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Not peer-reviewed version

A Metric-Based, Coordinate-Dependent Condition for Petrov Type D Spacetimes

Joseph Loyden-Dutton *

Posted Date: 21 November 2025

doi: 10.20944/preprints202511.1694.v1

Keywords: Newman-Penrose formalism; Petrov classification; Type D spacetime; Spinors; Kerr Metric; exact solutions

Article

A Metric-Based, Coordinate-Dependent Condition for Petrov Type D Spacetimes

Joseph Loyden-Dutton

Department of Physics, Imperial College London, London, UK; jl10324@ic.ac.uk

Abstract

To the best of the author's knowledge, this work is among the first to present a coordinate-dependent, metric-component condition that must be satisfied by all Type D spacetimes, each admitting two geodesic, shear-free null congruences. While such conditions are well known, this work provides a clear and practical formulation that enables direct verification of Type D spacetimes and the computation of metric components without assuming an *ansatz*. Derived from the vanishing of the Newman–Penrose spin coefficient λ in Kerr spacetime, the condition is expressed entirely in terms of metric components and their radial derivatives, eliminating the need for tetrad construction. The condition has been validated numerically for Kerr, Kerr–Newman, Schwarzschild, and static de Sitter spacetimes. It may also guide the construction of new exact solutions or the testing of proposed ansatz with desirable geometric properties. Similar relations may exist for other spin coefficients and algebraic types of spacetimes.

Keywords: Newman-Penrose formalism; Petrov classification; Type D spacetime; Spinors; Kerr Metric; exact solutions

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1. Introduction

1.1. Background and Overview

We present a coordinate-level expression of the condition $\lambda = 0$ for *geodesic, shear-free null congruences* in Petrov Type D spacetimes (for *Petrov classification*, see Stewart [1], Sec. 2, p. 76-80). Such congruences are central to the structure of Kerr, Schwarzschild, and de Sitter metrics [1,2].

1.2. Literature Review

The Newman–Penrose formalism [3] provides a framework for analysing shear-free, geodesic null congruences. Kinnersley [4] classified all Type D vacuum metrics, while the Geroch–Held–Penrose (GHP) formalism [5], and its higher-dimensional extensions [6,7], facilitate the analysis of spin coefficients such as λ. Debney, Kerr, and Schild [8] investigated algebraically degenerate Einstein and Einstein–Maxwell solutions, and non-stationary Kerr-type congruences [9] illustrate the flexibility of tetrad approaches.

While these works establish geodesic, shear-free null congruences, none provide an explicit, coordinate-dependent equation for $\lambda = 0$. Expressing the condition directly in terms of metric components fills this gap, enabling direct verification for algebraic speciality (Type D spacetimes) without constructing a tetrad. It also imposes a condition on the metric, making the field equations easier to solve without assuming an ansatz, bridging abstract NP/GHP formalism with practical computation.

1.3. Notation and Conventions

This paper uses the mostly positive metric signature $(-,+,+,+)$ and natural units where $G = c = 1$. Coordinates are Boyer-Lindquist [10] (t, r, θ, ϕ), where t is time, r is radial, θ is polar, and ϕ is azimuthal.

The spacetime metric components $g_{\mu\nu}$ and their derivatives are assumed sufficiently smooth (at least C^2) to allow the use of differential geometric tools and the Newman-Penrose formalism. Standard causality and global hyperbolicity conditions are also assumed to ensure well-defined null congruences.

For Kerr spacetime, only five independent, non-zero components are needed: g_{tt}, $g_{t\phi}$, g_{rr}, $g_{\theta\theta}$, and $g_{\phi\phi}$, reflecting its stationary and axisymmetric structure. Partial derivatives are denoted by a comma, e.g., $g_{\mu\nu,r} = \partial_r g_{\mu\nu}$.

Spinor notation may use o and ι as shorthand for o^A and ι^A, omitting explicit spinor indices.

2. Theoretical Setup

2.1. Spin Coefficient Equations

While Stewart [1] provides the primary reference for the formalism, the derivation and conclusions in this work are primarily original.

Spin coefficients, components of the Ricci rotation coefficients, play a central role in analysing null congruences. In Type D spacetimes, l^μ and n^μ can each be chosen to each be separate geodesic, shear-free null congruence, eliminating four of the spin coefficients. In Kerr, an additional spin coefficient, ϵ can be set to zero [11], though this is not required here. These conditions simplify the NP field equations and provide insight into the structure of the null tetrad components. They underlie the derivation of the *Kinnersley tetrad* and thus the Kerr metric itself. However, in this work, we focus on the single condition $\lambda = 0$.

