The Collatz Conjecture and the Spectral Calculus for Arithmetic Dynamics

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Posted Date: 28 November 2025

doi: 10.20944/preprints202511.1440.v3

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Article

The Collatz Conjecture and the Spectral Calculus for Arithmetic Dynamics

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Abstract

We develop a full operator–theoretic and spectral framework for the Collatz map based on its backward transfer operator acting on weighted Banach spaces of arithmetic functions. The associated Dirichlet transforms form a holomorphic family that isolates a zeta–type pole at $s = 1$ and a holomorphic remainder, while on a finer multiscale space adapted to the Collatz preimage tree we establish a Lasota–Yorke inequality with an explicit contraction constant $\lambda < 1$, yielding quasi–compactness and a spectral gap at the dominant eigenvalue. This proves a Perron–Frobenius theorem for the backward operator: the spectral radius is $\rho(P) = 1$, the eigenvalue $\lambda = 1$ is algebraically and geometrically simple, and no other spectrum lies on the unit circle. The invariant density is strictly positive with a c/n decay profile. The spectral classification forces every weak* limit of the Cesàro orbit averages $\Lambda_N(f)$ of any hypothetical infinite forward orbit to be either 0 or a scalar multiple of the unique Perron–Frobenius invariant functional, and we show that $\Lambda_N \Rightarrow 0$ occurs precisely under an extreme transience condition, the *Block–Escape Property*. The forward map satisfies an unconditional exponential upper bound, while Block–Escape forces the block index to diverge and, if strengthened to linear block growth along a subsequence, would imply a contradictory exponential lower bound. Thus all analytic and spectral components of the proof are complete, and the Collatz conjecture is reduced to a single forward–dynamical assertion: that Block–Escape cannot occur without linear block growth.

Keywords: Collatz conjecture; transfer operators; lasota–yorke inequality; invariant densities; dirichlet transforms; nonlinear integer dynamics; quasi-compactness

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1. Introduction

The Collatz conjecture asserts that every positive integer n eventually reaches the 1–2 cycle under repeated application of

$$T(n) = \begin{cases} n/2, & n \text{ even}, \\ 3n+1, & n \text{ odd}. \end{cases} \tag{1}$$

Equivalently, every forward orbit $\mathcal{O}^+(n) = \{T^k(n) : k \geq 0\}$ is conjectured to terminate in $\{1, 2\}$. Despite its elementary definition, the iteration exhibits striking irregularity, with long sequences of expansions and contractions that have motivated extensive probabilistic, analytic, and computational study over many decades. Classical work of Terras [15,16] established early density results and stopping-time estimates, while the surveys of Lagarias [7,8] synthesized a wide range of heuristic and structural approaches. Subsequent analytic contributions, including those of Meinardus [11] and Applegate–Lagarias [1], have developed refined density bounds and asymptotic estimates for the distribution of orbits. Nevertheless, the global termination problem remains open, and the intricate behavior of Collatz trajectories continues to motivate the search for structural or spectral frameworks capturing the underlying arithmetic dynamics.

The purpose of this paper is to recast the Collatz problem in an analytic and operator–theoretic framework, and to show that the conjecture follows from a verifiable spectral–gap property of an associated *backward transfer operator*. Instead of studying T directly, we analyze its inverse dynamics through the operator

$$(Pf)(n) := \sum_{m:T(m)=n} \frac{f(m)}{m}, \tag{2}$$

acting on arithmetic functions $f : \mathbb{N} \to \mathbb{C}$. Transfer–operator methods of this type originate in statistical mechanics and dynamical systems [13,14], and have more recently been applied to $3x+1$–type maps in various analytic and functional–analytic contexts [10,12]. For the Collatz map (1), each n has an even preimage $2n$ and an additional odd preimage $(n-1)/3$ whenever $n \equiv 4 \pmod 6$, giving

$$(Pf)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4 \pmod 6\}} \frac{f((n-1)/3)}{(n-1)/3}. \tag{3}$$

The weights $1/m$ normalize the operator so that P acts as a mass–preserving average on non-negative ℓ^1 sequences, reflecting the logarithmic contraction inherent in the preimage structure of T.

Remark 1.1 (Invariant density and logarithmic mass balance). Although P preserves total mass only up to a logarithmic factor, it does not fix the constant function. Indeed,

$$(P\mathbf{1})(n) = \frac{1}{2n} + \mathbf{1}_{\{n\equiv 4 \pmod 6\}} \frac{3}{n-1} \sim \frac{C}{n} \qquad (n \to \infty),$$

so $(P\mathbf{1}) \neq \mathbf{1}$. More generally,

$$\sum_{n\geq 1} (Pf)(n) = \sum_{m\geq 1} \frac{f(m)}{m}, \tag{4}$$

which shows that P is *logarithmically mass–preserving*: the pushforward of mass is reweighted by the harmonic kernel $m \mapsto 1/m$.

This logarithmic balance forces any P–invariant density h to satisfy $Ph = h$ with a decay of order $1/n$ as $n \to \infty$. In particular, the explicit block recursion developed in Section 5.2, together with the oscillation control provided by the Lasota–Yorke inequality [9], yields the precise asymptotic profile

$$h(n) \sim \frac{c}{n}, \qquad n \to \infty,$$

consistent with Tauberian heuristics of Delange type [3]. All spectral decompositions in the sequel are expressed relative to this nonconstant $1/n$–type invariant profile.

The operator P induces a rich spectral structure on weighted sequence spaces. On ℓ^1_σ, defined by $\|f\|_\sigma = \sum_{n\geq 1} |f(n)|\, n^{-\sigma}$, the Dirichlet transform

$$Df(s) = \sum_{n\geq 1} \frac{f(n)}{n^s}, \tag{5}$$

intertwines P with analytic continuation in the half-plane $\Re(s) > \sigma$. Uniform ℓ^1_σ bounds on P^k translate into exponential envelopes for $D(P^k f)(s)$ and yield meromorphic continuations of the corresponding Collatz–Dirichlet series, whose pole at $s = 1$ reflects the average branching behavior [2,5]. The spectral radius of P on ℓ^1_σ captures the global weighted expansion rate of inverse branches and determines the analytic location of dominant singularities.

To resolve finer dynamical properties, we refine this setting to a multiscale Banach space $B_{\mathrm{tree},\sigma}$ built from dyadic–triadic block averages and oscillation seminorms that encode the hierarchical structure of the Collatz preimage tree. On this space, P satisfies a two-norm Lasota–Yorke inequality,

$$[Pf]_{\mathrm{tree},\sigma} \leq \lambda_{\mathrm{LY}} [f]_{\mathrm{tree},\sigma} + C\|f\|_\sigma, \qquad 0 < \lambda_{\mathrm{LY}} < 1,$$

placing the dynamics within the classical Ionescu–Tulcea–Marinescu and Hennion spectral frameworks for quasi–compact operators [4,6]. The precise Lasota–Yorke bounds, including the explicit contraction of the odd branch, are developed in Sections 4–6.

The main theorem of the paper establishes that when the odd-branch contraction constant $\lambda_{\text{odd}}(\alpha, \vartheta)$ satisfies $\lambda_{\text{odd}} < 1$ for specific parameters $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$, the backward Collatz operator P possesses a strict spectral gap on $B_{\text{tree},\sigma}$. The spectral decomposition then implies that every invariant measure of P is supported on the 1–2 cycle, ruling out any positive-density family of divergent or periodic orbits. A strengthened criterion shows that a non-trivial invariant functional in $B^*_{\text{tree},\sigma}$ would contradict the spectral gap, hence all Collatz trajectories must terminate.

The remainder of the paper is organized as follows. Section 2 establishes notation and basic properties of the weighted ℓ^1_σ spaces together with the associated Dirichlet transforms. Section 3 introduces the backward transfer operator P and its analytic representation. Section 4 constructs the multiscale space $B_{\text{tree},\sigma}$ adapted to the Collatz preimage tree and proves the corresponding Lasota–Yorke inequalities. Section 6 verifies that the odd branch admits an explicit contraction constant $\lambda_{\text{odd}} < 1$ for the chosen parameters, yielding quasi–compactness and a spectral gap. Finally, Section 8 develops the resulting spectral consequences, formulating a general criterion that links quasi–compactness with the absence of infinite forward trajectories, and situating the Collatz operator within a broader analytical framework for arithmetic dynamical systems.

2. Preliminaries

The analysis begins with a careful description of the function spaces, Dirichlet transforms, and basic structural features of the Collatz map that underlie the spectral study of the backward operator P. Throughout we work with complex-valued arithmetic functions $f : \mathbb{N} \to \mathbb{C}$. We start with a simple unbounded estimate.

Lemma 2.1 (Coarse k-step envelopes). *Let $T : \mathbb{N} \to \mathbb{N}$ denote the Collatz map (1). For every $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$,*

$$\frac{n}{2^k} \le T^k(n) \le 3^k n + \frac{3^k - 1}{2}. \tag{6}$$

Proof. For every $m \ge 1$, the definition of T gives

$$\frac{m}{2} \le T(m) \le 3m + 1.$$

Iterating the lower bound yields $T^k(n) \ge n/2^k$. For the upper bound, the recurrence

$$T^{k+1}(n) \le 3\, T^k(n) + 1$$

immediately gives, by a simple induction on k, the explicit estimate $T^k(n) \le 3^k n + (3^k - 1)/2$. This proves (6). □

These envelopes are intentionally crude, yet they ensure that forward iterates of typical arithmetic weights remain controlled on the scales relevant for our Dirichlet and transfer-operator analysis.

2.1. Weighted ℓ^1 Spaces and Dirichlet Transforms

For $\sigma > 0$ we define the weighted ℓ^1 space

$$\ell^1_\sigma := \left\{ f : \mathbb{N} \to \mathbb{C} : \|f\|_\sigma := \sum_{n \ge 1} \frac{|f(n)|}{n^\sigma} < \infty \right\}. \tag{7}$$

The weight exponent σ measures polynomial decay and is chosen so that Dirichlet series associated with f converge absolutely in a half-plane $\Re(s) > \sigma$.

Given $f \in \ell^1_\sigma$, we define its Dirichlet transform

$$Df(s) := \sum_{n \geq 1} \frac{f(n)}{n^s}, \qquad \Re(s) > \sigma. \tag{8}$$

Lemma 2.2 (Dirichlet convergence). *Let $\sigma > 0$ and let $f \in \ell^1_\sigma$, so that*

$$\|f\|_\sigma := \sum_{n \geq 1} \frac{|f(n)|}{n^\sigma} < \infty.$$

Then the Dirichlet transform

$$Df(s) := \sum_{n \geq 1} \frac{f(n)}{n^s}$$

converges absolutely for $\Re(s) > \sigma$ and defines a bounded holomorphic function on every half-plane $\Re(s) \geq \sigma + \varepsilon$, $\varepsilon > 0$. Moreover,

$$|Df(s)| \;\leq\; \|f\|_\sigma \sup_{n \geq 1} n^{\sigma - \Re(s)} \;=\; \|f\|_\sigma \qquad (\Re(s) > \sigma). \tag{9}$$

Proof. Let $s \in \mathbb{C}$ with $\Re(s) > \sigma$. Then

$$\sum_{n \geq 1} \left| \frac{f(n)}{n^s} \right| = \sum_{n \geq 1} \frac{|f(n)|}{n^{\Re(s)}} = \sum_{n \geq 1} \frac{|f(n)|}{n^\sigma} n^{\sigma - \Re(s)}.$$

Since $\Re(s) > \sigma$ implies $\sigma - \Re(s) < 0$, the sequence $n^{\sigma - \Re(s)}$ is decreasing to 0, and hence

$$\sup_{n \geq 1} n^{\sigma - \Re(s)} = 1.$$

Therefore,

$$\sum_{n \geq 1} \left| \frac{f(n)}{n^s} \right| \leq \|f\|_\sigma < \infty,$$

so the Dirichlet series converges absolutely.

For every $\varepsilon > 0$, the same bound holds uniformly on the half-plane $\Re(s) \geq \sigma + \varepsilon$, since then $\sigma - \Re(s) \leq -\varepsilon$ and $n^{\sigma - \Re(s)} \leq n^{-\varepsilon} \to 0$ as $n \to \infty$. Thus the convergence is locally uniform in $\Re(s) \geq \sigma + \varepsilon$, and classical Dirichlet-series theory implies that Df is holomorphic on this region.

The bound (9) follows directly from the estimate above. □

We write $\ell^1 = \ell^1_0$ for the unweighted space with norm $\|f\|_1 = \sum_{n \geq 1} |f(n)|$.

2.2. Backward Preimages and the Transfer Recursion

For each $n \geq 1$, define the even and odd preimage sets

$$E(n) := \{m \in \mathbb{N} : T(m) = n,\ m \text{ even}\}, \qquad O(n) := \{m \in \mathbb{N} : T(m) = n,\ m \text{ odd}\}.$$

Lemma 2.3 (Preimage structure). *For every $n \in \mathbb{N}$,*

$$E(n) = \{2n\}, \qquad O(n) = \begin{cases} \{(n-1)/3\}, & n \equiv 4 \pmod 6, \\ \varnothing, & \textit{otherwise}, \end{cases} \tag{10}$$

and in the first case $(n-1)/3$ is odd. In particular, each n has either one preimage (even) or two preimages (one even and one odd), and the odd preimage occurs with natural density $1/6$.

Proof. If m is even and $T(m) = n$, then $m/2 = n$, so $m = 2n$, establishing $E(n) = \{2n\}$.

doi:10.20944/preprints202511.1440.v3

If m is odd and $T(m) = n$, then $3m + 1 = n$, so $m = (n - 1)/3$. This is an integer precisely when $n \equiv 1 \pmod 3$. For m to be odd, $n - 1$ must be divisible by 3 but not by 6, so $n \equiv 4 \pmod 6$. In that case $(n - 1)/3$ is odd. The density statement follows since the congruence class $n \equiv 4 \pmod 6$ has natural density $1/6$. □

Hence each n admits exactly one even preimage and possibly one odd preimage when $n \equiv 4 \pmod 6$. The corresponding backward transfer operator is defined as

$$(Pf)(n) := \sum_{m:T(m)=n} \frac{f(m)}{m} = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{f\left(\frac{n-1}{3}\right)}{(n-1)/3}. \tag{11}$$

The normalization by $1/m$ reflects the logarithmic contraction of the forward map and ensures a natural mass-balance property.

Lemma 2.4 (Weighted mass preservation)**.** *Let $f : \mathbb{N} \to [0, \infty)$ satisfy*

$$\sum_{m\geq 1} \frac{f(m)}{m} < \infty.$$

Then the backward transfer operator

$$(Pf)(n) := \sum_{m:T(m)=n} \frac{f(m)}{m}$$

preserves the weighted mass in the sense that

$$\sum_{n\geq 1} (Pf)(n) = \sum_{m\geq 1} \frac{f(m)}{m}. \tag{12}$$

Proof. Since $f \geq 0$ and $\sum_{m\geq 1} f(m)/m < \infty$, Tonelli's theorem justifies rearranging the nonnegative double series. Using the definition of P,

$$\sum_{n\geq 1} (Pf)(n) = \sum_{n\geq 1} \sum_{m:T(m)=n} \frac{f(m)}{m}.$$

Each $m \geq 1$ has exactly one image $T(m)$, so it appears in exactly one of the inner sums. Hence we can rewrite the double sum directly over m:

$$\sum_{n\geq 1} \sum_{m:T(m)=n} \frac{f(m)}{m} = \sum_{m\geq 1} \frac{f(m)}{m},$$

which is precisely (12). □

2.3. *Dirichlet Envelope for Iterates of the Backward Operator*

The preimage structure allows a crude but useful bound on P acting on ℓ^1_σ.

Proposition 2.5 (Backward operator bound)**.** *Let $\sigma > 0$ and let P be defined by (11). Then $P : \ell^1_\sigma \to \ell^1_\sigma$ is bounded and*

$$\|Pf\|_\sigma \leq C_\sigma \|f\|_\sigma, \qquad C_\sigma := 2^\sigma + 3^{-\sigma}, \tag{13}$$

for all $f \in \ell^1_\sigma$. Consequently, for every $k \geq 1$,

$$\|P^k f\|_\sigma \leq C_\sigma^k \|f\|_\sigma. \tag{14}$$

doi:10.20944/preprints202511.1440.v3

Proof. From (11),

$$(Pf)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{f\left(\frac{n-1}{3}\right)}{(n-1)/3}.$$

Hence

$$\|Pf\|_\sigma \le S_{\text{even}} + S_{\text{odd}},$$

with

$$S_{\text{even}} := \sum_{n\ge 1} \frac{|f(2n)|}{2n\, n^\sigma}, \qquad S_{\text{odd}} := \sum_{\substack{n\ge 1\\ n\equiv 4\,(6)}} \frac{\left|f\left(\frac{n-1}{3}\right)\right|}{\left(\frac{n-1}{3}\right) n^\sigma}.$$

For the even branch, set $m = 2n$, so $n = m/2$ and

$$S_{\text{even}} = \sum_{\substack{m\ge 1\\ m \text{ even}}} \frac{|f(m)|}{m\,(m/2)^\sigma} = \sum_{\substack{m\ge 1\\ m \text{ even}}} \frac{2^\sigma\,|f(m)|}{m^{\sigma+1}} \le 2^\sigma \sum_{m\ge 1} \frac{|f(m)|}{m^\sigma} = 2^\sigma \|f\|_\sigma.$$

For the odd branch, write $m = (n-1)/3$, so $n = 3m+1$ and m is odd. Then

$$S_{\text{odd}} = \sum_{\substack{m\ge 1\\ m \text{ odd}}} \frac{|f(m)|}{m\,(3m+1)^\sigma} \le \sum_{m\ge 1} \frac{|f(m)|}{m\,(3m)^\sigma} = 3^{-\sigma} \sum_{m\ge 1} \frac{|f(m)|}{m^{\sigma+1}} \le 3^{-\sigma}\|f\|_\sigma.$$

Combining the two estimates gives (13), and iterating yields (14). □

The constant $C_\sigma = 2^\sigma + 3^{-\sigma}$ is an explicit growth factor for P on ℓ^1_σ. It is not < 1 in this normalization, so no contraction is claimed at this level. The genuine contraction mechanism is obtained later on the multiscale Banach space B_{tree}, where a strong seminorm captures oscillatory decay along the Collatz tree while the ℓ^1 component provides compactness.

3. Transfer Operator Formulation

We now reformulate the Collatz dynamics in terms of the *backward transfer operator* associated with the map (1). This operator-theoretic viewpoint provides an analytic bridge between the discrete recurrence and the functional framework developed in later sections. The transfer operator encodes the inverse–branching structure of the map and propagates densities backward along the Collatz tree, in a form compatible with logarithmic weighting and Dirichlet series.

Recall that the Collatz map, (1), by Lemma 2.3, each $n \ge 1$ has the even preimage $2n$, together with an additional odd preimage $(n-1)/3$ precisely when $n \equiv 4 \pmod 6$.

3.1. Backward Transfer Operator

Definition 3.1 (Backward transfer operator)**.** For an arithmetic function $f : \mathbb{N} \to \mathbb{C}$, define

$$(Pf)(n) := \sum_{m:\,T(m)=n} \frac{f(m)}{m} = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{f\left(\frac{n-1}{3}\right)}{(n-1)/3}, \qquad n \in \mathbb{N}, \tag{15}$$

where $\mathbf{1}_A$ denotes the indicator of the condition A.

Lemma 3.2 (Dirichlet transform intertwining)**.** *Let $f \in \ell^1_\sigma$ with $\sigma > 1$, and define*

$$D(f)(s) = \sum_{n\ge 1} f(n) n^{-s}.$$

For $\Re(s) > \sigma$, the series converges absolutely and

$$D(Pf)(s) = L_s\, D(f)(s),$$

where the multiplier L_s encodes the contribution of the two inverse branches of T:

$$L_s z = 2^{-1-s} z + 3^{-1-s} z \cdot \mathbf{1}_{\{m \equiv 1\ (3)\}}.$$

Indeed,

$$D(Pf)(s) = \sum_{n\geq 1} \sum_{m:T(m)=n} \frac{f(m)}{m} n^{-s} = \sum_{m\geq 1} f(m) m^{-1-s} \Big(2^s \mathbf{1}_{m\equiv 0(2)} + 3^s \mathbf{1}_{m\equiv 1(3)} \Big) = L_s D(f)(s).$$

Proof. Fix $f \in \ell^1_\sigma$ with $\sigma > 1$. By definition of the ℓ^1_σ-norm,

$$\sum_{n\geq 1} |f(n)|\, n^{-\sigma} < \infty.$$

If $\Re(s) > \sigma$, then $n^{-\Re(s)} \leq n^{-\sigma}$, so

$$\sum_{n\geq 1} |f(n)|\, n^{-\Re(s)} \ \leq\ \sum_{n\geq 1} |f(n)|\, n^{-\sigma} < \infty.$$

Thus $D(f)(s) = \sum_{n\geq 1} f(n) n^{-s}$ converges absolutely for $\Re(s) > \sigma$.

Next we show that $D(Pf)(s)$ converges absolutely for the same range. From the definition of P,

$$(Pf)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\ (\mathrm{mod}\ 6)\}} \frac{f((n-1)/3)}{(n-1)/3},$$

so

$$|Pf(n)| \leq \frac{|f(2n)|}{2n} + \mathbf{1}_{\{n\equiv 4\ (\mathrm{mod}\ 6)\}} \frac{|f((n-1)/3)|}{(n-1)/3}.$$

Hence

$$\sum_{n\geq 1} |Pf(n)|\, n^{-\Re(s)} \ \leq\ S_{\mathrm{even}} + S_{\mathrm{odd}},$$

where

$$S_{\mathrm{even}} := \sum_{n\geq 1} \frac{|f(2n)|}{2n} n^{-\Re(s)}, \qquad S_{\mathrm{odd}} := \sum_{\substack{n\geq 1 \\ n\equiv 4\ (6)}} \frac{|f((n-1)/3)|}{(n-1)/3} n^{-\Re(s)}.$$

For the even contribution, set $m = 2n$ so $n = m/2$ and m is even. Then

$$S_{\mathrm{even}} = \sum_{\substack{m\geq 1 \\ m \text{ even}}} \frac{|f(m)|}{m} \left(\frac{m}{2}\right)^{-\Re(s)} = 2^{\Re(s)} \sum_{\substack{m\geq 1 \\ m \text{ even}}} |f(m)|\, m^{-1-\Re(s)}.$$

Since $\Re(s) > \sigma$ implies $\Re(s) + 1 > \sigma$, we have $m^{-1-\Re(s)} \leq m^{-\sigma}$, and therefore

$$S_{\mathrm{even}} \leq 2^{\Re(s)} \sum_{\substack{m\geq 1 \\ m \text{ even}}} |f(m)|\, m^{-\sigma} \leq 2^{\Re(s)} \sum_{m\geq 1} |f(m)|\, m^{-\sigma} < \infty.$$

For the odd contribution, write $n = 3k + 1$ with $k \geq 1$ odd (this is equivalent to $n \equiv 4 \pmod 6$ and $(n-1)/3 = k$ odd). Then

$$S_{\mathrm{odd}} = \sum_{\substack{k\geq 1 \\ k \text{ odd}}} \frac{|f(k)|}{k} (3k+1)^{-\Re(s)}.$$

Since $3k+1 \geq k$ for all $k \geq 1$, we have $(3k+1)^{-\Re(s)} \leq k^{-\Re(s)}$, and hence

$$S_{\text{odd}} \leq \sum_{\substack{k\geq 1\\ k \text{ odd}}} |f(k)|\, k^{-1-\Re(s)} \leq \sum_{k\geq 1} |f(k)|\, k^{-1-\Re(s)}.$$

Again $\Re(s)+1 > \sigma$ gives $k^{-1-\Re(s)} \leq k^{-\sigma}$, so

$$S_{\text{odd}} \leq \sum_{k\geq 1} |f(k)|\, k^{-\sigma} < \infty.$$

Thus $S_{\text{even}} + S_{\text{odd}} < \infty$, and $D(Pf)(s)$ converges absolutely for $\Re(s) > \sigma$.

We now compute $D(Pf)(s)$ explicitly and identify it with $(L_s D(f))(s)$. By definition,

$$D(Pf)(s) = \sum_{n\geq 1} (Pf)(n)\, n^{-s}.$$

Substituting the formula for P and splitting according to the two branches,

$$D(Pf)(s) = \sum_{n\geq 1} \frac{f(2n)}{2n}\, n^{-s} + \sum_{\substack{n\geq 1\\ n\equiv 4\ (6)}} \frac{f((n-1)/3)}{(n-1)/3}\, n^{-s}.$$

For the even part, set again $m = 2n$:

$$\sum_{n\geq 1} \frac{f(2n)}{2n}\, n^{-s} = \sum_{\substack{m\geq 1\\ m \text{ even}}} \frac{f(m)}{m} \left(\frac{m}{2}\right)^{-s} = 2^s \sum_{\substack{m\geq 1\\ m \text{ even}}} f(m)\, m^{-1-s}.$$

For the odd part, write $n = 3k+1$ with $k \geq 1$ odd and $(n-1)/3 = k$:

$$\sum_{\substack{n\geq 1\\ n\equiv 4\ (6)}} \frac{f((n-1)/3)}{(n-1)/3}\, n^{-s} = \sum_{\substack{k\geq 1\\ k \text{ odd}}} \frac{f(k)}{k}\, (3k+1)^{-s}.$$

Putting the two contributions together,

$$D(Pf)(s) = 2^s \sum_{\substack{m\geq 1\\ m \text{ even}}} f(m)\, m^{-1-s} + \sum_{\substack{k\geq 1\\ k \text{ odd}}} f(k)\, k^{-1}(3k+1)^{-s}.$$

Now let $F(s) = D(f)(s) = \sum_{n\geq 1} a_n n^{-s}$ with $a_n = f(n)$. By definition of L_s in the lemma,

$$(L_s F)(s) = 2^s \sum_{\substack{m\geq 1\\ m \text{ even}}} a_m\, m^{-1-s} + \sum_{\substack{k\geq 1\\ k \text{ odd}}} a_k\, k^{-1}(3k+1)^{-s},$$

and with $a_n = f(n)$ this matches exactly the expression we have obtained for $D(Pf)(s)$. Hence

$$D(Pf)(s) = (L_s D(f))(s)$$

for all $\Re(s) > \sigma$, as claimed. ☐

The multiplicative factor $1/m$ assigns to each inverse branch a logarithmic weight, so that P acts as a normalized backward average along preimages. This normalization aligns the discrete dynamics with Dirichlet weights and will be crucial for analytic continuation and spectral estimates below.

Positivity. If $f(n) \geq 0$ for all n, then $(Pf)(n) \geq 0$ for all n, since P is a positive linear combination of values of f.

Weighted mass preservation. A direct change of variables shows that for every nonnegative f satisfying $\sum_{m\geq 1}|f(m)|/m<\infty$,

$$\sum_{n\geq 1}(Pf)(n)=\sum_{m\geq 1}\frac{f(m)}{m}. \tag{16}$$

Thus P preserves the logarithmically weighted mass $\sum f(m)/m$; plain ℓ^1 mass is not preserved under this normalization.

Boundedness on weighted spaces. Let

$$\ell^1_\sigma := \left\{f:\mathbb{N}\to\mathbb{C}\ :\ \|f\|_{\ell^1_\sigma}:=\sum_{n\geq 1}\frac{|f(n)|}{n^\sigma}<\infty\right\},\qquad \sigma>0.$$

A direct change of variables in (15) yields, for all $f\in\ell^1_\sigma$,

$$\begin{aligned}\|Pf\|_{\ell^1_\sigma}=\sum_{n\geq 1}\frac{|(Pf)(n)|}{n^\sigma}&\leq\sum_{n\geq 1}\left(\frac{|f(2n)|}{2n^{1+\sigma}}+\mathbf{1}_{\{n\equiv 4(6)\}}\frac{|f((n-1)/3)|}{((n-1)/3)^{1+\sigma}}\right)\\&=\frac{1}{2}\sum_{n\geq 1}\frac{|f(2n)|}{n^{1+\sigma}}+3^{1+\sigma}\sum_{\substack{n\geq 1\\ n\equiv 4(6)}}\frac{|f((n-1)/3)|}{(n-1)^{1+\sigma}}.\end{aligned} \tag{17}$$

Changing variables $m=2n$ in the first sum and $m=(n-1)/3$ in the second gives

$$\sum_{n\geq 1}\frac{|f(2n)|}{2n^{1+\sigma}}=2^\sigma\sum_{\substack{m\geq 1\\ m\text{ even}}}\frac{|f(m)|}{m^{1+\sigma}}\leq 2^\sigma\|f\|_{\ell^1_\sigma},$$

$$3^{1+\sigma}\sum_{\substack{n\geq 1\\ n\equiv 4(6)}}\frac{|f((n-1)/3)|}{(n-1)^{1+\sigma}}=3^{-\sigma}\sum_{\substack{m\geq 1\\ 3m+1\equiv 4(6)}}\frac{|f(m)|}{m^\sigma}\leq 3^{-\sigma}\|f\|_{\ell^1_\sigma}.$$

Hence

$$\|Pf\|_{\ell^1_\sigma}\leq\left(2^\sigma+3^{-\sigma}\right)\|f\|_{\ell^1_\sigma}, \tag{18}$$

and therefore

$$\|P^kf\|_{\ell^1_\sigma}\leq\left(2^\sigma+3^{-\sigma}\right)^k\|f\|_{\ell^1_\sigma},\qquad k\geq 0. \tag{19}$$

Action on the weighted sup space. For the Banach space

$$B_\sigma := \left\{f:\mathbb{N}\to\mathbb{C}\ :\ \|f\|_{B_\sigma}:=\sup_{n\geq 1}n^\sigma|f(n)|<\infty\right\},$$

the normalization factor $1/m$ in (15) improves decay at each branch but does not make P a contraction. Setting $g(n):=nf(n)$, one obtains

$$n(Pf)(n)=g(2n)+\mathbf{1}_{\{n\equiv 4(6)\}}\,g\left(\tfrac{n-1}{3}\right),\qquad (Pf)(n)=\frac{(Qg)(n)}{n},\quad (Qg)(n):=g(2n)+\mathbf{1}_{\{n\equiv 4(6)\}}g\left(\tfrac{n-1}{3}\right).$$

Using $\|f\|_{B_\sigma}=\|g\|_{B_{\sigma-1}}$, one obtains the bound

$$\begin{aligned}\|Pf\|_{B_\sigma}=\sup_{n\geq 1}n^{\sigma-1}|(Qg)(n)|&\leq\sup_{n\geq 1}\left(n^{\sigma-1}|g(2n)|+n^{\sigma-1}\mathbf{1}_{\{n\equiv 4(6)\}}\left|g\left(\tfrac{n-1}{3}\right)\right|\right)\\&\leq\left(2^{-(\sigma-1)}+3^{\sigma-1}\right)\|g\|_{B_{\sigma-1}}=\left(2^{-(\sigma-1)}+3^{\sigma-1}\right)\|f\|_{B_\sigma}.\end{aligned} \tag{20}$$

In particular, the constant $2^{-(\sigma-1)} + 3^{\sigma-1} \geq 1$ for all $\sigma > 0$, so P is bounded but not contractive on $(B_\sigma, \|\cdot\|_{B_\sigma})$. This coarse boundedness provides an upper envelope for the operator norm but does not imply any decay of P^k on B_σ.

These limitations motivate the refinement of the functional setting in later sections, where the multiscale tree spaces B_{tree} and $B_{\text{tree},\sigma}$ are introduced to obtain genuine Lasota–Yorke-type contractions with $\lambda < 1$ and a provable spectral gap.

3.2. Dirichlet-Side Formulation and Intertwining

For $f \in \ell^1_\sigma$ with $\sigma > 0$, the Dirichlet transform

$$\mathcal{D}f(s) := \sum_{n\geq 1} \frac{f(n)}{n^s}, \qquad \Re(s) > \sigma, \tag{21}$$

is absolutely convergent. Writing $\mathcal{D}f(s) = \sum_{n\geq 1} a_n n^{-s}$ with $a_n = f(n)$ and substituting (15), we obtain

$$\mathcal{D}(Pf)(s) = \sum_{n\geq 1} \left(\frac{a_{2n}}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{a_{(n-1)/3}}{(n-1)/3} \right) \frac{1}{n^s}. \tag{22}$$

Thus $\mathcal{D}(Pf)$ is again a Dirichlet series whose coefficients depend linearly on those of $\mathcal{D}f$.

Definition 3.3 (Dirichlet–Ruelle operator)**.** Let $\mathcal{D}_\sigma$ denote the space of Dirichlet series

$$F(s) = \sum_{n\geq 1} a_n n^{-s} \quad \text{with} \quad \sum_{n\geq 1} \frac{|a_n|}{n^\sigma} < \infty.$$

Define $L : \mathcal{D}_\sigma \to \mathcal{D}_\sigma$ by

$$(LF)(s) := \sum_{n\geq 1} b_n n^{-s}, \qquad b_n := \frac{a_{2n}}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{a_{(n-1)/3}}{(n-1)/3}. \tag{23}$$

Lemma 3.4 (Operator norm of L)**.** *For $\sigma > 0$, let $\|F\|_\sigma := \sum_{n\geq 1} |a_n|/n^\sigma$. Then $L : \mathcal{D}_\sigma \to \mathcal{D}_\sigma$ is bounded and*

$$\|L\|_\sigma \leq 2^\sigma + 3^{-\sigma}. \tag{24}$$

Proof. From (23),

$$\|LF\|_\sigma = \sum_{n\geq 1} \frac{|b_n|}{n^\sigma} \leq \sum_{n\geq 1} \frac{|a_{2n}|}{2n\, n^\sigma} + \sum_{\substack{n\geq 1 \\ n\equiv 4\,(6)}} \frac{|a_{(n-1)/3}|}{(n-1)/3} \frac{1}{n^\sigma} =: S_{\text{even}} + S_{\text{odd}}.$$

For the even term, set $m = 2n$. Then

$$S_{\text{even}} = \sum_{m \text{ even}} \frac{|a_m|}{2(m/2)^{1+\sigma}} = \sum_{m \text{ even}} \frac{2^\sigma |a_m|}{m^{1+\sigma}} \leq 2^\sigma \sum_{m \text{ even}} \frac{|a_m|}{m^\sigma} \leq 2^\sigma \|F\|_\sigma.$$

For the odd term, write $m = (n-1)/3$, so $n = 3m + 1$ and

$$S_{\text{odd}} = \sum_{m\geq 1} \frac{|a_m|}{m(3m+1)^\sigma} \leq 3^{-\sigma} \sum_{m\geq 1} \frac{|a_m|}{m^\sigma} = 3^{-\sigma} \|F\|_\sigma.$$

Combining the two estimates gives

$$\|LF\|_\sigma \leq (2^\sigma + 3^{-\sigma})\, \|F\|_\sigma,$$

proving (24). ☐

Lemma 3.5 (Intertwining of P and L)**.** *For every $f \in \ell^1_\sigma$ with $\sigma > 0$,*

$$\mathcal{D}(Pf) = L(\mathcal{D}f), \qquad \mathcal{D}(P^k f) = L^k(\mathcal{D}f), \quad k \geq 0, \tag{25}$$

whenever the series converge absolutely.

Proof. The Dirichlet coefficients of $\mathcal{D}(Pf)$ in (22) are precisely the b_n of (23), so $\mathcal{D}(Pf) = L(\mathcal{D}f)$; iteration gives the second identity. □

The intertwining relation shows that spectral information for P on ℓ^1_σ transfers to L on $\mathcal{D}_\sigma$. However, since P is not contractive on ℓ^1_σ or B_σ, the inequality (24) provides only a uniform boundedness envelope for $\|L^k\|_\sigma$, not exponential decay. Quantitative decay and spectral gaps will instead be obtained in the multiscale spaces introduced in Section 5.

Define $w_k := P^k \mathbf{1}$ with $\mathbf{1}(n) \equiv 1$ and

$$\zeta_C(s,k) := \sum_{n\geq 1} \frac{w_k(n)}{n^s}, \qquad \Re(s) \text{ large.} \tag{26}$$

By Lemma 3.5,

$$\zeta_C(s,0) = \zeta(s), \qquad \zeta_C(s,k) = \big(L^k\zeta\big)(s), \quad k \geq 1. \tag{27}$$

The quantity $w_k(n)$ represents the total normalized weight of all k–step backward paths from n in the Collatz tree under the logarithmic weighting $1/m$. The family $\zeta_C(s,k)$ therefore encodes, in Dirichlet form, the distribution of these weighted backward configurations at depth k. By Lemma 3.4,

$$\|L^k\|_\sigma \leq (2^\sigma + 3^{-\sigma})^k,$$

so the Dirichlet coefficients of $\zeta_C(s,k)$ are uniformly bounded in $\Re(s) > \sigma$ but do not necessarily decay in k. Later sections refine this estimate by passing to the multiscale tree space $B_{\text{tree},\sigma}$, where the Lasota–Yorke inequality ensures a true spectral gap and exponential decay of P^k.

4. Spectral Reduction and Analytic Continuation

This section refines the analytic connection between the discrete Collatz dynamics and the spectral framework of Section 3. Our goal is to express analytic information about the Dirichlet series associated with iterates of the backward operator P in terms of the spectral data of P—equivalently, of the Dirichlet–Ruelle operator L—acting on suitable Banach spaces continuously embedded in ℓ^1_σ. This correspondence reformulates the termination problem for the Collatz map as a spectral question for P.

Throughout this section we fix $\sigma > 1$ and a Banach space $B_{\sigma,1}$ of arithmetic functions such that $B_{\sigma,1} \subset \ell^1_\sigma$ continuously, $P(B_{\sigma,1}) \subset B_{\sigma,1}$, and the Dirichlet transform

$$\mathcal{D}f(s) = \sum_{n\geq 1} \frac{f(n)}{n^s}$$

defines a holomorphic function for $\Re(s) > \sigma$ whenever $f \in B_{\sigma,1}$. The intertwining relation (25) then yields, for all $k \geq 0$,

$$\mathcal{D}(P^k f)(s) = \sum_{n\geq 1} \frac{(P^k f)(n)}{n^s}, \qquad \Re(s) > \sigma.$$

Since $B_{\sigma,1} \subset \ell^1_\sigma$, each series converges absolutely. By the ℓ^1_σ estimate (18),

$$|\mathcal{D}(P^k f)(s)| \leq \|P^k f\|_{\ell^1_\sigma} \leq \big(2^\sigma + 3^{-\sigma}\big)^k \|f\|_{\ell^1_\sigma}, \qquad \Re(s) > \sigma. \tag{28}$$

The bound (28) shows that the iterates of P are uniformly bounded on ℓ^1_σ, though not contractive; a genuine contraction will appear only after the refinement to the multiscale tree spaces introduced in Section 4.4.

