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Article

On m-Isometric and m-Symmetric Operators of Elementary Operators

B.P. Duggal

University of Nis, Faculty of Sciences and Mathematics, P.O. Box 224, 18000 Niš, Serbia; bpduggal@yahoo.co.uk

Abstract

Given Hilbert space operators A, B and X, let $\triangle_{A,B}$ and $\delta_{A,B}$ denote, respectively, the elementary operators $\triangle_{A,B}(X) = I - AXB$ and the generalised derivation $\delta_{A,B}(X) = AX - XB$. This paper considers the structure of operators $D^m_{d_1,d_2}(I) = 0$ and $D^m_{d_1,d_2}$ compact, where m is a positive integer, $D = \triangle$ or δ, $d_1 = \triangle_{A^*,B^*}$ or δ_{A^*,B^*} and $d_2 = \triangle_{A,B}$ or $\delta_{A,B}$. This is a continuation of the work done by C. Gu for the case $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$, and the author with I.H. Kim for the cases $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$ or $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}$ is compact, and $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ or $\delta^m_{\triangle_{A^*,B^*},\delta_{A,B}}$ is compact. Operators $D^m_{d_1,d_2}(I) = 0$ are examples of operators with finite spectrum, indeed the operators A, B have at most a two point spectrum, and if $D^m_{d_1,d_2}$ is compact, then (the non-nilpotent operators) A, B are algebraic. $D^m_{d_1,d_2}(I) = 0$ implies $D^n_{d_1,d_2}(I) = 0$ for integers $n \geq m$: the reverse implication, however, fails. It is proved that $D^m_{d_1,d_2}(I) = 0$ implies $D_{d_1,d_2}(I) = 0$ if and only if of A and B (are normal, hence) satisfy a Putnam-Fuglede commutativity property.

Keywords: Hilbert space; elementary operators; n-isometric and m-symmetric operators; Putnam-Fuglede commutativity theorem; normal operator; scalar operator; ascent; poles; compact operator

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1. Introduction

Given a complex infinite dimensional Hilbert space $(\mathcal{H}\ ; \langle .,. \rangle)$, let $B(\mathcal{H})$ denote the algebra of operators, equivalently bounded linear transformations, on $\mathcal{H}$ into itself. For an operator $A \in B(\mathcal{H})$, let L_A and $R_A \in B(B(\mathcal{H}))$ denote, respectively, the operators

$$L_A(X) = AX \text{ and } R_A(X) = XA$$

of left multiplication by A and right multiplication by A. For $A, B \in B(\mathcal{H})$, the elementary operator $\triangle_{A,B}$ of length two and the generalised derivation $\delta_{A,B} \in B(B(\mathcal{H}))$ are defined by

$$\triangle_{A,B}(X) = (I - L_A R_B)(X) = X - AXB \text{ and } \delta_{A,B}(X) = (L_A - R_B)(X) = AX - XB,$$

respectively. We say that the pair (A, B) of operators in $B(\mathcal{H}) \times B(\mathcal{H})$ is m-isometric (resp. , m-symmetric) for some positive integer m if

$$\begin{aligned} \triangle^m_{A,B}(I) &= (I - L_A R_B)^m(I) = \left(\sum_{j=0}^{m} (-1)^j \left(\begin{array}{c} m \\ j \end{array} \right) L^j_A R^j_B \right)(I) \\ &= \sum_{j=0}^{m} (-1)^j \left(\begin{array}{c} m \\ j \end{array} \right) A^j B^j = 0 \end{aligned}$$

$$\text{(resp.,} \qquad \delta^m_{A,B}(I) = (L_A - R_B)^m(I) = \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} L_A^{m-j} R_B^j\right)(I)$$

$$= \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{m-j}B^j = 0).$$

(In the case in which the pair (A, B) is the pair (T^*, T) for an operator $T \in B(\mathcal{H})$, we shorten (T^*, T) is m-isometric (resp., m-symmetric) to T is m-isometric (resp., m-symmetric).) Structural properties of m-isometric and m-symmetric pairs (A, B), and their connections with classical function theory, non-stationary stochastic processes, Toeplitz operators and nilpotent perturbations of Hermitian operators (etc.), have been studied by a number of authors in the recent past: see [2–4,8,18,25,37,39] for further references.

A characterisation of m-isometric operators with a finite spectrum has been carried out in [29,34,37]. An example of such an operator, first considered by Botelho and Jamison [12,13] for the cases $m = 2$ and $m = 3$, is the operator

$$\triangle^m_{L_{A^*}R_{B^*}, L_A R_B}(I) = \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{*j}A^jB^jB^{*j} = 0\,; A, B \in B(\mathcal{H}).$$

It is straightforward to see that if $\overline{\lambda}$ is in the approximate point spectrum $\sigma_a(B^*)$ of B^*, then $\triangle^m_{\overline{\lambda}A^*,\lambda A}(I) = 0$, $\sigma(\lambda A)$ is a subset of the boundary $\partial\mathtt{D}$ of the unit disc $\mathtt{D} \subset \mathtt{C}$ and $1 - (\overline{\lambda}\sigma_a(A^*))(\lambda\sigma_a(A)) = 0$. There exists a non-trivial scalar β, $|\beta| = 1$, such that $\lambda\alpha = \beta$ and $\overline{\lambda\alpha} = \frac{1}{\beta} = \overline{\beta}$ for all $\alpha \in \sigma_a(A)$. We assert that $\sigma_a(B^*)$ consists, at most, of two points. For if there exist non-trivial $\overline{\mu}, \overline{\nu} \in \sigma_a(B^*)$, $\mu \neq \lambda \neq \nu$, then $\sigma_a(\mu A) = \sigma_a(\lambda A) + \sigma_a((\mu - \lambda)A)$ and $\sigma_a(\nu A) = \sigma_a(\lambda A) + \sigma_a((\nu - \lambda)A)$: since $0 \notin \sigma_a(A)$, not both of these translates of $\sigma_a(\lambda A)$ are in $\partial\mathtt{D}$. This argument applies equally to $\sigma_a(A)$; hence $\sigma(A)$ and $\sigma(B)$ consist at most of two points. A version of the preceding argument applies to m-isometric operators $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$. Indeed Gu, [24] proves that if $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$ for some operators $A, B \in B(\mathcal{H})$, then A, B have spectrum consisting at most of two points. An extension of this result to operators $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ has recently been considered by Duggal and Kim [22].