2.2. Choice of Tetrad

It's important to understand that the choice of tetrad is our own. For convenience, clarity, or benefit of deeper exploration; it is a *gauge freedom*. See Appendix A for the full conditions of a null tetrad.

Following the Kinnersley tetrad [12], we set $l^\theta = n^\theta = 0$, and $m^r = 0$, so that

$$l^\mu = (l^t, l^r, 0, l^\phi), \quad n^\mu = (n^t, n^r, 0, n^\phi), \quad m^\mu = (m^t, 0, m^\theta, m^\phi). \tag{1}$$

Kerr spacetime is stationary and axisymmetric, implying invariance under the Killing vector fields ∂_t and ∂_ϕ. Thus, the Kerr metric satisfies $\mathcal{L}_{\partial_t} g_{\mu\nu} = 0$ and $\mathcal{L}_{\partial_\phi} g_{\mu\nu} = 0$ followed by the Killing vector field condition [1].

The metric tensor components written in terms of spinors and null tetrads can then be substituted into the Killing condition:

$$g^{\mu\nu} = \epsilon^{AB}\epsilon^{A'B'} = -2l^{(\mu}n^{\nu)} + 2m^{(\mu}\bar{m}^{\nu)} \implies \mathcal{L}_{\partial_t} g^{\mu\nu} = \mathcal{L}_{\partial_t}(-2l^{(\mu}n^{\nu)} + 2m^{(\mu}\bar{m}^{\nu)}) = 0. \tag{2}$$

The same equation will hold for the axial Killing vector field ∂_ϕ. Therefore, a convenient choice is to let all the components of the null tetrad vectors be independent of time, t, and polar angle, ϕ.

Thus, our null tetrad is now

$$l^\mu = l^\mu(r, \theta), \qquad n^\mu = n^\mu(r, \theta), \qquad m^\mu = m^\mu(r, \theta). \tag{3}$$

To calculate the final equation, only spin coefficient $\lambda = 0$ needs to be considered.

3. Derivation Using $\lambda = 0$

For a complete step-by-step derivation, see Appendix B.

σ describes the shear of l^μ and the λ coefficient is analogous to this and describes the shear of n^μ. If n^μ is along a geodesic, shear-free null congruence, which is the desired setup, then $\lambda = 0$. Since $\lambda = 0$, the conjugate $\bar{\lambda} = 0$ holds. Starting with the equation:

$$\bar{\lambda} = m^\mu m^\nu \nabla_\nu n_\mu = 0. \tag{4}$$

To derive the equations final form, we manipulate the expression to ensure it is independent of tetrad components or functions, which both guarantees tetrad invariance and facilitates determining metric components without making additional ansatz.

It can be calculated that:

$$g_{t\phi}\Gamma^{\phi}_{tr} + g_{tt}\Gamma^{t}_{tr} = \tfrac{1}{2}g_{tt,r} \qquad g_{\phi\phi}\Gamma^{\phi}_{\phi r} + g_{t\phi}\Gamma^{t}_{\phi r} = \tfrac{1}{2}g_{\phi\phi,r}\,, \tag{5}$$

and

$$g_{t\phi}\Gamma^{\phi}_{\phi r} + g_{\phi\phi}\Gamma^{\phi}_{tr} + g_{t\phi}\Gamma^{t}_{tr} + g_{tt}\Gamma^{t}_{\phi r} = g_{t\phi,r}\,. \tag{6}$$

Note that $\Gamma^a_{bc} = \Gamma^a_{cb}$ for the Levi-Civita connection due to its torsion-free property.

The next step is to express the final equation purely in *metric components* and their derivatives. Note that $(g_{t\phi})^2 - g_{tt}g_{\phi\phi} = \triangle \sin^2\theta$ and $\triangle = \frac{g_{\theta\theta}}{g_{rr}}$.

Incorporating these into the equation, produces the final equation.

The explicit form of the final equation is given by:

$$\boxed{g_{\theta\theta,r} - \frac{(g_{t\phi})^2 g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt}g_{\phi\phi})(1+g_{tt})^2}g_{tt,r} + \frac{2g_{t\phi}g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt}g_{\phi\phi})(1+g_{tt})}g_{t\phi,r} - \frac{g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt}g_{\phi\phi})}g_{\phi\phi,r} = 0} \tag{7}$$

where $g_{\mu\nu}$ denotes the general metric components and $g_{\mu\nu,r} \equiv \frac{\partial g_{\mu\nu}}{\partial r}$ denotes the radial derivatives.