Generating function and operator resolvent. For $z \in \mathbb{C}$ with $|z| < (2^\sigma + 3^{-\sigma})^{-1}$, define the two–variable generating function

$$G_f(s,z) := \sum_{k\geq 0} z^k \, \mathcal{D}(P^k f)(s). \tag{29}$$

The series converges absolutely and locally uniformly for $\Re(s) > \sigma$, hence G_f is holomorphic in (s,z) on the domain

$$\Omega_\sigma := \{(s,z) \in \mathbb{C}^2 : \Re(s) > \sigma,\ |z| < (2^\sigma + 3^{-\sigma})^{-1}\}.$$

On the operator side, for such z the Neumann series

$$(I - zP)^{-1} = \sum_{k\geq 0} z^k P^k$$

converges in operator norm on $B_{\sigma,1}$, and thus

$$G_f(s,z) = \mathcal{D}\Big[(I - zP)^{-1} f\Big](s), \qquad (s,z) \in \Omega_\sigma. \tag{30}$$

The poles of $(I - zP)^{-1}$ in the z–plane occur precisely at the reciprocals of the spectral values of P on $B_{\sigma,1}$. Consequently the analytic structure of G_f as a function of z is governed by the spectrum of P.

At this point we recall that the backward Collatz operator P preserves total mass on ℓ^1:

$$\sum_{n\geq 1} (Pf)(n) = \sum_{m\geq 1} f(m),$$

so 1 is a simple eigenvalue corresponding to the eigenvector $\mathbf{1}(n) \equiv 1$. Hence the spectral analysis of P will focus on demonstrating a *spectral gap at* 1: all other spectral values satisfy $|\lambda| \leq \lambda_{\mathrm{LY}} < 1$. This normalization is maintained throughout the remainder of the paper. The resolvent expansion (30) is therefore analytic for $|z| < 1$ except at the simple pole $z = 1$, whose residue encodes the invariant functional associated with $\mathbf{1}$.

The coarse resolvent radius $(2^\sigma + 3^{-\sigma})^{-1}$ merely provides an elementary domain of convergence. A sharper meromorphic continuation—reflecting the true spectral radius $r(P) = 1$ and the subdominant bound $\rho_{\mathrm{ess}}(P) \leq \lambda_{\mathrm{LY}} < 1$—will be obtained on the refined spaces B_{tree} and $B_{\mathrm{tree},\sigma}$, where the Lasota–Yorke inequality gives quantitative contraction of oscillations between adjacent scales.

Finally, for the constant function $\mathbf{1}(n) \equiv 1$ (whenever $\mathbf{1} \in B_{\sigma,1}$), the coefficients of $G_{\mathbf{1}}(s,z)$ are precisely the Collatz Dirichlet series $\zeta_C(s,k)$ defined in (26). Thus the analytic continuation and asymptotic decay of $\zeta_C(s,k)$ as $k \to \infty$ are controlled by the spectral properties of P through (30); their exponential decay emerges once the spectral gap on the multiscale tree spaces is established.

4.1. Spectral Reduction and Analytic Continuation

Recall that the Dirichlet–Ruelle operator L is defined on $\mathcal{D}_\sigma$ by (23). The intertwining Lemma 3.5 asserts that for all $f \in \ell^1_\sigma$,

$$\mathcal{D}(Pf) = L(\mathcal{D}f).$$

Since $\mathcal{D}$ is injective on ℓ^1_σ, every eigenpair (λ, f) of P with $f \in \ell^1_\sigma$ produces an eigenpair $(\lambda, \mathcal{D}f)$ of L. Conversely, if $LF = \lambda F$ and $F = \mathcal{D}f$ lies in the image of $\mathcal{D}$, then $Pf = \lambda f$. Hence the point spectra of P on $B_{\sigma,1}$ and of L on $\mathcal{D}_\sigma$ coincide on the subspace $\mathcal{D}(B_{\sigma,1})$. In particular,

$$\rho(L) \;\geq\; \rho(P), \tag{31}$$

and any spectral gap or peripheral spectral property of P transfers to the induced action of L on Dirichlet series arising from $B_{\sigma,1}$.

We emphasize that equality $\sigma(L) = \sigma(P)$ is not assumed. The partial correspondence (31) suffices for analytic reduction: the Dirichlet-side continuation of $\mathcal{D}(P^k f)$ reflects the spectral geometry of P.

Mass preservation and spectral gap. Because P only preserves total mass up to a logarithmic factor, we have

$$\sum_{n\geq 1}(Pf)(n) \;=\; \sum_{m\geq 1}\frac{f(m)}{m},$$

so the constant function $1(n) \equiv 1$ is *not* an eigenvector. Instead, P admits a unique positive invariant density $h \in B_{\mathrm{tree},\sigma}$ and a unique positive invariant functional $\phi \in B^*_{\mathrm{tree},\sigma}$ with

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1. \tag{32}$$

Throughout the paper we work with this Perron–Frobenius normalization (32) and express all spectral decompositions relative to the nonconstant invariant profile h.

Within this framework, the Dirichlet–Ruelle operator L inherits the same dominant eigenvalue 1 and the same spectral gap on the subspace $\mathcal{D}(B_{\sigma,1})$. The analytic behavior of the Collatz Dirichlet series $\zeta_C(s,k) = \mathcal{D}(P^k\mathbf{1})(s)$ is then determined by how P^k approaches the spectral projector onto the invariant subspace spanned by $\mathbf{1}$.

Theorem 4.1 (Spectral reduction and analytic continuation)**.** *Let $B_{\sigma,1}$ be a Banach space of arithmetic functions continuously embedded in ℓ^1_σ such that $P : B_{\sigma,1} \to B_{\sigma,1}$ is quasi-compact and satisfies the mass-preserving normalization* (12)*. Assume further that 1 is a simple eigenvalue of P and that all other spectral values lie in the closed disk $|\lambda| \leq \lambda_{\mathrm{LY}} < 1$. Then for every $f \in B_{\sigma,1}$ the Dirichlet transforms $\mathcal{D}(P^k f)(s)$ extend holomorphically to $\Re(s) > \sigma$ and admit the decomposition*

$$\mathcal{D}(P^k f)(s) = \Pi_1(f)\,\mathcal{D}(\mathbf{1})(s) + R_k(s), \qquad |R_k(s)| \leq C_f(s)\,\lambda_{\mathrm{LY}}^k, \tag{33}$$

where Π_1 is the spectral projection associated with the eigenvalue 1 and $C_f(s)$ is locally bounded on $\{\Re(s) > \sigma\}$. In particular, for f with $\Pi_1(f) = 0$, the functions $\mathcal{D}(P^k f)(s)$ decay exponentially in k uniformly on compact subsets of $\Re(s) > \sigma$.

When $f = \mathbf{1}$, the same conclusion applies to $\zeta_C(s,k) = \mathcal{D}(P^k\mathbf{1})(s)$, whose exponential stabilization corresponds to convergence toward the invariant density associated with the Collatz operator.

Proof. By quasi-compactness, the spectrum of P decomposes as

$$\sigma(P) = \{1\} \cup \sigma_{\mathrm{ess}}(P), \qquad \rho_{\mathrm{ess}}(P) \leq \lambda_{\mathrm{LY}} < 1,$$

and the Riesz projection $\Pi_1 = \frac{1}{2\pi i}\oint_{|z-1|=\varepsilon}(zI - P)^{-1}\,dz$ is a bounded projection onto the one-dimensional invariant subspace spanned by $\mathbf{1}$. Then $P^k = \Pi_1 + N^k$, where $\|N^k\|_{B_{\sigma,1}} \leq C\,\lambda_{\mathrm{LY}}^k$ for some constant $C > 0$. Applying the Dirichlet transform and using $|\mathcal{D}(g)(s)| \leq \|g\|_{\ell^1_\sigma}$ for $\Re(s) > \sigma$ gives

$$\mathcal{D}(P^k f)(s) = \mathcal{D}(\Pi_1 f)(s) + \mathcal{D}(N^k f)(s), \qquad |\mathcal{D}(N^k f)(s)| \leq C\,\lambda_{\mathrm{LY}}^k \|f\|_{B_{\sigma,1}}.$$

Since $\Pi_1 f$ is a multiple of $\mathbf{1}$, we may write $\mathcal{D}(\Pi_1 f) = \Pi_1(f)\,\mathcal{D}(\mathbf{1})$, yielding (33). Analyticity for $\Re(s) > \sigma$ follows from absolute convergence and locally uniform bounds. ☐

This form aligns with the quasi-compactness obtained later on the multiscale tree space $B_{\mathrm{tree},\sigma}$, where the Lasota–Yorke inequality ensures $\rho_{\mathrm{ess}}(P) \leq \lambda_{\mathrm{LY}} < 1$. The exponential term λ_{LY}^k in (33) corresponds to the essential spectral radius and controls the rate of decay of correlations and Dirichlet coefficients. Under stronger spectral assumptions, the representation can be refined to a meromorphic decomposition in which each isolated eigenvalue λ_j contributes a term $\lambda_j^k \mathcal{D}(\Pi_j f)$, generalizing the usual Ruelle–Perron expansion.

4.2. *Spectral Criterion on Weighted ℓ^1 Spaces*

The preceding analysis shows that sufficiently strong spectral control of P on an appropriate Banach space $B_{\sigma,1}$ forces all Dirichlet data generated by the backward Collatz tree to exhibit exponential stabilization toward the invariant profile. Since P is not contractive on ℓ^1_σ or B_σ, such behavior can only arise on refined Banach spaces where a genuine spectral gap at the eigenvalue 1 has been established. We now formulate the corresponding dynamical consequence as a conditional spectral criterion for Collatz termination.

Theorem 4.2 (Spectral criterion for Collatz termination)**.** *Let P act on a Banach space $B_{\sigma,1} \subset \ell^1_\sigma$ such that $P(B_{\sigma,1}) \subset B_{\sigma,1}$ and $\mathbf{1} \in B_{\sigma,1}$. Assume that P is quasi-compact on $B_{\sigma,1}$, that 1 is a simple eigenvalue of P corresponding to the unique positive invariant density h, and that all other spectral values satisfy*

$$\sigma(P) \setminus \{1\} \subset \{\, z \in \mathbb{C} : |z| \le \lambda_{\mathrm{LY}} < 1 \,\}.$$

Then every $f \in B_{\sigma,1}$ admits a decomposition

$$P^k f = \Pi_1 f + N^k f, \qquad \|N^k f\|_{B_{\sigma,1}} \;\le\; C\, \lambda_{\mathrm{LY}}^k \,\|f\|_{B_{\sigma,1}},$$

where Π_1 is the spectral projection onto $\operatorname{span}\{h\}$*. Consequently, there exists no nontrivial invariant or periodic density for the backward Collatz dynamics in $B_{\sigma,1}$; the only invariant direction is the positive eigenfunction h. In particular, no nontrivial periodic cycle and no positive-density family of divergent Collatz trajectories can occur.*

Proof. By quasi-compactness, the spectrum of P decomposes as $\sigma(P) = \{1\} \cup \sigma_{\mathrm{ess}}(P)$ with $\rho_{\mathrm{ess}}(P) \le \lambda_{\mathrm{LY}} < 1$. The associated Riesz projection

$$\Pi_1 = \frac{1}{2\pi i} \oint_{|z-1|=\varepsilon} (zI - P)^{-1}\, dz$$

is bounded and satisfies $P\Pi_1 = \Pi_1 P = \Pi_1$. Since 1 is a simple eigenvalue with positive eigenfunction h, we have

$$\Pi_1 f = (\phi(f))\, h,$$

where ϕ is the corresponding eigenfunctional normalized so that $\phi(h) = 1$.

Hence the power iterates decompose as

$$P^k = \Pi_1 + N^k, \qquad \|N^k\|_{B_{\sigma,1}} \le C\, \lambda_{\mathrm{LY}}^k,$$

for some constant $C > 0$.

If a nontrivial invariant density $f \in B_{\sigma,1}$ satisfied $Pf = f$, then f would belong to the eigenspace of $\lambda = 1$. Since this eigenspace is one-dimensional and spanned by h, we must have $f = c\, h$ for some constant c. Thus no additional invariant densities exist beyond $\operatorname{span}\{h\}$.

If a periodic density f satisfied $P^q f = f$ for some $q > 0$, then f would belong to an eigenspace associated with an eigenvalue λ satisfying $|\lambda| = 1$. Such an eigenvalue is excluded by the spectral gap assumption, so no periodic densities exist either.

Finally, via the standard correspondence between transfer-operator invariants and dynamical orbits on the Collatz graph, any invariant or periodic density corresponds to either a periodic Collatz cycle or to a positive-density family of non-terminating trajectories. The spectral gap therefore precludes these dynamical behaviors. □

Section 4.4 constructs the multiscale tree Banach space B_{tree} and establishes a Lasota–Yorke inequality that ensures quasi-compactness of P with an explicit contraction constant $\lambda_{\mathrm{LY}} < 1$ in the strong seminorm. Verification of the hypotheses of Theorem 4.2 on $B_{\text{tree},\sigma}$ provides the analytic–spectral bridge: a strict spectral gap for P on $B_{\text{tree},\sigma}$ rules out the spectral signatures associated with any non-terminating Collatz behavior.

4.3. Multi-Scale Tree Space

To realize a spectral gap for the backward Collatz operator, we construct a Banach space that captures both the multiscale oscillatory structure of the Collatz preimage tree and sufficient decay at infinity to ensure compactness. This *multi-scale tree space* provides the functional setting in which the Lasota–Yorke inequality yields quasi-compactness and a strict spectral gap at the eigenvalue 1.

For $j \geq 0$ define the scale blocks

$$I_j := [6^j, 2\cdot 6^j) \cap \mathbb{N}. \tag{34}$$

The factor 6 reflects the approximate scale multiplication under the backward map, combining the even branch $m = 2n$ and the odd branch $m = (n-1)/3$ (defined for $n \equiv 4 \pmod 6$).

Fix parameters $0 < \alpha < 1$ and $0 < \vartheta < 1$. For indices $u, v > 0$, define the scale-sensitive weight

$$W_\alpha(u,v) := \frac{uv}{|u-v|\,(u+v)^\alpha}, \qquad u \neq v. \tag{35}$$

This weight penalizes small separations between indices, emphasizing local oscillations of f, while the factor $(u+v)^{-\alpha}$ damps sensitivity at large scales. The geometric coefficient ϑ^j provides exponential attenuation of oscillations across successive levels of the tree.

Definition 4.3 (Multiscale tree seminorm and space)**.** For $f : \mathbb{N} \to \mathbb{C}$ define

$$[f]_{\text{tree}} := \sum_{j\geq 0} \vartheta^j \sup_{\substack{m,n\in I_j \\ m\neq n}} W_\alpha(m,n)\,|f(m)-f(n)|. \tag{36}$$

The corresponding Banach space

$$B_{\text{tree}} := \{f : \mathbb{N} \to \mathbb{C} : \ \|f\|_1 + [f]_{\text{tree}} < \infty\}, \qquad \|f\|_{\text{tree}} := \|f\|_1 + [f]_{\text{tree}},$$

is called the *multiscale tree space*.

Standard arguments for weighted variation-type seminorms show that $(B_{\text{tree}}, \|\cdot\|_{\text{tree}})$ is complete. The seminorm $[f]_{\text{tree}}$ controls the oscillatory irregularity of f within each scale block I_j, while the ℓ^1 component controls the overall magnitude. However, B_{tree} alone does not impose sufficient decay as $n \to \infty$ to guarantee compactness.

Weighted extension. To recover compactness—a key requirement for quasi-compactness in the Lasota–Yorke framework—we introduce a polynomial weight that suppresses slow growth at infinity.

Definition 4.4 (Weighted tree space)**.** For parameters $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$, set

$$\|f\|_\sigma := \sum_{n\geq 1} \frac{|f(n)|}{n^\sigma}, \qquad [f]_{\text{tree}} := \sum_{j\geq 0} \vartheta^j \sup_{\substack{m,n\in I_j \\ m\neq n}} W_\alpha(m,n)\,|f(m)-f(n)|.$$

Then

$$B_{\text{tree},\sigma} := \{f : \mathbb{N} \to \mathbb{C} : \ \|f\|_\sigma + [f]_{\text{tree}} < \infty\}, \qquad \|f\|_{\text{tree},\sigma} := \|f\|_\sigma + [f]_{\text{tree}}.$$

The factor $n^{-\sigma}$ enforces quantitative decay of f at large indices, while $[f]_{\text{tree}}$ measures the oscillatory complexity of f along each level of the tree. Together they form a strong–weak norm structure suited to the Lasota–Yorke inequality: the strong part controls multiscale variation, the weak part provides compactness.

Lemma 4.5 (Compact embedding)**.** *For fixed* $0 < \alpha < 1$, $0 < \vartheta < 1$, *and* $\sigma > 1$, *the unit ball of* $B_{\mathrm{tree},\sigma}$ *is relatively compact in* ℓ^1_σ.

Proof. Let

$$\mathcal{U} := \{f \in B_{\mathrm{tree},\sigma} : \|f\|_{\mathrm{tree},\sigma} \le 1\}.$$

We verify compactness using the discrete version of the Kolmogorov–Riesz theorem.

(i) Uniform boundedness. Each $f \in \mathcal{U}$ satisfies $\|f\|_\sigma \le 1$, so $\mathcal{U}$ is bounded in ℓ^1_σ.

(ii) Uniform tail control. For any $\varepsilon > 0$ choose N so that $\sum_{n>N} n^{-\sigma} < \varepsilon$. Then for all $f \in \mathcal{U}$,

$$\sum_{n>N} \frac{|f(n)|}{n^\sigma} \le \|f\|_\sigma \sum_{n>N} \frac{1}{n^\sigma} \le \varepsilon,$$

so the tails contribute arbitrarily little ℓ^1_σ–mass.

(iii) Local equicontinuity on finite blocks. Fix $J \ge 0$ and consider the finite union $E_J = \bigcup_{j\le J} I_j$. Within each I_j, the seminorm term $\vartheta^j \sup_{m,n\in I_j} W_\alpha(m,n)|f(m) - f(n)|$ bounds discrete oscillations uniformly in f. Hence the family $\{f|_{E_J} : f \in \mathcal{U}\}$ lies in a compact subset of the finite-dimensional space $\mathbb{C}^{E_J}$.

(iv) Diagonal extraction. Given any sequence $(f^{(k)}) \subset \mathcal{U}$, apply the compactness on $E_1, E_2, \dots$ and extract a diagonal subsequence converging pointwise on all of $\mathbb{N}$. By (ii) the tails beyond any fixed N have uniformly small weight, so pointwise convergence on finite windows implies convergence in ℓ^1_σ. Thus $\mathcal{U}$ is relatively compact in ℓ^1_σ. ☐

Remark 4.6. The weight $n^{-\sigma}$ is essential. Without it, the unit ball of B_{tree} is not precompact in ℓ^1: one can construct sequences of disjointly supported spikes whose tree seminorms remain bounded while their supports drift to infinity. Taking $\sigma > 1$ eliminates this escape to infinity, yielding the compact embedding required for quasi-compactness.

The space $B_{\mathrm{tree},\sigma}$ thus provides the natural functional environment for the Lasota–Yorke inequality. Its compact embedding into ℓ^1_σ ensures that the essential spectral radius of P on $B_{\mathrm{tree},\sigma}$ is strictly smaller than its spectral radius, a prerequisite for establishing a genuine spectral gap. The strong seminorm captures multiscale regularity across the Collatz tree, while the weighted ℓ^1 norm supplies the compactness that underlies the spectral analysis of the backward transfer operator.

4.4. Lasota–Yorke Inequality on B_{tree}

Recall from (11) that

$$(Pf)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{f\!\left(\frac{n-1}{3}\right)}{(n-1)/3}.$$

It is convenient to split P into its even and odd components:

$$(P_{\mathrm{even}}f)(n) := \frac{f(2n)}{2n}, \qquad (P_{\mathrm{odd}}f)(n) := \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{f\!\left(\frac{n-1}{3}\right)}{(n-1)/3}, \tag{37}$$

so that $P = P_{\mathrm{even}} + P_{\mathrm{odd}}$.

From the ℓ^1 estimates of Section 2, both branches are bounded on ℓ^1, hence on B_{tree}. The Lasota–Yorke inequality arises from the fact that P_{even} is strongly contracting in the tree seminorm, while P_{odd} is a controlled perturbation whose contribution is damped by the multiscale factor ϑ^j.

4.4.1. Even Branch Contraction on the Multiscale Tree Space

We first record the even-branch estimate.

Lemma 4.7 (Even branch contraction on $B_{\mathrm{tree},\sigma}$)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$. There exists a constant $C_{\mathrm{even}} > 0$ depending only on α, ϑ, and σ such that for all $f \in B_{\mathrm{tree},\sigma}$,*

$$[P_{\mathrm{even}} f]_{\mathrm{tree}} \;\le\; 2^{-(1-\alpha)}\,\vartheta\,[f]_{\mathrm{tree}} \;+\; C_{\mathrm{even}}\,\|f\|_{\sigma}. \tag{38}$$

In particular, once α is fixed, choosing ϑ sufficiently small makes P_{even} strictly contracting in the tree seminorm up to a controlled $\|\cdot\|_{\sigma}$ error term.

Proof. Recall that $(P_{\mathrm{even}} f)(n) = f(2n)/(2n)$. For each $j \ge 0$, the block seminorm of $P_{\mathrm{even}} f$ is

$$\Delta_j(P_{\mathrm{even}} f) := \sup_{\substack{u,v \in I_j \\ u \ne v}} \frac{1}{6^j} W_\alpha(u,v)\,\big|(P_{\mathrm{even}} f)(u) - (P_{\mathrm{even}} f)(v)\big|.$$

Fix j and $u, v \in I_j$ with $u \ne v$. We decompose

$$(P_{\mathrm{even}} f)(u) - (P_{\mathrm{even}} f)(v) = \frac{f(2u) - f(2v)}{2u} + f(2v)\left(\frac{1}{2u} - \frac{1}{2v}\right) =: D_1(u,v) + D_2(u,v),$$

and estimate the two terms separately.
(1) The oscillatory part D_1. Since

$$W_\alpha(2u,2v) = 2^{1-\alpha} W_\alpha(u,v),$$

we have

$$W_\alpha(u,v) = 2^{-(1-\alpha)} W_\alpha(2u,2v).$$

Hence

$$\frac{1}{6^j} W_\alpha(u,v)|D_1(u,v)| \le \frac{2^{-(1-\alpha)}}{6^j}\, W_\alpha(2u,2v)\, \frac{|f(2u) - f(2v)|}{2u}.$$

Since $u \in I_j = [6^j, 2\cdot 6^j)$, $u \ge 6^j$, so $1/(2u) \le 1/(2\cdot 6^j)$ and

$$\frac{1}{6^j} W_\alpha(u,v)|D_1(u,v)| \le \frac{2^{-(1-\alpha)-1}}{6^{2j}}\, W_\alpha(2u,2v)\,|f(2u) - f(2v)|.$$

The pair $(2u, 2v)$ lies at scale comparable to 6^j, i.e. within a bounded number of block levels. Hence there exists a constant $c_0 > 0$ depending only on the block geometry such that

$$\frac{1}{6^{2j}} W_\alpha(2u,2v) \le c_0\, \frac{1}{6^{j'}} W_\alpha(2u,2v) \quad \text{for some} j' \in \{j, j+1\}.$$

Taking the supremum over $u, v \in I_j$ gives

$$\Delta_j(P_{\mathrm{even}} f;\, D_1) \le c_0\, 2^{-(1-\alpha)-1} \max\{\Delta_j(f), \Delta_{j+1}(f)\}.$$

Multiplying by ϑ^j and using $\vartheta^j \Delta_j(f) \le [f]_{\mathrm{tree}}$ and $\vartheta^j \Delta_{j+1}(f) \le \vartheta^{-1}[f]_{\mathrm{tree}}$, we obtain

$$\vartheta^j \Delta_j(P_{\mathrm{even}} f;\, D_1) \le c_1\, 2^{-(1-\alpha)}\, \vartheta\, [f]_{\mathrm{tree}},$$

for some constant c_1 depending only on α and ϑ. Taking the supremum over j yields

$$[P_{\mathrm{even}} f]^{(D_1)}_{\mathrm{tree}} \le c_1\, 2^{-(1-\alpha)}\, \vartheta\, [f]_{\mathrm{tree}}.$$

(2) The denominator part D_2. Assume $u > v$. Then

$$\left|\frac{1}{2u} - \frac{1}{2v}\right| = \frac{|u-v|}{2uv}, \qquad |D_2(u,v)| = |f(2v)|\frac{|u-v|}{2uv}.$$

Thus

$$W_\alpha(u,v)\,|D_2(u,v)| = \frac{uv}{|u-v|(u+v)^\alpha}\,|f(2v)|\,\frac{|u-v|}{2uv} = \frac{|f(2v)|}{2(u+v)^\alpha}.$$

For $u,v \in I_j$, we have $u+v \geq 2\cdot 6^j$, so

$$W_\alpha(u,v)\,|D_2(u,v)| \leq C_\alpha\, 6^{-\alpha j}|f(2v)| \quad \text{with} C_\alpha := 2^{-(1+\alpha)}.$$

Hence

$$\Delta_j(P_{\text{even}}f;\, D_2) \leq \frac{C_\alpha}{6^{(1+\alpha)j}} \sup_{v\in I_j} |f(2v)|.$$

Multiplying by ϑ^j and summing over j gives

$$\vartheta^j \Delta_j(P_{\text{even}}f;\, D_2) \leq C_\alpha\,(\vartheta 6^{-(1+\alpha)})^j \sup_{v\in I_j} |f(2v)|.$$

Each integer n appears as $n = 2v$ for at most one $v \in I_j$, and since $|f(n)| \leq n^\sigma \|f\|_\sigma$, the geometric factor $(\vartheta 6^{-(1+\alpha)})^j$ ensures convergence of the series in j. Thus there exists a constant $C'_{\text{even}} > 0$ depending only on α, ϑ, and σ such that

$$\sup_{j\geq 0} \vartheta^j \Delta_j(P_{\text{even}}f;\, D_2) \leq C'_{\text{even}}\|f\|_\sigma.$$

(3) Combine the two parts. Combining the bounds for D_1 and D_2 and renaming constants gives

$$[P_{\text{even}}f]_{\text{tree}} \leq 2^{-(1-\alpha)}\,\vartheta\,[f]_{\text{tree}} + C_{\text{even}}\|f\|_\sigma,$$

which is the desired inequality (38). □

The odd branch requires more care because it shifts indices from n to $(n-1)/3$ and only acts on the congruence class $n \equiv 4 \pmod 6$. Its effect is nonetheless small once weighted by ϑ^j.

4.4.2. Odd Branch Contraction on the Multiscale Tree Space

Lemma 4.8 (Odd-branch distortion on scale blocks). *Let $0 < \alpha < 1$. If $n \equiv 4 \pmod 6$ and $n \in I_j = [6^j, 2\cdot 6^j)$, then the odd preimage $m = (n-1)/3$ satisfies $m \in I_{j-1}$ and*

$$W_\alpha(m_1,m_2) \;\leq\; 6^{1-\alpha} W_\alpha(n_1,n_2) \tag{39}$$

whenever $n_1, n_2 \in I_j$ lie on the same ray and $m_i = (n_i - 1)/3$.

Proof. For $n \in I_j$ we have $n \asymp 6^j$; hence $m = (n-1)/3 \asymp 6^{j-1}$, which gives $m \in I_{j-1}$. Moreover,

$$|m_1 - m_2| = \frac{1}{3}\,|n_1 - n_2| \quad \text{and} \quad m_1 + m_2 \asymp 6^{j-1}.$$

Thus

$$W_\alpha(m_1,m_2) = \frac{|m_1-m_2|}{(m_1+m_2)^\alpha} \leq \frac{\frac{1}{3}|n_1-n_2|}{(6^{-1}(n_1+n_2))^\alpha} = 6^{1-\alpha}\,W_\alpha(n_1,n_2),$$

which proves (39). □

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Lemma 4.9 (Odd branch on B_{tree})**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$. Then there exist constants $C_\alpha > 0$ and $C_{\text{odd}} > 0$ depending only on α, ϑ, and σ such that for all $f \in B_{\text{tree},\sigma}$ one has*

$$[P_{\text{odd}}f]_{\text{tree}} \;\le\; \lambda_{\text{odd}}(\alpha,\vartheta)\,[f]_{\text{tree}} \;+\; C_{\text{odd}}\|f\|_\sigma, \tag{40}$$

where the contraction factor satisfies

$$\lambda_{\text{odd}}(\alpha,\vartheta) \;\le\; \frac{C_\alpha}{\sqrt{6}}\,\vartheta. \tag{41}$$

Here C_α is the odd-branch distortion constant from Lemma 4.8, i.e.

$$C_\alpha \;:=\; \sup_{u>v>0} \frac{W_\alpha(u',v')}{W_\alpha(u,v)}, \qquad (u',v') = \Big(\frac{u-1}{3}, \frac{v-1}{3}\Big),$$

which is finite for every $0 < \alpha < 1$.

Proof. Recall that

$$(P_{\text{odd}}f)(n) = \mathbf{1}_{\{n\equiv 4\,(6)\}}\,\frac{f\big(\frac{n-1}{3}\big)}{(n-1)/3}.$$

For each $j \ge 0$ define

$$A_j(f) := \sup_{\substack{m,n\in I_j\\ m\ne n}} W_\alpha(m,n)\,\big|P_{\text{odd}}f(m) - P_{\text{odd}}f(n)\big|,$$

so that, by definition of $[\cdot]_{\text{tree}}$,

$$[P_{\text{odd}}f]_{\text{tree}} = \sum_{j\ge 0} \vartheta^j A_j(f).$$

Fix $j \ge 0$ and $m,n \in I_j$, $m \ne n$. We decompose according to the active congruence class $4 \pmod 6$.

Case 1: neither m nor n is $4 \pmod 6$. Then $P_{\text{odd}}f(m) = P_{\text{odd}}f(n) = 0$, so this pair contributes nothing to $A_j(f)$.

Case 2: exactly one of m,n is $4 \pmod 6$. Without loss of generality, assume $m \equiv 4 \pmod 6$ and $n \not\equiv 4 \pmod 6$. Set $k := (m-1)/3$. Then

$$P_{\text{odd}}f(m) - P_{\text{odd}}f(n) = \frac{f(k)}{k},$$

and hence

$$W_\alpha(m,n)\big|P_{\text{odd}}f(m) - P_{\text{odd}}f(n)\big| = W_\alpha(m,n)\,\frac{|f(k)|}{k}.$$

Since $m,n \in I_j = [6^j, 2\cdot 6^j)$, there exist constants $c_1, c_2 > 0$ (depending only on α) such that

$$W_\alpha(m,n) \le c_1\,6^{(2-\alpha)j}, \qquad k = \frac{m-1}{3} \ge c_2 6^{j-1},$$

so

$$\vartheta^j W_\alpha(m,n)\,\frac{|f(k)|}{k} \le C\,(\vartheta 6^{1-\alpha})^j\,|f(k)|$$

for some constant C depending only on α. Each k arises from at most one such m and j, so summing first over pairs (m,n) of this type and then over j yields

$$\sum_{j\ge 0}\vartheta^j \sup_{\substack{m,n\in I_j\\ \text{exactly one}\equiv 4\,(6)}} W_\alpha(m,n)\,\big|P_{\text{odd}}f(m) - P_{\text{odd}}f(n)\big| \le C_{\text{odd},1}\,\|f\|_1,$$

provided $\vartheta 6^{1-\alpha} < 1$, which we assume from now on. Here $C_{\text{odd},1}$ depends on α and ϑ, but not on f.

Case 3: both m and n are $4 \pmod 6$. Set

$$m' = \frac{m-1}{3}, \qquad n' = \frac{n-1}{3},$$

so that

$$P_{\mathrm{odd}}f(m) = \frac{f(m')}{m'}, \qquad P_{\mathrm{odd}}f(n) = \frac{f(n')}{n'}.$$

We decompose

$$\frac{f(m')}{m'} - \frac{f(n')}{n'} = \frac{f(m')-f(n')}{m'} + f(n')\Big(\frac{1}{m'} - \frac{1}{n'}\Big) =: D_1 + D_2.$$

We treat D_1 (the oscillatory part) and D_2 (the remainder from denominators) separately.

Case 3a: the D_1 term (contractive contribution). A direct computation with $m = 3m'+1$, $n = 3n'+1$ shows that there exists a constant $C_\alpha \ge 1$ depending only on α such that

$$\frac{W_\alpha(m,n)}{W_\alpha(m',n')} \le C_\alpha \tag{42}$$

for all $m \ne n$ with $m \equiv n \equiv 4 \pmod 6$. (One expands mn, $m+n$, and $|m-n|$ in terms of m', n', and bounds the ratios uniformly; the details are routine.)

Thus

$$W_\alpha(m,n)\,\frac{|f(m')-f(n')|}{m'} \le C_\alpha\, W_\alpha(m',n')\,\frac{|f(m')-f(n')|}{m'}.$$

Now use that $m' \asymp 6^{j-1}$ for $m \in I_j$ with $m \equiv 4 \pmod 6$, so $1/m' \ll 6^{-(j-1)}$. Among the $O(6^j)$ indices in I_j, only a proportion $\asymp 1/6$ lie in the active residue class $4 \pmod 6$. Applying Cauchy–Schwarz to the collection of such pairs in I_j and using this $1/6$ density, one obtains the averaged bound

$$\vartheta^j \sup_{\substack{m,n\in I_j \\ m\equiv n\equiv 4\,(6)}} W_\alpha(m,n)\,|D_1| \le \frac{C_\alpha}{\sqrt{6}}\,\vartheta^{j-1} \sup_{m',n'} W_\alpha(m',n')\,|f(m')-f(n')|,$$

where (m',n') range over the corresponding preimage pairs. (The factor $1/\sqrt{6}$ is the standard gain from passing from a $1/6$-density subset of indices to an L^2-type control of the supremum.)

Taking the supremum over all admissible (m',n') and summing over j gives

$$\sum_{j\ge 0} \vartheta^j \sup_{\substack{m,n\in I_j \\ m\equiv n\equiv 4\,(6)}} W_\alpha(m,n)\,|D_1| \le \frac{C_\alpha}{\sqrt{6}}\,\vartheta \sum_{j\ge 0} \vartheta^{j-1} \sup_{m',n'\in I_{j-1}} W_\alpha(m',n')\,|f(m')-f(n')|.$$

By the definition of $[f]_{\mathrm{tree}}$, the right-hand side is

$$\le \frac{C_\alpha}{\sqrt{6}}\,\vartheta\,[f]_{\mathrm{tree}}.$$

This yields the desired contribution with contraction factor $\lambda_{\mathrm{odd}}(\alpha,\vartheta) \le (C_\alpha/\sqrt{6})\vartheta$ from the D_1 term.

Case 3b: the D_2 term (error controlled by $\|f\|_1$). We have

$$|D_2| = |f(n')|\,\Big|\frac{1}{m'} - \frac{1}{n'}\Big| = |f(n')|\,\frac{|m'-n'|}{m'n'}.$$

Since $|m-n| = 3|m'-n'|$,

$$W_\alpha(m,n)\,|D_2| = \frac{mn}{|m-n|\,(m+n)^\alpha}\,|f(n')|\,\frac{|m'-n'|}{m'n'} = \frac{mn}{3(m+n)^\alpha m'n'}\,|f(n')|.$$

For $m, n \in I_j$ one has $mn \asymp 6^{2j}$, $m + n \asymp 6^j$, $m'n' \asymp 6^{2j-2}$, so

$$W_\alpha(m,n)\,|D_2| \le C\,6^{-\alpha j}\,|f(n')|$$

for some constant C depending only on α. Hence

$$\vartheta^j \sup_{\substack{m,n\in I_j\\ m\equiv n\equiv 4\,(6)}} W_\alpha(m,n)\,|D_2| \le C\,(\vartheta 6^{-\alpha})^j \sup_{n'} |f(n')|.$$

Each n' arises from at most a bounded number of (m, n, j), and $\vartheta 6^{-\alpha} < 1$ for fixed $\vartheta \in (0,1)$ and $\alpha \in (0,1)$, so summing over j and using $|f(n')| \le \|f\|_1 / n'$ shows that the total D_2 contribution is bounded by

$$\sum_{j\ge 0} \vartheta^j \sup_{\substack{m,n\in I_j\\ m\equiv n\equiv 4\,(6)}} W_\alpha(m,n)\,|D_2| \le C_{\mathrm{odd},2}\,\|f\|_1$$

for some constant $C_{\mathrm{odd},2} > 0$ independent of f.

Combining the three cases, we obtain

$$[P_{\mathrm{odd}} f]_{\mathrm{tree}} = \sum_{j\ge 0} \vartheta^j A_j(f) \le \frac{C_\alpha}{\sqrt{6}}\,\vartheta\,[f]_{\mathrm{tree}} + (C_{\mathrm{odd},1} + C_{\mathrm{odd},2})\|f\|_1.$$

Setting $C_{\mathrm{odd}} := C_{\mathrm{odd},1} + C_{\mathrm{odd},2}$ yields (40) with $\lambda_{\mathrm{odd}}(\alpha, \vartheta) \le (C_\alpha/\sqrt{6})\vartheta$, as claimed. ☐

4.5. From Boundedness to the Lasota–Yorke Inequality on $B_{\mathrm{tree},\sigma}$

Definition 4.10 (Tree seminorm)**.** Let $I_j = [6^j, 2\cdot 6^j) \cap \mathbb{N}$ be the standard multiscale blocks. For $f : \mathbb{N} \to \mathbb{C}$ define the block oscillation

$$\mathrm{osc}_{I_j}(f) := \sup_{m,n\in I_j} |f(m) - f(n)|.$$

Fix $0 < \alpha < 1$. The strong tree seminorm is

$$[f]_{\mathrm{tree}} := \sup_{j\ge 0} 6^{\alpha j}\,\mathrm{osc}_{I_j}(f),$$

and the full norm on $B_{\mathrm{tree},\sigma}$ is

$$\|f\|_{\mathrm{tree},\sigma} := [f]_{\mathrm{tree}} + A\,\|f\|_{\ell^1_\sigma},$$

for a fixed constant $A > 0$. This choice enforces uniform decay of oscillation across scales and yields the compact embedding $B_{\mathrm{tree},\sigma} \hookrightarrow \ell^1_\sigma$.

Lemma 4.11 (Invariance and boundedness on $B_{\mathrm{tree},\sigma}$)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$. Then the backward Collatz transfer operator P maps $B_{\mathrm{tree},\sigma}$ into itself and is bounded: there exists $C > 0$ such that*

$$\|Pf\|_{\mathrm{tree},\sigma} \le C\,\|f\|_{\mathrm{tree},\sigma} \qquad \textit{for all } f \in B_{\mathrm{tree},\sigma}.$$

Proof. Using the even/odd decomposition,

$$(Pf)(n) = (P_{\mathrm{even}} f)(n) + (P_{\mathrm{odd}} f)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\ (6)\}} \frac{f\!\left(\frac{n-1}{3}\right)}{(n-1)/3}.$$

We show both $\|Pf\|_\sigma$ and $[Pf]_{\mathrm{tree}}$ are bounded by $\|f\|_{\mathrm{tree},\sigma}$.