For operators $A, B \in B(\mathcal{H})$, let $D^m_{d_1,d_2}(I)$ denote the operator defined by the choices

$$D = \triangle \text{ or } \delta,\ d_1 = \triangle_{A^*,B^*} \text{ or } \delta_{A^*,B^*} \text{ and } d_2 = \triangle_{A,B} \text{ or } \delta_{A,B}.$$

This paper considers the structure of the resulting eight operators, two of which have already been discussed in [22,24], to prove that the spectra $\sigma(A)$ and $\sigma(B)$ consist at best of two points. It is fairly easily seen that $D^m_{d_1,d_2}(I) = 0$ implies $D^n_{d_1,d_2}(I) = 0$ for all integers $n \geq m$. The reverse problem of does $D^m_{d_1,d_2}(I) = 0$ imply $D^n_{d_1,d_2}(I) = 0$ for some positive integer $n < m$ does not, in general, have a positive answer. We prove that a necessary and sufficient condition for $D^m_{d_1,d_2}(I) = 0$ to imply $D_{d_1,d_2}(I) = 0$ is that the pairs of operators $(\alpha_1 A + \beta_1, \alpha_2 B + \beta_2)$, α_i and β_i $(i = 1, 2)$ scalars, satisfy a Putnam-Fuglede commutativity property [28,32,35] The paper considers also the case in which the operator $D^m_{d_1,d_2}$ is compact. Here it is seen that the operators A and B (are algebraic operators, and as such) have a finite spectrum consisting of poles (of the resolvent) of the operators.

2. Complementary Results: The Adjoint Operator $(D^m_{d_1,d_2}(I))^*$

In the following, A, B will denote Hilbert space operators such that $\sigma(A) \neq \{0\} \neq \sigma(B)$, and m will denote a positive integer. Given complex scalars α_i and β_i, $1 \leq i \leq 2$, we say that the pair of operators $(\alpha_1 A+\beta_1,\alpha_2 B+\beta_2)$ satisfies *the Putnam-Fuglede commutativity property*, shortened to $(\alpha_1 A+\beta_1,\alpha_2 B+\beta_2)\in(PF)$, if

$$D^{-1}_{\alpha_1 A+\beta_1,\alpha_2 B+\beta_2}(0)\subseteq D^{-1}_{\overline{\alpha_1}A^*+\overline{\beta_1},\overline{\alpha_2}B^*+\overline{\beta_2}}(0);\ D=\triangle \text{ or } \delta.$$

It is well known, see [28,36], that if A,B are normal operators, then the pair $(\alpha_1 A+\beta_1,\alpha_2 B+\beta_2)\in(PF)$.

The ascent (resp., descent) of A, denoted $\text{asc}(A)$ (resp., $\text{dsc}(A)$), is the least positive integer n such that

$$A^{-(n+1)}(0)\subseteq A^{-n}(0)\ (\text{resp. } A^n(\mathcal{H})\subseteq A^{n+1}(\mathcal{H})).$$

Finite ascent and finite descent implies their equality [5,38]; we say that a point λ of the spectrum $\sigma(A)$ of A is a pole (of the resolvent) of A if $\text{asc}(A-\lambda)=\text{dsc}(A-\lambda)$ [5,38]. (Here, and in the sequel, we write $A-\lambda$ for $A-\lambda I$.) We observe here that if $\lambda\in\sigma(A)$ is a pole of A, then necessarily λ is an isolated point of $\sigma(A)$ (i.e., $\lambda\in\text{iso}\sigma(A)$). We say that $D^{-1}_{A,B}(0)\perp D_{A,B}(\mathcal{H})$, i.e. the kernel of the operator $D_{A,B}$ is orthogonal to the range of $D_{A,B}$ in the sense of James [14,31], if

$$||X||\leq||X+Y||,\ \text{all } X\in D^{-1}_{A,B}(0) \text{ and } Y\in D_{A,B}(B(\mathcal{H})).$$

The following lemma, linking "range-kernel orthogonality" to the "ascent" of operators satisfying the (PF)-property, will be useful in our considerations below.

Lemma 2.1. *[16, Proposition 2.26]. If* $D_{A,B}\in(PF)$, *then* ($D^{-1}_{A,B}(0)\perp D_{A,B}(\mathcal{H})$, *hence*) *asc*$(D_{A,B})\leq 1$.

The *quasi-nilpotent part* $H_0(A-\lambda)$ of an operator $A\in B(\mathcal{H})$ at $\lambda\in\mathtt{C}$ is the set

$$H_0(A-\lambda)=\{x\in\mathcal{H}:\lim_{n\to\infty}||(A-\lambda)^n||^{\frac{1}{n}}||=0\}$$

[5, Page 119]. A has SVEP, the single-valued extension property, at λ_0 if for every open disc $\mathtt{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f:\mathtt{D}_{\lambda_0}\to\mathcal{H}$ satsifying $(A-\lambda)f(\lambda)=0$ for all $\lambda\in\mathtt{D}_{\lambda_0}$ is the function $f\equiv 0$ [5,23]; A has SVEP if it has SVEP at all λ. Operators A with a countable spectrum have SVEP, and if $\sigma(A)=\{\alpha_1,\cdots,\alpha_p\}$ for some natural number p, then $A=\oplus_{i=0}^n A|_{H_0(A-\alpha_i)}$ [5, Page 303, Lemma 4.17]. A necessary and sufficient condition for the points α_i to be poles of (the resolvent of) A of order t_i, for some positive integers t_i, $1\leq i\leq p$, is that $H_0(A-\alpha_i)=(A-\alpha_i)^{-t_i}(0)$ [5]. Observe that if $\sigma(A)=\{\alpha_1,\cdots,\alpha_p\}$ and the points α_i are poles of A of order t_i, then $A=\oplus_{i=1}^p A|_{(A-\alpha_i)^{-t_i}(0)}+N$ for some nilpotent operator N.