This is the main result: a purely metric-level condition equivalent to $\lambda = 0$, for Type D spacetimes. Through the use of the Kinnersley tetrad and metric identities, this condition is expressed entirely in terms of metric components and their radial derivatives, providing a tetrad-independent test of algebraic speciality and a direct constraint on the metric without requiring an ansatz.

4. Extension to Other Spacetimes

This equation provides a constraint relating the metric components, which in specific cases (e.g., Schwarzschild) can be used to determine them, which is illustrated below.

4.1. Deriving Schwarzschild Metric Components

For the Schwarzschild metric [13], the cross terms become zero, resulting in a simplified equation:

$$g_{\theta\theta,r} + \frac{g_{\theta\theta}g_{\phi\phi,r}}{g_{rr}g_{tt}g_{\phi\phi}} = 0. \tag{8}$$

Considering the spherical symmetry of the spacetime, we generate the condition:

$$g_{tt} = -\frac{1}{g_{rr}}, \tag{9}$$

recovering the standard Schwarzschild relation without assuming an *ansatz*. The same procedure applies to the static de Sitter metric.

4.2. Results for Other Spacetimes

Numerical evaluation of the equation for several well-known spacetimes confirms its validity. For Kerr, Schwarzschild, static de Sitter, and Kerr–Newman metrics ([2,13–15]), the equation evaluates to zero or values on the order of 10^{-15}, consistent with machine precision.

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For FLRW ([16–19]), and Minkowski [20] spacetimes, the equation also vanishes, due both to the absence of cross terms and the condition $1 + g_{tt} = 0$.

Overall, the equation serves as a diagnostic condition that is satisfied in Type D spacetimes, which are characterised by the existence of two geodesic, shear-free null congruences. As such, it provides a practical, coordinate-based criterion for identifying or verifying such spacetimes, without requiring prior knowledge of their full geometric classification.

doi:10.20944/preprints202511.1694.v1

Table 1. Numerical evaluation results for various spacetimes.

Each row corresponds to a distinct parameter set used to numerically evaluate the derived equation.

Note: All "Result" values are either numerical zeros or effectively zero within floating-point precision, with magnitudes on the order of $\mathcal{O}(10^{-15})$.

Table 2. Static de Sitter

Set #	r	θ	M	Λ	Result
1	12	$\pi/4$	7	30	0
2	8	$\pi/3$	5	20	10^{-15}
3	20	$\pi/2$	12	10	10^{-15}
4	5	$\pi/6$	3	35	10^{-15}
5	16	$\pi/8$	9	5	0

Table 3. Kerr-Newman

Set #	r	θ	M	a	Q	Result
1	12	$\pi/4$	7	40	27	0
2	8	$\pi/3$	5	15	14	10^{-15}
3	20	$\pi/2$	12	5	22	0
4	5	$\pi/6$	3	25	18	0
5	16	$\pi/8$	9	10	30	0

Table 4. Kerr Metric

Set #	r	θ	M	a	Result
1	12	$\pi/4$	7	40	0
2	8	$\pi/3$	5	15	10^{-15}
3	20	$\pi/2$	12	5	0
4	5	$\pi/6$	3	25	0
5	16	$\pi/8$	9	10	10^{-15}

Table 5. Schwarzschild

Set #	r	θ	M	Result
1	12	$\pi/4$	7	0
2	8	$\pi/3$	5	0
3	20	$\pi/2$	12	0
4	5	$\pi/6$	3	0
5	16	$\pi/8$	9	0

5. Discussion

The derived equation represents a potentially powerful tool in General Relativity, providing a purely metric-component condition that holds specifically for spacetimes admitting two geodesic, shear-free null congruences, as in Petrov Type D geometries. Crucially, its formulation in terms of metric components offers a path toward recovering full spacetime metrics without imposing an *ansatz*, potentially simplifying the exploration and analysis of new exact solutions of Einstein's field equations. The fact that the equation continues to hold for Kerr–Newman, despite $R_{\mu\nu} \neq 0$, shows that its validity is not restricted to vacuum solutions. The equation may also admit generalisations, for example by extending the structure to additional cross terms beyond $g_{t\phi}$.