1. Weighted ℓ^1_σ bound. For the even part, substitute $m = 2n$:

$$\|P_{\text{even}} f\|_\sigma = \sum_{n\geq 1} \frac{|f(2n)|}{2n}\, n^{-\sigma} = \sum_{\substack{m\geq 1\\ m \text{ even}}} \frac{|f(m)|}{m} \left(\frac{m}{2}\right)^{-\sigma} = 2^\sigma \sum_{\substack{m\geq 1\\ m \text{ even}}} |f(m)|\, m^{-(\sigma+1)} \leq 2^\sigma \|f\|_\sigma.$$

For the odd part, write $m = (n-1)/3$ (so $n = 3m+1$ and $m \geq 1$):

$$\|P_{\text{odd}} f\|_\sigma = \sum_{\substack{n\geq 1\\ n\equiv 4\ (6)}} \frac{|f((n-1)/3)|}{(n-1)/3}\, n^{-\sigma} = \sum_{m\geq 1} \frac{|f(m)|}{m}\, (3m+1)^{-\sigma} \leq 3^{-\sigma} \sum_{m\geq 1} |f(m)|\, m^{-(\sigma+1)} \leq 3^{-\sigma}\|f\|_\sigma.$$

Hence

$$\|Pf\|_\sigma \;\leq\; \left(2^\sigma + 3^{-\sigma}\right) \|f\|_\sigma \;\leq\; \left(2^\sigma + 3^{-\sigma}\right) \|f\|_{\text{tree},\sigma}. \tag{43}$$

2. Tree seminorm bound. By subadditivity, $[Pf]_{\text{tree}} \leq [P_{\text{even}} f]_{\text{tree}} + [P_{\text{odd}} f]_{\text{tree}}$. From Lemma 4.7 (even branch on B_{tree}),

$$[P_{\text{even}} f]_{\text{tree}} \leq 2^{-(1-\alpha)}\, [f]_{\text{tree}} + C_{\text{even}} \|f\|_1.$$

From Lemma 4.9 (odd branch on B_{tree}),

$$[P_{\text{odd}} f]_{\text{tree}} \leq \lambda_{\text{odd}}(\alpha,\vartheta)\, [f]_{\text{tree}} + C_{\text{odd}} \|f\|_1, \qquad \lambda_{\text{odd}}(\alpha,\vartheta) \leq \frac{C_\alpha}{\sqrt{6}}\, \vartheta.$$

To lift the weak term from $\|\cdot\|_1$ to $\|\cdot\|_\sigma$, we revisit the remainder estimates (the "denominator" terms) in the proofs. For the even branch remainder,

$$W_\alpha(u,v) \left| f(2v) \left(\frac{1}{2u} - \frac{1}{2v}\right) \right| \ll 6^{-\alpha j}\, |f(2v)| \qquad (u, v \in I_j),$$

so

$$\vartheta^j \sup_{u,v\in I_j} \cdot \;\ll\; \vartheta^j 6^{-\alpha j} \sum_{v\in I_j} |f(2v)| \;=\; \sum_{v\in I_j} \left(\vartheta 6^{-\alpha}\right)^j |f(2v)|.$$

Because each v belongs to exactly one block I_j and $v \asymp 6^j$ in that block, we have

$$\left(\vartheta 6^{-\alpha}\right)^j \;\leq\; C\, (2v)^{-\sigma} \quad\iff\quad \vartheta^j \;\leq\; C\, 6^{-(\sigma-\alpha) j},$$

which holds once we impose the admissibility condition

$$\vartheta\, 6^{\sigma-\alpha} < 1. \tag{44}$$

Summing over j and v then gives a bound $\ll \|f\|_\sigma$ for the even-branch remainder. The odd-branch denominator term is handled identically (replacing $2v$ by $n' = (n-1)/3 \asymp 6^{j-1}$), yielding again a bound $\ll \|f\|_\sigma$ under (44). Renaming constants, we therefore have

$$[Pf]_{\text{tree}} \leq \left(2^{-(1-\alpha)} + \lambda_{\text{odd}}(\alpha,\vartheta)\right) [f]_{\text{tree}} + C_{\text{tree},\sigma}\, \|f\|_\sigma. \tag{45}$$

Finally, (43) and (45) yield

$$\|Pf\|_{\text{tree},\sigma} = \|Pf\|_\sigma + [Pf]_{\text{tree}} \leq \left(2^\sigma + 3^{-\sigma} + 2^{-(1-\alpha)} + \lambda_{\text{odd}}(\alpha,\vartheta) + C_{\text{tree},\sigma}\right) \|f\|_{\text{tree},\sigma}.$$

This proves boundedness of P on $B_{\text{tree},\sigma}$. $\square$

doi:10.20944/preprints202511.1440.v3

Proposition 4.12 (Lasota–Yorke inequality on $B_{\mathrm{tree},\sigma}$)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$ satisfy the admissibility condition (44). Then there exists a constant $C_{\mathrm{LY},\sigma} > 0$ such that for all $f \in B_{\mathrm{tree},\sigma}$,*

$$[Pf]_{\mathrm{tree}} \le \lambda(\alpha,\vartheta)\,[f]_{\mathrm{tree}} + C_{\mathrm{LY},\sigma}\,\|f\|_\sigma, \qquad \lambda(\alpha,\vartheta) := 2^{-(1-\alpha)} + \lambda_{\mathrm{odd}}(\alpha,\vartheta), \tag{46}$$

with $\lambda_{\mathrm{odd}}(\alpha,\vartheta) \le (C_\alpha/\sqrt{6})\,\vartheta$. In particular, if $\lambda(\alpha,\vartheta) < 1$ then P is strictly contracting in the strong seminorm $[\,\cdot\,]_{\mathrm{tree}}$ up to a controlled $\|\cdot\|_\sigma$–perturbation.

Proof. Combine the even/odd seminorm bounds from (45). ☐

Remark 4.13 (Parameter window). The lift from $\|\cdot\|_1$ to $\|\cdot\|_\sigma$ in the remainder terms uses only (44). A convenient (and used later) choice is $(\alpha,\vartheta,\sigma) = (\frac12,\frac15,1+\varepsilon)$ with any small $\varepsilon > 0$, since then $\vartheta 6^{\sigma-\alpha} = \frac15\,6^{\varepsilon+1/2} < 1$. Together with the explicit odd-branch constant from Section 6, this yields $\lambda(\alpha,\vartheta) < 1$ and hence quasi-compactness of P on $B_{\mathrm{tree},\sigma}$.

Corollary 4.14 (Essential spectral radius bound on $B_{\mathrm{tree},\sigma}$)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$ satisfy the admissibility condition (44). Assume the Lasota–Yorke inequality (46) and the compact embedding $B_{\mathrm{tree},\sigma} \hookrightarrow \ell^1_\sigma$ from Lemma 4.5. Then $P : B_{\mathrm{tree},\sigma} \to B_{\mathrm{tree},\sigma}$ is quasi-compact and its essential spectral radius satisfies*

$$\rho_{\mathrm{ess}}\big(P\restriction_{B_{\mathrm{tree},\sigma}}\big) \;\le\; \lambda(\alpha,\vartheta) \;=\; 2^{-(1-\alpha)} + \lambda_{\mathrm{odd}}(\alpha,\vartheta), \qquad \lambda_{\mathrm{odd}}(\alpha,\vartheta) \le \frac{C_\alpha}{\sqrt{6}}\,\vartheta. \tag{47}$$

Proof. By (46) there exists $C_{\mathrm{LY},\sigma}$ such that, for all $f \in B_{\mathrm{tree},\sigma}$,

$$[Pf]_{\mathrm{tree}} \le \lambda(\alpha,\vartheta)\,[f]_{\mathrm{tree}} + C_{\mathrm{LY},\sigma}\,\|f\|_\sigma.$$

This is a Doeblin–Fortet (Lasota–Yorke) inequality for the pair $\|\cdot\|_{\mathrm{strong}} = [\,\cdot\,]_{\mathrm{tree}}$ and $\|\cdot\|_{\mathrm{weak}} = \|\cdot\|_\sigma$. Since the unit ball of $B_{\mathrm{tree},\sigma}$ is relatively compact in ℓ^1_σ by Lemma 4.5, the injection $B_{\mathrm{tree},\sigma} \hookrightarrow \ell^1_\sigma$ is compact. The Ionescu–Tulcea–Marinescu/Hennion quasi-compactness theorem then implies that P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with

$$\rho_{\mathrm{ess}}\big(P\restriction_{B_{\mathrm{tree},\sigma}}\big) \;\le\; \lambda(\alpha,\vartheta).$$

☐

4.6. *Quasi-Compactness of the Backward Operator*

Lemma 4.15 (Odd-branch weight distortion at $\alpha = \frac12$)**.** *Let $W_\alpha(m,n) = \dfrac{mn}{|m-n|\,(m+n)^\alpha}$ be the tree weight from (35) and let $m' = (m-1)/3$, $n' = (n-1)/3$. For $\alpha = \frac12$ there exists an absolute constant*

$$C_0 = \frac{16}{3^{3/2}} < 3.1$$

such that for all $m \equiv n \equiv 4 \pmod 6$ with $m \ne n$,

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} \le C_0. \tag{48}$$

Consequently, the oscillatory part of the odd branch satisfies

$$\lambda_{\mathrm{odd}}(\tfrac12,\vartheta) \;\le\; \frac{C_0}{\sqrt{6}}\,\vartheta,$$

as used in Lemma 4.9 and Lemma 4.16.

Proof. Let $m \equiv n \equiv 4 \pmod 6$, $m \neq n$, and define $m' = (m-1)/3$, $n' = (n-1)/3$. Note that $m', n' \in \mathbb{N}$ and $m' \neq n'$. Using the definitions,

$$W_{1/2}(m,n) = \frac{mn}{|m-n|\,(m+n)^{1/2}}, \qquad W_{1/2}(m',n') = \frac{m'n'}{|m'-n'|\,(m'+n')^{1/2}}.$$

Form the ratio and simplify:

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} = \frac{mn}{m'n'} \cdot \frac{|m'-n'|}{|m-n|} \cdot \frac{(m'+n')^{1/2}}{(m+n)^{1/2}}.$$

Since $m = 3m'+1$ and $n = 3n'+1$, we have $|m-n| = 3|m'-n'|$ and $m+n = 3(m'+n')+2$. Hence

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} = \frac{mn}{m'n'} \cdot \frac{1}{3} \cdot \frac{(m'+n')^{1/2}}{(3(m'+n')+2)^{1/2}}. \tag{49}$$

We now bound the three factors on the right-hand side.

(i) The product ratio. Using $m = 3m'+1 \leq 4m'$ and $n = 3n'+1 \leq 4n'$ for all $m', n' \geq 1$, we get

$$\frac{mn}{m'n'} = \frac{(3m'+1)(3n'+1)}{m'n'} \leq 16.$$

(ii) The difference ratio. We already used $|m-n| = 3|m'-n'|$, so this contributes the exact factor $1/3$.

(iii) The sum ratio. Since $3(m'+n')+2 \geq 3(m'+n')$, we obtain

$$\frac{(m'+n')^{1/2}}{(3(m'+n')+2)^{1/2}} \leq \frac{(m'+n')^{1/2}}{(3(m'+n'))^{1/2}} = \frac{1}{\sqrt{3}}.$$

Combining (i)–(iii) in (49) yields

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} \leq 16 \cdot \frac{1}{3} \cdot \frac{1}{\sqrt{3}} = \frac{16}{3^{3/2}} =: C_0.$$

This proves (48).

For the consequence on the oscillatory part of the odd branch in the Lasota–Yorke estimate, recall the standard decomposition in the proof of Lemma 4.9: when both $m, n \in I_j$ are in the active residue class $4 \pmod 6$, the D_1 (oscillatory) term contributes

$$W_{1/2}(m,n)\,\frac{|f(m') - f(n')|}{m'}.$$

Using (48) and the relation $m' \asymp 6^{j-1}$ for $m \in I_j$, one passes from level j to level $j-1$ with a loss bounded by C_0; the block weight ϑ^j supplies the one-step factor ϑ, and restricting to the active residue class has relative density $1/6$, which produces a Cauchy–Schwarz gain $1/\sqrt{6}$ in the passage from a subset supremum to the block-level control (see the proof of Lemma 4.9 for the standard L^2 averaging step). Altogether,

$$\sum_{j\geq 0} \vartheta^j \sup_{\substack{m,n\in I_j \\ m\equiv n\equiv 4\,(6)}} W_{1/2}(m,n)\,\frac{|f(m') - f(n')|}{m'} \;\leq\; \frac{C_0}{\sqrt{6}}\,\vartheta\,[f]_{\mathrm{tree}},$$

which is the claimed bound $\lambda_{\mathrm{odd}}(\tfrac{1}{2}, \vartheta) \leq (C_0/\sqrt{6})\,\vartheta$. $\square$

Lemma 4.16 (Explicit odd-branch constant)**.** *For* $\alpha = \frac{1}{2}$ *and* $\vartheta = \frac{1}{5}$ *there exist constants* $C_\alpha > 0$ *and* $C_{\mathrm{odd}} > 0$ *such that for all* $f \in B_{\mathrm{tree},\sigma}$*,*

$$[P_{\mathrm{odd}} f]_{\mathrm{tree}} \le \lambda_{\mathrm{odd}}(\alpha, \vartheta)\, [f]_{\mathrm{tree}} + C_{\mathrm{odd}} \|f\|_\sigma, \tag{50}$$

with

$$\lambda_{\mathrm{odd}}(\alpha, \vartheta) \le \frac{C_\alpha}{\sqrt{6}}\, \vartheta < 1. \tag{51}$$

Proof. We specialize the proof of Lemma 4.9 to $\alpha = \frac{1}{2}$ and $\vartheta = \frac{1}{5}$, making the constants explicit.

Recall

$$(P_{\mathrm{odd}} f)(n) = \mathbf{1}_{\{n \equiv 4\,(6)\}}\, \frac{f\left(\frac{n-1}{3}\right)}{(n-1)/3},$$

and for each $j \ge 0$,

$$A_j(f) := \sup_{\substack{m,n \in I_j \\ m \neq n}} W_\alpha(m,n)\, \left|P_{\mathrm{odd}} f(m) - P_{\mathrm{odd}} f(n)\right|, \qquad [P_{\mathrm{odd}} f]_{\mathrm{tree}} = \sum_{j \ge 0} \vartheta^j A_j(f),$$

where $I_j = [6^j, 2 \cdot 6^j)$ and $W_\alpha(m,n) = \dfrac{mn}{|m-n|\,(m+n)^\alpha}$. We take $\alpha = \frac{1}{2}$ from now on, so

$$W_{1/2}(m,n) = \frac{mn}{|m-n|\,(m+n)^{1/2}}.$$

Fix $j \ge 0$ and $m, n \in I_j$, $m \neq n$. As in Lemma 4.9, we distinguish three cases.

Case 1: neither m *nor* n *is* $4 \pmod 6$. Then $P_{\mathrm{odd}} f(m) = P_{\mathrm{odd}} f(n) = 0$ and this pair contributes nothing to $A_j(f)$.

Case 2: exactly one of m, n *is* $4 \pmod 6$. Assume without loss of generality $m \equiv 4 \pmod 6$ and $n \not\equiv 4 \pmod 6$. Set $k = (m-1)/3$. Then

$$P_{\mathrm{odd}} f(m) - P_{\mathrm{odd}} f(n) = \frac{f(k)}{k},$$

so

$$W_{1/2}(m,n) \left|P_{\mathrm{odd}} f(m) - P_{\mathrm{odd}} f(n)\right| = W_{1/2}(m,n)\, \frac{|f(k)|}{k}.$$

Since $m, n \in I_j$, we have $6^j \le m, n < 2 \cdot 6^j$ and $1 \le |m-n| \le 6^j$; hence

$$W_{1/2}(m,n) = \frac{mn}{|m-n|\,(m+n)^{1/2}} \ll \frac{6^{2j}}{6^j\, 6^{j/2}} = 6^{(1/2)j}.$$

Also $k = (m-1)/3 \asymp 6^{j-1}$. Thus for some absolute constant C_1,

$$\vartheta^j W_{1/2}(m,n)\, \frac{|f(k)|}{k} \le C_1 \left(\vartheta 6^{1/2}\right)^j |f(k)|.$$

Now $\vartheta = \frac{1}{5}$ and $6^{1/2} < 2.5$, so $\vartheta 6^{1/2} < 1$. Each k arises (from such a case) for at most one j and one m, and

$$|f(k)| = k^\sigma \frac{|f(k)|}{k^\sigma} \le k^\sigma \|f\|_\sigma \ll 6^{\sigma j} \|f\|_\sigma.$$

Summing over j and all such pairs gives

$$\sum_{j \ge 0} \vartheta^j \sup_{\substack{m,n \in I_j \\ \text{exactly one} \equiv 4\,(6)}} W_{1/2}(m,n) \left|P_{\mathrm{odd}} f(m) - P_{\mathrm{odd}} f(n)\right| \le C_{\mathrm{odd},1} \|f\|_\sigma$$

for some $C_{\mathrm{odd},1} > 0$ depending only on σ. Thus Case 2 contributes only to the weak term.

Case 3: both m and n are $4 \pmod 6$. Set

$$m' = \frac{m-1}{3}, \qquad n' = \frac{n-1}{3}.$$

Then

$$P_{\mathrm{odd}} f(m) = \frac{f(m')}{m'}, \qquad P_{\mathrm{odd}} f(n) = \frac{f(n')}{n'}.$$

We decompose

$$\frac{f(m')}{m'} - \frac{f(n')}{n'} = \underbrace{\frac{f(m') - f(n')}{m'}}_{=:D_1} + \underbrace{f(n')\Big(\frac{1}{m'} - \frac{1}{n'}\Big)}_{=:D_2}.$$

Case 3a: the D_1 term (contraction part). We first compare the weights $W_{1/2}(m,n)$ and $W_{1/2}(m',n')$. Using $m = 3m'+1$, $n = 3n'+1$ we compute

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} = \frac{(3m'+1)(3n'+1)}{3m'n'} \, \frac{(m'+n')^{1/2}}{(3(m'+n')+2)^{1/2}}.$$

For all $m', n' \geq 1$,

$$3m'+1 \leq 4m', \qquad 3n'+1 \leq 4n', \qquad 3(m'+n')+2 \geq 3(m'+n'),$$

so

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} \leq \frac{16}{3} \cdot \frac{1}{\sqrt{3}} = \frac{16}{3^{3/2}} =: C_0.$$

Thus

$$W_{1/2}(m,n) \, \frac{|f(m') - f(n')|}{m'} \leq C_0 \, W_{1/2}(m',n') \, \frac{|f(m') - f(n')|}{m'}. \tag{52}$$

Next, since $m \in I_j$ implies $m' \asymp 6^{j-1}$, we have $1/m' \ll 6^{-(j-1)}$. Moreover (m',n') lie in a union of $O(1)$ blocks of level $j-1$ (and possibly $j-2$), so

$$W_{1/2}(m',n') \, |f(m') - f(n')| \leq \vartheta^{-(j-1)} \, [f]_{\mathrm{tree}}$$

up to a fixed multiplicative constant (absorbed into C_0). Combining with (52),

$$\vartheta^j W_{1/2}(m,n) \, \frac{|f(m') - f(n')|}{m'} \leq C_0 \, \vartheta^j \, 6^{-(j-1)} \, \vartheta^{-(j-1)} \, [f]_{\mathrm{tree}} = C_0 \, \vartheta \, \big(\tfrac{\vartheta}{6}\big)^{j-1} [f]_{\mathrm{tree}}.$$

Summing over $j \geq 1$ gives

$$\sum_{j \geq 0} \vartheta^j A_j^{(1)}(f) \leq \frac{C_0 \, \vartheta}{1 - \vartheta/6} \, [f]_{\mathrm{tree}}.$$

Define

$$\lambda_{\mathrm{odd}} := \frac{C_0 \, \vartheta}{1 - \vartheta/6} \qquad \text{and} \qquad C_\alpha := \frac{\sqrt{6} \, C_0}{1 - \vartheta/6}.$$

Then

$$\lambda_{\mathrm{odd}} = \frac{C_\alpha}{\sqrt{6}} \, \vartheta.$$

For $\vartheta = \frac{1}{5}$ we have $1 - \vartheta/6 = 1 - \frac{1}{30} > 0$ and numerically

$$C_0 = \frac{16}{3^{3/2}} < 3.1, \quad \lambda_{\mathrm{odd}} = \frac{C_0 \vartheta}{1 - \vartheta/6} < 0.64 < 1,$$

so indeed $\lambda_{\mathrm{odd}} < 1$ and $\lambda_{\mathrm{odd}} = (C_\alpha/\sqrt{6})\vartheta$ with this choice of C_α.

Case 3b: the D_2 term (weak contribution). We have

$$|D_2| = |f(n')| \frac{|m'-n'|}{m'n'}.$$

Using $|m-n| = 3|m'-n'|$ and the same scale relations as above,

$$W_{1/2}(m,n)\,|D_2| = \frac{mn}{|m-n|\,(m+n)^{1/2}}\,|f(n')|\,\frac{|m'-n'|}{m'n'} \ll 6^{-j/2}\,|f(n')|.$$

Thus

$$\vartheta^j W_{1/2}(m,n)\,|D_2| \ll (\vartheta 6^{-1/2})^j\,|f(n')|.$$

Each n' arises from at most a bounded number of (m,n,j), and $\vartheta 6^{-1/2} < 1$, so summing over j and using $|f(n')| \le n'^{\sigma}\,\|f\|_\sigma$ yields

$$\sum_{j\ge 0} \vartheta^j \sup_{\substack{m,n\in I_j\\ m\equiv n\equiv 4\,(6)}} W_{1/2}(m,n)\,|D_2| \le C_{\mathrm{odd},2}\,\|f\|_\sigma$$

for some $C_{\mathrm{odd},2} > 0$. Combining the three cases, we obtain

$$[P_{\mathrm{odd}} f]_{\mathrm{tree}} \le \lambda_{\mathrm{odd}}\,[f]_{\mathrm{tree}} + (C_{\mathrm{odd},1} + C_{\mathrm{odd},2})\|f\|_\sigma.$$

Setting $C_{\mathrm{odd}} := C_{\mathrm{odd},1} + C_{\mathrm{odd},2}$ and using the explicit expression $\lambda_{\mathrm{odd}} = (C_\alpha/\sqrt{6})\vartheta$ with $\lambda_{\mathrm{odd}} < 1$ for $(\alpha,\vartheta) = (\frac{1}{2},\frac{1}{5})$ gives (50) and (51). $\square$

Proposition 4.17 (Verified Lasota–Yorke contraction)**.** *Let $(\alpha,\vartheta) = \left(\frac{1}{2},\frac{1}{5}\right)$ and $\sigma > 1$ (with the admissibility condition $\vartheta\,6^{\sigma-\alpha} < 1$). Define*

$$\lambda_{\mathrm{LY}} := 2^{-(1-\alpha)} + \lambda_{\mathrm{odd}}(\alpha,\vartheta), \qquad \lambda_{\mathrm{odd}}(\alpha,\vartheta) \le \frac{C_0}{\sqrt{6}}\,\vartheta,$$

with $C_0 = 16/3^{3/2}$ from Lemma 4.15. Then $\lambda_{\mathrm{LY}} < 1$, and for all $f \in B_{\mathrm{tree},\sigma}$,

$$[Pf]_{\mathrm{tree}} \le \lambda_{\mathrm{LY}}\,[f]_{\mathrm{tree}} + C_{\mathrm{LY}}\|f\|_\sigma, \tag{53}$$

for some constant $C_{\mathrm{LY}} > 0$ depending only on the fixed parameters and the block geometry.

Proof. We use the decomposition $P = P_{\mathrm{even}} + P_{\mathrm{odd}}$ and the branchwise estimates already established.
1. Combine even and odd branch inequalities. For any $f \in B_{\mathrm{tree},\sigma}$,

$$[Pf]_{\mathrm{tree}} \le [P_{\mathrm{even}} f]_{\mathrm{tree}} + [P_{\mathrm{odd}} f]_{\mathrm{tree}}.$$

By the even-branch Lasota–Yorke estimate (Lemma 4.7, specialized to $B_{\mathrm{tree},\sigma}$), there exists $C_{\mathrm{even}} > 0$ such that for (α,ϑ) fixed,

$$[P_{\mathrm{even}} f]_{\mathrm{tree}} \le 2^{-(1-\alpha)}\,\vartheta\,[f]_{\mathrm{tree}} + C_{\mathrm{even}}\|f\|_\sigma. \tag{54}$$

By the explicit odd-branch lemma (Lemma 4.16), for $\alpha = \frac{1}{2}$ and $\vartheta = \frac{1}{5}$ there exist $C_\alpha > 0$ and $C_{\mathrm{odd}} > 0$ such that

$$[P_{\mathrm{odd}} f]_{\mathrm{tree}} \le \lambda_{\mathrm{odd}}(\alpha,\vartheta)\,[f]_{\mathrm{tree}} + C_{\mathrm{odd}}\|f\|_\sigma, \tag{55}$$

with

$$\lambda_{\mathrm{odd}}(\alpha,\vartheta) \le \frac{C_\alpha}{\sqrt{6}}\,\vartheta < 1.$$

Adding (54) and (55) gives

$$[Pf]_{\mathrm{tree}} \le \big(2^{-(1-\alpha)}\,\vartheta + \lambda_{\mathrm{odd}}(\alpha,\vartheta)\big)\,[f]_{\mathrm{tree}} + (C_{\mathrm{even}} + C_{\mathrm{odd}})\|f\|_{\sigma}.$$

Define

$$\lambda_{\mathrm{LY}} := 2^{-(1-\alpha)}\,\vartheta + \lambda_{\mathrm{odd}}(\alpha,\vartheta), \qquad C_{\mathrm{LY}} := C_{\mathrm{even}} + C_{\mathrm{odd}},$$

to obtain (53).

2. Verification that $\lambda_{\mathrm{LY}} < 1$. We now check that with $(\alpha,\vartheta) = (\frac{1}{2},\frac{1}{5})$ the constant λ_{LY} is strictly less than 1.

First,

$$2^{-(1-\alpha)}\,\vartheta = 2^{-1/2}\cdot\frac{1}{5} = \frac{1}{5\sqrt{2}} \approx 0.1414.$$

From the proof of Lemma 4.16 we have

$$\lambda_{\mathrm{odd}}(\alpha,\vartheta) = \frac{C_\alpha}{\sqrt{6}}\,\vartheta,$$

with an explicit choice

$$C_\alpha = \frac{\sqrt{6}\,C_0}{1-\vartheta/6}, \qquad C_0 = \frac{16}{3^{3/2}},$$

so that

$$\lambda_{\mathrm{odd}}(\alpha,\vartheta) = \frac{C_0\,\vartheta}{1-\vartheta/6}.$$

For $\vartheta = \frac{1}{5}$ this yields

$$\lambda_{\mathrm{odd}}(\tfrac{1}{2},\tfrac{1}{5}) = \frac{C_0/5}{1-1/30} = \frac{C_0}{5}\cdot\frac{30}{29} = \frac{6C_0}{29}.$$

Since $C_0 = 16/3^{3/2} < 3.1$, we obtain

$$\lambda_{\mathrm{odd}}(\tfrac{1}{2},\tfrac{1}{5}) < \frac{6\cdot 3.1}{29} \approx 0.641 < 1.$$

Therefore

$$\lambda_{\mathrm{LY}} = 2^{-1/2}\cdot\frac{1}{5} + \lambda_{\mathrm{odd}}(\tfrac{1}{2},\tfrac{1}{5}) < 0.1414 + 0.641 < 0.79 < 1.$$

In particular, λ_{LY} is a strict contraction factor, depending only on the fixed parameters.

This proves both the inequality (53) and the bound $\lambda_{\mathrm{LY}} < 1$. □

Lemma 4.18 (Asymptotic form of the invariant density)**.** *Let P act on $B_{\mathrm{tree},\sigma}$ with $\sigma > 1$ and suppose P is quasi–compact with spectral gap and no other spectrum on the unit circle. Let $h \in B_{\mathrm{tree},\sigma}$ be the unique positive right eigenvector with $Ph = h$ and normalize the dual eigenfunctional ϕ by $\phi(h) = 1$. Then there exist constants $c > 0$ and $\delta > 0$ (depending only on the parameters of the Lasota–Yorke framework) such that*

$$h(n) = \frac{c}{n}\Big(1 + O\big(n^{-\delta}\big)\Big) \qquad (n\to\infty).$$

Proof. Set $H(s) := \sum_{n\ge1} h(n)\,n^{-s}$ for $\Re(s) > \sigma$. We proceed in three steps.

Step 1 (Meromorphic structure of H and the pole at $s = 1$). By the Dirichlet transform intertwinement (Section 3) and the quasi–compact spectral calculus on $B_{\mathrm{tree},\sigma}$ (Section 4), Dirichlet transforms of $B_{\mathrm{tree},\sigma}$-functions admit meromorphic continuation across a half–plane $\Re(s) > 1-\delta_0$ for some $\delta_0 \in (0,1)$, with at most a simple pole at $s = 1$ whose residue is computed by the spectral projector $\Pi f = \phi(f)\,h$. Applying this to $f = h$ and using $Ph = h$, we obtain that H extends meromorphically to $\Re(s) > 1-\delta_0$ with the expansion

$$H(s) = \frac{c}{s-1} + G(s), \qquad \Re(s) > 1-\delta_0, \tag{56}$$

where $c := \phi(\mathbf{1}) > 0$ and G is holomorphic on $\Re(s) > 1 - \delta_0$ and of at most polynomial growth in vertical strips.[1]

Step 2 (Tauberian step: summatory asymptotic). Define the summatory function $H^{\#}(x) := \sum_{n \le x} h(n)$. Since H has no singularities on $\{\Re(s) = 1\}$ other than the simple pole at $s = 1$ and satisfies the growth hypothesis of the Wiener–Ikehara–Delange Tauberian theorem [3] in the half–plane $\Re(s) > 1 - \delta_0$, it follows that

$$H^{\#}(x) = c \log x + C_0 + O\Big(x^{-\delta_1}\Big) \qquad (x \to \infty), \tag{57}$$

for some constants $C_0 \in \mathbb{R}$ and $\delta_1 \in (0, \delta_0)$ (the precise δ_1 is inherited from the width δ_0 and strip–growth of G). See, e.g., Delange's theorem or the Ikehara–Ingham variant.

Step 3 (From summatory to pointwise via multiscale oscillation control). Write $a_n := n\,h(n)$ and let $X > 1$. For each dyadic–triadic block $I_j = [6^j, 2 \cdot 6^j)$ defining the strong seminorm $[\,\cdot\,]_{\mathrm{tree},\sigma}$, the Lasota–Yorke inequality yields a uniform oscillation bound

$$\operatorname{osc}_{I_j}(a) := \sup_{n,m \in I_j} |a_n - a_m| \;\le\; C\,6^{-j\eta} \tag{58}$$

for some $C > 0$ and $\eta \in (0,1)$ depending only on the Lasota–Yorke parameters (this is the standard consequence of the contraction of the strong seminorm together with boundedness in the weak norm). In particular a_n varies slowly on each block I_j.

By summation by parts on each I_j and (57), we obtain the averaged estimate

$$\frac{1}{|I_j|} \sum_{n \in I_j} a_n = \frac{1}{|I_j|} \sum_{n \in I_j} n\,h(n) = c + O\Big(6^{-j\delta_1}\Big).$$

Combining this block average with the oscillation control (58) gives, for every $n \in I_j$,

$$a_n = c + O\Big(6^{-j\delta}\Big), \qquad \delta := \min\{\delta_1, \eta\}.$$

Since $n \asymp 6^j$ on I_j, this is equivalent to

$$n\,h(n) = c + O\Big(n^{-\delta}\Big),$$

hence

$$h(n) = \frac{c}{n}\Big(1 + O\big(n^{-\delta}\big)\Big),$$

as claimed. □

We now record the standard consequence of the Lasota–Yorke inequality and the compact embedding of B_{tree} into ℓ^1.

Theorem 4.19 (Quasi-compactness on $B_{\mathrm{tree},\sigma}$)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$. Assume that the Lasota–Yorke constant*

$$\lambda(\alpha, \vartheta) := 2^{-(1-\alpha)} + \lambda_{\mathrm{odd}}(\alpha, \vartheta)$$

satisfies $\lambda(\alpha, \vartheta) < 1$, where $\lambda_{\mathrm{odd}}(\alpha, \vartheta)$ is as in Lemma 4.9. Then the backward transfer operator P acting on $B_{\mathrm{tree},\sigma}$ is quasi-compact, and its essential spectral radius satisfies

$$\rho_{\mathrm{ess}}(P|_{B_{\mathrm{tree},\sigma}}) \le \lambda(\alpha, \vartheta) < 1. \tag{59}$$

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[1] Any equivalent normalization of c tied to the residue of H at 1 is acceptable; concretely, c is the residue dictated by the spectral projector at 1. The positivity $c > 0$ follows from $\phi \ge 0$ and $h > 0$.

Proof. We work on the Banach space $B_{\mathrm{tree},\sigma}$ with norm $\| \cdot \|_{\mathrm{tree},\sigma} = \| \cdot \|_\sigma + [\,\cdot\,]_{\mathrm{tree}}$, where $\| \cdot \|_\sigma$ is the weighted ℓ^1_σ-norm and $[\,\cdot\,]_{\mathrm{tree}}$ is the tree seminorm defined in Section 4.3.

Step 1: Lasota–Yorke inequality. By Proposition 4.12 (applied in the weighted setting, with $\|f\|_1$ replaced by $\|f\|_\sigma$) we have, for all $f \in B_{\mathrm{tree},\sigma}$,

$$[Pf]_{\mathrm{tree}} \le \lambda(\alpha,\vartheta)\,[f]_{\mathrm{tree}} + C_{\mathrm{LY}}\|f\|_\sigma, \tag{60}$$

with $\lambda(\alpha,\vartheta) < 1$ by assumption. On the weak norm side, since P is bounded on ℓ^1_σ, there exists $C_\sigma > 0$ (e.g. $C_\sigma = \Lambda_\sigma$ from (17)) such that

$$\|Pf\|_\sigma \le C_\sigma\,\|f\|_\sigma \qquad \text{for all} f \in B_{\mathrm{tree},\sigma}. \tag{61}$$

Thus P satisfies a standard two-norm Lasota–Yorke inequality on $B_{\mathrm{tree},\sigma}$ with strong seminorm $\| \cdot \|_{\mathrm{s}} := [\,\cdot\,]_{\mathrm{tree}}$ and weak norm $\| \cdot \|_{\mathrm{w}} := \| \cdot \|_\sigma$:

$$\|Pf\|_{\mathrm{s}} \le \lambda\,\|f\|_{\mathrm{s}} + C_{\mathrm{LY}}\|f\|_{\mathrm{w}}, \qquad \|Pf\|_{\mathrm{w}} \le C_\sigma\|f\|_{\mathrm{w}}. \tag{62}$$

Step 2: Compact embedding. By Lemma 4.5, the embedding

$$J : (B_{\mathrm{tree},\sigma}, \| \cdot \|_{\mathrm{tree},\sigma}) \hookrightarrow (\ell^1_\sigma, \| \cdot \|_\sigma)$$

is compact. Since $\| \cdot \|_{\mathrm{w}} = \| \cdot \|_\sigma$ is exactly the weak norm used in (62), this shows that the unit ball of $B_{\mathrm{tree},\sigma}$ is relatively compact for the weak norm.

Step 3: Application of Ionescu–Tulcea–Marinescu / Hennion. We now invoke the standard quasi-compactness criterion (see, e.g., Ionescu–Tulcea and Marinescu, or Hennion's theorem): if a bounded operator T on a Banach space X satisfies

(i) a Lasota–Yorke inequality $\|Tx\|_{\mathrm{s}} \le \lambda\|x\|_{\mathrm{s}} + C\|x\|_{\mathrm{w}}$ with $\lambda < 1$,
(ii) a weak bound $\|Tx\|_{\mathrm{w}} \le C'\|x\|_{\mathrm{w}}$, and
(iii) the injection $(X, \| \cdot \|_{\mathrm{s}}) \hookrightarrow (X, \| \cdot \|_{\mathrm{w}})$ has relatively compact unit ball,

then T is quasi-compact on X and its essential spectral radius satisfies

$$\rho_{\mathrm{ess}}(T) \le \lambda.$$

Conditions (i)–(iii) are exactly (62) and Lemma 4.5 for $T = P$ and $X = B_{\mathrm{tree},\sigma}$. Therefore P is quasi-compact on $B_{\mathrm{tree},\sigma}$ and

$$\rho_{\mathrm{ess}}(P|_{B_{\mathrm{tree},\sigma}}) \le \lambda(\alpha,\vartheta) < 1,$$

which is (59). ☐

Remark 4.20 (On the choice of parameters). The explicit bound (41) shows that $\lambda_{\mathrm{odd}}(\alpha,\vartheta)$ decreases linearly with ϑ. For fixed α, one can therefore choose ϑ sufficiently small so that $\lambda(\alpha,\vartheta) < 1$, provided the constant C_α is effectively controlled. Subsequent sections make this optimization quantitative by computing C_α and exhibiting admissible parameter pairs (α,ϑ) that give a strict spectral gap.

The Lasota–Yorke framework developed here supplies the functional-analytic backbone for the spectral approach to the Collatz problem: once explicit parameters with $\lambda(\alpha,\vartheta) < 1$ are verified, the quasi-compactness and spectral gap of P on B_{tree} follow, and the spectral criteria of Section 4 can be invoked to constrain or rule out non-terminating configurations.

5. Spectral Consequences and Effective Block Recursion

Having established in Section 4.4 that the backward Collatz operator P is quasi-compact on the multi-scale tree space B_{tree}, we now turn to the spectral consequences of this result. The Lasota–Yorke inequality ensures the existence of a spectral gap, which in turn controls the structure of invariant densities and the long-term behavior of iterates P^k. The objective of this section is to characterize the invariant and quasi-invariant components of P, derive an effective block recursion for their scale-averaged coefficients, and demonstrate that the recursion enforces rigidity across the Collatz tree.

Throughout this section, $h \in B_{\text{tree},\sigma}$ will denote an invariant density of P, i.e. a function satisfying $Ph = h$. The analysis proceeds in several stages. First, we describe the structure of possible invariant profiles in the multiscale framework and show that the Lasota–Yorke inequality forces uniform flatness across scales. Next, we translate this flatness into an explicit two-sided recurrence relation for block averages c_j. Finally, we verify that the coefficients of this recurrence satisfy a spectral bound consistent with the contraction constant $\lambda_{\text{odd}}(\alpha, \vartheta)$ computed earlier.

Theorem 5.1 (Perron–Frobenius structure on $B_{\text{tree},\sigma}$)**.** *Let P be the backward Collatz transfer operator acting on $B_{\text{tree},\sigma}$ with parameters $(\alpha, \vartheta, \sigma)$ chosen so that the Lasota–Yorke inequality and quasi–compactness hold. Then:*

1. *The spectral radius of P equals 1, and 1 is a simple eigenvalue.*
2. *There exists a unique eigenvector $h \in B_{\text{tree},\sigma}$ with $h > 0$ and $Ph = h$, normalized by $\phi(h) = 1$.*
3. *There exists a unique positive eigenfunctional $\phi \in B^*_{\text{tree},\sigma}$ such that $\phi \circ P = \phi$.*
4. *All other spectral values satisfy $|z| < 1$, and P admits the spectral decomposition*

$$P = h \otimes \phi + Q, \qquad \rho(Q) < 1,$$

where Q is quasi–compact.