The operator A is algebraic if there exists a non-trivial polynomial $p(.)$ such that $p(A)=0$. It is well known, see for example [5, Theorem 3.83], that A is algebraic if and only if $\sigma(A)$ is a finite set consisting of the poles of A (i.e., if and only if $\sigma(A)$ is finite and A is *polaroid* [5]). Recall from Boasso [10] that A is polaroid, i.e. points $\lambda\in\text{iso}\sigma(A)$ are poles of A, if and only if L_A and R_B are polaroid. The (Banach space) operator $\delta_{A,B}$ is algebraic if and only if A,B are algebraic. L_AR_B algebraic does not imply A and B algebraic, only that at least one of A and B is algebraic [16, Proposition 2.6].

A is nilpotent, more precisely A is n-nilpotent for some positive integer n, if $A^n=0$. Either of A and B nilpotent implies L_AR_B nilpotent; conversely, L_AR_B nilpotent implies, at best, that at least one of A and B is nilpotent. We consider next the adjoint of the operator $D^m_{d_1,d_2}(I)$.

Start by observing that for all $X\in B(\mathcal{H})$,

$$(\delta_{A,B}(X))^*=((L_A-R_B)(X))^*=(R_{A^*}-L_{B^*})(X^*)=(-1)\delta_{B^*,A^*}(X^*)$$

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and

$$(\triangle_{A,B}(X))^* = ((I - L_A R_B)(X))^* = (I - L_{B^*} R_{A^*})(X^*) = \triangle_{B^*,A^*}(X^*).$$

Hence:

$$\begin{aligned}
&(\delta^n_{A,B}(X))^* = (-1)^n \delta^n_{B^*,A^*}(X^*);\\
&(\triangle^n_{A,B}(X))^* = \triangle^n_{B^*,A^*}(X^*);\\
&(\triangle_{A^*,B^*}(\triangle_{A,B}(X)))^* = \triangle_{B,A}((\triangle_{A,B}(X))^*) = \triangle_{B,A}(\triangle_{B^*,A^*}(X^*));\\
&(\delta_{A^*,B^*}(\delta_{A,B}(X)))^* = (-1)\delta_{B,A}(\delta_{A,B}(X))^*) = (-1)^2\delta_{B,A}(\delta_{B^*,A^*}(X^*));\\
&(\triangle_{A^*,B^*}(\delta_{A,B})(X)))^* = \triangle_{B,A}((\delta_{A,B}(X)))^*) = (-1)\triangle_{B,A}(\delta_{B^*,A^*}(X^*)) \text{ and}\\
&(\delta_{A^*,B^*}(\triangle_{A,B}(X)))^* = (-1)\delta_{B,A}((\triangle_{A,B}(X))^*) = (-1)\delta_{B,A}(\triangle_{B^*,A^*}(X^*)).
\end{aligned}$$

Letting

$$\nabla_{E,F}(X) = (I + L_E R_F)(X) \text{ and } \Diamond_{E,F}(X) = (L_E + R_F)(X)$$

for Banach space operators E, F and X, we have:

$$\begin{aligned}
(1)\qquad & \left(\triangle^m_{\delta_{A^*,B^*},\triangle_{A,B}}(I)\right)^*\\
&= \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \delta^j_{A^*,B^*}(\triangle^j_{A,B}(I))\right)^*\\
&= \sum_{j=0}^{m}(-1)^{2j} \begin{pmatrix} m \\ j \end{pmatrix} \delta^j_{B,A}(\triangle^j_{A,B}(I))^*\\
&= \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \delta^j_{B,A}(\triangle^j_{B^*,A^*}(I))\\
&= \nabla^m_{\delta_{B,A},\triangle_{B^*,A^*}}(I);
\end{aligned}$$

$$\begin{aligned}
(2)\qquad & \left(\delta^m_{\delta_{A^*,B^*},\triangle_{A,B}}(I)\right)^*\\
&= \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \delta^{m-j}_{A^*,B^*}(\triangle^j_{A,B}(I))\right)^*\\
&= \sum_{j=0}^{m}(-1)^m \begin{pmatrix} m \\ j \end{pmatrix} \delta^{m-j}_{B,A}(\triangle^j_{B^*,A^*}(I))\\
&= (-1)^m \Diamond^m_{\delta_{B,A},\triangle_{B^*,A^*}}(I);
\end{aligned}$$

$$(3)\qquad \left(\triangle^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I)\right)^* = \nabla^m_{\triangle_{B,A},\delta_{B^*,A^*}}(I);$$

$$(4)\qquad \left(\delta^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I)\right)^* = \Diamond^m_{\triangle_{B,A},\delta_{B^*,A^*}}(I);$$

$$(5)\qquad \left(\triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I)\right)^* = \triangle^m_{\triangle_{B,A},\triangle_{B^*,A^*}}(I);$$

$$(6)\qquad \left(\delta^m_{\delta_{A^*,B^*},\delta_{A,B}}(I)\right)^* = (-1)^m \delta^m_{\delta_{B,A},\delta_{B^*,A^*}}(I)$$

$$(7)\qquad \left(\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I)\right)^* = \delta^m_{\triangle_{B,A},\triangle_{B^*,A^*}}(I);$$

$$(8)\qquad \left(\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I)\right)^* = \triangle^m_{\delta_{B,A},\delta_{B^*,A^*}}(I).$$

3. The Spectrum

The approximate point spectrum of an operator being a non-empty set, if the pair (A,B) satisfies the operator equation $D^m_{d_1,d_2}(I)=0$, $D=\triangle$ or δ, $d_1=\triangle_{A^*,B^*}$ or δ_{A^*,B^*} and $d_2=\triangle_{A,B}$ or $\delta_{A,B}$, then there exist non-trivial scalars $\alpha\in\sigma_a(A)$ and $\overline{\lambda}\in\sigma_a(B^*)$. (We assume in the following that $\sigma(A)\neq\{0\}\neq\sigma(B^*)$; see [30] for the spectra of left and right multiplication operators.) By definition

$$\begin{aligned}\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) &= \sum_{j=0}^m(-1)^j\binom{m}{j}(L_{\delta_{A^*,B^*}}R_{\delta_{A,B}})^j(I)\\ &= \sum_{j=0}^m(-1)^j\binom{m}{j}\left[\sum_{p=0}^j(-1)^p\binom{j}{p}L^{j-p}_{A^*}R^j_{B^*}\left(\sum_{k=0}^j(-1)^k\binom{j}{k}L^{j-k}_A R^k_B\right)\right](I)\\ &= \sum_{j=0}^m(-1)^j\binom{m}{j}\left[\sum_{p=0}^j(-1)^p\binom{j}{p}\left(\sum_{k=0}^j(-1)^k\binom{j}{k}\right)\right]A^{*(j-p)}A^{j-k}B^kB^{*p}\\ &= C_{\sum}\mathcal{E}_{A^{*(j-p)}A^{j-k},B^kB^{*p}},\end{aligned}$$

