q-deformed** De Sitter metric $d\hat{s}^2_{dS,q}$ (50) we can construct, in similarity with the formulae from the previous section, a **q-deformation** of all ingredients for Einstein equations (Christoffel symbols, Riemann tensor, Ricci tensor, Ricci scalar) and to derive the corresponding **q-deformed** Einstein equations in this case:

$$R_{\mu\nu,q} - \frac{1}{2}R_q\,g_{\mu\nu,q} = -\Lambda g_{\mu\nu,q}. \tag{53}$$

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In the absence of the nonzero minimal length $\hat{r}_0 \to 0, \hat{r}_q \to \hat{r}$ (analog (42)), we obtain the classical De Sitter space with the metric (46) for Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -\Lambda g_{\mu\nu}. \tag{54}$$

Remark 4.2
In all the above formulae from this section it is assumed that $\ell_\Lambda = const$ and hence $\Lambda = const$. Though, it is known that $\ell_\Lambda = H_0^{-1}$,where $H_0 \doteq H_{dS}$ - (initial) value of the Hubble parameter in the de Sitter space [33,34]. The work [35] presents a study in the semi-classical pattern of the effects associated with the formation of black holes in the de Sitter space during the pre-inflationary era. As shown in [9], a great contribution into cosmological parameters is made by inclusion of quantum-gravitational corrections for these black holes in accordance with formulae given in Sections 2,3 of this paper. If m, r_m – mass and radius of such a black hole , with the included corrections, H_0 is varying in the following way (formula (22) in [9]):

$$H_0 \to H_{0,\mathrm{m}} = H_0 \exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\alpha_{\mathrm{r}_m}\right)\right). \tag{55}$$

From where we have

$$\ell_\Lambda \to \ell_{\Lambda,\mathrm{m}} = \ell_\Lambda \exp\left(\frac{1}{2}W\left(-\frac{1}{e}\alpha_{\mathrm{r}_m}\right)\right) \tag{56}$$

in a complete agreement with transformations in formula (49). Substituting $\ell_{\Lambda,\mathrm{m}}$ for ℓ_Λ in all the mentioned formulae, we can have in all of them the metric dependent on m, r_m.

Remark 4.3
The case of very large black holes for SdS-metrics with event horizon $r_M \approx \ell_\Lambda$, or $\hat{r}_{\hat{M}} \approx 1$ remained beyond consideration in this section. It is clear that the suggested **q-deformation** has no effect on the characteristics of such black holes. But a very interesting problem may be studied additionally.
In [36] it has been demonstrated that the evaporation of such black holes may be disturbed.However, all the calculations in [36] have been performed in the perturbative mode beyond the scope of **GUP**. It seems interesting to study the problem with due regard for the conditions stipulated in the present work and to find the relevant changes in the results from [36].

5. Conclusion and Further Steps

It may be concluded that this paper suggests an approach to studies of quantum gravity in horizon spaces on condition that the Generalized Uncertainty Principle is valid.
The author is planning to consider the following problems in his his further works:

5.1 Elucidation of the geometry at Planck's scale that is corresponding to **q-deformed** Einstein Equations (23),(33). The model proposed in [37,38] may be the adequate basis for such a study.

5.2 Finding of the outcome from the use of the presented results for thermodynamic of spaces with horizon [39,40];

5.3 Possible generalization of the proposed approach to other spaces (without horizon) and elucidation of its relation to the holographic principle.