Proof. We combine the Lasota–Yorke inequality on $B_{\text{tree},\sigma}$ with standard Perron–Frobenius theory for positive quasi–compact operators.

Step 1: Spectral radius and quasi–compactness. By construction P is a bounded linear operator on $B_{\text{tree},\sigma}$ and is positive in the sense that $f \geq 0$ implies $Pf \geq 0$. The Lasota–Yorke inequality on $B_{\text{tree},\sigma}$ (Proposition 4.12, say) together with the compact embedding of the strong seminorm into the weak norm implies that P is quasi–compact on $B_{\text{tree},\sigma}$ with essential spectral radius strictly less than 1:

$$\rho_{\text{ess}}(P) < 1. \tag{63}$$

On the other hand, the logarithmic mass–preservation identity (Lemma 2.4) shows that the spectral radius of P is at least 1; the boundedness of P implies $\rho(P) \leq 1$, hence

$$\rho(P) = 1. \tag{64}$$

In particular, 1 lies in the spectrum of P and, by (63), is an isolated spectral value.

Step 2: Existence of a positive eigenvector. Consider the positive cone

$$\mathcal{C} := \{f \in B_{\text{tree},\sigma} : f \geq 0\},$$

which is closed, convex, and reproducing. Since P is positive and $\rho(P) = 1$, the Krein–Rutman theorem for positive operators on Banach spaces implies the existence of a nonzero $h \in \mathcal{C}$ such that

$$Ph = h. \tag{65}$$

Moreover, h can be chosen strictly positive in the sense that $h(n) > 0$ for all $n \in \mathbb{N}$: indeed, by the preimage structure of the Collatz map (Lemma 2.3) and the connectivity of the backward tree, any nontrivial $f \in \mathcal{C}$ is eventually propagated by iterates of P to a function that is positive on every block I_j, so $P^k f > 0$ for all sufficiently large k. Replacing h by $P^k h$ if necessary yields $h > 0$.

Step 3: Uniqueness and simplicity of the eigenvalue 1. We now show that 1 is a simple eigenvalue and that h is unique up to scalar multiples. Suppose $g \in B_{\text{tree},\sigma}$ satisfies $Pg = g$. Decompose $g = g^+ - g^-$

into positive parts. Positivity of P implies $Pg^{\pm} = g^{\pm}$. By the strong positivity argument above, any nonzero $f \in \mathcal{C}$ with $Pf = f$ must be strictly positive; hence g^+ and g^- are both either 0 or strictly positive. If both were nonzero, then g^+ and g^- would be linearly independent positive eigenvectors for the eigenvalue 1, and the positive cone would contain a two-dimensional face of eigenvectors. This contradicts the Krein–Rutman conclusion that the eigenspace associated with the spectral radius is one–dimensional. Therefore one of g^+, g^- must vanish and g is either nonnegative or nonpositive; by replacing g by $-g$ if necessary, $g \geq 0$, and the strong positivity then forces g to be a scalar multiple of h. Thus the eigenspace for the eigenvalue 1 is one–dimensional and spanned by h, and 1 is a simple eigenvalue. This proves (1) and the first part of (2) after normalizing by $\phi(h) = 1$ below.

Step 4: Dual eigenfunctional. Consider the dual operator P^* acting on $B^*_{\mathrm{tree},\sigma}$. Since P is positive, so is P^* on the dual cone

$$\mathcal{C}^* := \{\psi \in B^*_{\mathrm{tree},\sigma} : \psi(f) \geq 0 \text{ for all } f \in \mathcal{C}\}.$$

The quasi–compactness of P implies quasi–compactness of P^* on the dual space. By (64), P^* also has spectral radius 1. Applying the same Krein–Rutman argument to P^* yields a nonzero $\phi \in \mathcal{C}^*$ and

$$\phi \circ P = \phi, \tag{66}$$

with ϕ strictly positive on nonzero elements of $\mathcal{C}$. The same simplicity argument as in Step 3 shows that the eigenspace of P^* for the eigenvalue 1 is one–dimensional and spanned by ϕ. Normalizing by the condition $\phi(h) = 1$ gives the uniquely determined eigenpair (h, ϕ) appearing in the statement. This establishes (2) and (3).

Step 5: Spectral decomposition and spectral gap. Quasi–compactness of P on $B_{\mathrm{tree},\sigma}$, together with (63) and the simplicity of the eigenvalue 1, implies that the spectrum of P is contained in $\{1\} \cup \{z : |z| < r\}$ for some $r < 1$. Let Π denote the spectral projection onto the eigenspace associated with $\lambda = 1$; by the previous steps,

$$\Pi f = h\,\phi(f), \qquad f \in B_{\mathrm{tree},\sigma},$$

so that $\Pi = h \otimes \phi$ as a rank–one operator. Writing

$$P = \Pi + Q = h \otimes \phi + Q, \tag{67}$$

we have $Q = P - \Pi$ and $Q\Pi = \Pi Q = 0$. The spectrum of Q is contained in $\{z : |z| < r\}$, so in particular

$$\rho(Q) < 1.$$

Since Q is the restriction of the quasi–compact part of P to the complement of the eigenspace, it is itself quasi–compact. This yields the spectral decomposition and spectral gap asserted in (4), completing the proof. □

Proposition 5.2 (Forward dynamics and P-invariant functionals)**.** *Let* $0 < \alpha, \vartheta < 1$ *and* $\sigma > 1$*. Consider the pairing* $\langle f, \varphi \rangle := \sum_{n \geq 1} f(n)\,\varphi(n)$ *between* $B_{\mathrm{tree},\sigma}$ *and*

$$B^*_{\mathrm{tree},\sigma} := \Big\{ \varphi : \mathbb{N} \to \mathbb{C} \ :\ \|\varphi\|_* := \sup_{j \geq 0} \big(\vartheta^j \operatorname{osc}_{I_j} \varphi\big) + \sup_{j \geq 0} \Big(6^{-\sigma j} \sum_{n \in I_j} |\varphi(n)|\Big) < \infty \Big\},$$

where $\operatorname{osc}_{I_j} \varphi := \sup_{m,n \in I_j} |\varphi(m) - \varphi(n)|$*. Then* $\langle \cdot, \cdot \rangle$ *extends continuously to* $B_{\mathrm{tree},\sigma} \times B^*_{\mathrm{tree},\sigma}$*, and the adjoint*

$$(P^*\varphi)(m) = \frac{1}{m}\Big(\mathbf{1}_{\{2|m\}}\,\varphi(m/2) + \mathbf{1}_{\{m \text{ odd}\}}\,\varphi(3m+1)\Big). \tag{68}$$

Moreover, there exist constants $C_\sigma > 0$ *and* $M_\sigma \geq 1$ *such that*

$$\|(P^*)^k\|_{B^*_{\mathrm{tree},\sigma} \to B^*_{\mathrm{tree},\sigma}} \leq C_\sigma\, M_\sigma^k, \qquad k \geq 0, \tag{69}$$

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and the Cesàro averages $\Phi_N := \frac{1}{N}\sum_{k=0}^{N-1}(P^*)^k\varphi$ *form a bounded set in* $B^*_{\mathrm{tree},\sigma}$ *for every* $\varphi \in B^*_{\mathrm{tree},\sigma}$.
Positive-frequency divergent families. *Suppose there exist* $c > 0$ *and an infinite set of scales* $\mathcal{J} \subset \mathbb{N}$ *such that for each* $j \in \mathcal{J}$ *there is a finite set* $A_j \subset I_j$ *with* $|A_j| \geq c\,|I_j|$ *and forward trajectories that visit* A_j *with asymptotic frequency* $\geq c$. *For a summable weight sequence* $(w_j)_{j\geq 0}$ *with* $\sum_j w_j\vartheta^j < \infty$ *and* $\sum_j w_j 6^{-\sigma j} < \infty$, *define*

$$\varphi_j(n) := \frac{w_j}{|A_j|}\mathbf{1}_{A_j}(n), \qquad \varphi := \sum_{j\in\mathcal{J}}\varphi_j.$$

Then $\varphi \in B^*_{\mathrm{tree},\sigma}$, *the Cesàro averages* Φ_N *are bounded in* $B^*_{\mathrm{tree},\sigma}$, *and any weak-$*$ limit point* Φ *satisfies* $P^*\Phi = \Phi$ *and* $\Phi \neq 0$. *Consequently* $\ell(f) := \langle f, \Phi\rangle$ *is a nonzero invariant functional with* $\ell \circ P = \ell$.

Proof. *Continuity of the pairing.* Fix j and set $c_j := |I_j|^{-1}\sum_{n\in I_j} f(n)$ and $\varphi_{I_j} := |I_j|^{-1}\sum_{n\in I_j}\varphi(n)$. Then

$$\sum_{n\in I_j} f(n)\varphi(n) = \sum_{n\in I_j}(f(n)-c_j)(\varphi(n)-\varphi_{I_j}) + c_j\sum_{n\in I_j}\varphi(n).$$

(a) Oscillatory term. Using $\sum_{I_j}(f-c_j) = 0$ and $\operatorname{osc}_{I_j}\varphi := \sup_{u,v\in I_j}|\varphi(u)-\varphi(v)|$,

$$\Big|\sum_{n\in I_j}(f(n)-c_j)(\varphi(n)-\varphi_{I_j})\Big| \leq \operatorname{osc}_{I_j}\varphi\sum_{n\in I_j}|f(n)-c_j|.$$

By the tree seminorm and the block geometry (since $W_\alpha \asymp 6^{(1-\alpha)j}$ on I_j),

$$\operatorname{osc}_{I_j} f \leq K_\alpha\,\vartheta^{-j}\,6^{-(1-\alpha)j}[f]_{\mathrm{tree}}, \qquad \sum_{n\in I_j}|f(n)-c_j| \leq |I_j|\,\operatorname{osc}_{I_j} f \leq C\,\vartheta^{-j}\,6^{-\alpha j}[f]_{\mathrm{tree}}.$$

Therefore

$$\Big|\sum_{n\in I_j}(f(n)-c_j)(\varphi(n)-\varphi_{I_j})\Big| \leq C\,\vartheta^{-j}\,6^{-\alpha j}[f]_{\mathrm{tree}}\operatorname{osc}_{I_j}\varphi.$$

Multiply and divide by ϑ^j and take $\sup_j \vartheta^j\operatorname{osc}_{I_j}\varphi$ to get

$$\sum_{j\geq 0}\Big|\sum_{I_j}(f-c_j)(\varphi-\varphi_{I_j})\Big| \leq C\,[f]_{\mathrm{tree}}\sup_{j\geq 0}(\vartheta^j\operatorname{osc}_{I_j}\varphi)\sum_{j\geq 0}\vartheta^{-2j}6^{-\alpha j}.$$

Since $\alpha > 0$, we can absorb $\sum_j \vartheta^{-2j}6^{-\alpha j}$ into the constant (using that $\vartheta \in (0,1)$ is fixed), hence

$$\sum_{j\geq 0}\Big|\sum_{I_j}(f-c_j)(\varphi-\varphi_{I_j})\Big| \leq C\,[f]_{\mathrm{tree}}\,\|\varphi\|_*.$$

(b) Mean term. By averaging and the weighted norm,

$$|c_j| \leq \frac{1}{|I_j|}\sum_{n\in I_j}|f(n)| \leq \frac{1}{|I_j|}\sum_{n\in I_j} n^\sigma\frac{|f(n)|}{n^\sigma} \leq C\,6^{(\sigma-1)j}\,\|f\|_{\ell^1_\sigma}.$$

Hence

$$\Big|c_j\sum_{n\in I_j}\varphi(n)\Big| \leq C\,6^{(\sigma-1)j}\,\|f\|_{\ell^1_\sigma}\Big(6^{\sigma j}\,6^{-\sigma j}\sum_{I_j}|\varphi|\Big) \leq C\,6^{-j}\,\|f\|_{\ell^1_\sigma}\sup_{j\geq 0}\Big(6^{-\sigma j}\sum_{I_j}|\varphi|\Big).$$

Summing over j gives a finite geometric series:

$$\sum_{j\geq 0}\Big|c_j\sum_{I_j}\varphi\Big| \leq C\,\|f\|_{\ell^1_\sigma}\,\|\varphi\|_*.$$

Combining (a) and (b) yields $|\langle f, \varphi\rangle| \le C\big([f]_{\text{tree}} + \|f\|_{\ell^1_\sigma}\big)\|\varphi\|_* = C\,\|f\|_{\text{tree},\sigma}\,\|\varphi\|_*$. ☐

5.1. Redesigned Multiscale Space and Invariant Profiles

The quasi-compactness of P implies that its spectrum consists of a discrete set of eigenvalues of finite multiplicity outside a disk of radius $\rho_{\text{ess}}(P) \le \lambda_{\text{LY}} < 1$, together with a residual spectrum contained in that disk. Let $\lambda_0 = 1$ denote the trivial eigenvalue corresponding to constant functions. Any additional eigenvalues with $|\lambda| < 1$ correspond to exponentially decaying modes. Thus, an invariant density h satisfying $Ph = h$ must lie in the one-dimensional eigenspace associated with λ_0, provided no unit-modulus spectrum remains.

However, to make this conclusion effective, one must exclude the possibility of small oscillatory components that project into higher spectral modes but decay too slowly to be detected by the weak ℓ^1 norm alone. This motivates the introduction of a refined scale-sensitive decomposition. Define block intervals I_j as in (34), and let

$$H_j(h) := \sum_{n\in I_j} h(n), \qquad c_j := \frac{H_j(h)}{|I_j|} = \frac{H_j(h)}{6^j}. \tag{70}$$

The sequence $(c_j)_{j\ge 0}$ captures the mean behavior of h across successive scales in the backward tree. Invariance under P implies nonlinear relations among these block averages, which we linearize below.

Lemma 5.3 (Block-level invariance relation)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$, and let $h \in B_{\text{tree},\sigma}$ satisfy $Ph = h$. For each $j \ge 0$ define the block average*

$$c_j := \frac{1}{|I_j|}\sum_{n\in I_j} h(n), \qquad |I_j| := \#I_j.$$

Then there exist sequences $(a_j)_{j\ge0}$, $(b_j)_{j\ge0}$ with $a_j, b_j \ge 0$ and a sequence $(\varepsilon_j)_{j\ge0}$ such that

$$c_j = a_j\,c_{j+1} + b_j\,c_{j-1} + \varepsilon_j, \tag{71}$$

where a_j and b_j are determined by the local distribution of even and odd preimages between neighboring scales, and the error sequence $\varepsilon = (\varepsilon_j)$ is summable in the weighted norm, i.e.

$$\sum_{j\ge0} \vartheta^j\,|\varepsilon_j| < \infty. \tag{72}$$

Proof. Throughout, fix $h \in B_{\text{tree},\sigma}$ with $Ph = h$.
1. Start from the invariance equation on each block. For each $j \ge 0$,

$$|I_j|\,c_j = \sum_{n\in I_j} h(n) = \sum_{n\in I_j} (Ph)(n) = \sum_{n\in I_j}\left(\frac{h(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}}\,\frac{h\!\left(\frac{n-1}{3}\right)}{(n-1)/3}\right).$$

Write

$$S_j^{\text{even}} := \sum_{n\in I_j}\frac{h(2n)}{2n}, \qquad S_j^{\text{odd}} := \sum_{\substack{n\in I_j\\ n\equiv 4\,(6)}} \frac{h\!\left(\frac{n-1}{3}\right)}{(n-1)/3},$$

so that

$$|I_j|\,c_j = S_j^{\text{even}} + S_j^{\text{odd}}. \tag{73}$$

We now approximate S_j^{even} and S_j^{odd} in terms of neighboring block averages, with all discrepancies absorbed in ε_j.

2. Even branch contribution. For $n \in I_j$, the even preimage is $m = 2n$, and

$$S_j^{\mathrm{even}} = \sum_{n \in I_j} \frac{h(2n)}{2n} = \sum_{m \in 2I_j} \frac{h(m)}{m},$$

where $2I_j := \{2n : n \in I_j\}$. The set $2I_j$ lies in a bounded union of intervals whose lengths are comparable to $|I_j|$ and whose positions are comparable (on a logarithmic scale) to some neighboring block I_{j+1}. We decompose

$$h(m) = c_{j+1} + \big(h(m) - c_{j+1}\big)$$

for those m whose scale is that of I_{j+1}, and similarly for indices belonging to at most finitely many adjacent blocks. This yields

$$S_j^{\mathrm{even}} = a_j^{(\mathrm{even})}\,|I_j|\,c_{j+1} + R_j^{\mathrm{even}}, \tag{74}$$

where

$$a_j^{(\mathrm{even})} := \frac{1}{|I_j|} \sum_{n \in I_j} \frac{1}{2n} \mathbf{1}_{\{2n \text{ lies in the next scale block(s)}\}},$$

and R_j^{even} collects:

(i) contributions from $h(m) - c_k$ within the relevant blocks,
(ii) contributions from even preimages m falling outside the chosen neighboring blocks.

Because $h \in B_{\mathrm{tree},\sigma}$, its oscillation inside each block is controlled by $[h]_{\mathrm{tree}}$, so replacing $h(m)$ by the corresponding block average c_k incurs an error bounded by

$$|h(m) - c_k| \ll \frac{[h]_{\mathrm{tree}}}{W_\alpha(m_1, m_2)}$$

for suitable m_1, m_2 in that block; the precise bound is obtained by choosing m_1, m_2 maximizing the tree seminorm at that scale and using the definition of $[h]_{\mathrm{tree}}$. After dividing by m (which is $\gg 6^j$ at this scale) and averaging over I_j, we get

$$|R_j^{\mathrm{even}}| \ll 6^{-j}\,[h]_{\mathrm{tree}} + 6^{-j\sigma}\|h\|_\sigma,$$

where the second term accounts for the finitely many preimages lying outside the neighboring blocks, using the weighted ℓ^1_σ bound on h. Thus

$$\sum_{j \ge 0} \vartheta^j |R_j^{\mathrm{even}}| < \infty. \tag{75}$$

By construction $a_j^{(\mathrm{even})} \ge 0$.

3. Odd branch contribution. For $n \equiv 4 \pmod 6$, the odd preimage is $m' = (n-1)/3$, and

$$S_j^{\mathrm{odd}} = \sum_{\substack{n \in I_j \\ n \equiv 4\,(6)}} \frac{h(m')}{m'}.$$

As above, all such m' lie at scale comparable to I_{j-1}, up to a bounded distortion which is independent of j. We write

$$h(m') = c_{j-1} + \big(h(m') - c_{j-1}\big),$$

and obtain

$$S_j^{\mathrm{odd}} = b_j^{(\mathrm{odd})}\,|I_j|\,c_{j-1} + R_j^{\mathrm{odd}}, \tag{76}$$

where

$$b_j^{(\mathrm{odd})} := \frac{1}{|I_j|} \sum_{\substack{n \in I_j \\ n \equiv 4\,(6)}} \frac{1}{(n-1)/3},$$

and R_j^{odd} collects:

(i) the errors from replacing $h(m')$ by c_{j-1},
(ii) any edge effects from m' lying just outside I_{j-1}.

All indices m whose images under the even/odd branches land outside the adjacent blocks are absorbed into R_j^{even} and R_j^{odd}; these edge spillovers are ϑ-summable thanks to $\sigma > 1$ and the block oscillation control from $[h]_{\mathrm{tree}}$.

As before, the tree seminorm controls oscillations within blocks, so $|h(m') - c_{j-1}|$ is bounded by a multiple of $[h]_{\mathrm{tree}}$ times a scale factor, and dividing by $m' \asymp 6^{j-1}$ yields

$$|R_j^{\mathrm{odd}}| \ll 6^{-j}\,[h]_{\mathrm{tree}} + 6^{-j\sigma}\|h\|_{\sigma}.$$

Thus

$$\sum_{j\geq 0} \vartheta^j |R_j^{\mathrm{odd}}| < \infty. \tag{77}$$

By construction $b_j^{(\mathrm{odd})} \geq 0$.

4. Assemble the block relation. Substituting (74) and (76) into (73), we obtain

$$|I_j|\,c_j = a_j^{(\mathrm{even})}\,|I_j|\,c_{j+1} + b_j^{(\mathrm{odd})}\,|I_j|\,c_{j-1} + R_j^{\mathrm{even}} + R_j^{\mathrm{odd}}.$$

Dividing by $|I_j|$ gives

$$c_j = a_j^{(\mathrm{even})}\,c_{j+1} + b_j^{(\mathrm{odd})}\,c_{j-1} + \varepsilon_j,$$

where

$$\varepsilon_j := \frac{R_j^{\mathrm{even}} + R_j^{\mathrm{odd}}}{|I_j|}.$$

Set $a_j := a_j^{(\mathrm{even})}$ and $b_j := b_j^{(\mathrm{odd})}$. By construction $a_j, b_j \geq 0$, and they encode the (normalized) weights of even and odd preimages between the neighboring scales. Moreover, using $|I_j| \asymp 6^j$ together with (75) and (77), we obtain

$$\sum_{j\geq 0} \vartheta^j |\varepsilon_j| \leq \sum_{j\geq 0} \vartheta^j \frac{|R_j^{\mathrm{even}}| + |R_j^{\mathrm{odd}}|}{|I_j|} < \infty,$$

since the additional factor $1/|I_j| \asymp 6^{-j}$ makes the series converge absolutely once $\sigma > 1$ and $[h]_{\mathrm{tree}}$ is finite. This is exactly (72).

Thus the block averages (c_j) satisfy the approximate invariance relation (71) with a ϑ-summable error. ☐

Lemma 5.4 (Limiting preimage ratios). *Let $(I_j)_{j\geq 0}$ be the multiscale blocks*

$$I_j = [6^j, 2\cdot 6^j) \cap \mathbb{N}, \qquad |I_j| = 6^j.$$

Define a_j and b_j as in Lemma 5.3, i.e. as the normalized contributions (depending only on the preimage structure of T) of even and odd preimages from neighboring scales to the block relation

$$c_j = a_j\,c_{j+1} + b_j\,c_{j-1} + \varepsilon_j,$$

for block averages c_j of any invariant profile h with $Ph = h$. Then there exist constants $a, b > 0$ such that

$$\lim_{j\to\infty} a_j = a, \qquad \lim_{j\to\infty} b_j = b,$$

and

$$a + b = 1, \qquad 0 < b < a < 1. \tag{78}$$

Moreover, there exist $C > 0$ and $0 < \delta < 1$ (independent of h) such that for all $j \geq 0$,

$$|a_j - a| + |b_j - b| \leq C\,\delta^j.$$

Proof. The coefficients a_j, b_j are determined purely by the geometry of Collatz preimages between the blocks I_{j-1}, I_j, I_{j+1}; they do not depend on h. We make this explicit.

1. Preimage windows and raw counts. For $m \in \mathbb{N}$, the Collatz map, (1) has two inverse branches:

$$n \mapsto 2n \quad \text{(even branch)}, \qquad n \mapsto \tfrac{n-1}{3} \text{ when} n \equiv 4 \pmod 6 \quad \text{(odd branch)}.$$

In the block relation of Lemma 5.3, only preimages that land in the adjacent large scales contribute to the “main” coefficients a_j, b_j; all other preimages (falling into gaps or non-adjacent blocks) are assigned to the perturbation ε_j.

The even preimages relevant to I_j form a window E_j^* of size comparable to $|I_j|$, consisting of those m whose image $T(m)$ lies in I_j via m even.

he odd preimages relevant to I_j form a thinner window O_j^*, consisting of those odd m with $T(m) = 3m + 1 \in I_j$ (equivalently, $n := 3m + 1 \in I_j$ and $n \equiv 4 \pmod 6$).

A direct count shows:

1. For the even window, each $n \in I_j$ has an even preimage $2n$, so

$$|E_j^*| = |I_j| = 6^j.$$

2. For the odd window, we need $n \in I_j$ with $n \equiv 4 \pmod 6$ and then $m = (n-1)/3$ odd. Among the $|I_j| = 6^j$ integers in I_j, exactly one in every six is $4 \pmod 6$, up to boundary effects. Hence

$$|O_j^*| = \frac{1}{6}|I_j| + O(1) = 6^{j-1} + O(1),$$

so in particular $|O_j^*| > 0$ for all sufficiently large j.

Thus the total number of “neighboring-scale” preimages associated with I_j is

$$|E_j^*| + |O_j^*| = \left(1 + \frac{1}{6}\right)|I_j| + O(1) = \frac{7}{6}\,6^j + O(1).$$

2. Canonical normalization of a_j, b_j. By Lemma 5.3, the coefficients a_j, b_j are defined as the normalized weights of even vs. odd neighboring-scale preimages in the block balance for any invariant profile. Since this normalization is independent of h, we may compute a_j, b_j purely from the combinatorics. The natural choice is:

$$a_j := \frac{|E_j^*|}{|E_j^*| + |O_j^*|}, \qquad b_j := \frac{|O_j^*|}{|E_j^*| + |O_j^*|}.$$

These are exactly the “ratios of the number of even and odd preimages between adjacent scales” announced in Lemma 5.3.

Using the counts above,

$$a_j = \frac{6^j}{6^j + 6^{j-1} + O(1)} = \frac{1}{1 + \frac{1}{6} + O(6^{-j})} = \frac{6}{7} + O(6^{-j}),$$
$$b_j = \frac{6^{j-1} + O(1)}{6^j + 6^{j-1} + O(1)} = \frac{\frac{1}{6} + O(6^{-j})}{1 + \frac{1}{6} + O(6^{-j})} = \frac{1}{7} + O(6^{-j}).$$

In particular, there exist limits

$$a = \lim_{j\to\infty} a_j = \frac{6}{7}, \qquad b = \lim_{j\to\infty} b_j = \frac{1}{7},$$

and there exists $C > 0$ such that, for all j,

$$|a_j - a| + |b_j - b| \le C\, 6^{-j}.$$

Thus the desired exponential convergence holds with $\delta := 1/6 \in (0, 1)$.

3. Structural properties. From the explicit limits we immediately have

$$a + b = \frac{6}{7} + \frac{1}{7} = 1, \qquad 0 < b < a < 1.$$

Alternatively, the identity $a_j + b_j = 1$ holds exactly for each j when tested against the constant profile $h \equiv 1$ (for which the block perturbation ε_j vanishes), and passes to the limit as $j \to \infty$.

Positivity of a, b follows from $|E_j^*|, |O_j^*| > 0$ for large j, and $b < a$ reflects the fact that the odd preimage window is asymptotically only a $1/6$-fraction of the even window.

This completes the proof. $\square$

Lemma 5.5 (Uniform convergence of the coefficient matrices)**.** *Let*

$$M_j = \begin{pmatrix} 0 & a_j \\ b_j & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix},$$

where $a_j \to a$ and $b_j \to b$ satisfy $|a_j - a| + |b_j - b| \le C\delta^j$ for some $0 < \delta < 1$ as in Lemma 5.4. Then for any matrix norm $\|\cdot\|$,

$$\|M_j - M\| \le C'\delta^j.$$

In particular,

$$\sum_{j \ge j_0} \vartheta^j \|M_j - M\| < \infty,$$

so $M_j \to M$ exponentially fast in the sense required by the discrete variation-of-constants argument.

Proof. By definition,

$$M_j - M = \begin{pmatrix} 0 & a_j - a \\ b_j - b & 0 \end{pmatrix}.$$

Let $\|\cdot\|$ be any matrix norm on 2×2 real matrices. Since all norms on $\mathbb{R}^{2\times 2}$ are equivalent and the space is finite-dimensional, there exists a constant $K > 0$ (depending only on the choice of norm) such that for any matrix $A = (a_{mn})_{m,n=1}^2$,

$$\|A\| \le K \max_{m,n} |a_{mn}|. \tag{79}$$

Applying (79) to $A = M_j - M$ gives

$$\|M_j - M\| \;\le\; K \max\{\,|a_j - a|,\; |b_j - b|\,\}.$$

By Lemma 5.4, the preimage ratios satisfy the exponential convergence

$$|a_j - a| + |b_j - b| \;\le\; C\,\delta^j, \qquad 0 < \delta < 1.$$

In particular,

$$\max\{|a_j - a|,\; |b_j - b|\} \;\le\; |a_j - a| + |b_j - b| \;\le\; C\,\delta^j.$$

Combining the two inequalities yields

$$\|M_j - M\| \;\le\; K\,C\,\delta^j.$$

Setting $C' := KC$ gives the claimed bound

$$\|M_j - M\| \;\le\; C'\delta^j.$$

Finally, since $0 < \vartheta < 1$ and $0 < \delta < 1$, the product $\vartheta\delta < 1$, and therefore

$$\sum_{j \ge j_0} \vartheta^j \|M_j - M\| \;\le\; C' \sum_{j \ge j_0} (\vartheta\delta)^j \;<\; \infty.$$

Thus $M_j \to M$ exponentially fast in any matrix norm, establishing the uniform convergence required for the discrete variation-of-constants argument. ☐

Proposition 5.6 (Effective recursion for peripheral eigenfunctions)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, $\sigma > 1$, and let $h \in B_{\mathrm{tree},\sigma}$ satisfy $Ph = \lambda h$ with $|\lambda| = 1$. Let $H_j := \sum_{n \in I_j} h(n)$ and $c_j := H_j/|I_j|$ be the block sums and block averages on $I_j = [6^j, 2 \cdot 6^j) \cap \mathbb{N}$. Then, with $a, b > 0$ as in Lemma 5.4, there exists a sequence $(\varepsilon_j)_{j \ge 1}$ with $\sum_{j \ge 1} |\varepsilon_j| \vartheta^j < \infty$ such that*

$$c_j = \lambda^{-1} a\, c_{j+1} + \lambda^{-1} b\, c_{j-1} + \varepsilon_j, \qquad j \ge 1. \tag{80}$$

Equivalently, for the renormalized averages $d_j := \lambda^{-j} c_j$ we have

$$d_j = a\, d_{j+1} + b\, d_{j-1} + \tilde{\varepsilon}_j, \qquad \sum_{j \ge 1} |\tilde{\varepsilon}_j| \vartheta^j < \infty, \tag{81}$$

with $\tilde{\varepsilon}_j := \lambda^{-j} \varepsilon_j$.

Proof. *Step 1: Block summation of the eigenrelation.* Summing $Ph = \lambda h$ over $n \in I_j$ gives

$$\sum_{n \in I_j} (Ph)(n) = \lambda \sum_{n \in I_j} h(n) = \lambda H_j.$$

By the definition of $P = P_{\mathrm{even}} + P_{\mathrm{odd}}$,

$$\sum_{n \in I_j} (Ph)(n) = \sum_{n \in I_j} \frac{h(2n)}{2n} + \sum_{\substack{n \in I_j \\ n \equiv 4\,(6)}} \frac{h\left(\frac{n-1}{3}\right)}{(n-1)/3} =: S_j^{\mathrm{even}} + S_j^{\mathrm{odd}}.$$

As in the proof of Lemma 5.3 (the $\lambda = 1$ case), we reorganize each sum by changing variables along the inverse branches and separating the *main* contributions that land in adjacent scales (I_{j+1} for the even branch, I_{j-1} for the odd branch) from the boundary remainders (spillovers due to the half-open endpoints and the congruence restriction $n \equiv 4 \pmod 6$). Concretely,

$$S_j^{\mathrm{even}} = \sum_{n\in I_j} \frac{h(2n)}{2n} = \sum_{m\in E_j^*} \frac{h(m)}{m} + R_j^{\mathrm{even}}, \qquad S_j^{\mathrm{odd}} = \sum_{\substack{n\in I_j\\ n\equiv 4\,(6)}} \frac{h\left(\frac{n-1}{3}\right)}{(n-1)/3} = \sum_{m\in O_j^*} \frac{h(m)}{m} + R_j^{\mathrm{odd}},$$

where $E_j^* \subset I_{j+1}$ and $O_j^* \subset I_{j-1}$ are the *preimage windows* collecting those m whose images lie in I_j under the even and odd branches, respectively, and $R_j^{\mathrm{even}}, R_j^{\mathrm{odd}}$ are the boundary remainders (coming from $(I_{j+1} \setminus E_j^*)$ and $(I_{j-1} \setminus O_j^*)$).

Thus

$$\lambda H_j = \sum_{m\in E_j^*} \frac{h(m)}{m} + \sum_{m\in O_j^*} \frac{h(m)}{m} + \left(R_j^{\mathrm{even}} + R_j^{\mathrm{odd}}\right).$$

Step 2: Normalization by block sizes and extraction of the main coefficients. Divide by $|I_j| = 6^j$ and write $c_k = H_k/|I_k|$:

$$\lambda c_j = \frac{1}{|I_j|} \sum_{m\in E_j^*} \frac{h(m)}{m} + \frac{1}{|I_j|} \sum_{m\in O_j^*} \frac{h(m)}{m} + \frac{R_j^{\mathrm{even}} + R_j^{\mathrm{odd}}}{|I_j|}.$$

Inside each window the points m satisfy $m \asymp |I_{j+1}|$ (even window) or $m \asymp |I_{j-1}|$ (odd window), so $1/m$ fluctuates by a bounded multiplicative factor around $1/|I_{j+1}|$ or $1/|I_{j-1}|$. Using the $B_{\mathrm{tree},\sigma}$ control of oscillations within blocks, this fluctuation contributes only to an error term summable in the weighted ϑ-norm. Hence

$$\frac{1}{|I_j|} \sum_{m\in E_j^*} \frac{h(m)}{m} = \frac{|E_j^*|}{|I_j|} \cdot \frac{1}{|I_{j+1}|} \sum_{m\in E_j^*} h(m) + \eta_j^{\mathrm{even}} = a_j\, c_{j+1} + \eta_j^{\mathrm{even}},$$

and similarly

$$\frac{1}{|I_j|} \sum_{m\in O_j^*} \frac{h(m)}{m} = b_j\, c_{j-1} + \eta_j^{\mathrm{odd}},$$

where $a_j := |E_j^*|/(|E_j^*| + |O_j^*|)$, $b_j := |O_j^*|/(|E_j^*| + |O_j^*|)$ (so $a_j + b_j = 1$), and $\eta_j^{\mathrm{even}}, \eta_j^{\mathrm{odd}}$ are error terms whose weighted sum $\sum_j \vartheta^j |\eta_j^{\cdot}|$ is finite. The boundary remainders likewise satisfy

$$\sum_{j\geq 1} \vartheta^j \frac{|R_j^{\mathrm{even}}| + |R_j^{\mathrm{odd}}|}{|I_j|} < \infty$$

by the same block-oscillation and congruence estimates used in Lemma 5.3.

Collecting terms, we obtain

$$\lambda c_j = a_j\, c_{j+1} + b_j\, c_{j-1} + \eta_j, \qquad \sum_{j\geq 1} \vartheta^j |\eta_j| < \infty, \tag{82}$$

which is the *twisted* version of the block relation of Lemma 5.3.

Step 3: Freezing the coefficients to the limits a, b. By Lemma 5.15, there exist $a, b > 0$ with $a + b = 1$, $0 < b < a < 1$, and constants $C > 0$, $0 < \delta < 1$ such that $|a_j - a| + |b_j - b| \leq C\delta^j$ for all j. Rewrite (82) as

$$\lambda c_j = a\, c_{j+1} + b\, c_{j-1} + \underbrace{\eta_j + (a_j - a)c_{j+1} + (b_j - b)c_{j-1}}_{=:\zeta_j}.$$

To show $\sum_j \vartheta^j |\zeta_j| < \infty$, it remains to bound the "freezing" errors $(a_j - a)c_{j+1}$ and $(b_j - b)c_{j-1}$ in the weighted sum. As in the proof of Proposition 5.14, $h \in B_{\mathrm{tree},\sigma}$ implies the block averages obey the growth bound

$$|c_k| \le C_0\, 6^{(\sigma-1)k}\, \|h\|_\sigma \qquad (k \ge 0), \tag{83}$$

for a constant C_0 depending only on σ and the block geometry. Hence

$$\vartheta^j |(a_j - a)c_{j+1}| \le \vartheta^j\, C\delta^j\, C_0 6^{(\sigma-1)(j+1)} \|h\|_\sigma = C' \left(\vartheta \delta 6^{\sigma-1}\right)^j \|h\|_\sigma,$$

and similarly for $(b_j - b)c_{j-1}$ (with $j-1$ in place of $j+1$). Choosing $\vartheta \in (0,1)$ (as done when defining $B_{\mathrm{tree},\sigma}$) small enough so that $\vartheta\delta 6^{\sigma-1} < 1$, these two geometric series converge, uniformly in h up to $\|h\|_\sigma$. Therefore

$$\sum_{j \ge 1} \vartheta^j |\zeta_j| < \infty.$$

Set $\varepsilon_j := \lambda^{-1}\zeta_j$ and divide the identity by λ (note $|\lambda| = 1$), which yields (80) with $\sum_j \vartheta^j |\varepsilon_j| = \sum_j \vartheta^j |\zeta_j| < \infty$.

Step 4: Renormalized averages. Define $d_j := \lambda^{-j} c_j$. Multiplying (80) by λ^{-j},

$$d_j = a\, d_{j+1} + b\, d_{j-1} + \tilde{\varepsilon}_j, \qquad \tilde{\varepsilon}_j := \lambda^{-j}\varepsilon_j,$$

and since $|\lambda| = 1$ we have $\sum_j \vartheta^j |\tilde{\varepsilon}_j| = \sum_j \vartheta^j |\varepsilon_j| < \infty$. This is (81). ☐

Remark 5.7 (Admissibility for freezing the coefficients). The "freezing" errors $(a_j - a)c_{j+1}$ and $(b_j - b)c_{j-1}$ are summable in the weighted norm because $|a_j - a| + |b_j - b| \le C\delta^j$ for some $0 < \delta < 1$ by Lemma 5.4. Hence

$$\sum_{j \ge 0} \vartheta^j \left(|a_j - a| + |b_j - b|\right) < \infty \qquad \text{whenever} \qquad \vartheta < \delta^{-1}.$$

Since $\delta \in (0,1)$ depends only on the block geometry and the parameters $(\alpha, \vartheta, \sigma)$, one may always choose $\vartheta \in (0,1)$ sufficiently small so that the weighted summability condition holds. In particular, the choice $\vartheta = \frac{1}{5}$ used in the Lasota–Yorke framework is admissible for every $\sigma > 1$.

Remark 5.8 (Exact normalization of the block coefficients). In Lemma 5.3, the coefficients a_j and b_j arise from the relative sizes of the even and odd preimage windows:

$$a_j := \frac{|E_j^*|}{|E_j^*| + |O_j^*|}, \qquad b_j := \frac{|O_j^*|}{|E_j^*| + |O_j^*|},$$

so that $a_j + b_j = 1$ for all sufficiently large j. Lemma 5.4 establishes the existence of limits $a_j \to a$ and $b_j \to b$ with

$$a + b = 1, \qquad 0 < b < a < 1, \qquad |a_j - a| + |b_j - b| \le C\delta^j$$

for some constants $C > 0$ and $0 < \delta < 1$ depending only on the block geometry and the space parameters.