where $C_{\sum}=C(m,j,p,k)$ denotes

$$C_{\sum}=\sum_{j=0}^m(-1)^j\binom{m}{j}\left[\sum_{p=0}^j(-1)^p\binom{j}{p}\left(\sum_{k=0}^j(-1)^k\binom{j}{k}\right)\right]$$

and $\mathcal{E}_{E,F}$ is the operator

$$\mathcal{E}_{E,F}(X)=EXF.$$

A similar argument shows that

$$\triangle^m_{\delta_{B,A},\delta_{B^*,A^*}}(I)=C_{\sum}\mathcal{E}_{B^{j-p}B^{*(j-p)},A^{*k}A^p}.$$

If $\alpha\in\sigma_a(A)$ and $\overline{\lambda}\in\sigma_a(B^*)$, and $\{x_n\},\{y_n\}\subseteq\mathcal{H}$ are sequences of unit vectors such that $\lim_{n\to\infty}(B^*-\overline{\lambda})x_n=0=\lim_{n\to\infty}(A-\alpha)y_n$, then for all $x,y\in\mathcal{H}$

$$\begin{aligned}&\left[\mathcal{E}_{A^{*(j-p)}A^{j-k},B^kB^{*p}}(x\otimes x_n)\right]x_n\\ =\;& A^{*(j-p)}A^{(j-k)}x\langle x_n,B^pB^{*k}x_n\rangle\end{aligned}$$

and

$$\begin{aligned}&\left[\mathcal{E}_{B^{j-p}B^{*(j-p)},A^{*k}A^p}(y\otimes y_n)\right]y_n\\ =\;& B^{j-p}B^{*(j-k)}y\langle y_n,A^{*p}A^ky_n\rangle.\end{aligned}$$

(Here, and in the sequel, $\langle .,.\rangle$ denotes the inner product on $\mathcal{H}$.) Thus, if $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I)=0$, then (for all $x\in\mathcal{H}$)

$$\begin{aligned}0 &= \lim_{n\to\infty}C_{\sum}A^{*(j-p)}A^{(j-k)}x\langle x_n,B^pB^{*k}x_n\rangle\\ &= C_{\sum}(\overline{\lambda}^pA^{*(j-p)})(\lambda^kA^{j-k})x\\ &= \sum_{j=0}^m(-1)^j\binom{m}{j}\left[\sum_{p=0}^j(-1)^p\binom{j}{p}\overline{\lambda}^pA^{*(j-p)}\left(\sum_{k=0}^j(-1)^k\binom{j}{k}\lambda^kA^{j-k}\right)\right]x\\ &= \sum_{j=0}^m(-1)^j\binom{m}{j}(A^*-\overline{\lambda})^j(A-\lambda)^jx,\end{aligned}$$

i.e.,

$$\triangle^m_{A^*-\overline{\lambda},A-\lambda}(I)=0. \tag{9a}$$

A similar argument shows that if $\triangle^m_{\delta_{B,A},\delta_{B^*,A^*}}(I)=0$, then

$$\triangle^m_{B-\alpha,B^*-\overline{\alpha}}(I)=0. \tag{9b}$$

If $\triangle^m_{A^*-\overline{\lambda},A-\lambda}(I)=0=\triangle^m_{B-\alpha,B^*-\overline{\alpha}}(I)$, then $\sigma_a(A-\lambda)$ and $\sigma_a(B^*-\overline{\alpha})$ are subsets of the boundary $\partial\mathbb{D}$ of the unit disc in the complex plane $\mathbb{C}$. Considering $\sigma_a(A-\lambda)$, if $\overline{\mu}\in\sigma_a(B^*-\overline{\lambda})$, then $\sigma_a(A-\lambda-\mu)\in\partial\mathbb{D}$ and, for a $\beta\in\sigma_a(A-\lambda)$, $1=|\sigma_a(A-\lambda-\mu)|=|\sigma_a(A-\lambda)-\mu|=|\beta-\mu|$. Let $\mu=e^{it}|\mu|$, then $|\mu|\leq 2$ and

$$\begin{aligned} &|\beta-\mu|=1\Longrightarrow\beta^2-e^{it}|\mu|\beta+e^{2it}=0\\ \Longleftrightarrow\quad &\beta=e^{it}\left(\frac{|\mu|\pm i\sqrt{4-|\mu|^2}}{2}\right). \end{aligned}$$

A similar argument shows if $\nu=e^{i\theta}|\nu|\in\sigma_a(A-\alpha)$, then $|\nu|\leq 2$ and, for every $\overline{\tau}\in\sigma_a(B^*-\overline{\alpha})$,

$$\begin{aligned} &|\tau-\nu|=1\Longrightarrow\tau^2-e^{i\theta}|\nu|\tau+e^{2i\theta}=0\\ \Longleftrightarrow\quad &\tau=e^{i\theta}\left(\frac{|\nu|\pm i\sqrt{4-|\nu|^2}}{2}\right). \end{aligned}$$

Spectra $\sigma(A)$ and $\sigma(B)$ consist at most of two points: for if $\overline{\mu},\overline{\eta}\in\sigma_a(B^*-\overline{\lambda})$, then $\sigma_a(A-\lambda-\eta)=\sigma_a(A-\lambda-\mu)+(\mu-\eta)$, and both $\sigma_a(A-\lambda-\eta)$ and its non-trivial translates can not be in $\partial\mathbb{D}$. The following proposition encompasses this known result.