More broadly, the existence of such a coordinate-level relation suggests that analogous metric-component equations may exist for spacetimes admitting only a single geodesic, shear-free null congruence, although such formulas have not yet been identified. The main utility of this result lies in expressing the covariant geometric condition in a simplified, coordinate-dependent form, illuminating the geometric structure of Kerr, Schwarzschild, and de Sitter spacetimes, all of which admit two such congruences (characteristic of Type D).

6. Conclusions

To conclude, this work presents a coordinate-based expression for the condition satisfied by geodesic, shear-free null congruences in Type D spacetimes. Open questions include whether similarly simplified relations exist in other coordinate systems, or if the observed structure is specific to Boyer–Lindquist coordinates. It also remains to be explored whether analogous simplified equations can be derived for other Newman–Penrose spin coefficient conditions and for spacetimes of different Petrov types.

Funding: The author received no specific funding for this work.

Institutional Review Board Statement: This work does not involve human participants, animal subjects, or sensitive data requiring ethical approval.

Data Availability Statement: No datasets were generated or analysed during the current study.

Acknowledgments: This research was conducted independently.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. Known Mathematical Formulations

Appendix A.1. Conditions on a Null Tetrad

A null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ satisfies the following conditions:

$$l^a l_a = n^a n_a = m^a m_a = \bar{m}^a \bar{m}_a = 0, \quad \text{(each vector is null)} \tag{A1}$$

$$l^a n_a = -1, \tag{A2}$$

$$m^a \bar{m}_a = 1, \tag{A3}$$

$$l^a m_a = l^a \bar{m}_a = n^a m_a = n^a \bar{m}_a = 0, \quad \text{(orthogonality conditions)} \tag{A4}$$

Here, the metric signature is $(-+++)$.

Appendix B. Full Derivation of the $\lambda = 0$ Condition

Appendix B.1. $\lambda = 0$

See the derivations for spin coefficients in Newman and Penrose [3], although these expressions are not always fully expanded in covariant form. σ describes the shear of l^μ and the λ coefficient is analogous to this and describes the shear of n^μ. If n^μ is along a geodesic, shear-free null congruence,

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which is the desired setup, then $\lambda = 0$. Since $\lambda = 0$, the conjugate $\bar{\lambda} = 0$ holds. The conjugate will be used simply because it allows us to use m^μ instead of the conjugate $\bar{m}^\mu$. Starting with the equation:

$$\bar{\lambda} = m^\mu m^\nu \nabla_\nu n_\mu = 0. \tag{A5}$$

We focus on this contraction because it isolates the spin coefficient λ in terms of the metric. The Kinnersley tetrad is a widely known choice of tetrad for the Kerr spacetime because it satisfies the null tetrad conditions: all skew-symmetric products are zero except for $l^\mu n_\mu = 1$ and $m^\mu \bar{m}_\mu = 1$.

Only certain components of the Kinnersley tetrad (Eqs. 1, 3) will be selected without invoking the full tetrad.

See Loutrel [12], Section V, for the un-rotated version of the Kinnersley tetrad.

$$l^\mu = \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad n^\mu = \frac{1}{2\Sigma}\left(r^2+a^2, -\Delta, 0, a\right), \quad m^\mu = \frac{1}{\sqrt{2}\Gamma}(ia\sin\theta, 0, 1, i\csc\theta), \tag{A6}$$

Where:

- $\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta, \quad \Gamma = r + ia\cos\theta.$

Here:

- M is the mass of the black hole,
- a is the specific angular momentum (angular momentum per unit mass),
- r is the radial Boyer–Lindquist coordinate,
- θ is the polar angle from the symmetry axis.

Notice that $l^\theta = n^\theta = m^r = 0$, as explained above.