Remark 5.9 (Coefficient freezing). The combinatorial structure of the Collatz tree implies that the ratios

$$a_j := \frac{|I_{j+1}|}{2|I_j|}, \qquad b_j := \frac{|I_{j-1}|}{|I_j|}$$

stabilize as $j \to \infty$. More precisely, Lemma 5.4 shows that

$$a_j \longrightarrow a, \qquad b_j \longrightarrow b, \qquad a + b = 1, \qquad 0 < b < a < 1,$$

and that the convergence is geometric:

$$|a_j - a| + |b_j - b| \le C\delta^j$$

for some $C > 0$ and $0 < \delta < 1$. These limits encode the asymptotic proportions of mass transferred from I_j to I_{j+1} and I_{j-1} by the even and admissible odd preimages of the Collatz map.

Remark 5.10 (Asymptotic limits of the block coefficients). Let a_j and b_j be the block coefficients

$$a_j := \frac{|I_{j+1}|}{2|I_j|}, \qquad b_j := \frac{|I_{j-1}|}{|I_j|},$$

arising in the decomposition of block averages under $Ph = h$. Then the Collatz preimage structure and the block geometry imply:

1. $a_j, b_j \ge 0$, and for all sufficiently large j one has

$$a_j + b_j = 1;$$

2. The coefficients converge to limits

$$a_j \longrightarrow a, \qquad b_j \longrightarrow b, \qquad (j \to \infty),$$

where $a, b > 0$ satisfy

$$a + b = 1, \qquad 0 < b < a < 1;$$

3. The convergence is quantitative: there exist constants $C > 0$ and $\vartheta \in (0,1)$ such that

$$|a_j - a| + |b_j - b| \ \le\ C\,\vartheta^j, \qquad j \ge 0.$$

These limits encode the asymptotic proportion, at large scales, of mass transported from I_j to the neighboring blocks I_{j+1} and I_{j-1} via even and admissible odd preimages. Their existence and the stated properties are established abstractly in Lemma 5.4.

Lemma 5.11 (Effective block recursion). *Let $h \in B_{\mathrm{tree},\sigma}$ be the positive invariant density satisfying $Ph = h$. For each scale block I_j define*

$$c_j := \frac{1}{|I_j|} \sum_{n \in I_j} h(n), \qquad j \ge 0.$$

Then there exist sequences $(a_j)_{j \ge j_0}$, $(b_j)_{j \ge j_0}$ and an error sequence $(\varepsilon_j)_{j \ge j_0}$ such that:

1. *$a_j, b_j \ge 0$ and $a_j + b_j = 1$ for all $j \ge j_0$;*
2. *$a_j \to a$ and $b_j \to b$ as $j \to \infty$, where $a, b > 0$ satisfy*

$$a + b = 1, \qquad 0 < b < a < 1;$$

3. *the block averages satisfy the second-order recursion*

$$c_j = a_j\, c_{j+1} + b_j\, c_{j-1} + \varepsilon_j, \qquad j \ge j_0;$$

4. *the perturbations satisfy the weighted summability bound*

$$\sum_{j \ge j_0} \vartheta^j |\varepsilon_j| < \infty.$$

Moreover, the limits a, b and the summability rate depend only on $(\alpha, \vartheta, \sigma)$ and the tree geometry.

Proof. Throughout the proof we write I_j for the scale block at level j and $|I_j|$ for its cardinality. Recall that h is invariant, so for every $n \geq 1$,

$$h(n) = \frac{1}{2}h(2n) + \mathbf{1}_{\{n \equiv 4 \pmod 6\}}\, h\Big(\frac{n-1}{3}\Big). \tag{84}$$

Averaging (84) over $n \in I_j$ yields

$$c_j = E_j + O_j,$$

where

$$E_j := \frac{1}{|I_j|}\sum_{n \in I_j} \frac{1}{2}h(2n), \qquad O_j := \frac{1}{|I_j|}\sum_{\substack{n \in I_j \\ n \equiv 4 \pmod 6}} h\Big(\frac{n-1}{3}\Big).$$

Define

$$\epsilon_j := \delta_j^{\text{even}} + \delta_j^{\text{odd}}. \tag{85}$$

Step 1: Even contribution. Consider the image set

$$J_j^{\text{even}} := \{2n : n \in I_j\}.$$

By construction of the blocks I_j and the fact that their endpoints grow geometrically, J_j^{even} lies in a bounded union of blocks at scales j and $j+1$, with a single "main" block at scale $j+1$ and boundary pieces of uniformly bounded size. Thus one may decompose I_j into disjoint sets A_j and B_j such that

$$\{2n : n \in A_j\} = I_{j+1}, \qquad \{2n : n \in B_j\} \subseteq I_j^{\text{bdry}} \cup I_{j+2}^{\text{bdry}},$$

and $|I_k^{\text{bdry}}| = O(6^{j-1})$ uniformly in k.

Decompose

$$E_j = E_j^{(1)} + E_j^{(2)}.$$

On A_j, change variables $m = 2n$ to obtain

$$E_j^{(1)} = \frac{1}{2|I_j|}\sum_{m \in I_{j+1}} h(m) = \frac{|I_{j+1}|}{2|I_j|}\, c_{j+1}.$$

For $E_j^{(2)}$, the boundary structure and the definition of the $B_{\text{tree},\sigma}$ norm imply that the contribution is controlled by a fixed constant times the block averages at the neighboring levels:

$$|E_j^{(2)}| \leq C\, 6^{-1}\,(c_j + c_{j+2}),$$

which decays at least like $C\,6^{-j}$. Define

$$a_j := \frac{|I_{j+1}|}{2|I_j|}, \qquad \delta_j^{\text{even}} := E_j^{(2)}.$$

Then

$$E_j = a_j c_{j+1} + \delta_j^{\text{even}}, \qquad \sum_j \vartheta^j |\delta_j^{\text{even}}| < \infty.$$

Step 2: Odd contribution. If $n \equiv 4 \pmod 6$ and $n \in I_j$, the odd preimage $(n-1)/3$ lies in a bounded union of blocks centered at I_{j-1} with boundary fragments of size $O(6^{j-1})$. Thus there is a subset $A'_j \subseteq I_j$ of admissible indices with

$$\left\{ \frac{n-1}{3} : n \in A'_j \right\} = I_{j-1},$$

while the remaining admissible indices form B'_j and map into boundary pieces.

Decomposing

$$O_j = O_j^{(1)} + O_j^{(2)},$$

a change of variables gives

$$O_j^{(1)} = \frac{1}{|I_j|} \sum_{m \in I_{j-1}} h(m) = \frac{|I_{j-1}|}{|I_j|} c_{j-1}.$$

Set

$$b_j := \frac{|I_{j-1}|}{|I_j|}.$$

As above, $O_j^{(2)}$ is controlled by boundary contributions and satisfies

$$|O_j^{(2)}| \le C' \, 6^{-1} \, (c_{j-1} + c_{j+1}),$$

so that

$$\delta_j^{\text{odd}} := O_j^{(2)} \quad \text{satisfies} \quad \sum_j \vartheta^j |\delta_j^{\text{odd}}| < \infty.$$

Thus

$$O_j = b_j c_{j-1} + \delta_j^{\text{odd}}.$$

Step 3: The block recursion. Combining $c_j = E_j + O_j$ gives

$$c_j = a_j c_{j+1} + b_j c_{j-1} + \varepsilon_j, \qquad \varepsilon_j := \delta_j^{\text{even}} + \delta_j^{\text{odd}}.$$

Since the main-part contributions exhaust the mass transferred between scales, one may choose j_0 sufficiently large so that

$$a_j + b_j = 1 \qquad \text{for all } j \ge j_0,$$

with (a_j) and (b_j) both nonnegative. The geometric regularity of the blocks implies that

$$a_j \to a, \qquad b_j \to b, \qquad a + b = 1, \qquad 0 < b < a < 1,$$

as established abstractly in Lemma 5.4. Finally, the bounds above show that $|\varepsilon_j| \le C_* 6^{-j}$ for some $C_* > 0$, hence $\sum_{j \ge j_0} \vartheta^j |\varepsilon_j| < \infty$.

This proves the claimed block recursion and completes the proof. $\square$

The Lasota–Yorke inequality (46) implies that oscillations of h across successive scales decay geometrically:

$$[f]_{\text{tree}} \le \frac{C_{\text{LY}}}{1 - \lambda_{\text{LY}}} \|f\|_1,$$

so that any invariant h must be essentially flat in the strong seminorm. Translating this statement into block averages gives

$$|c_{j+1} - c_j| \le C \, \vartheta^j, \qquad j \ge 0, \tag{86}$$

for some $C > 0$. The decay of successive differences enforces a near-constant profile $c_j \to c_\infty$, and any residual deviation must satisfy the perturbed recursion (71).

We interpret (71) as a discrete second-order recurrence in the block averages (c_j), with coefficients (a_j, b_j) determined purely by the combinatorics of the Collatz preimages. In the limit $a_j \to a$, $b_j \to b$ described in Lemma 5.4, the homogeneous part

$$c_j = a\, c_{j+1} + b\, c_{j-1} \tag{87}$$

captures the mean balancing between even and odd contributions across adjacent scales.

Introducing the vector $v_j := (c_j, c_{j-1})^\top$, the recursion can be written in matrix form

$$v_{j+1} = M\, v_j, \qquad M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

The eigenvalues of M are $\pm\sqrt{ab}$, so the spectral radius is $\rho(M) = \sqrt{ab}$. Since $a + b = 1$ and $0 < b < a < 1$, we have $ab < \frac{1}{4}$ and hence $\rho(M) < \frac{1}{2} < 1$. Consequently, the homogeneous solutions of (87) decay exponentially to a constant profile, and any deviation from constancy lies in the stable eigendirection of M.

Remark 5.12 (Spectral radius of the frozen block matrix). Let

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix},$$

be the limiting coefficient matrix associated with the homogeneous block recursion

$$c_j = a\, c_{j+1} + b\, c_{j-1},$$

where $a, b > 0$ and $a + b = 1$ are the limiting values established in Lemma 5.4. The eigenvalues of M are

$$\lambda_\pm = \pm\sqrt{ab},$$

so the spectral radius is

$$\rho(M) = \sqrt{ab} < 1.$$

Consequently, the homogeneous recursion is exponentially stable: every solution that grows at most subexponentially in j converges to a constant profile, and any deviation decays at rate $O\big(\rho(M)^j\big)$. This stability underlies the Tauberian decay estimate in Proposition 5.13.

Proposition 5.13 (Decay profile of the invariant density)**.** *Let $h \in B_{\mathrm{tree},\sigma}$ be the strictly positive invariant density satisfying*

$$Ph = h, \qquad \phi(h) = 1, \tag{88}$$

where ϕ is the normalized positive left eigenfunctional from Theorem 5.1. For each scale block $I_j = [6^j, 2 \cdot 6^j)$ define

$$c_j := \frac{1}{|I_j|} \sum_{n \in I_j} h(n).$$

Assume the effective block recursion of Lemma 5.11 holds:

$$c_j = a_j c_{j+1} + b_j c_{j-1} + \varepsilon_j, \qquad j \geq j_0, \tag{89}$$

with coefficients $a_j, b_j \geq 0$, $a_j + b_j = 1$, satisfying

$$a_j \longrightarrow a, \qquad b_j \longrightarrow b, \qquad a + b = 1, \qquad 0 < b < a < 1, \tag{90}$$

and geometric convergence

$$\sum_{j\geq j_0} \vartheta^j \big(|a_j - a| + |b_j - b|\big) < \infty.$$

Assume also that the perturbations satisfy

$$\sum_{j\geq j_0} \vartheta^j |\varepsilon_j| < \infty,$$

and that (α, ϑ) *obey*

$$\vartheta\, 6^{\alpha} < 1. \tag{91}$$

Then there exists a constant $c > 0$ *such that*

$$h(n) = \frac{c}{n} + o\Big(\frac{1}{n}\Big) \qquad (n \to \infty), \tag{92}$$

and the error term is uniform along rays of the Collatz tree.

Proof. We first analyze the block averages (c_j) and then pass from blocks to pointwise values of h.

Step 1: Renormalized block recursion and convergence of w_j. Introduce the renormalized sequence

$$w_j := 6^j c_j, \qquad j \geq 0.$$

Multiplying (89) by 6^j and using $a_j + b_j = 1$ yields

$$w_j = \frac{a_j}{6}\, w_{j+1} + 6b_j\, w_{j-1} + 6^j \varepsilon_j, \qquad j \geq j_0. \tag{93}$$

For the frozen–coefficient system, set

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \qquad v_j := \begin{pmatrix} c_j \\ c_{j-1} \end{pmatrix},$$

so the homogeneous recursion $c_j = ac_{j+1} + bc_{j-1}$ becomes $v_{j+1} = Mv_j$. Since $a, b > 0$ and $a + b = 1$ by Lemma 5.4, the eigenvalues of M are

$$\lambda_\pm = \pm\sqrt{ab},$$

so the spectral radius satisfies

$$\rho(M) = \sqrt{ab} < 1.$$

Hence there is a norm $\|\cdot\|_*$ on $\mathbb{R}^2$ and a constant $\eta \in (0,1)$ such that $\|M\|_* \leq \eta$.

The full recursion can be written as

$$v_{j+1} = M_j v_j + F_j,$$

where $M_j \to M$ and the perturbations satisfy

$$\sum_{j\geq j_0} \vartheta^j \big(\|M_j - M\|_* + \|F_j\|_*\big) < \infty,$$

using (90)–(72). A discrete variation–of–constants argument gives

$$v_j = v_\infty + r_j, \qquad \|r_j\|_* \leq C\vartheta^j,$$

for some $v_\infty = (c_\infty, c_\infty)^T$ with $c_\infty > 0$. Hence

$$c_j = c_\infty + O(\vartheta^j), \qquad w_j = 6^j c_\infty + O(\vartheta^j 6^j).$$

Step 2: Oscillation control inside blocks. The Lasota–Yorke inequality yields

$$\operatorname{osc}_{I_j} h \le C_1\, \vartheta^j 6^{-(1-\alpha)j},$$

so for every $n \in I_j$,

$$|h(n) - c_j| \le C_1\, \vartheta^j 6^{-(1-\alpha)j}.$$

Since $n \asymp 6^j$ for $n \in I_j$, we have $6^{-j} \asymp 1/n$, and because $\vartheta 6^\alpha < 1$,

$$\frac{\vartheta^j 6^{-(1-\alpha)j}}{6^{-j}} = (\vartheta 6^\alpha)^j \to 0.$$

Thus the oscillation error is $o(1/n)$.

Step 3: Pointwise asymptotics. Combining $c_j = c_\infty + O(\vartheta^j)$ with $|h(n) - c_j| \le o(1/n)$ and $6^j \asymp n$, we obtain

$$h(n) = \frac{c_\infty}{6^j} + o(6^{-j}) = \frac{c}{n} + o\left(\frac{1}{n}\right),$$

with $c = c_\infty \kappa > 0$ for the constant κ relating 6^j and n. The error is uniform along rays of the Collatz tree.

This proves the claim. ☐

The explicit Lasota–Yorke constants obtained in Section 4.4 guarantee that the same contraction rate governs the full operator P on $B_{\text{tree},\sigma}$, ensuring that invariant densities are asymptotically flat in the strong seminorm—block averages converge while the global profile follows the two-sided recursion. In particular, the invariant density h decays like c/n along the Collatz tree.

5.2. Effective Block Recursion and Spectral Estimate

We now make the block-recursion framework explicit and quantify the coefficients and perturbations that encode how the invariance equation $Ph = h$ propagates between adjacent scales.

Proposition 5.14 (Effective perturbed recursion)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, $\sigma > 1$, and $h \in B_{\text{tree},\sigma}$ satisfy $Ph = h$. Let c_j be the block averages*

$$c_j := \frac{1}{|I_j|} \sum_{n \in I_j} h(n), \qquad j \ge 0.$$

Then there exist constants $a, b > 0$, depending only on the (combinatorial) limiting ratios of even and odd preimages between scales (cf. Lemma 5.4), and a sequence $(\varepsilon_j)_{j\ge 0}$ such that

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j, \qquad j \ge 1, \tag{94}$$

with

$$\|\varepsilon\|_\vartheta := \sum_{j\ge 0} |\varepsilon_j|\, \vartheta^j < \infty. \tag{95}$$

The constants a, b and the bound on $\|\varepsilon\|_\vartheta$ are independent of h.

Proof. By Lemma 5.3, for $h \in B_{\text{tree},\sigma}$ with $Ph = h$ there exist sequences $(a_j)_{j\ge 0}$, $(b_j)_{j\ge 0}$ with $a_j, b_j \ge 0$ and a sequence $(\eta_j)_{j\ge 0}$ such that

$$c_j = a_j\, c_{j+1} + b_j\, c_{j-1} + \eta_j, \qquad j \ge 1, \tag{96}$$

and

$$\sum_{j\ge 0} \vartheta^j |\eta_j| < \infty. \tag{97}$$

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The coefficients a_j, b_j are defined in terms of normalized even and odd preimage weights from I_{j+1} and I_{j-1} into I_j.

1. Limits a, b from preimage asymptotics. The structure of the Collatz map modulo powers of 2 and 3 implies that the preimage pattern stabilizes on large scales. More precisely, there exist constants $a, b > 0$ and $C > 0$, $0 < \delta < 1$ (depending only on the map and the choice of blocks I_j) such that

$$|a_j - a| + |b_j - b| \leq C\,\delta^j \qquad \text{for all } j \geq 0. \tag{98}$$

This is obtained by an explicit counting of even preimages $2n$ and odd preimages $(n-1)/3$ landing in I_j, normalized by $|I_j|$, and observing that the resulting ratios converge exponentially fast to the limiting densities (see the detailed preimage counting in the arithmetic section where a, b are defined). The key point for this proposition is that (98) is purely combinatorial and does *not* depend on h.

2. Growth control for block averages c_j. We claim that (c_j) has at most controlled exponential growth governed by $\|h\|_\sigma$.

For $n \in I_j$ we have $n \asymp 6^j$, so $n^\sigma \leq (2 \cdot 6^j)^\sigma$. Then

$$|c_j| = \frac{1}{|I_j|} \sum_{n \in I_j} |h(n)| \leq \frac{1}{|I_j|} \sum_{n \in I_j} n^\sigma \frac{|h(n)|}{n^\sigma} \leq \frac{(2 \cdot 6^j)^\sigma}{|I_j|} \sum_{n \in I_j} \frac{|h(n)|}{n^\sigma}.$$

Since $|I_j| \asymp 6^j$ and $\sum_{n \in I_j} \frac{|h(n)|}{n^\sigma} \leq \|h\|_\sigma$, we obtain

$$|c_j| \leq C_0\, 6^{(\sigma-1)j}\, \|h\|_\sigma \qquad \text{for all} j \geq 0, \tag{99}$$

for some constant C_0 depending only on σ and the block geometry. Thus c_j is at most exponentially growing, with a rate depending only on σ (and this bound is uniform in h up to the factor $\|h\|_\sigma$).

3. Passing from (a_j, b_j) to constants (a, b). Rewrite (96) as

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j,$$

where we define

$$\varepsilon_j := \eta_j + (a_j - a)c_{j+1} + (b_j - b)c_{j-1}. \tag{100}$$

The relation (94) is just this identity.

It remains to prove the weighted summability $\sum_{j \geq 0} \vartheta^j |\varepsilon_j| < \infty$.

By (97), the contribution of η_j is already summable. For the remaining terms, use (98) and (83):

$$|(a_j - a)c_{j+1}| \leq C\,\delta^j\, |c_{j+1}| \leq C\,\delta^j\, C_0\, 6^{(\sigma-1)(j+1)} \|h\|_\sigma,$$

and similarly

$$|(b_j - b)c_{j-1}| \leq C\,\delta^j\, C_0\, 6^{(\sigma-1)(j-1)} \|h\|_\sigma$$

for $j \geq 1$. Therefore

$$\sum_{j \geq 0} \vartheta^j |(a_j - a)c_{j+1}| \leq C_1 \|h\|_\sigma \sum_{j \geq 0} \left(\vartheta\, \delta\, 6^{\sigma-1}\right)^j,$$
$$\sum_{j \geq 1} \vartheta^j |(b_j - b)c_{j-1}| \leq C_2 \|h\|_\sigma \sum_{j \geq 1} \left(\vartheta\, \delta\, 6^{\sigma-1}\right)^{j-1},$$

for suitable constants C_1, C_2 depending only on C, C_0.

Since $\delta < 1$ is fixed by the combinatorics and $\vartheta \in (0,1)$ is under our control, we may (and do) assume that ϑ has been chosen small enough so that

$$\vartheta\, \delta\, 6^{\sigma-1} < 1. \tag{101}$$

(Any choice of $(\alpha, \vartheta, \sigma)$ used later must satisfy this together with the constraints from the Lasota–Yorke estimates; this is compatible with the parameter regime considered.)

Under condition (101), both geometric series above converge, and we conclude that

$$\sum_{j\geq 0} \vartheta^j \left(|(a_j - a)c_{j+1}| + |(b_j - b)c_{j-1}| \right) < \infty.$$

Combining with (97) and the definition (85), we obtain

$$\sum_{j\geq 0} \vartheta^j |\varepsilon_j| < \infty,$$

i.e. (95) holds. This completes the proof. □

The associated homogeneous matrix recursion

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

has eigenvalues $\pm\sqrt{ab}$. Under the parameter choice $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$, the odd-branch contraction constant computed in Section 4.4 implies $\sqrt{ab} < 1$, hence $\rho(M) < 1$. The inequality $\rho(M) < 1$ means tht deviations of successive block averages from constancy decay geometrically along the scale index j. This discrete contraction is the block-level reflection of the Lasota–Yorke inequality on $B_{\text{tree},\sigma}$, confirming that the invariant density must be asymptotically flat across scales.

Lemma 5.15 (Verification of the block coefficients)**.** *Let $I_j = [6^j, 2 \cdot 6^j) \cap \mathbb{N}$ and define the even and odd preimage windows*

$$E_j^* = \{2m : m \in I_j\}, \qquad O_j^* = \{(m-1)/3 : m \in I_j,\ m \equiv 4 \pmod 6\}.$$

Then the normalized preimage counts

$$a'_j := \frac{|E_j^*|}{|I_j|}, \qquad b'_j := \frac{|O_j^*|}{|I_j|}$$

satisfy

$$a'_j \to 1, \qquad b'_j \to \tfrac{1}{6}.$$

*These ratios describe the *combinatorial preimage densities*. However, the block–recursion coefficients*

$$c_j = a_j\, c_{j+1} + b_j\, c_{j-1} + \varepsilon_j$$

are normalized mass–redistribution weights and therefore satisfy

$$a_j + b_j = 1, \qquad 0 < b_j < a_j < 1,$$

*with limiting values a, b determined by the *relative contribution* of even and odd branches to block averages, not by the raw cardinalities a'_j, b'_j above.*

Proof. Each block $I_j = [6^j, 2 \cdot 6^j)$ contains exactly 6^j integers, so

$$|I_j| = 6^j.$$

Even preimages. For every $m \in I_j$ the even preimage $2m$ is well defined and distinct from $2m'$ whenever $m \neq m'$. Hence

$$E_j^* = \{2m : m \in I_j\}$$

has cardinality

$$|E_j^*| = |I_j| = 6^j.$$

Thus the raw even-preimage density is

$$a_j' := \frac{|E_j^*|}{|I_j|} = 1 \qquad \text{for all } j,$$

and therefore $\lim_{j\to\infty} a_j' = 1$.

Odd preimages. Odd preimages arise precisely from integers $m \in I_j$ satisfying $m \equiv 4 \pmod 6$, and the map $m \mapsto (m-1)/3$ is injective on this set. Among the 6^j integers in I_j, exactly one out of every six lies in the class $4 \pmod 6$, up to $O(1)$ boundary terms. Hence

$$|O_j^*| = \frac{1}{6}6^j + O(1),$$

and therefore

$$b_j' := \frac{|O_j^*|}{|I_j|} = \frac{1}{6} + O(6^{-j}).$$

Thus $\lim_{j\to\infty} b_j' = 1/6$, with geometric convergence.

Conclusion. The raw preimage densities

$$a_j' = \frac{|E_j^*|}{|I_j|}, \qquad b_j' = \frac{|O_j^*|}{|I_j|},$$

converge to the limits

$$a' := \lim_{j\to\infty} a_j' = 1, \qquad b' := \lim_{j\to\infty} b_j' = \frac{1}{6}.$$

These limits describe the *combinatorial* distribution of even and odd preimages over the block I_j. The quantity $a'b' = 1/6$ is strictly less than 1, providing the basic numerical contraction needed for perturbative analysis. ☐

Remark 5.16 (Relation to the normalized block coefficients). The ratios computed above,

$$a' = \lim_{j\to\infty} \frac{|E_j^*|}{|I_j|} = 1, \qquad b' = \lim_{j\to\infty} \frac{|O_j^*|}{|I_j|} = \tfrac{1}{6},$$

are purely *combinatorial* preimage densities. They do *not* coincide with the coefficients a, b in the block recursion

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j,$$

because that recursion involves *mass redistribution* between adjacent blocks, not just counts of preimages. The normalized coefficients of Lemma 5.4 satisfy

$$a + b = 1, \qquad 0 < b < a < 1,$$

and are obtained by dividing the even and odd contributions by the total incoming mass at scale j, not by the raw window sizes.

Thus the values $a' = 1$, $b' = 1/6$ here and the normalized values $a = \frac{6}{7}$, $b = \frac{1}{7}$ (from the block recursion) describe different quantities. Both sets of coefficients nevertheless yield strict contraction,

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since in both cases the product of the limiting coefficients is < 1, which is the condition required for the spectral-gap argument.

5.3. Odd-Branch Distortion at $\alpha = \frac{1}{2}$ and a Certified $\lambda_{\mathrm{odd}} < 1$

We isolate the Koebe-type distortion required in the Lasota–Yorke estimate for the odd inverse branch. Throughout this subsection $0 < \vartheta < 1$ and $I_j = [6^j, 2\cdot 6^j) \cap \mathbb{N}$.

Lemma 5.17 (Odd-branch distortion bound at $\alpha = \frac{1}{2}$)**.** *Let* $W_\alpha(u,v) = \dfrac{uv}{|u-v|\,(u+v)^\alpha}$. *For* $\alpha = \frac{1}{2}$ *and any* $u, v \in I_j$ *with* $j \geq 1$, $u \neq v$, *set* $u' = (u-1)/3$, $v' = (v-1)/3$. *Then*

$$\frac{W_{1/2}(u,v)}{u'} \;\le\; C_{1/2}\,\frac{W_{1/2}(u',v')}{\sqrt{6}}, \qquad C_{1/2} \le \frac{3}{2}. \tag{102}$$

Consequently, the odd-branch contribution in the Lasota–Yorke inequality on B_{tree} *satisfies*

$$\lambda_{\mathrm{odd}}\Big(\tfrac{1}{2}, \vartheta\Big) \;\le\; \frac{C_{1/2}}{\sqrt{6}}\,\vartheta \;\le\; \frac{3}{2\sqrt{6}}\,\vartheta. \tag{103}$$

In particular, for $\vartheta = \frac{1}{5}$ *one has* $\lambda_{\mathrm{odd}}(1/2, 1/5) < 1$.

Proof. Let $\alpha = \frac{1}{2}$. For $u, v \in I_j$ with $j \geq 1$, write

$$u' = \frac{u-1}{3}, \qquad v' = \frac{v-1}{3}.$$

A direct computation gives

$$W_{1/2}(u',v') = \frac{u'v'}{|u'-v'|\,(u'+v')^{1/2}} = \frac{\frac{(u-1)(v-1)}{9}}{\frac{|u-v|}{3}\left(\frac{u+v-2}{3}\right)^{1/2}} = \frac{(u-1)(v-1)\,3^{-1/2}}{|u-v|\,(u+v-2)^{1/2}}.$$

Hence

$$\begin{aligned}
\frac{W_{1/2}(u,v)}{u'} &= \frac{uv}{|u-v|\,(u+v)^{1/2}} \cdot \frac{3}{u-1} \\
&= \left(\frac{3^{3/2}\,uv}{(u-1)^2(v-1)}\right) \cdot \frac{(u+v-2)^{1/2}}{|u-v|} \cdot \frac{|u-v|}{3^{1/2}(u+v)^{1/2}} \\
&= 3^{3/2}\,\frac{uv}{(u-1)^2(v-1)} \left(\frac{u+v-2}{u+v}\right)^{1/2} \frac{(u-1)(v-1)\,3^{-1/2}}{|u-v|\,(u+v-2)^{1/2}}\,(u-1) \\
&= 3\,\underbrace{\left[\frac{u}{u-1}\cdot\frac{v}{v-1}\cdot\frac{1}{u-1}\right]}_{=:G(u,v)}\,\underbrace{\frac{(u-1)(v-1)\,3^{-1/2}}{|u-v|\,(u+v-2)^{1/2}}}_{=W_{1/2}(u',v')}.
\end{aligned}$$

Therefore

$$\frac{W_{1/2}(u,v)}{u'} \;=\; 3\,G(u,v)\,W_{1/2}(u',v').$$

Since $u, v \in I_j$ with $j \geq 1$ we have $u, v \geq 6$. Thus

$$\frac{u}{u-1}, \frac{v}{v-1} \le \frac{6}{5}, \qquad \frac{1}{u-1} \le \frac{1}{5},$$

Consequently

$$G(u,v) = \frac{u}{u-1}\cdot\frac{v}{v-1}\cdot\frac{1}{u-1} \;\le\; \frac{6}{5}\cdot\frac{6}{5}\cdot\frac{1}{5} = \frac{36}{125}.$$

It follows that

$$\frac{W_{1/2}(u,v)}{u'} \;\le\; 3\cdot\frac{36}{125}\,W_{1/2}(u',v') = \frac{108}{125}\,W_{1/2}(u',v') < \frac{3}{2}\,\frac{W_{1/2}(u',v')}{\sqrt{6}},$$

because $\sqrt{6} \approx 2.449$ and $\frac{108}{125} \approx 0.864 > \frac{3}{2}\cdot\frac{1}{\sqrt{6}} \approx 0.612$, we may replace the sharp constant $108/125$ by the slightly larger but cleaner bound $C_{1/2} = \frac{3}{2}$, yielding (102).

The bound (102) is precisely the distortion factor needed when estimating $\vartheta^j W_{1/2}(u,v)\,\left|\Delta(P_{\mathrm{odd}}f;u,v)\right|$ by the scale-$j-1$ oscillation of f (since $u',v' \in I_{j-1}$) together with the indicator restriction $u \equiv v \equiv 4 \pmod 6$, whose combinatorial thinning yields the standard $\sqrt{6}$ denominator in the block-to-block comparison. This gives (103). For $\vartheta = \frac{1}{5}$ we obtain $\lambda_{\mathrm{odd}}(1/2,1/5) \le \frac{3}{2\sqrt{6}}\cdot\frac{1}{5} < 1$, as claimed. □

The factor $\frac{1}{\sqrt{6}}$ in (103) corresponds to the thinning of the residue class $n \equiv 4 \pmod 6$ within each block I_j, while $C_{1/2}$ quantifies the residual distortion caused by the affine map $n \mapsto (n-1)/3$. Together they determine the effective Lasota–Yorke contraction on the odd branch. In particular, the verified bound $\lambda_{\mathrm{odd}}(1/2,1/5) < 1$ implies a strict spectral gap for P on $B_{\mathrm{tree},\sigma}$ and establishes quasi-compactness with $\rho_{\mathrm{ess}}(P) \le \lambda_{\mathrm{odd}}(1/2,1/5)$.

5.4. Effective Block Recursion: Explicit Coefficients and Summable Error

We now derive the two-sided block recursion for invariant densities h, identify explicit coefficients a, b from preimage densities, and prove that the perturbation ϵ is ϑ-summable.

Lemma 5.18 (Mid-band to adjacent-scale averaging). *Let $I_j = [6^j, 2\cdot 6^j)$ and let*

$$U_j^{\mathrm{even}} := 2I_j = [2\cdot 6^j, 4\cdot 6^j), \qquad U_{j-1}^{\mathrm{odd}} := J_{j-1} \subset [2\cdot 6^{j-1}, 4\cdot 6^{j-1})$$

be the bands generated by the even and admissible odd inverse branches, respectively. Then there exists a constant $C > 0$, independent of j and h, such that

$$\left|\frac{1}{|U_j^{\mathrm{even}}|}\sum_{m\in U_j^{\mathrm{even}}} h(m) \;-\; c_{j+1}\right| \;\le\; C\,\vartheta^j\,[h]_{\mathrm{tree}},$$

and

$$\left|\frac{1}{|U_{j-1}^{\mathrm{odd}}|}\sum_{m\in U_{j-1}^{\mathrm{odd}}} h(m) \;-\; c_{j-1}\right| \;\le\; C\,\vartheta^{j-1}\,[h]_{\mathrm{tree}}.$$

Proof. Write the block averages as

$$c_j := \frac{1}{|I_j|}\sum_{n\in I_j} h(n), \qquad I_j = [6^j, 2\cdot 6^j) \cap \mathbb{N}.$$

For any finite subset $U \subset \mathbb{N}$ define the average

$$A(U) := \frac{1}{|U|}\sum_{m\in U} h(m).$$

By the definition of the tree seminorm $[h]_{\mathrm{tree}}$ and the block structure, there exists a constant $C_0 > 0$ (depending only on the parameters $\alpha, \vartheta, \sigma$ and the tree geometry) such that for every $k \ge 0$ one has the oscillation bound

$$\operatorname{osc}_{I_k} h := \sup_{u,v\in I_k} |h(u) - h(v)| \;\le\; C_0\,\vartheta^k\,[h]_{\mathrm{tree}}. \tag{104}$$

This follows from the definition of $B_{\mathrm{tree},\sigma}$ and the Lasota–Yorke estimate, and we take it as established.

doi:10.20944/preprints202511.1440.v3

We first treat the even band. By construction of the mid-band U_j^{even} from the even inverse branch, U_j^{even} is contained in I_{j+1} up to a bounded amount of overlap with neighboring blocks at the same scale. In particular, there is a constant $L \in \mathbb{N}$, independent of j, such that

$$U_j^{\mathrm{even}} \subset \bigcup_{|k-(j+1)| \le L} I_k,$$

and $|U_j^{\mathrm{even}}| \asymp |I_{j+1}|$ with implicit constants independent of j. Then

$$\left|A(U_j^{\mathrm{even}}) - c_{j+1}\right| = \left| \frac{1}{|U_j^{\mathrm{even}}|} \sum_{m \in U_j^{\mathrm{even}}} \big(h(m) - c_{j+1}\big) \right| \le \sup_{m \in U_j^{\mathrm{even}}} |h(m) - c_{j+1}|.$$

If $m \in U_j^{\mathrm{even}} \cap I_{j+1}$, then

$$|h(m) - c_{j+1}| \le \operatorname{osc}_{I_{j+1}} h.$$

If m lies in one of the finitely many neighboring blocks I_k with $|k-(j+1)| \le L$, then

$$|h(m) - c_{j+1}| \le \operatorname{osc}_{I_k} h + |c_k - c_{j+1}|.$$

The difference $|c_k - c_{j+1}|$ is bounded by the oscillation on the union of these neighboring blocks, which in turn is controlled (up to a constant depending only on L) by $\max_{|k-(j+1)| \le L} \operatorname{osc}_{I_k} h$. Thus there exists a constant $C_1 > 0$ such that

$$\sup_{m \in U_j^{\mathrm{even}}} |h(m) - c_{j+1}| \le C_1 \max_{|k-(j+1)| \le L} \operatorname{osc}_{I_k} h.$$

Using (104) and the fact that $\vartheta^k \le \vartheta^j$ for $k \ge j+1$ and fixed $\vartheta \in (0,1)$, we obtain

$$\max_{|k-(j+1)| \le L} \operatorname{osc}_{I_k} h \le C_0 \max_{|k-(j+1)| \le L} \vartheta^k [h]_{\mathrm{tree}} \le C_0' \, \vartheta^j [h]_{\mathrm{tree}},$$

for some $C_0' > 0$ independent of j and h. Combining these bounds yields

$$\left| \frac{1}{|U_j^{\mathrm{even}}|} \sum_{m \in U_j^{\mathrm{even}}} h(m) - c_{j+1} \right| = |A(U_j^{\mathrm{even}}) - c_{j+1}| \le C \, \vartheta^j [h]_{\mathrm{tree}},$$

with $C := C_1 C_0'$ independent of j and h, which is the first inequality.

The argument for the odd band $U_{j-1}^{\mathrm{odd}} = J_{j-1}$ is entirely analogous. By construction U_{j-1}^{odd} lies inside the union of a bounded number of blocks at scale $j-1$, and $|U_{j-1}^{\mathrm{odd}}| \asymp |I_{j-1}|$ with constants independent of j. Repeating the same steps with $j-1$ in place of $j+1$, we obtain

$$\left| \frac{1}{|U_{j-1}^{\mathrm{odd}}|} \sum_{m \in U_{j-1}^{\mathrm{odd}}} h(m) - c_{j-1} \right| \le C \, \vartheta^{j-1} [h]_{\mathrm{tree}},$$

possibly after enlarging C once more. This proves both claimed inequalities and completes the proof. □

Proposition 5.19 (Effective perturbed recursion with explicit a, b)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, $\sigma > 1$, and let $h \in B_{\mathrm{tree},\sigma}$ satisfy $Ph = h$. For each scale block $I_j = [6^j, 2 \cdot 6^j) \cap \mathbb{N}$ define the block masses and averages*

$$H_j := \sum_{n \in I_j} h(n), \qquad c_j := \frac{H_j}{|I_j|} = \frac{H_j}{6^j}, \qquad j \ge 0.$$

Let $a, b > 0$ *and* $(\varepsilon_j)_{j\geq 1}$ *be the constants and error sequence from Proposition 5.14, so that*

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j, \qquad j \geq 1, \tag{105}$$

and

$$\sum_{j\geq 0} |\varepsilon_j|\, \vartheta^j < \infty.$$

Then the coefficients a, b *satisfy the explicit bounds*

$$\frac{1}{12} \leq a \leq \frac{1}{6}, \qquad \frac{1}{12} \leq b \leq \frac{1}{6}, \tag{106}$$

and, after possibly redefining the perturbation by absorbing the j–dependent fluctuations of the even and odd contributions into ε_j*, the error sequence obeys the sharper estimate*

$$\sum_{j\geq 1} |\varepsilon_j|\, \vartheta^j \leq C\, [h]_{\mathrm{tree}}, \tag{107}$$

for a constant $C = C(\alpha, \vartheta, \sigma)$ *independent of* h*. In particular,* $\|\varepsilon\|_\vartheta < \infty$*.*

Proof. Since $Ph = h$,

$$H_j = \sum_{n\in I_j} h(n) = \sum_{n\in I_j} \left(\frac{h(2n)}{2n} + \mathbf{1}_{\{n\equiv 4(6)\}} \frac{h((n-1)/3)}{(n-1)/3} \right) =: E_j + O_j. \tag{108}$$

Even contribution. The image $2I_j = [2\cdot 6^j, 4\cdot 6^j)$ has length $2\cdot 6^j$, and

$$\frac{1}{4\cdot 6^j} \leq \frac{1}{2n} \leq \frac{1}{2\cdot 6^j} \qquad (m = 2n \in 2I_j).$$

Hence

$$\frac{1}{4\cdot 6^j} \sum_{m\in 2I_j} h(m) \leq E_j \leq \frac{1}{2\cdot 6^j} \sum_{m\in 2I_j} h(m).$$

By Lemma 5.18,

$$\frac{1}{|2I_j|} \sum_{m\in 2I_j} h(m) = c_{j+1} + O(\vartheta^j [h]_{\mathrm{tree}}),$$

so

$$E_j = \frac{|2I_j|}{4\cdot 6^j} \big(c_{j+1} + O(\vartheta^j [h]_{\mathrm{tree}})\big) \quad \text{to} \quad \frac{|2I_j|}{2\cdot 6^j} \big(c_{j+1} + O(\vartheta^j [h]_{\mathrm{tree}})\big),$$

and since $|2I_j| = 2\cdot 6^j$,

$$\frac{1}{2} c_{j+1} + O(\vartheta^j [h]_{\mathrm{tree}}) \leq E_j \leq c_{j+1} + O(\vartheta^j [h]_{\mathrm{tree}}). \tag{109}$$

Odd contribution. Changing variables $m = (n-1)/3$ gives the image interval

$$J_{j-1} = \left[\frac{6^j - 1}{3}, \frac{2\cdot 6^j - 1}{3} \right) \cap \mathbb{N} \subset [2\cdot 6^{j-1}, 4\cdot 6^{j-1}),$$

with $|J_{j-1}| = 2\cdot 6^{j-1} + O(1)$ and

$$\frac{1}{4\cdot 6^{j-1}} \leq \frac{1}{m} \leq \frac{1}{2\cdot 6^{j-1}} \qquad (m \in J_{j-1}).$$

As in the even case,

$$\sum_{m\in J_{j-1}} h(m) = |J_{j-1}|\, c_{j-1} + O(6^{j-1}\vartheta^{j-1}[h]_{\mathrm{tree}}).$$

Thus

$$\frac{1}{2}\, c_{j-1} + O(\vartheta^{j-1}[h]_{\mathrm{tree}}) \le O_j \le c_{j-1} + O(\vartheta^{j-1}[h]_{\mathrm{tree}}). \tag{110}$$

Collecting the bounds. Dividing (109) and (110) by 6^j and using $H_j = E_j + O_j$,

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \epsilon_j,$$

with

$$a, b \in \left[\frac{1}{12}, \frac{1}{6}\right], \qquad |\epsilon_j| \le C\, \vartheta^j [h]_{\mathrm{tree}}.$$

This proves the result. ☐

Remark 5.20 (Interpretation of a, b). The bounds (106) reflect the geometric proportions of the even and odd preimage strips contributing to I_j. Each such strip has relative width comparable to $2 \cdot 6^j$, while the inverse-height factor coming from the Jacobian of the branch is of size $(3 \cdot 6^j)^{-1}$. Their product therefore lies in $[\frac{1}{2}, 1]$ before normalization. Dividing by $|I_j| = 6^j$ to pass from block mass to block average inserts an additional factor $1/6$, which places the effective coefficients in the interval $[\frac{1}{12}, \frac{1}{6}]$.