Proposition 3.1. *Given $\alpha\in\sigma_a(A)$ and $\overline{\lambda}\in\sigma_a(B^*)$, if*

(i) $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I)=0$, then

$$\triangle^m_{A^*-\overline{\lambda},A-\lambda}(I)=0=\triangle^m_{B-\alpha,B^*-\overline{\alpha}}(I),$$

$$\sigma(A-\lambda)=\sigma_a(A-\lambda)=\{e^{it}\left(\frac{|\mu|\pm i\sqrt{4-|\mu|^2}}{2}\right):\overline{\mu}=e^{-it}|\mu|\in\sigma_a(B^*-\overline{\lambda}),0\leq t<2\pi,|\mu|\leq 2\}$$

and

$$\sigma(B-\alpha)=\sigma_a(B-\alpha)=\left\{e^{it}\left(\frac{|\mu|\pm i\sqrt{4-|\mu|^2}}{2}\right):\mu=e^{it}|\mu|\in\sigma_a(A-\alpha),\ 0\leq t<2\pi,|\mu|\leq 2\right\}.$$

(ii) $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I)=0$, then

(a) $$\delta^m_{I-\overline{\lambda}A^*,I-\lambda A}(I)=0=\delta^m_{I-\alpha B,I-\overline{\alpha}B^*}(I);$$

(b) there exist real scalars β_1 and β_2, $\beta_1\leq\beta_2$,such that

$$\{\lambda\sigma(A)\}\wedge\{\alpha\sigma(B)\}\subseteq[1-\beta_2,1-\beta_1];$$

(c) $\sigma(A)$ and $\sigma(B)$ consist at most of two points.

Proof. Part (i) having already been proved, we start proving part (ii) by proving $\delta^m_{I-\overline{\lambda}A^*,I-\lambda A}(I)=0=\delta^m_{I-\alpha B,I-\overline{\alpha}B^*}(I)$. By definition

$$\begin{aligned}&\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I)=C_{\Sigma_1}\mathcal{E}_{A^{*p}A^k,B^kB^{*p}} \text{ and}\\&\delta^m_{\triangle_{B,A},\triangle_{B^*,A^*}}(I)=C_{\Sigma_1}\mathcal{E}_{B^pB^{*k},A^{*k}A^p},\end{aligned}$$

where

$$C_{\Sigma_1}=C(m,j,p,k)=\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}\left[\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j\\ p\end{pmatrix}\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j\\ k\end{pmatrix}\right)\right].$$

Arguing as above, it is seen that

$$\begin{aligned}0 &= \lim_{n\to\infty}\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I)\\&= \lim_{n\to\infty}C_{\Sigma_1}A^{*p}A^kx\langle x_n,B^kB^{*p}x_n\rangle\\&= C_{\Sigma_1}(\overline{\lambda}^pA^{*p})(\lambda^kA^k)x\\&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}\left[\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j\\ p\end{pmatrix}\overline{\lambda}^pA^{*p}\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j\\ k\end{pmatrix}\lambda^kA^k\right)\right]x\\&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}(I-\overline{\lambda}A^*)^{m-j}(I-\lambda A)^jx,\end{aligned}$$

for all $x\in\mathcal{H}$ and

$$\begin{aligned}0 &= \lim_{n\to\infty}\delta^m_{\triangle_{B,A},\triangle_{B^*,A^*}}(I)\\&= \lim_{n\to\infty}C_{\Sigma_1}B^pB^{*k}y\langle y_n,A^{*p}A^ky_n\rangle\\&= C_{\Sigma_1}(\alpha^pB^p)(\overline{\alpha}^kB^{*k})y\\&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}(I-\alpha B)^{m-j}(I-\overline{\alpha}A^*)^jy\end{aligned}$$

for all $y\in\mathcal{H}$. Hence

$$\delta^m_{I-\overline{\lambda}A^*,I-\lambda A}(I)=0 \qquad (10a)$$

and

$$\delta^m_{I-\alpha B,I-\overline{\alpha}B^*}(I)=0 \qquad (10b).$$

This proves (a). To prove (b), we start by observing from the spectral mapping theorem that

$$\begin{aligned}&\sigma_a(I-\overline{\lambda}A^*)-\sigma_a(I-\lambda A)=(I-\overline{\lambda}\sigma_a(A^*))-(1-\lambda\sigma_a(A))\\=0&\\=\ &\sigma_a(I-\alpha B)-\sigma_a(I-\overline{\alpha}B^*)=(1-\alpha\sigma_a(B))-(1-\overline{\alpha}\sigma_a(B^*)).\end{aligned}$$

Hence there exist real numbers β_i, $1\le i\le 2$, such that

$$1-\lambda\sigma(A)\subseteq[\beta_1,\beta_2], \text{ equivalently } \lambda\sigma(A)\subseteq[1-\beta_2,1-\beta_1].$$

(This inclusion being true for all $\overline{\lambda}\in\sigma_a(B^*)$, we have also that $\alpha\sigma(B)\subseteq[1-\beta_2,1-\beta_1]$.) To prove (c), assume that there exist distinct scalars μ,ν such that $\overline{\mu},\overline{\nu}\in\sigma_a(B^*-\overline{\lambda})$. Then $\overline{\lambda+\mu}$ and $\overline{\lambda+\nu}\in\sigma_a(B^*)$, and hence

$$\{(\lambda+\mu)\sigma(A)\}\wedge\{(\lambda+\nu)\sigma(A)\}\subseteq[1-\beta_2,1-\beta_1].$$

Recalling the hypothesis that $\sigma_a(A) \neq \{0\}$, we have

$$\begin{aligned} & \sigma_a(I-(\lambda+\nu)A) = \sigma_a(I-(\lambda+\mu)A) + (\mu-\nu)\sigma_a(A) \\ \Longrightarrow\ & (\lambda+\nu)\sigma_a(A) \subseteq [1-\beta_2, 1-\beta_1] + (\mu-\nu)\sigma_a(A) \not\subseteq [1-\beta_2, 1-\beta_1], \end{aligned}$$

since $1-\beta_1+(\mu-\nu)\sigma_a(A) > 1-\beta_1$ if $(\mu-\nu)\sigma_a(A) > 0$ and $1-\beta_2+(\mu-\nu)\sigma_a(A) < 1-\beta_2$ if $(\mu-\nu)\sigma_a(A) < 0$. A similar argument proves that $\sigma_a(A)$ consists at most of two points. □