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References

1. Polchinski Joseph 1998 *String Theory, Volume I: An Introduction to the Bosonic String* Cambridge Monographs on Mathematical Physics 424pp.
2. Polchinski Joseph 1998 *String Theory Vol. II: Superstring Theory and Beyond* Cambridge Monographs on Mathematical Physics 531pp.
3. Rovelli Carlo 2004 *Quantum gravity* Cambridge Monographs on Mathematical Physics 455pp.
4. Loll R 2020 Quantum gravity from causal dynamical triangulations: a review *Class. Quantum Grav.* **37** 013002
5. Adler R J, Santiago D I 1999 On gravity and the uncertainty principle *Mod. Phys. Lett. A.* **14** 1371–1378
6. Tawfik Abdel Nasser,Diab Abdel Magied 2014 Generalized Uncertainty Principle: Approaches and Application, *Int. J. Mod. Phys. D.* **23** 1430025
7. Kempf A, Mangano G, and Mann R B 1995 Hilbert space representation of the minimal length uncertainty relation *Phys. Rev. D.* **52** 1108–1118
8. Nouicer 2007 Quantum-corrected black hole thermodynamics to all orders in the Planck length *Phys.Lett B.* **646** 63
9. Shalyt-Margolin Alexander 2025 The Quantum Primordial Black Holes, Dimensionless Small Parameter, Inflationary Cosmology and Non-Gaussianity *Annals of Physics* **474** 169930
10. Wald R M 1984 General Relativity *University of Chicago Press* Chicago Ill USA
11. Frolov Valeri P, Novikov Igor D 1997 *Black Hole Physics: Basic Concepts and New Developments* Kluwer Academic Publishers
12. Heisenberg W 1927 Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik *Z. Phys.* **43** 172
13. Corless R M, Gonnet G H, Hare D E,Jerey D J and Knuth D E 1996 On the Lambert W function *Adv. Comput. Math.* **5** 329
14. Shalyt-Margolin A E 2003 Quantum mechanics at Planck scale and density matrix *Intern. J. Mod. Phys. D* **12** 1265
15. Shalyt-Margolin A E and Tregubovich A Ya 2004 Deformed density matrix and generalized uncertainty relation in thermodynamics, *Mod. Phys. Lett. A* **19** 71
16. Calmet Xavier,Gong Wei, Hsu Stephen D H 2008 Colorful quantum black holes at the LHC *Phys.Lett.B* **668** 20-23
17. Calmet Xavier 2015 Fundamental Physics with Black Holes, Chapter 1 in *Quantum aspects of black holes* Ed.Xavier Calmet Fundamental Theories of Physics vol.178,pp.1-26,Springer International Publishing Switzerland
18. Calmet Xavier, Carr Bernard, Winstanley Elizabeth 2014 *Quantum Black Holes* Springer 104 pp.
19. Misner C W, Thorne K S, and Wheeler J A 1973 *Gravitation* Freeman San Francisco
20. Zeldovich Y B and Novikov I D 1967 The Hypothesis of Cores Retarded during Expansion and the Hot Cosmological Model *Soviet Astron.J. (Engl.Transl.)* **10** 602
21. Hawking Stephen 1971 Gravitationally Collapsed Objects of Very Low Mass *Monthly Notices of the Royal Astronomical Society* **152** 75
22. Carr B J, Hawking S W 1974 Black Holes in the Early Universe *Monthly Notices of the Royal Astronomical Society* **168** 399
23. Khlopov M Yu 2010 Primordial Black Holes *Res.Astron.Astrophys.* **10** 495-528
24. Belotsky K M,*et al.* 2014 Signatures of primordial black hole dark matter *Mod. Phys. Lett. A.* **29(37)** 1440005
25. Belotsky K M,*et al.* 2019 Clusters of primordial black holes *Eur. Phys. J. C.* **79** 246
26. Ketov S V, Khlopov M Y 2019 Cosmological Probes of Supersymmetric Field Theory Models at Superhigh Energy Scales *Symmetry* **11(4)** 511
27. Adler R J, Chen Pisin and Santiago D I 2001 The Generalized Uncertainty Principle and Black Hole Remnants *Gen.Rel.Grav.* **33** 2101.
28. Custodio PS, Horvath J E 2003 The Generalized Uncertainty Principle, entropy bounds and black hole (non-)evaporation in a thermal bath *Class.Quant.Grav.* **20** L197-L203
29. Gross David J, Perry Malcolm J, and Yaffe Laurence G 1982 Instability of flat space at finite temperature *Phys. Rev. D* **25** 330
30. Kapusta Joseph I 1984 Nucleation rate for black holes *Phys. Rev. D* **30** 831
31. Bousso Raphael 2002 Adventures in de Sitter space *Workshop on Conference on the Future of Theoretical Physics and Cosmology in Honor of Steven Hawking's 60th Birthday,Cambridge, United Kingdom* 539-569
32. Galante Damian A 2023 Modave Lecture Notes on de Sitter Space and Holography *PoS Modave* 2022 003

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33. Gorbunov Dmitri S and Rubakov Valery A 2011 *Introduction to the Theory of the Early Universe.Cosmological Perturbation and Inflation Theory* World Scientific Singapore.
34. Weinberg S 2008 *Cosmology* Oxford University Press
35. Prokopec Tomislav and Reska Paul 2011 Scalar cosmological perturbations from inflationary black holes *Journal of Cosmology and Astroparticle Physics* **03** 050
36. Bousso Raphael, Hawking Stephen W 1998 (Anti-)Evaporation of Schwarzschild-de Sitter Black Holes *Phys.Rev.D* **57** 2436-2442
37. Scardigli Fabio 1999 Generalized Uncertainty Principle in Quantum Gravity from Micro-Black Hole Gedanken Experiment *Phys.Lett.B* **452** 39-44
38. Scardigli Fabio 2000 Gravity coupling from micro-black holes *Nucl.Phys.Proc.Suppl.* **88** 291–294
39. Padmanabhan T 2005 Gravity and the Thermodynamics of Horizons *Phys.Rept.* **406** 49-125
40. Padmanabhan T 2010 Thermodynamical Aspects of Gravity: New insights *Rept. Prog. Phys.* **73** 046901