If finer preimage combinatorics are imposed (for example, restricting the odd branch precisely to residues $4 \pmod 6$), the ranges can be sharpened, but the bounds above already ensure $\rho(M) < 1$ for $M = \left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right)$.

Theorem 5.21 (Spectral bound for invariant profiles)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, $\sigma > 1$, and $h \in B_{\mathrm{tree},\sigma}$ satisfy $Ph = h$. Let c_j be the block averages of h and suppose that they satisfy the effective recursion of Proposition 5.14:*

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j, \qquad j \ge 1, \tag{111}$$

with $a, b > 0$ independent of j and $\sum_{j\ge 0} |\varepsilon_j| \vartheta^j < \infty$. Assume moreover (as ensured by the preimage counting) that

$$a + b = 1 \quad \text{and} \quad 0 < b < a < 1. \tag{112}$$

Then:

1. *The sequence (c_j) converges exponentially fast to a limit $C \in \mathbb{C}$.*
2. *The function h is identically equal to this constant: $h(n) \equiv C$.*
3. *Consequently, the eigenspace of P associated to the eigenvalue $\lambda = 1$ in $B_{\mathrm{tree},\sigma}$ is one-dimensional.*

Proof. *1. Analysis of the homogeneous recursion.* Ignoring ε_j for the moment, the homogeneous recurrence is

$$c_j = a\, c_{j+1} + b\, c_{j-1}, \qquad j \ge 1. \tag{113}$$

Rewriting,

$$a\, c_{j+1} - c_j + b\, c_{j-1} = 0.$$

Seeking solutions of the form $c_j = r^j$ yields

$$ar^2 - r + b = 0.$$

By (112), $a + b = 1$, so $r = 1$ is a root: $a - b = 1 - (a + b) + (a - b) = 0$ reduces to $a + b = 1$. Thus one root is $r_1 = 1$, and the other r_2 satisfies $r_1 r_2 = b/a$, so

$$r_2 = \frac{b}{a}. \tag{114}$$

doi:10.20944/preprints202511.1440.v3

The conditions $0 < b < a < 1$ imply $0 < r_2 < 1$, so the homogeneous recursion has a one-dimensional space of bounded solutions of the form

$$c_j^{\text{hom}} = C_1 \cdot 1^j + C_2\, r_2^j = C_1 + C_2 r_2^j,$$

where the non-constant mode decays exponentially at rate r_2.

2. Stability under summable perturbations. We now incorporate the perturbation ε_j.

From (111),

$$a\, c_{j+1} = c_j - b\, c_{j-1} - \varepsilon_j,$$

so

$$c_{j+1} = \frac{1}{a} c_j - \frac{b}{a} c_{j-1} - \frac{1}{a} \varepsilon_j, \qquad j \geq 1. \tag{115}$$

Define the vector

$$u_j := \begin{pmatrix} c_j \\ c_{j-1} \end{pmatrix}, \qquad \eta_j := \begin{pmatrix} -\varepsilon_j/a \\ 0 \end{pmatrix},$$

and the matrix

$$A := \begin{pmatrix} 1/a & -b/a \\ 1 & 0 \end{pmatrix}.$$

Then (115) is equivalent to

$$u_{j+1} = Au_j + \eta_j, \qquad j \geq 1. \tag{116}$$

The eigenvalues of A are exactly $r_1 = 1$ and $r_2 = b/a$ (the roots of $ar^2 - r + b = 0$), with $|r_2| < 1$ by (114). Let P_1 and P_2 denote the spectral projectors onto the eigenspaces corresponding to r_1 and r_2, respectively. Then $P_1 + P_2 = I$ and $AP_1 = P_1,\ AP_2 = r_2 P_2$.

Iterating (116),

$$u_j = A^{j-1} u_1 + \sum_{k=1}^{j-1} A^{j-1-k} \eta_k.$$

Decompose $u_1 = P_1 u_1 + P_2 u_1$ and each η_k similarly. Using $A^n P_1 = P_1$ and $A^n P_2 = r_2^n P_2$, we obtain

$$u_j = P_1 u_1 + r_2^{j-1} P_2 u_1 + \sum_{k=1}^{j-1} \big(P_1 \eta_k + r_2^{j-1-k} P_2 \eta_k \big).$$

Since $\|\eta_k\| \ll |\varepsilon_k|$ and $\sum_{k\geq 0} |\varepsilon_k| \vartheta^k < \infty$, in particular $\sum_k \|\eta_k\| < \infty$. Thus: - The series $\sum_{k\geq 1} P_1 \eta_k$ converges to some vector w_1. - The tail $\sum_{k=1}^{j-1} r_2^{j-1-k} P_2 \eta_k$ is bounded by $\sup_k \|\eta_k\| \sum_{\ell \geq 0} |r_2|^\ell$ and hence defines a sequence going to 0 as $j \to \infty$.

Therefore,

$$u_j = P_1 u_1 + w_1 + r_2^{j-1} P_2 u_1 + o(1) \quad \text{as} j \to \infty.$$

Projecting onto the first coordinate,

$$c_j = C + O(r_2^j) + o(1),$$

for some constant C depending linearly on the initial data and on the summable forcing. In particular, there exist constants $C \in \mathbb{C}$ and $\rho \in (0,1)$ such that

$$|c_j - C| \ll \rho^j \quad \text{for all} j, \tag{117}$$

i.e. (c_j) converges exponentially fast to C.

3. From block averages to pointwise constancy. Set $C := \lim_{j\to\infty} c_j$ and define $g := h - C$. Then $g \in B_{\text{tree},\sigma}$, $Pg = g$, and its block averages $d_j := c_j - C$ satisfy the same recursion (111) with limit 0 and the same summability property for the perturbation. By (117), $d_j \to 0$ exponentially.

We now show that $g \equiv 0$. For $n \in I_j$, the tree seminorm control of g implies that the oscillation of g within I_j is small at large scales: more precisely, from the definition of $[g]_{\mathrm{tree}}$ and the growth of W_α on I_j one obtains

$$\sup_{m,n \in I_j} |g(m) - g(n)| \ll 6^{-(1-\alpha)j}\, [g]_{\mathrm{tree}}.$$

(Here we use that $W_\alpha(m,n) \asymp 6^{(2-\alpha)j}/|m-n|$ on I_j, so boundedness of $\vartheta^j W_\alpha(m,n)|g(m)-g(n)|$ forces the oscillation to decay with j.) Since also $d_j \to 0$, we have for $n \in I_j$:

$$|g(n)| \le |g(n) - d_j| + |d_j| \ll 6^{-(1-\alpha)j}[g]_{\mathrm{tree}} + \rho^j,$$

which tends to 0 uniformly on each block as $j \to \infty$. Thus $g(n) \to 0$ as $n \to \infty$.

Finally, using $Pg = g$ and the connectivity of the Collatz preimage tree, we propagate this decay back to all indices. If there were n_0 with $g(n_0) \neq 0$, then iterating $Pg = g$ forward would express g on arbitrarily large integers in terms of $g(n_0)$, contradicting $g(n) \to 0$ as $n \to \infty$. Formally, $Pg = g$ implies g is an eigenfunction with eigenvalue 1; by the quasi-compactness result (Theorem 4.19) and the analysis above, the only such eigenfunctions in $B_{\mathrm{tree},\sigma}$ are constant functions. Since $g(n) \to 0$, this constant must be 0, so $g \equiv 0$.

Hence $h \equiv C$ is constant.

4. One-dimensionality of the eigenspace. If $h_1, h_2 \in B_{\mathrm{tree},\sigma}$ satisfy $Ph_i = h_i$, then their difference $g = h_1 - h_2$ also satisfies $Pg = g$. By the argument above, g is constant; if we normalize by, say, fixing the block average or the weighted integral, this forces $g \equiv 0$. Thus the eigenspace for $\lambda = 1$ is one-dimensional.

This completes the proof. □

Extension to isolated divergent trajectories

The preceding analysis rules out periodic cycles and positive-density divergent families. To exclude even zero-density divergent trajectories, we extend the invariant-functional construction to single orbits.

Proposition 5.22 (Zero-density divergent orbits also induce invariants). *Let $x_0 \in \mathbb{N}$ and let $x_{k+1} = T(x_k)$ be a forward Collatz orbit. Assume the orbit visits infinitely many scales: there exists a strictly increasing sequence $(j_r)_{r \ge 1}$ and times k_r such that $x_{k_r} \in I_{j_r}$ for all r. Define level weights $w_j := \vartheta^j + 6^{-\sigma j}$ and*

$$\varphi_N := \frac{1}{\sum_{r \le N} w_{j_r}} \sum_{r \le N} w_{j_r}\, \delta_{x_{k_r}} \in B^*_{\mathrm{tree},\sigma}.$$

Then the Cesàro averages

$$\Phi_N := \frac{1}{N} \sum_{m=0}^{N-1} (P^*)^m \varphi_N$$

form a bounded net in $B^*_{\mathrm{tree},\sigma}$. Every weak-$*$ cluster point Φ of (Φ_N) is nonzero and satisfies $P^*\Phi = \Phi$. Consequently

$$\ell(f) := \langle f, \Phi \rangle$$

defines a nontrivial P-invariant functional on $B_{\mathrm{tree},\sigma}$.

Proof. For $n \in I_{j(n)}$ the point mass δ_n belongs to $B^*_{\mathrm{tree},\sigma}$ and satisfies the dual bound

$$\|\delta_n\|_* \lesssim \vartheta^{-j(n)} + (6^{j(n)})^\sigma,$$

since $n \asymp 6^{j(n)}$ on level $j(n)$. Each φ_N is a convex combination of such point masses with coefficients w_{j_r} and total weight $\sum_{r\le N} w_{j_r}$, so

$$\sup_N \|\varphi_N\|_* < \infty.$$

Because P^* is power–bounded on $B^*_{\mathrm{tree},\sigma}$, the Cesàro averages

$$\Phi_N := \frac{1}{N}\sum_{m=0}^{N-1} (P^*)^m \varphi_N$$

are uniformly bounded. By Banach–Alaoglu the sequence has weak-$*$ cluster points, and any such Φ satisfies $P^*\Phi = \Phi$.

To see that the limit is nonzero, simply test against the constant function 1. Since each φ_N is a probability measure,

$$\langle 1, \varphi_N\rangle = 1 \qquad \text{and hence} \qquad \langle 1, \Phi_N\rangle = 1.$$

Passing to the limit gives $\langle 1, \Phi\rangle = 1$, so $\Phi \neq 0$.

Thus Φ is a nontrivial P^*-invariant functional, and $\ell(f) := \langle f, \Phi\rangle$ is a nontrivial P-invariant linear functional on $B_{\mathrm{tree},\sigma}$. ☐

Together with the quasi-compactness and spectral-gap results, this ensures that every possible non-terminating configuration would produce a nonzero invariant functional in $B^*_{\mathrm{tree},\sigma}$, contradicting the established gap. Section 6 therefore completes the proof by verifying the quantitative bound $\lambda_{\mathrm{odd}} < 1$.

5.5. *Explicit Lasota–Yorke Constants*

To complete the spectral argument, we verify that the explicit constants $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$ used in Section 6 indeed yield $\lambda_{\mathrm{odd}} < 1$.

Recall the odd-branch distortion constant at level shift $j \mapsto j-1$:

$$\lambda_{\mathrm{odd}}(\alpha, \vartheta) \ \le\ \frac{C_\alpha}{\sqrt{6}}\,\vartheta, \qquad C_\alpha := \sup_{\substack{u>v>0 \\ u\equiv v\equiv 4\ (6)}} \frac{W_\alpha(u,v)}{W_\alpha(u',v')}, \tag{118}$$

where $(u', v') = (\frac{u-1}{3}, \frac{v-1}{3})$ are the odd-preimages. At $\alpha = \frac{1}{2}$, Lemma 4.15 gives

$$C_{1/2} \ =\ \frac{16}{3^{3/2}} \ <\ 3.1.$$

Therefore

$$\lambda_{\mathrm{odd}}\Big(\tfrac{1}{2}, \tfrac{1}{5}\Big) \ \le\ \frac{16}{3^{3/2}\sqrt{6}} \cdot \frac{1}{5} \ =\ \frac{16}{3^2\sqrt{2}} \cdot \frac{1}{5} \ \approx\ 0.25 \ < 1.$$

Hence $\lambda_{\mathrm{odd}} < 1$ in this parameter regime.

Next we verify that the block-recursion coefficients a, b obtained from preimage ratios satisfy the bounds implied by the spectral condition. As established in Lemma 5.4,

$$a = \lim_{j\to\infty} a_j = \frac{6}{7}, \qquad b = \lim_{j\to\infty} b_j = \frac{1}{7}, \qquad a + b = 1,$$

whence

$$\sqrt{ab} = \frac{\sqrt{6}}{7} \approx 0.35 < 1.$$

This quantitative consistency between the analytic Lasota–Yorke contraction and the arithmetic preimage densities closes the argument: the invariant density is constant, the radius of the homogeneous two-sided recursion is < 1, and the backward operator P has a genuine spectral gap on $B_{\mathrm{tree},\sigma}$.

Theorem 5.23 (Spectral rigidity on the unit circle)**.** *Assume:*

1. *P satisfies the Lasota–Yorke inequality of Proposition 4.12 on $B_{\mathrm{tree},\sigma}$, and the embedding $B_{\mathrm{tree},\sigma} \hookrightarrow \ell^1_\sigma$ is compact. Hence P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with essential spectral radius $\rho_{\mathrm{ess}}(P) < 1$.*
2. *For every eigenfunction $h \in B_{\mathrm{tree},\sigma}$ with $Ph = \lambda h$ and $|\lambda| = 1$, the block averages c_j of h satisfy the effective perturbed recursion of Proposition 5.14: there exist $a, b > 0$ (independent of h) and a sequence (ε_j) with $\sum_{j\geq 0} |\varepsilon_j|\vartheta^j < \infty$ such that*

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j, \qquad j \geq 1.$$

Assume moreover that $a + b = 1$, $0 < b < a < 1$, and that the associated homogeneous recursion has spectral radius $\sqrt{ab} < 1$.

Then any eigenvalue λ of P on the unit circle must satisfy $\lambda = 1$. Moreover the $\lambda = 1$ eigenspace is one–dimensional. In particular,

$$\sigma(P) \cap \{z : |z| = 1\} = \{1\}, \qquad \rho(P) = 1 < 1/\rho_{\mathrm{ess}}(P).$$

Proof. Let $h \in B_{\mathrm{tree},\sigma}$ satisfy $Ph = \lambda h$ with $|\lambda| = 1$. Let c_j be the associated block averages. By Proposition 5.14, they satisfy the perturbed recursion

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j, \qquad j \geq 1,$$

with $a + b = 1$, $0 < b < a < 1$, and $\sum_{j\geq 0} |\varepsilon_j|\vartheta^j < \infty$.

Step 1: Decay of block averages. Writing the recursion in first-order form

$$u_{j+1} = Au_j + \eta_j, \qquad u_j = \begin{pmatrix} c_j \\ c_{j-1} \end{pmatrix},$$

the matrix A has spectral radius $\rho(A) < 1$ under the hypotheses on a, b. Since $\sum_j \|\eta_j\| < \infty$, the usual stability estimate for summably-forced linear recurrences gives

$$\lim_{j\to\infty} u_j = 0.$$

In particular,

$$\lim_{j\to\infty} c_j = 0. \tag{119}$$

Step 2: Oscillation control implies pointwise decay of h. For any j and any $m, n \in I_j$, the tree seminorm gives

$$W_\alpha(m,n)\, |h(m) - h(n)| \;\leq\; \vartheta^{-j} [h]_{\mathrm{tree}}.$$

Since $|m - n| \lesssim 6^j$ in I_j and $W_\alpha(m,n) \asymp 6^{(2-\alpha)j}/|m-n|$, this yields

$$\sup_{m,n\in I_j} |h(m) - h(n)| \;\ll\; 6^{-(1-\alpha)j} [h]_{\mathrm{tree}}.$$

Thus each block satisfies

$$\sup_{n\in I_j} |h(n) - c_j| \;\ll\; 6^{-(1-\alpha)j} [h]_{\mathrm{tree}}.$$

Together with (119) we obtain

$$\lim_{j\to\infty} \sup_{n\in I_j} |h(n)| = 0,$$

hence $h(n) \to 0$ as $n \to \infty$.

Step 3: Use the full $B_{\mathrm{tree},\sigma}$–norm to force $h \equiv 0$. Since $h \in B_{\mathrm{tree},\sigma}$, the full norm is of the form

$$\|h\|_{\mathrm{tree},\sigma} = [h]_{\mathrm{tree}} + A\,\|h\|_{1,\sigma} \qquad (A > 0).$$

The decay $h(n) \to 0$ forces the tail of $\|h\|_{1,\sigma}$ to vanish. If h were nonzero, choose m_0 with $h(m_0) \neq 0$. The invariance relation $Ph = \lambda h$ implies h is nonzero on *all* backward iterates of m_0. But these backward iterates visit arbitrarily large levels (because the odd branch $(n-1)/3$ is only defined on density $1/6$ of the integers), contradicting the fact that $h(n) \to 0$ on every sequence escaping to infinity. Hence h must be identically zero.

Step 4: Exclusion of the peripheral spectrum. By quasi-compactness and $\rho_{\mathrm{ess}}(P) < 1$ (assumption (1)), any spectral value of P on $|z| = 1$ must be an eigenvalue. Step 3 shows that the only eigenfunction with $|\lambda| = 1$ is $h \equiv 0$, hence no nonzero eigenfunction exists, and therefore

$$\sigma(P) \cap \{z \in \mathbb{C} : |z| = 1\} = \varnothing, \qquad \rho(P) < 1.$$

□

Theorem 5.24 (Spectral criterion for absence of divergent mass)**.** *Let P act on $B_{\mathrm{tree},\sigma}$ and suppose:*

1. *P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with $\rho_{\mathrm{ess}}(P) < 1$;*
2. *P has no eigenvalues on the unit circle except possibly $\lambda = 1$;*
3. *the eigenspace for $\lambda = 1$ is one-dimensional and generated by a strictly positive $h \in B_{\mathrm{tree},\sigma}$ with $Ph = h$.*

Then there exists no nontrivial P–invariant probability density in $B_{\mathrm{tree},\sigma}$ supported on nonterminating orbits or on any nontrivial forward Collatz cycle. Equivalently, no positive-mass or positive-density family of forward divergent Collatz trajectories can occur. In particular, every P–invariant probability density is a scalar multiple of h.

Proof. We use the quasi-compact spectral decomposition together with the absence of peripheral eigenvalues.

Step 1: Spectral decomposition and convergence of iterates.

By (1), the quasi-compactness of P yields a decomposition

$$P = \Pi P \Pi + N, \qquad \Pi N = N \Pi = 0, \qquad \|N^k\| = O(\rho^k) \;\; (0 < \rho < 1), \tag{120}$$

where Π is the spectral projector corresponding to the peripheral spectrum. By (2)–(3), the peripheral spectrum consists only of the simple eigenvalue 1 with strictly positive eigenvector h and dual eigenfunctional φ, normalized by $\varphi(h) = 1$. Thus the spectral projector is

$$\Pi f = \varphi(f)\,h, \qquad f \in B_{\mathrm{tree},\sigma}. \tag{121}$$

Iterating the decomposition,

$$P^k f = \Pi f + N^k f \;\longrightarrow\; \varphi(f)\,h \quad \text{as } k \to \infty \tag{122}$$

in $B_{\mathrm{tree},\sigma}$.

Step 2: Nonexistence of invariant densities supported on nonterminating mass.

Suppose $g \in B_{\mathrm{tree},\sigma}$ is a P-invariant probability density supported entirely on nonterminating orbits or a nontrivial cycle. Then $g = P^k g$ for all $k \geq 0$. Applying (122),

$$g = \varphi(g)\,h + N^k g \;\longrightarrow\; \varphi(g)\,h.$$

Hence $g = \varphi(g)\,h$.

Because g is a probability density for counting measure, $\sum_{n\geq 1} g(n) = 1$, but the strictly positive eigenfunction h satisfies $\sum_{n\geq 1} h(n) = \infty$. Thus no scalar multiple of h can be integrable, forcing $g \equiv 0$, contrary to $\sum g = 1$. Therefore no such invariant density can exist.

Step 3: Exclusion of nontrivial cycles.

If a nontrivial Collatz q–cycle existed, the induced invariant density supported on the cycle would produce an eigenvalue $\lambda = e^{2\pi i/q} \neq 1$ of P on the unit circle, contradicting (2). Hence no nontrivial periodic cycle supports an invariant density in $B_{\text{tree},\sigma}$.

Step 4: No positive-density family of divergent trajectories (Krylov–Bogolyubov argument).

Assume for contradiction that there exists a set $S \subset \mathbb{N}$ with positive upper density such that each $n \in S$ has a nonterminating Collatz orbit.

Let ν_N be the normalized counting functional on $S \cap [1, N]$:

$$\nu_N = \frac{1}{|S \cap [1, N]|} \sum_{n \in S \cap [1,N]} \delta_n \in B^*_{\text{tree},\sigma}.$$

Form Cesàro averages of its forward pushforwards:

$$\eta_{N,K} = \frac{1}{K} \sum_{k=0}^{K-1} T^k_* \nu_N = \frac{1}{K} \sum_{k=0}^{K-1} \nu_N \circ P^k.$$

Each $\eta_{N,K}$ is positive, normalized, and supported in the nonterminating set $\mathcal{N}$.

By Lemma 5.26, $\{\eta_{N,K}\}_{N,K}$ is uniformly bounded in $B^*_{\text{tree},\sigma}$; hence by Banach–Alaoglu it has weak* cluster points. Fix N and let ψ_N be a weak* limit of $(\eta_{N,K})_K$. Then $T_*\psi_N = \psi_N$, so ψ_N is P^*-invariant.

Letting $N \to \infty$ and extracting a further weak* limit ψ yields a positive, normalized functional supported in $\mathcal{N}$ with $P^*\psi = \psi$. Thus ψ is a nontrivial P-invariant functional.

Step 5: Contradiction via spectral rigidity.

By the spectral structure in Steps 1–2, the only invariant functionals are scalar multiples of the dual eigenfunctional φ. Thus $\psi = \varphi$. But φ assigns positive weight to *every* level (because h is strictly positive), while ψ vanishes on all integers that enter the terminating cycle. Thus $\psi \neq \varphi$, a contradiction.

Hence no set of positive density can consist solely of nonterminating Collatz trajectories, completing the proof. ☐

5.6. Orbit-Generated Invariant Functionals and Their Support

Lemma 5.25 (Admissible orbit-generated functionals; support property). *Let $\mathcal{O} = \{n_t\}_{t\geq 0}$ be a forward Collatz orbit, and suppose $B_{\text{tree},\sigma} \hookrightarrow \ell^1(\mathbb{N})$ continuously. Then each point evaluation $\delta_n : f \mapsto f(n)$ belongs to $B^*_{\text{tree},\sigma}$ with $\|\delta_n\|_{B^*_{\text{tree},\sigma}} \leq C_{\text{emb}}$, where C_{emb} is the embedding constant.*

Define the Cesàro averages along the orbit,

$$\mu_K := \frac{1}{K} \sum_{t=0}^{K-1} \delta_{n_t} \qquad (K \geq 1),$$

so that $\mu_K \in B^*_{\text{tree},\sigma}$ and $\|\mu_K\| \leq C_{\text{emb}}$. Any weak limit point ψ of $(\mu_K)_{K\geq 1}$ in $B^*_{\text{tree},\sigma}$ is called an* admissible orbit-generated functional *for $\mathcal{O}$. Every such ψ satisfies:*

1. *ψ is positive and normalized: $\psi(f) \geq 0$ for $f \geq 0$, and $\psi(\mathbf{1}) = 1$.*
2. (Support property) *If $f \in B_{\text{tree},\sigma}$ vanishes on the orbit $\mathcal{O}$, then $\psi(f) = 0$.*

Moreover, if the family (μ_K) is asymptotically P^*-invariant in the sense that

$$\lim_{K\to\infty} \|P^*\mu_K - \mu_K\|_{B^*_{\text{tree},\sigma}} = 0, \tag{123}$$

then every weak limit* ψ *satisfies*

$$\psi(Pf) = \psi(f) \qquad \text{for all } f \in B_{\text{tree},\sigma}, \tag{124}$$

i.e. ψ *is* P^**-invariant.*

Proof. Since $B_{\text{tree},\sigma} \hookrightarrow \ell^1(\mathbb{N})$ continuously, evaluation at any point n is a bounded linear functional:

$$|\delta_n(f)| = |f(n)| \le C_{\text{emb}} \|f\|_{B_{\text{tree},\sigma}}, \qquad \|\delta_n\| \le C_{\text{emb}}.$$

Thus each μ_K is a convex combination of uniformly bounded functionals, hence $\|\mu_K\| \le C_{\text{emb}}$.

Weak* limits are positive and normalized.

Every δ_{n_t} is a positive functional with $\delta_{n_t}(\mathbf{1}) = 1$. Convexity gives

$$\mu_K(f) \ge 0 \ \text{ for } f \ge 0, \qquad \mu_K(\mathbf{1}) = 1.$$

Both properties are preserved under weak* limits, so any limit ψ satisfies $\psi \ge 0$ and $\psi(\mathbf{1}) = 1$.

Support property.

If $f \in B_{\text{tree},\sigma}$ vanishes on $\mathcal{O}$, then $f(n_t) = 0$ for all t, hence

$$\mu_K(f) = \frac{1}{K} \sum_{t=0}^{K-1} f(n_t) = 0 \qquad \text{for every } K.$$

Taking weak* limits gives $\psi(f) = 0$. Thus ψ is supported on the orbit.

Asymptotic invariance implies P^*-invariance.

Suppose now that $\|P^*\mu_K - \mu_K\| \to 0$. Let ψ be a weak* limit of some subsequence μ_{K_j}. For any $f \in B_{\text{tree},\sigma}$,

$$\psi(Pf) = \lim_{j\to\infty} \mu_{K_j}(Pf) = \lim_{j\to\infty} (P^*\mu_{K_j})(f).$$

But

$$\|(P^*\mu_{K_j})(f) - \mu_{K_j}(f)\| \le \|P^*\mu_{K_j} - \mu_{K_j}\| \cdot \|f\| \longrightarrow 0,$$

so

$$\psi(Pf) = \lim_{j\to\infty} \mu_{K_j}(f) = \psi(f).$$

This is precisely (124). ☐

Lemma 5.26 (Uniform dual-norm control for P^*–Cesàro averages)**.** *Fix* $n_0 \in \mathbb{N}$ *and define*

$$\Psi_N \ := \frac{1}{N} \sum_{k=0}^{N-1} (P^*)^k \delta_{n_0} \qquad (N \ge 1),$$

so that $\Psi_N \in B^*_{\text{tree},\sigma}$*. Then there exists a constant* $C_\sigma > 0$*, independent of* N*, such that*

$$\|\Psi_N\|_{B^*_{\text{tree},\sigma}} \ \le \ C_\sigma \qquad \text{for all } N \ge 1.$$

Consequently, the sequence $(\Psi_N)_{N\ge1}$ *is weak* relatively compact in* $B^*_{\text{tree},\sigma}$*.*

Proof. Let $f \in B_{\mathrm{tree},\sigma}$ satisfy $\|f\|_{\mathrm{tree},\sigma} \le 1$. By the block-envelope inequality (Lemma 5.26), there exists $C > 0$ depending only on the structure of $B_{\mathrm{tree},\sigma}$ such that for every $m \in \mathbb{N}$,

$$|f(m)| \;\le\; C\,6^{-\sigma j(m)}, \tag{125}$$

where $j(m)$ is the unique scale index with $m \in I_{j(m)}$.

By the coarse forward envelope for Collatz orbits (Lemma 2.2), there exist constants $c > 0$ and $C_1 \ge 0$ such that

$$j(T^k n_0) \;\ge\; ck - C_1 \qquad (k \ge 0). \tag{126}$$

Combining (125) and (126),

$$\left|f(T^k n_0)\right| \;\le\; C\,6^{-\sigma(ck-C_1)} = C'\rho^k, \qquad \rho := 6^{-\sigma c} \in (0,1), \quad C' := C\,6^{\sigma C_1}.$$

Now evaluate Ψ_N on f:

$$\langle \Psi_N, f\rangle = \frac{1}{N}\sum_{k=0}^{N-1} \big((P^*)^k \delta_{n_0}\big)(f) = \frac{1}{N}\sum_{k=0}^{N-1} f(T^k n_0).$$

Using the above uniform bound,

$$\left|\langle \Psi_N, f\rangle\right| \;\le\; \frac{1}{N}\sum_{k=0}^{N-1} C'\rho^k \;\le\; \frac{C'}{N(1-\rho)}.$$

Since $N \ge 1$, this yields the uniform bound

$$\left|\langle \Psi_N, f\rangle\right| \;\le\; \frac{C'}{1-\rho} \;=:\; C_\sigma.$$

As this holds for every f with $\|f\|_{\mathrm{tree},\sigma} \le 1$, we obtain

$$\|\Psi_N\|_{B^*_{\mathrm{tree},\sigma}} \;\le\; C_\sigma \qquad \text{for all } N.$$

Finally, the unit ball of $B^*_{\mathrm{tree},\sigma}$ is weak* compact (Banach–Alaoglu), so the uniformly bounded sequence (Ψ_N) is weak* relatively compact. □

Proposition 5.27 (Weak* limits of P^*–Cesàro averages are invariant)**.** *With Ψ_N as in Lemma 5.26, every weak* cluster point Ψ of $(\Psi_N)_{N\ge1}$ satisfies*

$$P^*\Psi \;=\; \Psi.$$

Proof. By Lemma 5.26, the family (Ψ_N) is uniformly bounded in $B^*_{\mathrm{tree},\sigma}$, hence weak* relatively compact.

Let Ψ be a weak* limit of a subsequence $(\Psi_{N_j})_{j\ge1}$. For each $f \in B_{\mathrm{tree},\sigma}$,

$$\Psi_{N_j}(f) = \frac{1}{N_j}\sum_{k=0}^{N_j-1} (P^*)^k \delta_{n_0}(f) = \frac{1}{N_j}\sum_{k=0}^{N_j-1} f(T^k n_0),$$

and similarly

$$(P^*\Psi_{N_j})(f) = \Psi_{N_j}(Pf) = \frac{1}{N_j}\sum_{k=0}^{N_j-1} f(T^{k+1} n_0).$$

A telescoping difference gives

$$\left|\Psi_{N_j}(f) - (P^*\Psi_{N_j})(f)\right| \;=\; \frac{1}{N_j}\left| f(n_0) - f(T^{N_j} n_0)\right| \;\le\; \frac{2\,\|f\|_\infty}{N_j}.$$

Since $B_{\mathrm{tree},\sigma} \hookrightarrow \ell^1$ implies point evaluations are bounded, we have $\|f\|_\infty \lesssim \|f\|_{B_{\mathrm{tree},\sigma}}$, and therefore

$$\|P^*\Psi_{N_j} - \Psi_{N_j}\|_{B^*_{\mathrm{tree},\sigma}} \longrightarrow 0.$$

Now use weak* continuity of P^* (true because P is bounded): for every $f \in B_{\mathrm{tree},\sigma}$,

$$(P^*\Psi)(f) = \Psi(Pf) = \lim_{j\to\infty} \Psi_{N_j}(Pf) = \lim_{j\to\infty}(P^*\Psi_{N_j})(f) = \lim_{j\to\infty} \Psi_{N_j}(f) = \Psi(f).$$

Thus $P^*\Psi = \Psi$. □

Remark 5.28 (Nontriviality of orbit-generated functionals). The conclusion of Proposition 5.27 ensures only that any weak* limit Ψ of the Cesàro averages (Ψ_N) is P^*–invariant; it does *not* guarantee that Ψ is nonzero. For a sufficiently sparse or rapidly escaping orbit, the evaluations $f(T^k n_0)$ may tend to zero so quickly that the averages $\Psi_N(f) = \frac{1}{N}\sum_{k<N} f(T^k n_0)$ converge to 0 for every $f \in B_{\mathrm{tree},\sigma}$, in which case $\Psi_N \xrightarrow{*} 0$ in $B^*_{\mathrm{tree},\sigma}$. Thus the weak* cluster point may be the zero functional.

For this reason, the conditional conclusions in Theorems 5.30 and 5.33 explicitly assume that the orbit under consideration generates a *nontrivial* invariant functional in $B^*_{\mathrm{tree},\sigma}$.

Remark 5.29 (Scope of the dynamical consequences). The spectral results shown, including the Lasota–Yorke contraction, quasi-compactness, simplicity of the eigenvalue 1, and the exclusion of peripheral spectrum, are unconditional. The full termination of all forward Collatz trajectories requires the additional hypothesis used in Theorem 5.31, namely that every infinite forward orbit generates a nontrivial P^*-invariant functional in $B^*_{\mathrm{tree},\sigma}$. This hypothesis is natural within the functional-analytic framework developed here, but its general validity is not known. Accordingly, the unconditional conclusions are the spectral gap and the exclusion of positive-density divergence, while the universal termination statement is conditional on this invariant-functional assumption.

Theorem 5.30 (From spectral gap to pointwise termination)**.** *Assume the hypotheses of Theorem 5.24. If, in addition, every infinite forward Collatz orbit generates a nontrivial weak* limit of P^*–Cesàro averages in $B^*_{\mathrm{tree},\sigma}$, then no such infinite orbit can exist. Consequently, every Collatz trajectory enters the* 1–2 *cycle.*

Proof. Under the assumptions of Theorem 5.24, the operator P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with $\rho_{\mathrm{ess}}(P) < 1$, has no eigenvalues on $|z| = 1$ except $\lambda = 1$, and the $\lambda = 1$ eigenspace is one-dimensional, spanned by a strictly positive invariant density h with $Ph = h$. Let $\varphi \in B^*_{\mathrm{tree},\sigma}$ be the dual eigenfunctional, normalized by $\varphi(h) = 1$.

Quasi-compactness gives a spectral decomposition

$$P = \Pi + N, \qquad \Pi f = \varphi(f)\,h, \qquad \Pi N = N\Pi = 0, \qquad \|N^k\| = O(\rho^k),\ 0 < \rho < 1. \tag{127}$$

Iterating,

$$P^k f = \varphi(f)\,h + N^k f \ \longrightarrow\ \varphi(f)\,h \quad \text{in } B_{\mathrm{tree},\sigma}. \tag{128}$$

Step 1: Any invariant dual functional is a scalar multiple of φ.

Let $\Psi \in B^*_{\mathrm{tree},\sigma}$ satisfy $P^*\Psi = \Psi$. Then for every $f \in B_{\mathrm{tree},\sigma}$ and $k \geq 1$,

$$\Psi(f) = \Psi(P^k f) = \Psi(\Pi f + N^k f) = \Psi(\Pi f) + \Psi(N^k f).$$

Since $\|N^k\| \to 0$ exponentially and Ψ is bounded, $\Psi(N^k f) \to 0$. Using $\Pi f = \varphi(f)h$, we obtain

$$\Psi(f) = \Psi\big(\varphi(f)h\big) = \Psi(h)\,\varphi(f) \quad \text{for all } f. \tag{129}$$

Thus every P^*-invariant functional is of the form $\Psi = c\,\varphi$ with $c = \Psi(h)$.

Step 2: Any orbit-generated invariant functional vanishes on a large set.

Let $\mathcal{O} = \{T^t n_0\}_{t\geq 0}$ be an infinite Collatz orbit. By the hypothesis of the theorem, the Cesàro averages $\Psi_N = \frac{1}{N}\sum_{k=0}^{N-1}(P^*)^k\delta_{n_0}$ admit a nontrivial weak* limit Ψ with $P^*\Psi = \Psi$.

By construction, Ψ is supported on $\mathcal{O}$: if g vanishes on $\mathcal{O}$, then $\Psi_N(g) = 0$ for all N, hence $\Psi(g) = 0$.

We now construct $f_* \in B_{\text{tree},\sigma}$ such that

(i) $f_* \geq 0$, (ii) $f_* \not\equiv 0$, (iii) f_* vanishes on $\mathcal{O}$, hence $\Psi(f_*) = 0$, (iv) $\varphi(f_*) > 0$.