Since $m^r = 0$, the μ contraction becomes:

$$\bar{\lambda} = m^\mu m^\nu \nabla_\nu n_\mu = m^t m^\nu \nabla_\nu n_t + m^\theta m^\nu \nabla_\nu n_\theta + m^\phi m^\nu \nabla_\nu n_\phi = 0. \tag{A7}$$

Then, expanding the covariant component to attain a *covariant derivative* acting on a contravariant component. Recalling that $\nabla_a g_{\mu\nu} = 0$ for the *Levi-Civita connection* [1], this gives $\nabla_\mu n_\phi = g_{\phi\phi}\nabla n^\phi + g_{t\phi}\nabla n^t$.

$$\bar{\lambda} = g_{\theta\theta} m^\theta m^\nu \nabla_\nu n^\theta + (m^t g_{t\phi} + m^\phi g_{\phi\phi}) m^\nu \nabla_\nu n^\phi + (m^\phi g_{t\phi} + m^t g_{tt}) m^\nu \nabla_\nu n^t = 0. \tag{A8}$$

Then expand the covariant derivative:

$$\begin{aligned} \bar{\lambda} = \quad & g_{\theta\theta} m^\theta (m^t \Gamma^\theta_{tm} n^m + m^\phi \Gamma^\theta_{\phi m} n^m + m^\theta \Gamma^\theta_{\theta m} n^m) \\ + \quad & (m^t g_{t\phi} + m^\phi g_{\phi\phi})(m^t \Gamma^\phi_{tm} n^m + m^\theta n^\phi_{,\theta} + m^\theta \Gamma^\phi_{\theta m} n^m + m^\phi \Gamma^\phi_{\phi m} n^m) \\ + \quad & (m^t g_{tt} + m^\phi g_{t\phi})(m^t \Gamma^t_{tm} n^m + m^\theta n^t_{,\theta} + m^\theta \Gamma^t_{\theta m} n^m + m^\phi \Gamma^t_{\phi m} n^m) = 0. \end{aligned} \tag{A9}$$

Then, consider the m^μ components. The denominators with $\frac{1}{\sqrt{2}\Gamma}$ cancel out as a common factor for all the terms.

This allows us to focus on the m^μ components within the brackets $(ia\sin\theta, 0, 1, i\csc\theta)$ and focus on the real part of the equation. The separation into real and imaginary parts corresponds to the complex nature of λ; either part vanishing is necessary for algebraic speciality. Since $m^t m^t$, $m^t m^\phi$, $m^\phi m^\phi$ and $m^\theta m^\theta$ are all real-valued, the real part of the equation is

$$
\begin{aligned}
g_{\theta\theta} m^\theta m^\theta \Gamma^\theta_{\theta m} n^m + m^t m^t (\Gamma^\phi_{tm} n^m + g_{tt} \Gamma^t_{tm} n^m) \quad &+ \quad m^\phi m^\phi (g_{\phi\phi} \Gamma^\phi_{\phi m} n^m + g_{t\phi} \Gamma^t_{\phi m} n^m) \\
+ \quad m^t m^\phi (g_{t\phi} \Gamma^\phi_{\phi m} n^m + g_{\phi\phi} \Gamma^\phi_{tm} n^m &+ g_{t\phi} \Gamma^t_{tm} n^m + g_{tt} \Gamma^t_{\phi m} n^m) = 0.
\end{aligned} \tag{A10}
$$

Appendix B.2. Final Equation Form

Here we focus on manipulating the equation to ensure it is independent of tetrad components or functions, which both guarantees tetrad invariance and facilitates determining metric components without making additional ansatz.

The following non-zero *connection coefficients* that are of interest:

$$
\begin{array}{ll}
\Gamma^t_{tr} = \frac{1}{2} g^{tt} g_{tt,r} + \frac{1}{2} g^{t\phi} g_{t\phi,r} & \Gamma^t_{\phi r} = \frac{1}{2} g^{tt} g_{t\phi,r} + \frac{1}{2} g^{t\phi} g_{\phi\phi,r} \\
\Gamma^\phi_{tr} = \frac{1}{2} g^{\phi\phi} g_{t\phi,r} + \frac{1}{2} g^{t\phi} g_{tt,r} & \Gamma^\phi_{\phi r} = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,r} + \frac{1}{2} g^{t\phi} g_{t\phi,r}.
\end{array} \tag{A11}
$$

Also, recalling that $g_{t\phi} g^{t\phi} + g_{tt} g^{tt} = 1$ and $g_{t\phi} g^{tt} + g_{\phi\phi} g^{t\phi} = 0$, It can be calculated that:

$$
g_{t\phi} \Gamma^\phi_{tr} + g_{tt} \Gamma^t_{tr} = \frac{1}{2} g_{tt,r} \qquad g_{\phi\phi} \Gamma^\phi_{\phi r} + g_{t\phi} \Gamma^t_{\phi r} = \frac{1}{2} g_{\phi\phi,r}\,, \tag{A12}
$$

and

$$
g_{t\phi} \Gamma^\phi_{\phi r} + g_{\phi\phi} \Gamma^\phi_{tr} + g_{t\phi} \Gamma^t_{tr} + g_{tt} \Gamma^t_{\phi r} = g_{t\phi,r}\,. \tag{A13}
$$

Note that $\Gamma^a_{bc} = \Gamma^a_{cb}$ for the Levi-Civita connection due to its torsion-free property.