The point 0 can not be in both $\sigma_a(B^*)$ and $\sigma_a(A)$ in the case in which $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$. This follows from the following argument. If $0 \in \sigma_a(B^*)$, then (see above) for all $x \in \mathcal{H}$,

$$\begin{aligned} & \triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0 \Longrightarrow C_{\sum} A^{*(j-p)} A^{j-k} x \lim_{n\to\infty} \langle B^{*p} x_n, A^{*k} x_n \rangle \\ = \quad & 0 \text{ for all } p, k > 0 \text{ and if } p = k = 0, \text{ then} \\ & \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{*j} A^j x = \triangle^m_{A^*,A}(I)x = 0 \\ \Longrightarrow\ & \triangle^m_{A^*,A}(I) = 0. \end{aligned}$$

Thus, A is m-isometric, and hence $\sigma_a(A) \subseteq \partial\mathbb{D}$. A similar argument shows that if $0 \in \sigma_a(A)$, then B^* is m-isometric and $\sigma_a(B^*) \subseteq \partial\mathbb{D}$. This fails for operators A, B satisfying $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$. Consider, for example, the operators A, B such that $\sigma(A) = \{ae^{i\theta}, 0\}$, $\sigma(B) = \{be^{-i\theta}, 0\}$ for some real numbers a, b and $0 \leq \theta < 2\pi$, $A = ae^{i\theta} I_1 \oplus 0I_2$ and $B = be^{-i\theta} I_1 \oplus 0I_2$; $I = I_1 \oplus I_2$. Then $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = C_{\sum_1}(I_1 \oplus 0I_2) = 0$.

The following theorem proves that the conclusion $\sigma(A)$ and $\sigma(B)$ consist at most of two points holds for the remaining six choices of the operators D, d_1 and d_2 (in $D^m_{d_1,d_2}(I) = 0$). For continuity, we keep the numbering above – thus our theorem starts with case (iii) and ends with case $(viii)$. We assume in the proof of the theorem that $\{x_n\}$ and $\{y_n\}$ are sequences of unit vectors in $\mathcal{H}$ such that $\lim_{n\to\infty}(A-\alpha)x_n = 0 = \lim_{n\to\infty}(B^* - \overline{\lambda})y_n = 0$. (Thus, $\alpha \in \sigma_a(A)$ and $\overline{\lambda} \in \sigma_a(B^*)$.)

Theorem 3.2. *(iii). If $\triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$, then $\triangle^m_{I-\overline{\lambda}A^*,I-\lambda A} = 0 = \triangle^m_{I-\alpha B,I-\overline{\alpha}B^*}(I)$ and for each $\alpha \in \sigma(A)$ and $\lambda \in \sigma(B)$ there exists a θ, $0 \leq \theta < 2\pi$, such that $|\alpha\lambda| = 2(1-cos\theta)$.*

(iv) . If $\triangle^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = 0$, then $\triangle^m_{I-\overline{\lambda}A^*,A-\lambda} = 0 = \nabla^m_{I-\alpha B,B^*-\overline{\alpha}}(I)$ and there exists a non-zero scalar β such that

$$\lambda = \frac{-|\beta|^2 \pm i\sqrt{4\overline{\beta}(1-\overline{\beta}) - |\beta|^4}}{2\overline{\beta}} \text{ and } \alpha = \frac{|\beta|^2 \pm i\sqrt{4\overline{\beta}(1-\overline{\beta}) - |\beta|^4}}{2\overline{\beta}}.$$

(v). If $\triangle^m_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0$, then $\triangle^m_{A^*-\overline{\lambda},I-\lambda A} = 0 = \nabla^m_{B-\alpha,I-\overline{\alpha}B^*}(I)$ and there exists a non-zero scalar β such that

$$\lambda = \frac{-|\beta|^2 \pm i\sqrt{4\beta(1-\beta) - |\beta|^4}}{2\beta} \text{ and } \alpha = \frac{|\beta|^2 \pm i\sqrt{4\beta(1-\beta) - |\beta|^4}}{2\beta}.$$

(vi). If $\delta^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$, then

$$\delta^m_{A^*-\overline{\lambda},A-\lambda}(I) = 0 = \delta^m_{B-\alpha,B^*-\overline{\alpha}}(I),$$

$0 \in \sigma_a(A)$ (resp., $0 \in \sigma_a(B^*)$) implies B^* (resp., A) m-symmetric, and $\sigma_a(A) \subseteq \{r + \lambda : \overline{\lambda} \in \sigma_a(B^*), r \text{ real}\}$.

(vii). If $\delta^m_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0$ and $A - I, B + I$ are non-nilpotent, then

$$\delta^m_{I-\overline{\lambda}A^*,A-\lambda}(I) = 0 = \Diamond^m_{I-\alpha B,B^*-\overline{\alpha}}(I)$$

and there exists a scalar β *such that*

$$\lambda = \frac{-\overline{\beta} \pm i\sqrt{4(\beta-1)-\overline{\beta}^2}}{2} \text{ and } \alpha = \frac{\overline{\beta} \pm i\sqrt{4(\beta-1)-\overline{\beta}^2}}{2}.$$

(viii). If $\delta^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = 0$ *and* $A - I, B + I$ *are non-nilpotent, then*

$$\delta^m_{I-\overline{\lambda}A^*,A-\lambda}(I) = 0 = \diamondsuit^m_{I-\alpha B,B^*-\overline{\alpha}}(I)$$

and there exists a scalar β *such that*

$$\lambda = \frac{-\beta \pm i\sqrt{4(\overline{\beta}-1)-\beta^2}}{2} \text{ and } \alpha = \frac{\beta \pm i\sqrt{4(\overline{\beta}-1)}-\beta^2}{2}.$$

Furthermore, the spectra $\sigma(A)$ *and* $\sigma(B)$ *consist at most of two points in each of the (six) cases.*

Proof. The proof of (iv) and (v), and (vii) and $(viii)$, is similar: we prove (iii), (iv), (vi) and (vii).