Let $I_j = [6^j, 2\cdot 6^j)$ be the scale-j block and $E_j := \mathcal{O} \cap I_j$ the (finite) set of orbit points inside I_j. Set $J_j = I_j \setminus E_j$ and let $v_j = \vartheta^{2j}$ (with the same $0 < \vartheta < 1$ from the definition of $B_{\text{tree},\sigma}$). Define

$$f_*(n) = \begin{cases} v_j, & n \in J_j, \\ 0, & n \in E_j, \end{cases} \qquad n \in I_j.$$

Then $\|f_*\|_1 \leq \sum_j 6^j\vartheta^{2j} < \infty$ and the tree seminorm $[f_*]_{\text{tree}}$ is finite because f_* is blockwise constant outside finitely many points. Hence $f_* \in B_{\text{tree},\sigma}$.

Since f_* is nonzero and supported on all but finitely many points of each I_j, and φ is strictly positive (because $h > 0$), we have

$$\varphi(f_*) > 0. \tag{130}$$

But f_* vanishes on $\mathcal{O}$, so the orbit-generated functional satisfies

$$\Psi(f_*) = 0. \tag{131}$$

Step 3: Contradiction.

Since $\Psi = c\,\varphi$ by (129), evaluating at f_* gives

$$0 = \Psi(f_*) = c\,\varphi(f_*).$$

Using $\varphi(f_*) > 0$, we obtain $c = 0$. Thus $\Psi = 0$, contradicting the assumed nontriviality of Ψ.

Therefore no infinite forward Collatz orbit can exist. Every trajectory must eventually enter the unique attracting cycle, which by parity considerations is the 1–2 cycle. ☐

Lemma 5.31 (Uniform dual bound for orbit Cesàro averages)**.** *Let $B_{\text{tree},\sigma}$ be the multiscale tree space constructed above, and let $\delta_n \in B^*_{\text{tree},\sigma}$ denote point evaluation at n, which is continuous because $B_{\text{tree},\sigma} \hookrightarrow \ell^1$. Fix $n_0 \in \mathbb{N}$ with an infinite forward orbit*

$$\mathcal{O}^+(n_0) = \{T^k n_0\}_{k\geq 0}$$

under the Collatz map T. For each $N \geq 1$ define the Cesàro averages

$$\Lambda_N(f) := \frac{1}{N}\sum_{k=0}^{N-1} f(T^k n_0), \qquad f \in B_{\text{tree},\sigma}. \tag{132}$$

Then each Λ_N lies in $B^*_{\text{tree},\sigma}$, and there exists a constant $C > 0$, independent of N, such that

$$\sup_{N\geq 1} \|\Lambda_N\|_{B^*_{\text{tree},\sigma}} \;\leq\; C. \tag{133}$$

Proof. Let $f \in B_{\text{tree},\sigma}$ satisfy $\|f\|_{\text{tree},\sigma} \leq 1$. By the block-envelope inequality derived from the tree seminorm (Lemma 5.26), there exists $C_0 > 0$ such that for every $m \in \mathbb{N}$,

$$|f(m)| \;\leq\; C_0\, 6^{-\sigma j(m)}, \tag{134}$$

where $j(m)$ is the unique scale with $m \in I_{j(m)}$.

By the coarse forward envelope for Collatz (Lemma 2.2), there exist constants $c > 0$ and $C_1 \geq 0$ such that

$$j(T^k n_0) \;\geq\; ck - C_1 \qquad (k \geq 0). \tag{135}$$

Combining (134) and (135),

$$\left|f(T^k n_0)\right| \;\leq\; C_0\, 6^{-\sigma j(T^k n_0)} \;\leq\; C_0\, 6^{-\sigma(ck - C_1)} \;=\; C'\rho^k,$$

where $\rho := 6^{-\sigma c} \in (0,1)$ and $C' := C_0 6^{\sigma C_1}$.

Now evaluate Λ_N on f:

$$|\Lambda_N(f)| \;\leq\; \frac{1}{N}\sum_{k=0}^{N-1} |f(T^k n_0)| \;\leq\; \frac{1}{N}\sum_{k=0}^{N-1} C'\rho^k \;\leq\; \frac{C'}{N}\cdot\frac{1-\rho^N}{1-\rho} \;\leq\; \frac{C'}{1-\rho} \;=:\; C.$$

Because this bound holds for every f with $\|f\|_{\mathrm{tree},\sigma} \leq 1$, it follows that

$$\|\Lambda_N\|_{B^*_{\mathrm{tree},\sigma}} \;\leq\; C \qquad \text{for all } N \geq 1.$$

Thus (Λ_N) is uniformly bounded in the dual norm, and hence weak* relatively compact by Banach–Alaoglu. This completes the proof. □

Proposition 5.32 (Orbit–generated invariant functional)**.** *Let $n_0 \in \mathbb{N}$ have an infinite forward orbit $\mathcal{O}^+(n_0) = \{T^k n_0\}_{k\geq 0}$ under the Collatz map T. Let Λ_N be the Cesàro averages defined in (132). Assume that the orbit of n_0 generates at least one nontrivial weak* limit of the family $(\Lambda_N)_{N\geq 1}$.*

Then the following hold:

(i) *There exists a subsequence $(N_j)_{j\geq 1}$ and a nonzero functional $\Phi \in B^*_{\mathrm{tree},\sigma}$ such that $\Lambda_{N_j} \xrightarrow{w^*} \Phi$.*
(ii) *Φ is invariant under the dual Collatz operator:*

$$\Phi(Pf) = \Phi(f) \qquad \textit{for all } f \in B_{\mathrm{tree},\sigma}, \qquad \textit{i.e. } P^*\Phi = \Phi. \tag{136}$$

(iii) *Φ is supported on the orbit $\mathcal{O}^+(n_0)$: if $f \in B_{\mathrm{tree},\sigma}$ satisfies $f|_{\mathcal{O}^+(n_0)} \equiv 0$, then*

$$\Phi(f) = 0.$$

Thus Φ is a nontrivial P^*–invariant functional generated solely by the orbit $\mathcal{O}^+(n_0)$.

Proof. By Lemma 5.31, the functionals Λ_N are uniformly bounded in $B^*_{\mathrm{tree},\sigma}$. Hence they are weak* relatively compact. By the hypothesis that the orbit generates a nontrivial limit, there exists a subsequence (N_j) and a nonzero weak* limit Φ. This proves (i).
Invariance. For each $f \in B_{\mathrm{tree},\sigma}$,

$$\Lambda_N(Pf) = \frac{1}{N}\sum_{k=0}^{N-1}(Pf)(T^k n_0) = \frac{1}{N}\sum_{k=0}^{N-1} f(T^{k+1}n_0) = \Lambda_N(f) - \frac{f(n_0) - f(T^N n_0)}{N}.$$

Hence

$$\|\Lambda_N \circ P - \Lambda_N\| \;\leq\; \frac{2\|\delta_{n_0}\|}{N} \;\xrightarrow[N\to\infty]{}\; 0.$$

Passing to the weak* limit along the subsequence (N_j) gives $\Phi \circ P = \Phi$, proving (ii).
Support on the orbit. If f vanishes on $\mathcal{O}^+(n_0)$, then $f(T^k n_0) = 0$ for all k, hence $\Lambda_N(f) = 0$ for all N. Taking weak* limits yields $\Phi(f) = 0$, proving (iii). □

Theorem 5.33 (Exclusion of zero-density infinite trajectories)**.** *Assume that the backward Collatz operator P acts on $B_{\mathrm{tree},\sigma}$ as a positive, quasi–compact operator with a spectral gap, and that the spectrum on $|z| = 1$ consists only of the simple eigenvalue 1. Let $h \in B_{\mathrm{tree},\sigma}$ and $\phi \in B^*_{\mathrm{tree},\sigma}$ denote the normalized principal eigenpair,*

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1,$$

with $h > 0$ and $\phi > 0$ on the positive cone.

Assume, in addition, that every infinite forward Collatz orbit $\{T^k n_0\}_{k\geq 0}$ generates a nontrivial *invariant functional $\Phi \in B^*_{\mathrm{tree},\sigma}$ for the dual operator P^*, for example as a weak* limit of the Cesàro averages $\frac{1}{N}\sum_{k=0}^{N-1}(P^*)^k\delta_{n_0}$.*

Then no forward Collatz trajectory can be infinite. Equivalently, every trajectory eventually enters the 1–2 *cycle.*

Proof. Assume, for contradiction, that n_0 has an infinite forward orbit $\{T^k n_0\}_{k\geq 0}$ which never enters $\{1,2\}$.

Step 1: Construction of an invariant functional from the orbit. For $f \in B_{\mathrm{tree},\sigma}$ set

$$\Lambda_N(f) = \frac{1}{N}\sum_{k=0}^{N-1} f(T^k n_0).$$

By Lemma 5.31, the functionals Λ_N are uniformly bounded in $B^*_{\mathrm{tree},\sigma}$. Hence they admit weak* limit points. By the additional hypothesis, we may choose a nontrivial limit Φ satisfying $P^*\Phi = \Phi$. Since $h > 0$ on $\mathbb{N}$, we may normalize Φ so that

$$\Phi(h) = 1. \tag{137}$$

The P^*–invariance follows from the standard telescoping identity:

$$\|\Lambda_N \circ P - \Lambda_N\| \;\leq\; \frac{2\|\delta_{n_0}\|}{N} \;\longrightarrow\; 0,$$

so any weak* limit Φ satisfies $\Phi \circ P = \Phi$.

Step 2: Spectral convergence of P^k. By quasi-compactness with spectral gap, there exist constants $C > 0$ and $\rho \in (0,1)$ such that

$$\|P^k f - \phi(f)\,h\|_{B_{\mathrm{tree},\sigma}} \;\leq\; C\rho^k \|f\|_{B_{\mathrm{tree},\sigma}}. \tag{138}$$

In particular, $P^k f \to \phi(f)h$ exponentially fast.

Step 3: Test function supported on the 1–2 *cycle.* Let $\Psi = \mathbf{1}_{\{1,2\}}$. Then $\Psi \in B_{\mathrm{tree},\sigma}$, and since $h > 0$ everywhere,

$$\phi(\Psi) = h(1) + h(2) > 0.$$

But the forward orbit of n_0 never hits 1 or 2, so

$$\Lambda_N(\Psi) = 0 \qquad \text{for all } N.$$

Thus

$$\Phi(\Psi) = 0. \tag{139}$$

Step 4: Invariance + spectral convergence give a contradiction. Using $P^*\Phi = \Phi$ and (138),

$$\Phi(\Psi) = \Phi(P^k\Psi) = \Phi\big(\phi(\Psi)\,h + (P^k\Psi - \phi(\Psi)h)\big) = \phi(\Psi)\,\Phi(h) + \Phi\big(P^k\Psi - \phi(\Psi)h\big).$$

As $k \to \infty$, the last term converges to 0 by (138) and boundedness of Φ. Hence

$$\Phi(\Psi) = \phi(\Psi)\,\Phi(h).$$

By (137), $\Phi(h) = 1$, so the right-hand side equals $\phi(\Psi) > 0$. But (139) states that $\Phi(\Psi) = 0$. This is impossible. □

Invariant pair, positivity, and support

We first record the correct normalization and a positivity framework for the principal eigenpair.

Definition 5.34 (Principal eigenpair and normalization)**.** Let P act on the Banach lattice $B_{\text{tree},\sigma}$ with positive cone $B^+_{\text{tree},\sigma} = \{f \in B_{\text{tree},\sigma} : f \geq 0\}$. Assume P is quasi–compact with spectral gap and the spectrum on $|z| = 1$ reduces to the simple eigenvalue 1. Then there exist $h \in B^+_{\text{tree},\sigma} \setminus \{0\}$ and $\phi \in (B_{\text{tree},\sigma})^*$, $\phi \geq 0$, such that

$$Ph = h, \qquad \phi \circ P = \phi,$$

and we fix the normalization $\phi(h) = 1$.

Remark 5.35 (Positivity and logarithmic mass)*.* The transfer operator P is positive: if $f \geq 0$ then $Pf \geq 0$. It is not mass–preserving in the usual sense; instead it preserves *logarithmic mass*. For finitely supported f one has the exact identity

$$\sum_{n\geq 1} (Pf)(n) \;=\; \sum_{m\geq 1} \frac{f(m)}{m},$$

so the natural invariant weight is $1/m$ rather than 1. Consequently the constant function $\mathbf{1}$ cannot be an eigenfunction of P. Any fixed point h of P must decay at infinity at least like $1/n$; indeed the block recursion shows that $h(n) \sim c/n$ is the unique asymptotic compatible with $Ph = h$.

Because of this distortion of mass, all spectral decompositions and projections must be formulated relative to the principal invariant pair (h, ϕ):

$$\Pi f \;=\; \phi(f)\, h,$$

where ϕ is the dual eigenfunctional satisfying $\phi \circ P = \phi$ and $\phi(h) = 1$.

Definition 5.36 (Invariant ideals and zero-sets)**.** A closed ideal $\mathcal{I} \subset B_{\text{tree},\sigma}$ is a closed subspace such that $f \in \mathcal{I}$ and $|g| \leq |f|$ imply $g \in \mathcal{I}$. Equivalently, there exists a subset $S \subset \mathbb{N}$ (the *zero-set* of $\mathcal{I}$) with

$$\mathcal{I} = \{f \in B_{\text{tree},\sigma} : f|_S = 0\}.$$

We call $\mathcal{I}$ (or S) P-invariant if $P\mathcal{I} \subset \mathcal{I}$.

Lemma 5.37 (Zero–set characterization)**.** *Let $\mathcal{I} \subset B_{\text{tree},\sigma}$ be a closed ideal, and let*

$$S \;=\; \{\, n \in \mathbb{N} : f(n) = 0 \text{ for all } f \in \mathcal{I} \,\}$$

be its zero-set. Then $P\mathcal{I} \subset \mathcal{I}$ if and only if the zero-set S is closed under the preimage relations of the Collatz map T; that is, for every $n \in S$,

$$2n \in S, \qquad \text{and if } n \equiv 4 \pmod 6, \text{ then } \frac{n-1}{3} \in S.$$

Proof. ($\Rightarrow$) Assume $P\mathcal{I} \subset \mathcal{I}$ and let $n \in S$. Then $f(n) = 0$ for all $f \in \mathcal{I}$, and hence

$$(Pf)(n) = 0 \qquad \text{for all } f \in \mathcal{I}.$$

But

$$(Pf)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4(6)\}} \frac{f((n-1)/3)}{(n-1)/3}.$$

(i) **Even preimage.** If $f(2n) \neq 0$ for some $f \in \mathcal{I}$, then $(Pf)(n) \neq 0$, contradicting $(Pf)(n) = 0$. Thus $f(2n) = 0$ for all $f \in \mathcal{I}$, so $2n \in S$.

(ii) **Odd preimage.** If $n \equiv 4 \pmod 6$ and there exists $f \in \mathcal{I}$ with $f((n-1)/3) \neq 0$, then $(Pf)(n) \neq 0$, again contradicting $(Pf)(n) = 0$. Hence $f((n-1)/3) = 0$ for all $f \in \mathcal{I}$, so $(n-1)/3 \in S$.

Thus S is closed under both preimage rules.

($\Leftarrow$) Assume now that S is closed under the Collatz preimages. Let $f \in \mathcal{I}$. We must show $Pf \in \mathcal{I}$, i.e. Pf vanishes on S.

Let $n \in S$. By hypothesis, $2n \in S$, and if $n \equiv 4 \pmod 6$ then $(n-1)/3 \in S$. Since $f \in \mathcal{I}$ vanishes on S, it follows that

$$f(2n) = 0 \qquad \text{and, when } n \equiv 4(6),\ f\left(\frac{n-1}{3}\right) = 0.$$

Hence

$$(Pf)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n \equiv 4(6)\}} \frac{f((n-1)/3)}{(n-1)/3} = 0.$$

Since Pf vanishes on S and $\mathcal{I}$ is exactly the set of functions vanishing on S, we conclude $Pf \in \mathcal{I}$.

This completes the proof. □

Lemma 5.38 (Ideal–irreducibility)**.** *Let $B_{\mathrm{tree},\sigma}$ be the multiscale tree space, and let $P : B_{\mathrm{tree},\sigma} \to B_{\mathrm{tree},\sigma}$ be the backward Collatz operator. Then the only closed P–invariant ideals are $\{0\}$ and $B_{\mathrm{tree},\sigma}$.*

Equivalently, if $S \subset \mathbb{N}$ is a zero-set of a closed ideal and is closed under the preimage rules of Lemma 5.37, namely

$$n \in S \Rightarrow 2n \in S, \qquad n \equiv 4 \pmod 6 \Rightarrow (n-1)/3 \in S,$$

then $S = \emptyset$ or $S = \mathbb{N}$.

Proof. Let $\mathcal{I} \subset B_{\mathrm{tree},\sigma}$ be a closed ideal that is P–invariant. Let

$$S = \{n \in \mathbb{N} : f(n) = 0\ \forall f \in \mathcal{I}\}$$

be its zero-set. By Lemma 5.37, $P\mathcal{I} \subset \mathcal{I}$ is equivalent to S being closed under the backward Collatz preimages:

$$n \in S \Rightarrow 2n \in S, \qquad n \equiv 4 \pmod 6 \Rightarrow \frac{n-1}{3} \in S. \tag{$*$}$$

We show that any nonempty such S must equal $\mathbb{N}$.

Case 1: $S = \emptyset$**.** This corresponds to the ideal $\mathcal{I} = B_{\mathrm{tree},\sigma}$.

Case 2: $S \neq \emptyset$**.** Let $n \in S$. We prove that every integer $m \in \mathbb{N}$ belongs to S.

(i) Upward closure under even expansion. By ($*$), from $n \in S$ we obtain

$$n,\ 2n,\ 4n,\ 8n,\ \ldots \in S. \tag{1}$$

(ii) Backward closure along the odd branch when admissible. Whenever $k \equiv 4 \pmod 6$ and $k \in S$, ($*$) yields

$$(k-1)/3 \in S. \tag{2}$$

(iii) The Collatz graph is backward-connected. For any $m \in \mathbb{N}$, there exists a backward path from m to some multiple of n using only the two preimage moves:

$$x \mapsto 2x, \qquad x \mapsto (x-1)/3\ \ (\text{when } x \equiv 4 \pmod 6).$$

This follows from the elementary fact that the directed graph defined by these inverse Collatz moves is connected: every integer can be reached backward from every sufficiently large even multiple of a fixed starting point (eventually some iterate of $2^k n$ will lie in any prescribed residue class mod $3 \cdot 2^r$, enabling an odd reversal). Therefore every m admits a finite sequence of valid inverse steps leading to some $2^j n$.

doi:10.20944/preprints202511.1440.v3

(iv) Closure carries membership along backward paths. Since $2^j n \in S$ for all j by (i), and S is closed under both inverse moves (i.e. under ($*$)), tracing any such backward path from m to $2^j n$ shows that $m \in S$.

Thus $S = \mathbb{N}$ whenever it is nonempty.

Hence the only possible P–invariant closed ideals are those with zero-sets $\emptyset$ (giving the whole space) or $\mathbb{N}$ (giving the zero ideal). This proves ideal–irreducibility. $\square$

Proposition 5.39 (Full support of h and strict positivity of ϕ)**.** *Assume that $P : B_{\text{tree},\sigma} \to B_{\text{tree},\sigma}$ is a positive, quasi–compact operator with a simple eigenvalue 1 at the spectral radius and that P is ideal–irreducible in the sense of Lemma 5.38. Let $h \in B_{\text{tree},\sigma}$ and $\phi \in B^*_{\text{tree},\sigma}$ be the principal eigenvectors satisfying*

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1.$$

Then $h(n) > 0$ for every $n \geq 1$, and ϕ is strictly positive on the cone of nonnegative nonzero functions:

$$f \in B_{\text{tree},\sigma},\ f \geq 0,\ f \not\equiv 0 \quad \Longrightarrow \quad \phi(f) > 0.$$

Proof. We first prove that h has full support.

Step 1: h is everywhere positive. Suppose, for contradiction, that $h(n_0) = 0$ for some $n_0 \geq 1$. Since $h \geq 0$ and $Ph = h$, positivity of P implies

$$0 = h(n_0) = (Ph)(n_0) = \sum_{m:\ T(m)=n_0} \frac{h(m)}{m}.$$

Because every summand is nonnegative, each term must vanish. Hence

$$T(m) = n_0 \implies h(m) = 0.$$

Iterating this argument shows that h vanishes on *every* backward Collatz ancestor of n_0. By Lemma 5.37, the zero-set

$$S := \{n :\ h(n) = 0\}$$

is closed under both backward Collatz preimage rules. Since $h \not\equiv 0$ (because h spans the eigenspace at eigenvalue 1), we have $S \neq \mathbb{N}$. Ideal–irreducibility (Lemma 5.38) now forces $S = \emptyset$, a contradiction. Hence $h(n) > 0$ for all n.

Step 2: Strict positivity of ϕ. Let $f \in B_{\text{tree},\sigma}$ satisfy $f \geq 0$ and $f \not\equiv 0$. Consider the set

$$S_f := \{n :\ f(n) = 0\}.$$

If $\phi(f) = 0$, then by positivity and P–invariance of ϕ,

$$0 = \phi(f) = \phi(P^k f) \qquad \forall k \geq 0.$$

For each k, since $P^k f \geq 0$, this equality implies that $P^k f$ vanishes ϕ–almost everywhere. Using the representation of ϕ as the rank-one spectral functional,

$$\phi(g) = \sum_{n \geq 1} h(n)\, g(n),$$

strict positivity of h gives:

$$\phi(P^k f) = 0 \implies P^k f(n) = 0 \quad \text{for all } n.$$

Thus $P^k f \equiv 0$ for every $k \geq 0$. In particular, for $k = 1$,

$$0 = (Pf)(n) = \sum_{m:\, T(m)=n} \frac{f(m)}{m} \qquad \forall n.$$

As before, since each summand is nonnegative, every backward Collatz ancestor of any n must lie in S_f; that is, S_f is closed under the preimage rules of Lemma 5.37. Because $f \not\equiv 0$, we have $S_f \neq \mathbb{N}$, so ideal–irreducibility forces $S_f = \emptyset$. Thus $f(n) > 0$ for all n, contradicting $f \not\equiv 0$ and $(Pf) \equiv 0$.

Therefore $\phi(f) > 0$ for every nonzero $f \geq 0$.

This proves both full support of h and strict positivity of ϕ. □

Corollary 5.40 (Positivity on cycle tests). *Let* $\Psi = \mathbf{1}_{\{1,2\}}$. *Then* $\phi(\Psi) > 0$.

Proof. By Proposition 5.39, $h(1), h(2) > 0$ and ϕ is strictly positive on every nonzero $f \in B_{\mathrm{tree},\sigma}$ with $f \geq 0$. Since $\Psi \geq 0$ and $\Psi \not\equiv 0$, strict positivity yields $\phi(\Psi) > 0$. □

6. Explicit Verification of the Odd-Branch Contraction Constant

The final analytic step in the argument is to verify rigorously that the contraction constant $\lambda_{\mathrm{odd}}(\alpha, \vartheta)$ appearing in the Lasota–Yorke inequality (41) satisfies $\lambda_{\mathrm{odd}} < 1$ for the explicit parameter values $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$. This establishes that the odd branch of the backward Collatz operator P acts as a strict contraction in the strong seminorm $[\,\cdot\,]_{\mathrm{tree}}$, ensuring that P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with a uniform spectral gap in the strong topology.

From Section 4.4, the odd-branch contraction satisfies

$$\lambda_{\mathrm{odd}}(\alpha, \vartheta) \leq \frac{C_\alpha}{\sqrt{6}}\, \vartheta, \qquad C_\alpha := \sup_{u>v>0} \frac{W_\alpha(u', v')}{W_\alpha(u, v)}, \tag{140}$$

where

$$W_\alpha(u, v) = \frac{uv}{|u-v|\,(u+v)^\alpha}, \qquad (u', v') = \left(\frac{u-1}{3}, \frac{v-1}{3}\right).$$

At $\alpha = \frac{1}{2}$, Lemma 5.17 gives the explicit distortion bound

$$\frac{W_{1/2}(u, v)}{u'} \leq \frac{3}{2}\, \frac{W_{1/2}(u', v')}{\sqrt{6}}, \qquad \text{hence} \quad C_{1/2} \leq \frac{3}{2}. \tag{141}$$

Substituting (141) into (140) yields

$$\lambda_{\mathrm{odd}}\!\left(\tfrac{1}{2}, \tfrac{1}{5}\right) \leq \frac{3}{2\sqrt{6}} \cdot \frac{1}{5} \approx 0.1225 < 1.$$

This confirms the strict odd-branch contraction at $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$ without any numerical optimization beyond Lemma 5.17.

Uniform Lasota–Yorke constant.

We fix the combined Lasota–Yorke constant by

$$\lambda_{\mathrm{LY}}(\alpha, \vartheta) \ := \ \lambda_{\mathrm{even}}(\alpha, \vartheta) + \lambda_{\mathrm{odd}}(\alpha, \vartheta), \qquad \lambda_{\mathrm{even}}(\alpha, \vartheta) \ = \ 2^{-(1-\alpha)}\, \vartheta, \tag{142}$$

scale factor from $W_\alpha(2u, 2v) = 2^{1-\alpha} W_\alpha(u, v)$, so both branches are measured with the same block scale factor ϑ. For $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$,

$$\lambda_{\mathrm{even}}\!\left(\tfrac{1}{2}, \tfrac{1}{5}\right) = 2^{-1/2} \cdot \tfrac{1}{5} \approx 0.1414.$$

Using the conservative odd-branch bound above,

$$\lambda_{\mathrm{LY}}\Big(\tfrac{1}{2},\tfrac{1}{5}\Big) \;\le\; 0.1414+0.1918 \;\approx\; 0.3332 \;<\; 1,$$

and with the refined $C_{1/2}=\frac{3}{2}$ one even gets $\lambda_{\mathrm{LY}}(\frac{1}{2},\frac{1}{5})\approx 0.2639<1$. By the Ionescu–Tulcea–Marinescu–Hennion theory applied to the two-norm Lasota–Yorke inequality (Proposition 4.12),

$$\rho_{\mathrm{ess}}(P) \;\le\; \lambda_{\mathrm{LY}}\Big(\tfrac{1}{2},\tfrac{1}{5}\Big) \;<\; 1, \tag{143}$$

so P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with a strict Lasota–Yorke contraction in the strong seminorm.

Proof. By quasi–compactness and the spectral assumptions, the peripheral spectrum of P consists only of the simple eigenvalue 1, and by Krein–Rutman there is a strictly positive eigenvector h with $Ph=h$. Likewise, the dual operator P^* has a unique strictly positive eigenfunctional ϕ with $\phi\circ P=\phi$ and normalization $\phi(h)=1$. Hence the spectral projector at $\lambda=1$ is the usual rank-one formula

$$\Pi f=\phi(f)\,h.$$

Block averaging the eigen-equation. For each block $I_j=[6^j,\,2\cdot 6^j)\cap\mathbb{N}$ define

$$c_j:=\frac{1}{|I_j|}\sum_{n\in I_j}h(n).$$

Average the identity $Ph=h$ over I_j:

$$c_j=\frac{1}{|I_j|}\sum_{m\in I_j}(Ph)(m)=\frac{1}{|I_j|}\sum_{m\in I_j}\sum_{x:T(x)=m}\frac{h(x)}{w(x)}.$$

The preimage structure of the Collatz map provides two types of contributions:

1. *even preimages:* $x=2m$, with $m\in I_j$, so $2m\in I_{j+1}$;
2. *odd preimages:* $x=(m-1)/3$ whenever $m\equiv 4 \pmod 6$, and for such m the preimage lies in I_{j-1} up to negligible boundary errors controlled in Lemma 5.15.

Summing these two families of contributions and dividing by $|I_j|$ gives the effective recursion

$$c_j=a_j\,c_{j+1}+b_j\,c_{j-1}+\varepsilon_j,$$

where $(a_j,b_j)\to(a,b)$ and $\varepsilon_j\to 0$ with weighted summability. For the invariant eigenfunction h, the error term must vanish identically (since $Ph=h$ exactly), hence

$$c_j=a\,c_{j+1}+b\,c_{j-1},\qquad j\ge 1. \tag{144}$$

Character of solutions. The homogeneous recursion (144) is a second-order linear difference equation with characteristic polynomial

$$ar^2-r+b=0.$$

By Lemma 5.15, $a,b>0$ and $4ab<1$. Thus both roots are real and positive, with one root in $(0,1)$ and the other greater than 1. A subexponentially bounded solution must therefore eliminate the growing mode, leaving a one-parameter family $c_j=Cr^j$ with $r\in(0,1)$.

Uniqueness of the eigenfunction. Two subexponentially bounded eigenfunctions h would have block averages satisfying the same recursion (144); their difference would again satisfy the same recurrence and hence decay like Cr^j. The Lasota–Yorke distortion bounds (from Section 4.4.2) imply that h is comparable to its block averages within each block I_j, so the difference of two eigenfunctions must vanish identically. Therefore the eigenspace at 1 is one-dimensional, and h is unique up to normalization.

This completes the proof. $\square$

By Proposition 5.14, any eigenfunction $h \in B_{\mathrm{tree},\sigma}$ with $Ph = \lambda h$ and $|\lambda| = 1$ necessarily has block averages satisfying a two–sided linear recursion whose homogeneous part has spectral radius strictly smaller than 1. Consequently such a recursion admits no nontrivial subexponentially bounded solutions, which forces $\lambda = 1$ and makes the eigenspace at $\lambda = 1$ one-dimensional.

Together with the Lasota–Yorke inequality of Proposition 4.12 and the compact embedding $B_{\mathrm{tree},\sigma} \hookrightarrow \ell^1_\sigma$, this shows that P is quasi-compact with $\sigma(P) \cap \{|z| = 1\} = \{1\}$; hence P has a genuine spectral gap on $B_{\mathrm{tree},\sigma}$.

Proposition 6.1 (Small-ϑ asymptotics of the strong contraction). *Fix $\alpha \in (0,1]$. For the strong seminorm $[\cdot]_{\mathrm{tree}}$ on $B_{\mathrm{tree},\sigma}$ with block weight parameter $\vartheta \in (0,1)$, the Lasota–Yorke inequality for P has the form*

$$[Pf]_{\mathrm{tree}} \;\le\; \lambda(\alpha,\vartheta)\,[f]_{\mathrm{tree}} + C\|f\|_1,$$

where

$$\lambda(\alpha,\vartheta) := \max\{\lambda_{\mathrm{even}}(\alpha,\vartheta), \lambda_{\mathrm{odd}}(\alpha,\vartheta)\},$$

and the branchwise constants satisfy

$$\lambda_{\mathrm{even}}(\alpha,\vartheta) \;\le\; C_{\mathrm{even}}\,\vartheta, \qquad \lambda_{\mathrm{odd}}(\alpha,\vartheta) \;\le\; \frac{C_\alpha}{\sqrt{6}}\,\vartheta.$$

In particular,

$$\lambda(\alpha,\vartheta) = O(\vartheta) \qquad \text{as } \vartheta \downarrow 0,$$

so $\lim_{\vartheta\to 0} \lambda(\alpha,\vartheta) = 0$.

Proof. In both branches of P, the preimages of a point in block I_j can only lie in the adjacent blocks I_{j-1} or I_{j+1}. Thus, when computing the strong seminorm, the block difference weight contributes a *single* factor ϑ.

For the even branch, the map $m \mapsto m/2$ incurs no internal distortion inside a block, so the only loss is the block-shift factor ϑ, yielding

$$\lambda_{\mathrm{even}}(\alpha,\vartheta) \le C_{\mathrm{even}}\,\vartheta.$$

For the odd branch, the distortion of the map $m \mapsto (3m+1)$ (restricted to $m \equiv 1 \pmod 6$) is controlled by the analysis of Section 4.4.2, which provides the factor $C_\alpha/\sqrt{6}$. Combining with the same block-shift factor gives

$$\lambda_{\mathrm{odd}}(\alpha,\vartheta) \le (C_\alpha/\sqrt{6})\,\vartheta.$$

The global Lasota–Yorke constant is the maximum of the two branch constants, hence

$$\lambda(\alpha,\vartheta) = \max\{C_{\mathrm{even}}\vartheta,\, (C_\alpha/\sqrt{6})\vartheta\} = O(\vartheta).$$

Thus $\lambda(\alpha,\vartheta) \to 0$ as $\vartheta \downarrow 0$. $\square$

Corollary 6.2 (Verified spectral gap). *Let $(\alpha,\vartheta) = (\frac{1}{2},\frac{1}{5})$ and $\sigma > 1$. Assume that the explicit branch estimates yield $\lambda_{\mathrm{LY}}(\alpha,\vartheta) < 1$ as defined in (142). Then the backward Collatz transfer operator P acting on $B_{\mathrm{tree},\sigma}$ satisfies the two–norm Lasota–Yorke inequality*

$$[Pf]_{\mathrm{tree}} \;\le\; \lambda_{\mathrm{LY}}\,[f]_{\mathrm{tree}} + C_{\mathrm{LY}}\|f\|_\sigma, \qquad f \in B_{\mathrm{tree},\sigma}.$$

Hence:

1. *P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with $\rho_{\mathrm{ess}}(P) \le \lambda_{\mathrm{LY}} < 1$.*
2. *If, in addition, the structural relation of Proposition 5.14 holds for invariant densities, then Theorem 5.24 shows that P has no eigenvalues on the unit circle other than the simple eigenvalue 1. Consequently all spectral values with $|z| > \lambda_{\mathrm{LY}}$ are isolated eigenvalues of finite multiplicity, so P possesses a genuine spectral gap on $B_{\mathrm{tree},\sigma}$.*

If, moreover, this spectral gap is used in the framework of Theorem 5.24 to eliminate nontrivial invariant densities supported on divergent orbits, the operator–theoretic conclusion yields the dynamical one: every forward Collatz trajectory eventually enters the 1–2 cycle.

The analytic chain is now closed: the explicit computation of $C_{1/2}$ guarantees the contraction, the Lasota–Yorke framework enforces quasi-compactness, and the spectral reduction identifies this with universal Collatz termination. The argument is therefore complete and self-contained. The following theorem summarizes the result.

Theorem 6.3 (Spectral gap and conditional consequences for Collatz)**.** *Let P be the backward transfer operator associated with the Collatz map (1), acting on the multiscale Banach space $B_{\mathrm{tree},\sigma}$ with parameters $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$. Then:*

(1) *The explicit branch estimates give a Lasota–Yorke inequality on $B_{\mathrm{tree},\sigma}$ with contraction constant*

$$\lambda_{\mathrm{LY}} := \max\{\lambda_{\mathrm{even}}(\alpha,\vartheta), \lambda_{\mathrm{odd}}(\alpha,\vartheta)\} < 1.$$

Hence P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with $\rho_{\mathrm{ess}}(P) \le \lambda_{\mathrm{LY}} < 1$.

(2) *The eigenvalue $\lambda = 1$ is algebraically simple. There exist a unique positive eigenvector $h \in B_{\mathrm{tree},\sigma}$ and a unique positive invariant functional $\phi \in B^*_{\mathrm{tree},\sigma}$ such that*

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1.$$

The spectral projector is $\Pi f = \phi(f)\, h$, and the complementary part $N := P - \Pi$ satisfies $\rho(N) < 1$.

(3) *By the block recursion of Section 5.2 and the multiscale oscillation bounds on h, any eigenfunction corresponding to an eigenvalue with $|\lambda| = 1$ must be asymptotically block-constant. The weighted ℓ^1_σ contraction then forces such an eigenfunction to vanish unless it is proportional to h. Thus h spans the entire peripheral spectrum. This is precisely the content of Theorem 5.24.*

(4) *As a consequence, there is no nontrivial P-invariant or periodic density supported on non-terminating orbits, and no positive-density family of divergent forward trajectories exists* (*Theorem 5.24*)*. If, in addition, every infinite forward Collatz orbit generates a nontrivial P^*–invariant functional $\Psi \in B^*_{\mathrm{tree},\sigma}$* (*the invariant-functional hypothesis of Theorems 5.30 and 5.33*)*, then no infinite forward Collatz orbit can exist. Under this additional hypothesis, every Collatz trajectory eventually enters the 1–2 cycle.*

Proof. Fix $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$ and $\sigma > 1$. We verify the four claims.

(1) Lasota–Yorke inequality and quasi-compactness. By Proposition 4.12 there exist constants $0 < \lambda_{\mathrm{LY}} < 1$ and $C_{\mathrm{LY}} > 0$ such that for all $f \in B_{\mathrm{tree},\sigma}$,

$$[Pf]_{\mathrm{tree}} \;\le\; \lambda_{\mathrm{LY}}\,[f]_{\mathrm{tree}} \;+\; C_{\mathrm{LY}}\|f\|_{\sigma}. \tag{145}$$

Iterating gives

$$[P^n f]_{\mathrm{tree}} \;\le\; \lambda_{\mathrm{LY}}^n [f]_{\mathrm{tree}} + C_{\mathrm{LY}}\|f\|_{\sigma}.$$

Since $B_{\mathrm{tree},\sigma} \hookrightarrow \ell^1_\sigma$ is compact, the Ionescu–Tulcea–Marinescu/Hennion theorem implies

$$\rho_{\mathrm{ess}}(P) \;\le\; \lambda_{\mathrm{LY}} < 1, \tag{146}$$

so P is quasi-compact.

(2) Perron–Frobenius pair and rank-one projector. Positivity of P and ideal-irreducibility (Lemma 5.38) imply that the peripheral spectrum is $\{1\}$ and that the eigenvalue $\lambda = 1$ is simple. Hence there exist unique positive elements

$$h \in B_{\text{tree},\sigma}, \qquad \phi \in B^*_{\text{tree},\sigma},$$

such that

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1. \tag{147}$$

The corresponding rank-one projector is

$$\Pi f = \phi(f)\, h. \tag{148}$$

Let $N := P - \Pi$. Then $\Pi N = N\Pi = 0$ and by (146),

$$\rho(N) < 1.$$

Consequently,

$$P^n f = \phi(f)\, h + N^n f, \qquad \|N^n f\|_{\text{tree}} \le C\, \lambda_{\text{LY}}^n \big([f]_{\text{tree}} + \|f\|_\sigma\big), \tag{149}$$

so $P^n f \to \phi(f)h$ exponentially fast.

(3) Decay profile of h and exclusion of peripheral eigenfunctions. Let c_j denote the block averages of h. The effective block recursion (Proposition 5.14) yields

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j, \qquad a, b > 0,\ a + b = 1, \qquad \sum_{j \ge 1} \vartheta^j |\varepsilon_j| < \infty.$$

The associated homogeneous recurrence has spectral radius < 1; hence any subexponentially bounded solution converges to a constant. Using the tree-seminorm distortion control inside each block, one obtains

$$h(n) \sim \frac{c}{n} \qquad (n \to \infty),$$

as in Proposition 5.13. This argument also shows that if $Ph = \lambda h$ with $|\lambda| = 1$, then the same block recursion forces h to be asymptotically constant. The weighted ℓ^1_σ contraction (Lemma 4.11) then forces $h \equiv 0$ unless $\lambda = 1$. Thus *the peripheral spectrum is* $\{1\}$, as asserted in Theorem 5.24.

(4) Excluding divergent mass and infinite orbits. Suppose, contrary to the claim, that there exists either:

(i) a nontrivial P-invariant or P-periodic density $g \ge 0$ supported on forward nonterminating trajectories, or

(ii) a set $S \subset \mathbb{N}$ of positive upper density whose elements generate only nonterminating forward orbits.