The only non-zero contribution from contraction over index "m" is "r", as the connection coefficient is either zero or $n^\theta = 0$.

$$
\begin{aligned}
n^r [g_{\theta\theta} m^\theta m^\theta \Gamma^\theta_{\theta r} \quad + \quad m^t m^t (\Gamma^\phi_{tr} + g_{tt} \Gamma^t_{tr}) \quad + \quad m^\phi m^\phi (g_{\phi\phi} \Gamma^\phi_{\phi r} + g_{t\phi} \Gamma^t_{\phi r}) \\
+ \quad m^t m^\phi (g_{t\phi} \Gamma^\phi_{\phi r} + g_{\phi\phi} \Gamma^\phi_{tr} + g_{t\phi} \Gamma^t_{tr} + g_{tt} \Gamma^t_{\phi r})] = 0.
\end{aligned} \tag{A14}
$$

So n^r is a common factor.

Note that $\Gamma^\theta_{\theta r} = \frac{1}{2} g^{\theta\theta} g_{\theta\theta,r}$.

Recalling Equation (A12), and Equation (A13), these can be substituted into the equation and notice the factor of $\frac{1}{2}$ will cancel out.

$$
g_{\theta\theta} m^\theta m^\theta g^{\theta\theta} g_{\theta\theta,r} + m^t m^t (g_{tt,r}) + m^\phi m^\phi (g_{\phi\phi,r}) + 2 m^t m^\phi (g_{t\phi,r}) = 0. \tag{A15}
$$

Now consider the Kerr metric and the actual components of the m^μ vector in the Kinnersley tetrad.

First, recall the denominator is a common factor.

$$
\begin{aligned}
& m^\theta m^\theta \to 1, \quad m^t m^t \to -a^2 \sin^2\theta, \quad m^t m^\phi \to -a, \quad m^\phi m^\phi \to -\frac{1}{\sin^2\theta}, \\
& g_{\theta\theta} g^{\theta\theta} g_{\theta\theta,r} + -a^2 \sin^2\theta (g_{tt,r}) - \frac{1}{\sin^2\theta} (g_{\phi\phi,r}) - 2a (g_{t\phi,r}) = 0.
\end{aligned} \tag{A16}
$$

The next step is to express the final equation purely in *metric components* and their derivatives. Note that $(g_{t\phi})^2 - g_{tt} g_{\phi\phi} = \triangle \sin^2\theta$ and $\triangle = \frac{g_{\theta\theta}}{g_{rr}}$.

Therefore,

$$
\sin^2\theta = \frac{g_{rr}((g_{t\phi})^2 - g_{tt} g_{\phi\phi})}{g_{\theta\theta}}, \qquad a \sin^2\theta = -\frac{g_{t\phi}}{1 + g_{tt}}, \qquad a = -\frac{g_{t\phi} g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt} g_{\phi\phi})(1 + g_{tt})}. \tag{A17}
$$

Incorporating these into the equation, produces the final equation.

The explicit form of the final equation is given by:

$$\boxed{g_{\theta\theta,r} - \frac{(g_{t\phi})^2 g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt}g_{\phi\phi})(1+g_{tt})^2} g_{tt,r} + \frac{2g_{t\phi}g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt}g_{\phi\phi})(1+g_{tt})} g_{t\phi,r} - \frac{g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt}g_{\phi\phi})} g_{\phi\phi,r} = 0} \tag{A18}$$

where $g_{\mu\nu}$ denotes the general metric components and $g_{\mu\nu,r} \equiv \frac{\partial g_{\mu\nu}}{\partial r}$ denotes the radial derivatives.

Appendix C. Computational Details

Symbolic computation (using the Python library `SymPy`) was employed to define metric components for various spacetimes, compute their derivatives, and numerically evaluate the proposed metric-condition equation. Representative parameter values were substituted to verify that the equation vanishes within machine precision for Kerr, Kerr–Newman, Schwarzschild, and static de Sitter spacetimes. The full code is available from the author upon request.

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doi:10.20944/preprints202511.1694.v1

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