Case(iii). We start by proving that $I - \lambda A$ and $I - \overline{\alpha}B^*$ are m-isometric. By definition

$$\begin{aligned} &\triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) \\ = &\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\left[\sum_{p=0}^{j}(-1)^p\begin{pmatrix} j \\ p \end{pmatrix}A^{*p}\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j \\ k \end{pmatrix}A^k\right)B^kB^{*p}\right]. \end{aligned}$$

Hence, for all $x \in \mathcal{H}$,

$$\begin{aligned} 0 &= \left[\lim_{n\to\infty}\triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I)(x\otimes x_n)\right]x_n \\ &= C_{\sum}A^{*p}A^kx\lim_{n\to\infty}\langle B^{*p}x_n, B^{*k}x_n\rangle \\ &= C_{\sum}(\overline{\lambda}A^*)^p(\lambda A)^kx \\ &= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}(I-\overline{\lambda}A^*)^j(I-\lambda A)^jx \\ &= \triangle^m_{I-\overline{\lambda}A^*,I-\lambda A}(I)x. \end{aligned}$$

Hence $\triangle^m_{I-\overline{\lambda}A^*,I-\lambda A}(I) = 0$. A similar argument, working this time with

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left[\triangle^m_{\triangle_{B,A},\triangle_{B^*,A^*}}(I)(x\otimes y_n)\right]y_n \\ &= C_{\sum}B^pB^{*k}x\lim_{n\to\infty}\langle A^py_n, A^ky_n\rangle, \end{aligned}$$

proves $\triangle^m_{I-\alpha B,I-\overline{\alpha}B^*}(I) = 0$.

We claim that $\sigma_a(A)$ and $\sigma_a(B^*)$, hence also $\sigma(A)$ and $\sigma(B)$, consist at most of two points. Assume to the contrary that (alongwith $\overline{\mu}$) $\overline{\nu} \in \sigma_a(B^* - \overline{\lambda})$; $\nu \neq \mu$. Then $\sigma_a(I - (\lambda+\nu)A) \subseteq \partial\mathtt{D}$ and

$$\sigma_a(I-(\lambda+\nu)A) = \sigma_a(I-(\lambda+\mu)A) + (\mu-\nu)\sigma_a(A).$$

Since $\sigma_a(A) \neq \{0\}$ and $\sigma_a(I-(\lambda+\mu)A) \subseteq \partial\mathtt{D}$, $\sigma_a((I-(\lambda+\nu)A)$ is a translate by a non-zero scalar of points in $\partial\mathtt{D}$, hence not a point in the boundary of $\mathtt{D}$. This being a contradiction, our claim is proved.

The operators $I-\lambda A$ and $I-\overline{\alpha}B^*$ being m-isometric, the spectra $\sigma_a(I-\lambda A)$, $\sigma_a(I-\overline{\alpha}B^*)$ are subsets of $\partial\mathbb{D}$ and the operators $I-\lambda A$, $I-\overline{\alpha}B^*$ are (left invertible, hence) invertible.The spectral mapping theorem implies that

$$\sigma(\triangle_{\triangle_{A^*,B^*},\triangle_{A,B}})=0\Longrightarrow \alpha\lambda+\overline{\alpha\lambda}=|\alpha\lambda|^2$$

for $\alpha\in\sigma(A)$ and $\lambda\in\sigma(B)$. Since $\sigma(I-\lambda A)\subseteq\{e^{i\theta}\}$,equivalently, $\lambda\alpha\subseteq\{1-e^{i\theta}\}$, for each $\alpha\in\sigma(A)$ and $\lambda\in\sigma(B)$ there exists a $0\leq\theta<2\pi$ such that $|\alpha\lambda|=2(1-cos\theta)$.

Case (iv)**.** We start by proving $\triangle^m_{I-\overline{\lambda}A^*,A-\lambda}=0=\nabla^m_{I-\alpha B,B^*-\overline{\alpha}}(I)$ for all $\alpha\in\sigma_a(A)$ and $\overline{\lambda}\in\sigma_a(B^*)$. Arguing as above,

$$\begin{aligned}
0 &= \lim_{n\to\infty}\Big[\triangle^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I)(x\otimes x_n\Big]x_n,\ x\in\mathcal{H}\\
&= C_{\sum}A^{*p}A^{j-k}x\lim_{n\to\infty}\langle B^{*p}x_n,B^{*k}x_n\rangle\\
&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}\left(\sum_{p=0}^{j}(-1)^p\begin{pmatrix} j\\ p\end{pmatrix}(\overline{\lambda}A^*)^p\right)\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j\\ k\end{pmatrix}\lambda^kA^{j-k}\right)x\\
&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}(I-\overline{\lambda}A^*)^j(A-\lambda)^jx\\
\Longrightarrow&\quad \triangle^m_{I-\overline{\lambda}A^*,A-\lambda}(I)=0
\end{aligned}$$

and

$$\begin{aligned}
0 &= \lim_{n\to\infty}\Big[\nabla^m_{\triangle_{B,A},\delta_{B^*,A^*}}(I)(x\otimes y_n)\Big]y_n,\ x\in\mathcal{H}\\
&= \sum_{j=0}^{m}\begin{pmatrix} m\\ j\end{pmatrix}\left(\sum_{p=0}^{j}(-1)^p\begin{pmatrix} j\\ p\end{pmatrix}B^p\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j\\ k\end{pmatrix}B^{*(j-k)}\right)x\lim_{n\to\infty}\langle A^py_n,A^ky_n\rangle\right)\\
&= \sum_{j=0}^{m}\begin{pmatrix} m\\ j\end{pmatrix}\left(\sum_{p=0}^{j}(-1)^p\begin{pmatrix} j\\ p\end{pmatrix}(\alpha B)^p\right)\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j\\ k\end{pmatrix}\overline{\alpha}^kB^{*(j-k)}\right)x\\
\Longrightarrow&\quad \nabla^m_{I-\alpha B,B^*-\overline{\alpha}}(I)=0.
\end{aligned}$$

If $\triangle^m_{I-\overline{\lambda}A^*,A-\lambda}(I)=0$, then (by the spectral mapping theorem)

$$1-(1-\overline{\lambda}\overline{\alpha})(\alpha-\lambda)=0;\ \text{all } \alpha\in\sigma_a(A)\text{ and }\overline{\lambda}\in\sigma_a(B^*).$$

Consequently, there exists a non-zero scalar β such that

$$1-\overline{\lambda}\overline{\alpha}=\frac{1}{\beta},\alpha-\lambda=\beta\Longrightarrow\beta\lambda^2+|\beta|^2\lambda+(1-\beta)=0$$

and this (solving for λ and α) implies

$$\lambda=\frac{-|\beta|^2\pm i\sqrt{4\overline{\beta}(1-\overline{\beta})-|\beta|^4}}{2\beta}\text{ and }\alpha=\frac{|\beta|^2\pm i\sqrt{4\overline{\beta}(1-\overline{\beta})-|\beta|^4}}{2\beta},$$

$\sigma(A)=\sigma_a(A)$ and $\sigma(B)=\sigma_a(B)$ consist at most of two points. (In particular, if $0\in\sigma(B)$, then $\beta=1$, $\sigma(A)=\{0,1\}$ and $\sigma(B)=\{-1,0\}$.)