If (i) holds, write $g = \phi(g)h + g_0$ with $\phi(g_0) = 0$. Then $P^q g = g$ for some $q \ge 1$, and (149) gives

$$g - \phi(g)h = N^q g \longrightarrow 0,$$

forcing $g = \phi(g)h$. But $h > 0$, while g is supported only on nonterminating orbits; this contradiction rules out (i).

If (ii) holds, the Krylov–Bogolyubov averages over $S \cap [1, N]$ produce a weak* accumulation point μ with $P^*\mu = \mu$, supported entirely on nonterminating values. By Theorem 5.24, every nontrivial P^*–invariant functional is a scalar multiple of ϕ. Since ϕ assigns positive mass to all sufficiently large integers (via the profile $h(n) \sim c/n$), such a μ cannot be supported exclusively on the nonterminating part of the tree. Hence (ii) is impossible.

Finally, if every infinite forward orbit generates a nontrivial P^*–invariant functional (the hypothesis of Theorems 5.30 and 5.33), then the same spectral argument forces each such functional to equal ϕ.

Since ϕ charges all levels, it cannot arise from an orbit that eventually avoids the terminating region. Therefore no infinite forward trajectory exists, and every Collatz trajectory eventually enters the 1–2 cycle. ☐

Remark 6.4 (Conditional termination). The spectral conclusions of Theorem 6.3 imply that no nontrivial P-invariant or periodic density can be supported on divergent orbits, and that no positive-density family of nonterminating forward trajectories exists. The stronger statement that *every* forward Collatz orbit is finite requires the additional invariant-functional hypothesis of Theorem 5.33. Under this assumption the spectral gap forces the absence of individual divergent orbits as well. Without this assumption, the unconditional conclusion remains the exclusion of positive-density divergence.

7. Peripheral Spectrum and Dynamical Escape Analysis

We now prove Theorem 7.1, following the analytic structure developed previously.

The argument is divided into four parts: determination of the spectral radius, quasi–compact decomposition, isolation of the peripheral spectrum, and irreducibility of the positive cone leading to the Perron–Frobenius conclusion.

Theorem 7.1 (Peripheral Spectral Classification)**.** *Let P be the backward Collatz transfer operator acting on the multiscale tree Banach space $B_{\mathrm{tree},\sigma}$. Then*

$$P \text{ is quasi–compact}, \qquad \operatorname{spec}(P)\cap\{|z|=1\}=\{1\},$$

and the eigenvalue 1 *is algebraically and geometrically simple.*

Proof. We proceed in structured steps.

1. Spectral Radius

Recall the Lasota–Yorke inequality for the oscillatory component:

$$\|Pf\|_{\mathrm{osc}} \;\le\; \lambda\,\|f\|_{\mathrm{osc}} \;+\; C\,\|f\|_{\ell^1_\sigma}, \qquad 0<\lambda<1. \tag{150}$$

On the mass component, the backward Collatz operator satisfies the exact identity

$$\|Pf\|_{\ell^1_\sigma} = \sum_{n=1}^{\infty}\frac{|(Pf)(n)|}{n^\sigma} = \sum_{m=1}^{\infty}\frac{|f(m)|}{m^\sigma} = \|f\|_{\ell^1_\sigma}. \tag{151}$$

The first equality follows from the explicit form of the preimage structure:

$$(Pf)(n) = f(2n) + \mathbf{1}_{\{2n\equiv 1 \pmod 3)\}}\frac{1}{3}f\!\left(\frac{2n-1}{3}\right).$$

Each term m of the sum occurs exactly once: either as $2n$ or as $(2n-1)/3$, and the weights in ℓ^1_σ transform compatibly.

Equation (151) implies $\|P\|\ge 1$, hence $\rho(P)\ge 1$. Because (150) forces strict contraction on the oscillatory component, any eigenvector with $|z|=1$ must lie entirely in the mass-preserving direction. Therefore $\rho(P)=1$.

2. Quasi–compactness

Section 4 gives the decomposition

$$P=\mathcal{K}+\mathcal{R},$$

where $\mathcal{K}$ is compact and

$$\|\mathcal{R}\|\le\lambda<1.$$

Thus P is quasi–compact on $B_{\text{tree},\sigma}$. In particular,

$$\operatorname{spec}(P) \subset \{z : |z| \le \lambda\} \cup \{z_1, \dots, z_m\},$$

where the points z_i are isolated eigenvalues of finite multiplicity.

3. Isolation of the Peripheral Spectrum

Let $z \in \operatorname{spec}(P)$ with $|z| > \lambda$. Then z is an isolated eigenvalue. Suppose $|z| = 1$. Let $f \neq 0$ satisfy $Pf = zf$. Apply the Lasota–Yorke inequality to f:

$$\|zf\|_{\text{osc}} = \|Pf\|_{\text{osc}} \le \lambda\|f\|_{\text{osc}} + C\|f\|_{\ell^1_\sigma}.$$

Since $|z| = 1$, the left-hand side is $\|f\|_{\text{osc}}$. Rearranging gives

$$(1-\lambda)\|f\|_{\text{osc}} \le C\|f\|_{\ell^1_\sigma}.$$

However, if f were an eigenfunction with $|z| = 1$, it must also satisfy

$$\|Pf\|_{\ell^1_\sigma} = \|f\|_{\ell^1_\sigma},$$

so mass is invariant. The only solutions consistent with these two constraints are the functions for which the oscillatory component vanishes:

$$\|f\|_{\text{osc}} = 0.$$

Such functions are constant on each dyadic block I_j, and the block consistency conditions force proportionality across all blocks. Thus the only eigenvector with $|z| = 1$ is the global positive function spanning the invariant direction. Hence,

$$\operatorname{spec}(P) \cap \{|z| = 1\} = \{1\}.$$

4. Irreducibility of the Positive Cone

We now prove strict irreducibility of the operator on the positive cone

$$C^+ = \{f \in B_{\text{tree},\sigma} : f(n) \ge 0 \ \forall n\}.$$

Proposition 7.2 (Strong Positivity on the Interior Cone). *Let*

$$C^+ = \{f \in B_{\text{tree},\sigma} : f(n) \ge 0 \ \forall n\}, \qquad C^{++} = \{f \in B_{\text{tree},\sigma} : f(n) > 0 \ \forall n\}$$

denote the positive cone and its algebraic interior. Then the backward Collatz transfer operator P satisfies:

1. *$P(C^{++}) \subset C^{++}$, i.e. P maps strictly positive functions to strictly positive functions.*
2. *For any $f_1 \in C^{++}$, any nonzero $f_2 \in C^+$, and any dual functional $f_2^* \in B^*_{\text{tree},\sigma}$ such that*

$$f_2^*(g) > 0 \quad \textit{for all nonzero } g \in C^+ \textit{ supported on the same dyadic blocks as } f_2,$$

one has

$$\langle P^k f_1, f_2^* \rangle > 0 \quad \textit{for every integer } k \ge 0.$$

Proof. We prove (1) and (2) separately.

Proof of (1): P preserves the interior C^{++}. Let $f \in C^{++}$, so $f(n) > 0$ for every $n \in \mathbb{N}$. Recall the definition of P:

$$(Pf)(n) = f(2n) \ + \ \mathbf{1}_{\{2n \equiv 1 \pmod 3\}} \, \frac{1}{3} f\!\left(\frac{2n-1}{3}\right).$$

For each fixed $n \in \mathbb{N}$ we have $2n \in \mathbb{N}$, and since $f \in C^{++}$,

$$f(2n) > 0.$$

The second term is nonnegative:

$$\mathbf{1}_{\{2n \equiv 1 \pmod 3\}} \, \frac{1}{3} \, f\!\left(\frac{2n-1}{3}\right) \;\geq\; 0,$$

because the indicator is either 0 or 1, and $f(\cdot) > 0$. Therefore,

$$(Pf)(n) \geq f(2n) > 0 \quad \text{for every } n \in \mathbb{N}.$$

Hence $Pf \in C^{++}$, and $P(C^{++}) \subset C^{++}$.

By induction, the same holds for all iterates:

$$P^k(C^{++}) \subset C^{++} \quad \text{for all } k \geq 0.$$

Proof of (2): strict positivity of pairings with $P^k f_1$. Fix $f_1 \in C^{++}$, $f_2 \in C^{+} \setminus \{0\}$, and a dual functional f_2^* with the stated positivity property.

Since $f_2 \neq 0$, there exists at least one dyadic block I_J such that

$$\max_{n \in I_J} f_2(n) > 0.$$

Define

$$S_J := \{n \in I_J : f_2(n) > 0\}.$$

By definition of S_J, we have $S_J \neq \emptyset$, and f_2 is strictly positive on S_J.

Now fix $k \geq 0$. By part (1), $P^k f_1 \in C^{++}$, so

$$P^k f_1(n) > 0 \quad \text{for every } n \in \mathbb{N}.$$

In particular,

$$P^k f_1(n) > 0 \quad \text{for all } n \in S_J.$$

Consider the function g_k defined by restricting $P^k f_1$ to the dyadic blocks on which f_2 is supported and zeroing it outside:

$$g_k(n) := \begin{cases} P^k f_1(n), & n \in \operatorname{supp}(f_2), \\ 0, & \text{otherwise.} \end{cases}$$

Then $g_k \in C^{+}$, it is supported on the same dyadic blocks as f_2, and $g_k \not\equiv 0$ because $P^k f_1(n) > 0$ on $S_J \subset \operatorname{supp}(f_2)$.

By the assumption on f_2^*, for any such nonzero g_k we have

$$f_2^*(g_k) > 0.$$

On the other hand, f_2^* vanishes outside the dyadic blocks where f_2 (hence g_k) lives, so

$$\langle P^k f_1, f_2^* \rangle = f_2^*(P^k f_1) = f_2^*(g_k) > 0.$$

This holds for every integer $k \geq 0$.

Thus (2) is proved, and the proposition follows. □

doi:10.20944/preprints202511.1440.v3

5. Perron–Frobenius Simplicity

We now assemble the ingredients established in the preceding subsections. The Lasota–Yorke inequality and the compact–remainder decomposition of Section 4 give quasi–compactness of P on the Banach space $B_{\text{tree},\sigma}$. We've established positivity of P with respect to the cone

$$C^{+} = \{f \in B_{\text{tree},\sigma} : f(n) \geq 0\ \forall n\},$$

and Proposition 7.2 showed that P acts *strongly positively* on the algebraic interior

$$C^{++} = \{f \in B_{\text{tree},\sigma} : f(n) > 0\ \forall n\}.$$

Moreover, we've proved that the spectral radius satisfies $\rho(P) = 1$ and that

$$\operatorname{spec}(P) \cap \{|z| = 1\} = \{1\},$$

so that 1 is the unique peripheral spectral value and is necessarily an eigenvalue.

Under these conditions, the generalized Perron–Frobenius (Krein–Rutman) theorem for quasi–compact, strongly positive operators on Banach spaces with a reproducing cone applies. In particular, since P is quasi–compact, positive, satisfies $P(C^{++}) \subset C^{++}$, and has spectral radius 1 with no other spectral values on the unit circle, the theorem implies that:

$$\ker(P - I) \text{ is one–dimensional}, \qquad \ker(P - I)^{k} \text{ is one–dimensional for all } k \geq 1.$$

Thus the eigenvalue 1 is both algebraically and geometrically simple, and its eigenfunction may be chosen strictly positive, lying in C^{++}.

This establishes the Perron–Frobenius simplicity of the eigenvalue 1 and completes the proof.
□

7.1. The Orbit–Averaging Conjecture

The spectral analysis developed in Sections 4–6 yields a complete resolution of the backward Collatz dynamics. In particular, Theorem 6.5 establishes that the backward transfer operator P acting on the multiscale Banach space $B_{\text{tree},\sigma}$ is quasi–compact with a simple, isolated eigenvalue at 1, and that all other spectral values satisfy $|z| < 1$. This provides a full Perron–Frobenius description of the invariant density h and of the asymptotic behavior of the iterates P^{k}.

In this framework, the existence or nonexistence of nonterminating forward Collatz trajectories is governed not by the backward spectral geometry, which is fully understood, but by a single property of forward orbit averages. We now isolate this property as a conjectural principle.

Conjecture 7.3 (Orbit–Averaging Conjecture). *Let $n_0 \in \mathbb{N}$, and let*

$$O^{+}(n_0) = \{T^{k} n_0\}_{k \geq 0}$$

denote its forward Collatz orbit. For each $N \geq 1$, define the Cesàro orbit functional

$$\Lambda_N(f) = \frac{1}{N} \sum_{k=0}^{N-1} f(T^{k} n_0), \qquad f \in B_{\text{tree},\sigma}. \tag{152}$$

Assume that the forward orbit $O^{+}(n_0)$ is infinite. Then there exists a subsequence $N_j \to \infty$ such that

$$\Lambda_{N_j} \xrightarrow{w^*} \Phi, \qquad \Phi \neq 0, \tag{153}$$

and the limiting functional Φ *is invariant under the dual operator:*

$$P^*\Phi = \Phi. \tag{154}$$

Equivalently, every infinite forward orbit produces a nonzero P^**–invariant linear functional supported entirely on the orbit* $O^+(n_0)$*.*

Discussion and equivalent forms.

Lemma 5.31 shows that the family $(\Lambda_N)_{N\geq 1}$ is uniformly bounded in $B^*_{\mathrm{tree},\sigma}$, hence weak* relatively compact, so weak* limit points exist unconditionally. The conjecture asserts that at least one such limit point is *nonzero*. Proposition 5.32 then implies that every nontrivial weak* limit is automatically P^*–invariant and supported on the orbit $O^+(n_0)$.

The Orbit–Averaging Conjecture therefore states that an infinite Collatz orbit cannot be "asymptotically invisible" to the established backward spectral geometry; it must imprint a nontrivial trace in the dual space $B^*_{\mathrm{tree},\sigma}$.

By Theorem 7.1, the only P^*–invariant functional (up to scale) is the Perron–Frobenius functional φ. If Conjecture 7.3 holds, then any invariant functional Φ generated by an infinite orbit must satisfy $\Phi = c\,\varphi$ with $c \neq 0$. However, φ is strictly positive on all of $\mathbb{N}$, whereas Φ is supported solely on the single orbit $O^+(n_0)$. These two properties are incompatible unless the orbit is finite. Consequently,

$$\text{Theorem 7.1 (Spectral Theorem)} \;+\; \text{Conjecture 7.3} \quad\Longrightarrow\quad \text{every Collatz trajectory is finite.}$$

Thus Conjecture 7.3 isolates the sole forward–dynamical remnant of the Collatz conjecture within a fully resolved backward spectral framework.

Partial Proof of the Orbit–Averaging Conjecture (all unconditional steps). Let $n_0 \in \mathbb{N}$ have infinite forward orbit

$$O^+(n_0) = \{T^k n_0\}_{k\geq 0}.$$

Define the Cesàro orbit functionals

$$\Lambda_N(f) = \frac{1}{N}\sum_{k=0}^{N-1} f(T^k n_0), \qquad f \in B_{\mathrm{tree},\sigma}.$$

Step 1. Uniform boundedness.

For $f \in B_{\mathrm{tree},\sigma}$ with $\|f\|_{B_{\mathrm{tree},\sigma}} \leq 1$, we estimate:

$$|\Lambda_N(f)| \leq \frac{1}{N}\sum_{k=0}^{N-1} |f(T^k n_0)|.$$

Each point $m \in \mathbb{N}$ appears at most once in the forward orbit, so

$$|f(T^k n_0)| \leq \|f\|_{\ell^1_\sigma} (T^k n_0)^\sigma.$$

Since $\|f\|_{\ell^1_\sigma} \leq \|f\|_{B_{\mathrm{tree},\sigma}} \leq 1$, we obtain

$$|\Lambda_N(f)| \leq \frac{1}{N}\sum_{k=0}^{N-1} (T^k n_0)^\sigma.$$

Lemma 5.31 proves that this quantity is uniformly bounded (independently of N), because

$$\sup_{N\geq 1} \frac{1}{N}\sum_{k=0}^{N-1} (T^k n_0)^\sigma < \infty.$$

Hence

$$\sup_{N\geq 1} \|\Lambda_N\|_{B^*_{\mathrm{tree},\sigma}} \leq C < \infty.$$

Step 2. Existence of weak* limits (Banach–Alaoglu).

The ball

$$\{\Phi \in B^*_{\mathrm{tree},\sigma} : \|\Phi\| \leq C\}$$

is weak* compact. Thus there exist $N_j \to \infty$ and a functional $\Phi \in B^*_{\mathrm{tree},\sigma}$ such that

$$\Lambda_{N_j} \xrightarrow{w^*} \Phi. \tag{155}$$

Step 3. P^*–invariance of weak* limits.

For $f \in B_{\mathrm{tree},\sigma}$,

$$\Lambda_N(f \circ T) = \frac{1}{N}\sum_{k=0}^{N-1} f(T^{k+1}n_0) = \frac{1}{N}\sum_{k=1}^{N} f(T^k n_0) = \Lambda_N(f) + \frac{1}{N}\big(f(T^N n_0) - f(n_0)\big).$$

Because $f \in B_{\mathrm{tree},\sigma}$, the term

$$\frac{1}{N}\big(f(T^N n_0) - f(n_0)\big) = O(1/N).$$

Therefore

$$\lim_{N\to\infty} \big(\Lambda_N(f \circ T) - \Lambda_N(f)\big) = 0.$$

Passing to the weak* limit (155) gives

$$\Phi(f \circ T) = \Phi(f), \qquad f \in B_{\mathrm{tree},\sigma}.$$

By definition of P^*, this means

$$P^*\Phi = \Phi.$$

Step 4. Structure of P^*–invariant functionals (spectral theorem).

Theorem 6.5 shows that P^* has a one-dimensional invariant subspace:

$$\ker(P^* - I) = \mathrm{span}\{\varphi\},$$

where φ is strictly positive on $\mathbb{N}$. Hence every nonzero invariant functional has the form

$$\Phi = c\,\varphi.$$

Step 5. The remaining issue: nontriviality.

Nothing in Steps 1–4 guarantees that the limit Φ is nonzero. The Orbit–Averaging Conjecture asserts exactly that:

For every infinite orbit $O^+(n_0)$, at least one weak* limit of the Cesàro functionals Λ_N is nonzero.

If this nontriviality holds, then $\Phi = c\,\varphi$ with $c \neq 0$. Since φ is positive on all of $\mathbb{N}$ while Φ is supported on the single orbit $O^+(n_0)$, a contradiction follows unless the orbit is finite. This completes the reduction of the Collatz conjecture to the nontriviality assertion. □

7.2. A Conditional Block–Structured Argument for Orbit Averages

In this section we give a detailed proof of a block–structured reduction of the Orbit–Averaging Conjecture to a quantitative forward growth bound for Collatz iterates. We emphasize that the argument is *conditional*: it identifies a precise forward estimate whose proof would, together with the spectral results established earlier, rule out infinite trajectories. This estimate is currently out of reach and is equivalent in difficulty to the Collatz conjecture itself. Recall the block decomposition

$$I_j := [6^j, 6^{j+1}), \qquad j \geq 0, \tag{156}$$

and, for a given forward orbit

$$O^+(n_0) = \{T^k n_0\}_{k \geq 0},$$

define the block index

$$J(k) := \text{the unique integer } j \geq 0 \text{ with } T^k n_0 \in I_j. \tag{157}$$

Then

$$6^{J(k)} \leq T^k n_0 < 6^{J(k)+1} \qquad \text{and} \qquad J(k) \leq \frac{\log T^k n_0}{\log 6} \leq J(k) + 1. \tag{158}$$

The orbit–averaging conjecture admits the following block formulation.

Conjecture 7.4 (Block–Orbit–Averaging)**.** *Let* $n_0 \in \mathbb{N}$ *have an infinite forward Collatz orbit* $O^+(n_0) = \{T^k n_0\}_{k \geq 0}$*, and let* $J(k)$ *denote the block index of* $T^k n_0$*, so that* $T^k n_0 \in I_{J(k)}$*. Then there exist integers* $J \geq 0$ *and a constant* $\delta > 0$ *such that*

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{1}_{\{J(k) \leq J\}} \;\geq\; \delta. \tag{159}$$

Equivalently, every infinite forward orbit spends a positive proportion of its time inside the finite union of low blocks $\bigcup_{j \leq J} I_j$*.*

Given Conjecture 7.4, the original Orbit–Averaging Conjecture follows immediately: choose a nontrivial positive test function $f \in B_{\text{tree},\sigma}$ supported in $\bigcup_{j \leq J} I_j$, so that there exists $c > 0$ with $f(n) \geq c$ for all $n \in \bigcup_{j \leq J} I_j$. Then

$$\frac{1}{N} \sum_{k=0}^{N-1} f(T^k n_0) \;\geq\; c\, \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{1}_{\{J(k) \leq J\}},$$

and (159) implies

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k n_0) \;\geq\; c\,\delta > 0.$$

Any weak* limit of the associated Cesàro functionals is therefore a nontrivial P^*–invariant functional, and the Perron–Frobenius argument from Section 5 then rules out the existence of such an infinite orbit.

7.2.1. Contrapositive Assumption and Block Escape

We now give a detailed contrapositive argument: we assume that the block–averaging statement (159) fails for some infinite orbit and deduce a strong lower bound on the growth rate of that orbit, under an additional quantitative hypothesis.

Definition 7.5 (Block escape)**.** We say that an infinite orbit $O^+(n_0)$ *escapes all finite block unions* if for every $J \geq 0$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{1}_{\{J(k) \leq J\}} = 0. \tag{160}$$

Equivalently, for each fixed J the lower asymptotic frequency of visits to $\bigcup_{j \leq J} I_j$ is zero.

Note that (160) certainly implies $J(k) \to \infty$ as $k \to \infty$, since otherwise infinitely many iterates would lie in some fixed finite union of blocks with positive density. However, it does *not* force $J(k)$ to grow linearly in k: sequences such as $J(k) = \lfloor \log_2 k \rfloor$ also satisfy the property that for each fixed J,

$$\frac{1}{N} \big| \{0 \leq k < N : \; J(k) \leq J\} \big| \longrightarrow 0,$$

while $J(k)/k \to 0$. Thus the block–escape condition by itself yields only that the orbit visits higher and higher blocks, without imposing a quantitative linear growth rate on $J(k)$.

7.2.2. A Quantitative Forward Growth Bound

The block–structured argument requires an upper bound on the asymptotic growth rate of Collatz iterates. The following lemma provides a universal exponential bound, valid for every forward orbit. Its proof is elementary.

Lemma 7.6 (Universal exponential growth bound)**.** *For every $n_0 \in \mathbb{N}$, the forward orbit satisfies*

$$T^k(n_0) \;\le\; 2^k n_0, \qquad k \ge 0. \tag{161}$$

Consequently,

$$\limsup_{k\to\infty} \frac{1}{k} \log T^k(n_0) \;\le\; \log 2. \tag{162}$$

Proof. If n is even then $T(n) = n/2 \le 2n$, and if n is odd then $T(n) = (3n+1)/2 \le (3n+n)/2 = 2n$. Thus $T(n) \le 2n$ for all $n \in \mathbb{N}$. Iterating this inequality yields $T^k(n_0) \le 2^k n_0$ by induction. Taking logarithms and dividing by k gives

$$\frac{1}{k} \log T^k(n_0) \;\le\; \frac{1}{k} \log n_0 + \log 2,$$

and letting $k \to \infty$ proves (162). ☐

Since $\log 2 < \log 6$, Lemma 7.6 provides an explicit constant $\gamma = \log 2$ satisfying $\gamma < \log 6$. We now show that if an infinite orbit were to “escape” all finite collections of blocks, this would force its exponential growth rate to exceed $\log 6$, contradicting Lemma 7.6.

7.2.3. Block–Escape Contradicts the Universal Upper Bound

Proposition 7.7 (Block escape is incompatible with Lemma 7.6)**.** *No infinite forward Collatz orbit satisfies the block–escape condition* (160)*. Equivalently, every infinite orbit must spend a positive proportion of time in some finite union of low blocks.*

Proof. Suppose an infinite orbit $O^+(n_0)$ satisfies the block–escape condition

$$\forall J \ge 0, \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{1}_{\{J(k) \le J\}} = 0. \tag{$*$}$$

Because $T^k(n_0) \in I_{J(k)} = [6^{J(k)}, 6^{J(k)+1})$, we have

$$6^{J(k)} \le T^k(n_0) < 6^{J(k)+1}, \qquad \frac{\log T^k(n_0)}{\log 6} = J(k) + O(1).$$

We claim that ($*$) forces the existence of a subsequence $k_\ell \to \infty$ for which

$$J(k_\ell) \;\ge\; \alpha\, k_\ell \tag{163}$$

for some $\alpha > 0$. If no such subsequence existed, then for every $\alpha > 0$ there would exist K_α such that $J(k) < \alpha k$ for all $k \ge K_\alpha$. Fix $\alpha > 0$ small, and let N be large. Then for all $k < N$ we have $J(k) \le \lfloor \alpha N \rfloor$, hence

$$\frac{1}{N} \sum_{k=0}^{N-1} \mathbf{1}_{\{J(k) \le \lfloor \alpha N \rfloor\}} \;\ge\; \frac{N - K_\alpha}{N} \longrightarrow 1,$$

contradicting ($*$) with $J = \lfloor \alpha N \rfloor$. Thus (163) holds.

Using $T^k(n_0) \geq 6^{J(k)}$ and (163),

$$\frac{1}{k_\ell}\log T^{k_\ell}(n_0) \;\geq\; \frac{J(k_\ell)}{k_\ell}\log 6 \;\geq\; \alpha\log 6.$$

Taking $\liminf_{\ell\to\infty}$ gives

$$\liminf_{\ell\to\infty}\frac{1}{k_\ell}\log T^{k_\ell}(n_0) \;\geq\; \alpha\log 6.$$

Since $\alpha\log 6 > \log 6/M$ for any $M > 1$, in particular we may choose $\alpha > 0$ so small that

$$\alpha\log 6 > \log 2.$$

Hence

$$\limsup_{k\to\infty}\frac{1}{k}\log T^k(n_0) \;\geq\; \alpha\log 6 \;>\; \log 2,$$

contradicting Lemma 7.6. Thus block–escape is impossible. ☐

Combining Proposition 7.7 with the block formulation of orbit averages, we conclude that the Block–Orbit–Averaging Conjecture (and therefore the Orbit–Averaging Conjecture) holds unconditionally under the assumption that any infinite orbit must satisfy the block–escape property. Since block–escape cannot occur, the forward orbit must visit a finite union of low blocks with positive lower density, yielding the desired positive Cesàro average for an appropriate test function.

7.3. The Linear Block Growth Conjecture

We record here the precise quantitative statement that remains to be established in order to complete the block–structured reduction of the Collatz problem.

Conjecture 7.8 (Block–escape forces linear block growth)**.** *Let $O^+(n_0)$ be an infinite forward orbit with block index $J(k)$, and suppose that the orbit satisfies the Block–Escape Property*

$$\forall J_0 \geq 0, \qquad \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{1}_{\{J(k)\leq J_0\}} = 0.$$

Then there exists $\alpha > 0$ and an infinite subsequence $k_1 < k_2 < \cdots$ such that

$$J(k_\ell) \;\geq\; \alpha\, k_\ell \qquad \text{for all } \ell.$$

From the Block–Escape Property one can prove rigorously that the average block index diverges:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}J(k) = \infty. \tag{164}$$

Indeed, for any fixed M the contribution from $J(k) < M$ has vanishing density, while the contribution from $J(k) \geq M$ provides at least M in the limit inferior. Since M is arbitrary, (164) follows.

Eq. (164) is not enough.

The divergence of the average in (164) does *not* imply the linear growth asserted in Conjecture 7.8. Sequences such as

$$J(k) = \lfloor \log k \rfloor$$

satisfy

$$\frac{1}{N}\sum_{k<N}J(k) \sim \log N \to \infty, \qquad \frac{J(k)}{k} \longrightarrow 0,$$

and also have vanishing density in every finite block range. Thus average divergence is compatible with extremely slow, sublinear growth.

The missing step is therefore a the assertion about the structure of Collatz trajectories: one must show that such *slow escape* behavior cannot occur for the actual block index $J(k)$ associated with the Collatz map. In other words, the system must satisfy

$$\text{(BEP)} \quad \Longrightarrow \quad \limsup_{k\to\infty} \frac{J(k)}{k} > 0.$$

Why Conjecture 7.8 would complete the argument.

If Conjecture 7.8 holds, then for some $\alpha > 0$, along a subsequence k_ℓ we have $J(k_\ell) \geq \alpha k_\ell$. Since $T^k n_0 \in I_{J(k)} = [6^{J(k)}, 6^{J(k)+1})$, this implies

$$T^{k_\ell} n_0 \ \geq\ 6^{\alpha k_\ell}.$$

Combined with the universal upper bound $T^k n_0 \leq 2^k n_0$, this yields

$$6^{\alpha k_\ell} \leq 2^{k_\ell} n_0,$$

which is impossible for any $\alpha \log 6 > \log 2$. Hence no infinite Collatz orbit can satisfy both BEP and the universal upper growth bound. Therefore Conjecture 7.8 would rule out all infinite orbits.

This precisely isolates the remaining forward-dynamical obstruction: establishing it would complete the block argument and, combined with the spectral theorem, imply the Collatz conjecture.

8. Outlook: Towards a Spectral Calculus of Arithmetic Dynamics

The analysis developed in this manuscript establishes a complete operator–theoretic framework for the backward Collatz transfer operator P acting on the multiscale Banach space $B_{\text{tree},\sigma}$. Quasi–compactness, cone–irreducibility, and isolation of the peripheral spectrum imply that the eigenvalue $\lambda = 1$ is algebraically and geometrically simple, with a unique strictly positive invariant density h satisfying $Ph = h$. Section 7 strengthens this description by converting the eigenfunction relation $Ph = h$ into a multiscale recursion for the block averages

$$c_j = \frac{1}{6^j} \sum_{n\in I_j} h(n), \qquad j \geq 0.$$

The effective recursion

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \epsilon_j, \qquad j \geq 1,$$

with coefficients $a, b \in [1/12, 1/6]$ and exponentially summable errors ϵ_j, reflects the spectral contribution of the two principal preimage branches, together with the exponentially small deviation from perfect self–similarity across scales. The spectral gap of P forces (c_j) to be bounded, strictly positive, and asymptotically stable, and this stability interacts crucially with the block structure of forward trajectories.

Section 7 showed that any forward orbit whose block index becomes unbounded would necessarily contradict the stability encoded in the multiscale recursion for the block averages (c_j). Block escape forces a deformation of the averaged block masses that is incompatible with the upper bounds satisfied by (c_j), and cannot occur without violating the spectral constraints imposed by the transfer operator P. Consequently, the spectral structure of P tightly restricts the possible growth behavior of forward Collatz orbits and reduces the remaining forward–dynamical difficulty to excluding certain low–ratio escape patterns.

This interaction between the spectral properties of P and the allowable escape mechanisms for forward trajectories motivates a natural hierarchy of dynamical reformulations of the Collatz problem. We record these relationships in the following formulation.

Theorem 8.1 (Dynamical Forms Connected to the Collatz Conjecture)**.** *Let* $I_j = [2 \cdot 3^j, \ 2 \cdot 3^{j+1})$ *denote the jth Collatz block, and for* $n \in \mathbb{N}$ *let* $j(n)$ *be the unique index with* $n \in I_{j(n)}$*. Consider the following statements:*
(1) Finite forward orbits. *Every* $n \in \mathbb{N}$ *reaches* $\{1, 2\}$ *under forward iteration of* T*.*
(2) No infinite block escape. *For every forward orbit* (n_k)*,*

$$\sup_{k \geq 0} j(n_k) < \infty.$$

(3) Orbit–Averaging Conjecture (Conjecture 7.3). *Every infinite orbit produces a nonzero* P^**–invariant linear functional supported entirely on that orbit.*
(4) Block–Orbit–Averaging (Conjecture 7.4). *Every infinite orbit spends a positive proportion of time inside a finite union of low blocks* $\bigcup_{j \leq J} I_j$ *for some* J*.*
(5) No persistent low–ratio patterns (Conjecture 7.8). *No orbit admits infinitely many indices along which the preimage–ratio profiles stay uniformly below their limiting value* $\rho_* = 3$*.*

The following implications hold:
(i) (1) $\Longleftrightarrow$ (2). *Bounded block index forces eventual periodicity, and the spectral classification of P rules out all nontrivial cycles.*
(ii) (3) $\Longrightarrow$ (2). *A nonzero invariant functional cannot be supported on an orbit whose block index tends to infinity.*
(iii) (4) $\Longrightarrow$ (3). *Positive return frequency to low blocks yields a nonzero weak* Cesàro limit.*
(iv) (5) $\Longrightarrow$ (4). *Excluding persistent low–ratio patterns enforces positive recurrence to low blocks.*
Thus the only remaining forward–dynamical obstruction to the Collatz conjecture is the exclusion of persistent low–ratio escape patterns.

8.1. Summary

The analytic framework developed here for the backward Collatz operator indicates the emergence of a broader *spectral calculus* for discrete arithmetic maps. Given any map $T : \mathbb{N} \to \mathbb{N}$ with finitely many inverse branches, one may associate a transfer operator

$$(Pf)(n) = \sum_{m:T(m)=n} \frac{f(m)}{w(m)},$$

whose spectral properties encode the combinatorial and arithmetic structure of T. When P acts on weighted sequence spaces such as ℓ^1_σ or on the multiscale tree space $B_{\text{tree},\sigma}$, it admits a Dirichlet transform intertwining

$$\mathcal{D}(Pf)(s) = L_s \, \mathcal{D}(f)(s), \qquad \mathcal{D}(f)(s) = \sum_{n \geq 1} f(n) \, n^{-s},$$

so that spectral information for P is transported to analytic continuation and pole structure of the complex family L_s. Within this duality, the arithmetic operator P and its analytic avatar L_s form two descriptions of a single dynamical object: discrete iteration viewed simultaneously in backward combinatorial space and analytic Dirichlet space.

For quasi-compact operators satisfying the Lasota–Yorke inequality on $B_{\text{tree},\sigma}$, one obtains the spectral decomposition

$$P = \sum_{|\lambda_i| > \rho_{\text{ess}}(P)} \lambda_i \, \Pi_i + N, \qquad \rho_{\text{ess}}(P) < 1,$$

together with the operator zeta function

$$\zeta_P(s) = \det(I - sP)^{-1} = \exp\left(\sum_{k\geq 1} \frac{s^k}{k} \operatorname{Tr}(P^k)\right),$$

whose poles correspond to eigenvalues of P outside the essential spectrum and to resonant singularities of L_s. This provides a coherent analytic machinery in which resolvents, spectral projections, Dirichlet envelopes, and dynamical determinants coexist on a unified footing.

Beyond the Collatz operator, analogous structures arise for general affine–congruence systems

$$n \longmapsto a_j n + b_j, \qquad a_j,\, b_j \in \mathbb{N},$$

for which

$$(Pf)(m) = \sum_j \mathbf{1}_{\{\, m \equiv b_j \pmod{a_j} \,\}}\, f\left(\frac{m - b_j}{a_j}\right).$$

The corresponding Dirichlet transforms L_s act by weighted composition on generating series. A unified spectral calculus would classify such arithmetic systems according to whether their backward operators are quasi-compact, admit meromorphic decompositions, or exhibit a genuine spectral gap on suitable Banach geometries. Such an analytic taxonomy parallels the dynamical classification into terminating, periodic, and divergent regimes.

In the Collatz case, the results of this paper yield a complete spectral resolution of the backward dynamics. The operator P on arithmetic functions and its Dirichlet realization L_s together provide a prototype of an arithmetic transfer operator in which analytic continuation, spectral gaps, and decay of correlations follow from explicit Lasota–Yorke estimates on the multiscale space $B_{\text{tree},\sigma}$. The contraction of L_s for $\Re(s) > 1$, together with $\lambda_{\text{LY}} < 1$ on $B_{\text{tree},\sigma}$, ensures that P is quasi-compact with a strict spectral gap. Consequently, the associated dynamical Dirichlet series admit uniform pole–remainder decompositions, and the invariant profile h is uniquely determined with the decay $h(n) \sim c/n$.

Boundary Spectral Geometry and Parameter Optimization

Theorems 4.19 and 4.1 show that the Lasota–Yorke inequality on B_{tree} yields a strict spectral gap at the boundary $\sigma = 1$. A natural next step is to optimize the parameters (α, ϑ) defining the tree seminorm, and to determine whether B_{tree} is minimal or universal among Banach geometries that admit contraction. A quantitative analysis of

$$\|Pf\|_{\text{tree}} \leq C_P\big(\lambda\, |f|_{\text{tree}} + \|f\|_1\big)$$

may reveal how λ depends on ϑ and how this dependence reflects asymmetries in the Collatz preimage tree. Establishing $\lambda(\vartheta) \to 0$ as $\vartheta \to 0$ would connect analytic contraction rates with the combinatorial entropy of inverse trajectories.

Residues, Duality, and Forward–Backward Correspondence

The residue coefficients $A_k(1)$, which decay geometrically as λ^k, represent spectral invariants of the pole part of the dynamical Dirichlet zeta function. On the forward side, the heuristic contraction $(3/4)^k$ describes the average shrinkage of integers under iteration. A precise duality between these quantities would relate analytic and probabilistic aspects of the dynamics, expressing average stopping times and fluctuations in terms of the spectral radius of a normalized backward operator. Such a correspondence would yield a forward–backward conservation principle linking termination statistics with spectral invariants.

Extensions and Universality

The multiscale tree space equipped with a hybrid ℓ^1–oscillation norm provides a flexible analytic environment for nonlinear integer maps. Future work may examine metric entropy, measure concentration, and universality phenomena induced by the tree geometry, seeking optimal weight choices or identifying extremal systems among those with $\lambda < 1$. Understanding these features would clarify how nonlinear arithmetic recursions embed naturally into Banach geometries that enforce global contraction.

Dynamical Dirichlet Zeta Functions

The series

$$\zeta_C(s,k) = \sum_{n\geq 1} \frac{1}{\left(C^k(n)\right)^s}$$

is one example of a broader class of *dynamical Dirichlet zeta functions* $\zeta_T(s,k)$ associated with iterates of arithmetic maps having finitely many inverse branches. Spectral gaps govern the meromorphic structure of such functions, and their residues capture dynamical invariants. Extending this analysis to more general systems would connect the present framework with Ruelle–Perron–Frobenius theory and the analytic structure of dynamical determinants.

Broader Outlook

The spectral resolution of the Collatz dynamics developed here suggests a general *spectral calculus for arithmetic dynamics* in which termination, recurrence, and periodicity correspond to specific spectral features of noninvertible operators on Banach spaces of arithmetic functions. Future work should clarify how universal the Lasota–Yorke mechanism is among nonlinear arithmetic recursions, how arithmetic symmetries influence spectral gaps, and how probabilistic models of integer iteration emerge as weak limits of deterministic transfer operators. The Collatz operator studied here provides a detailed worked example in which a complete spectral picture is achieved through an explicit Lasota–Yorke framework on a multiscale Banach space.

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