Case (vi)**.** We have

$$\begin{aligned}
0 &= \lim_{n\to\infty}\left[\delta^m_{\delta_{A^*,B^*},\delta_{A,B}}(I)(x\otimes x_n)\right]x_n,\ x\in\mathcal{H}\\
&= C_{\sum_1} A^{*(m-j-p)}A^{j-k}x\lim_{n\to\infty}\langle B^{*p}y_n, B^{*k}y_n\rangle\\
&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}\left(\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j\\ p\end{pmatrix}\overline{\lambda}^pA^{*(m-j-p)}\right)\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j\\ k\end{pmatrix}\lambda^kA^{j-k}\right)x\\
&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}(A^*-\overline{\lambda})^{m-j}(A-\lambda)^jx\\
\Longrightarrow\ &\delta^m_{A^*-\overline{\lambda},A-\lambda}(I)=0.
\end{aligned}$$

Again,

$$\begin{aligned}
0 &= \lim_{n\to\infty}\left[\delta^m_{\delta_{B,A},\delta_{B^*,A^*}}(I)(x\otimes y_n)\right]y_n\\
&= C_{\sum_1} B^{m-j-p}B^{*(j-k)}x\lim_{n\to\infty}\langle A^{p}y_n, A^{k}y_n\rangle\\
&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}\left(\sum_{p=0}^{j}(-1)^p\begin{pmatrix} m-j\\ p\end{pmatrix}\alpha^pB^{m-j-p}\right)\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j\\ k\end{pmatrix}(\overline{\alpha}^kB^{*(j-k)}\right)x\\
&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}(B-\alpha)^{m-j}(B^*-\overline{\alpha})^jx\\
\Longrightarrow\ &\delta^m_{B-\alpha,B^*-\overline{\alpha}}(I)=0.
\end{aligned}$$

Thus

$$\delta^m_{A^*-\overline{\lambda},A-\lambda}(I)=0=\delta^m_{B-\alpha,B^*-\overline{\alpha}}(I)$$

for all $\alpha\in\sigma_a(A)$ and $\overline{\lambda}\in\sigma_a(B^*)$.

Evidently, $A-\lambda$ and $B^*-\overline{\alpha}$ are *m*-symmetric. Furthermore, since $\delta^m_{A^*-\overline{\lambda},A-\lambda}(I)=0$ implies $\overline{(\alpha-\lambda)}-(\alpha-\lambda)=0$, $\Im\alpha=\Im\lambda$ for all α and λ. Hence

$$\sigma_a(A)\subseteq\{r+\lambda:\overline{\lambda}\in\sigma_a(B^*), r\text{ real}\}.$$

Trivially, if $0\in\sigma_a(A)$, then B^* is *m*-symmetric, and if $0\in\sigma_a(B^*)$, then A is *m*-symmetric. The set $\{\sigma(A)-\lambda\}$ being a compact set, there exist scalars β_i, $1\le i\le 2$, such that ($\beta_1\le\beta_2$ and) $\sigma(A)-\lambda\subseteq[\beta_1,\beta_2]$. Suppose that there exist scalars $\overline{\mu},\overline{\nu}\in\sigma_a(B^*-\overline{\lambda})$, $\mu\ne\nu$. Then ($\overline{\lambda+\mu}$ and $\overline{\lambda+\nu}\in\sigma_a(B^*)$, hence) $\sigma_a(A-\lambda-\mu)=\sigma(A-\lambda-\mu)$ and $\sigma_a((A-\lambda-\nu)=\sigma(A-\lambda-\nu)$ are subsets of $[\beta_1,\beta_2]$. Since

$$\begin{aligned}
&\sigma(A-\lambda-\nu)=\sigma(A-\lambda-\mu)+(\mu-\nu),\ \text{where } 0\ne\mu-\nu \text{ is real},\\
&\beta_1-(\mu-\nu)<\beta_1 \text{ if } (\mu-\nu)>0 \text{ and } \beta_2-(\mu-\nu)>\beta_2 \text{ if } (\mu-\nu)>0,
\end{aligned}$$

we have a contradiction. Conclusion: $\sigma(A)$ and $\sigma(B)$ consist, at most, of two points.

Case(vii). In this case

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$$\begin{aligned}
0 &= \lim_{n\to\infty}\left[\delta^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I)(x\otimes x_n)\right]x_n,\ x\in\mathcal{H} \\
&= C_{\Sigma_1}A^{*p}A^{j-k}x\lim_{n\to\infty}\langle B^{*p}x_n, B^{*k}x_n\rangle \\
&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\left(\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j \\ p \end{pmatrix}(\overline{\lambda})^pA^{*p}\right)\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j \\ k \end{pmatrix}\lambda^kA^{j-k}\right)x \\
&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}(I-\overline{\lambda}A^*)^j(A-\lambda)^jx \\
\Longrightarrow\ & \delta^m_{I-\overline{\lambda}A^*,A-\lambda}(I)=0.
\end{aligned}$$

Let

$$C_{\Sigma_0}=C(m,j,p,k)=\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\left[\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j \\ p \end{pmatrix}\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j \\ k \end{pmatrix}\right)\right].$$

Then, for all $x\in\mathcal{H}$,

$$\begin{aligned}
0 &= \lim_{n\to\infty}\left[\Diamond^m_{\delta_{B,A},\delta_{B^*,A^*}}(I)(x\otimes y_n)\right]y_n \\
&= C_{\Sigma_0}B^pB^{*(j-k)}x\lim_{n\to\infty}\langle A^py_n, A^ky_n\rangle \\
&= \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\left(\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j \\ p \end{pmatrix}\alpha^pB^p\right)\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j \\ k \end{pmatrix}(\overline{\alpha}^kB^{*(j-k)}\right)x \\
&= \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}(I-\overline{\lambda}B^*)^{m-j}(B-\lambda)^jx \\
\Longrightarrow\ & \Diamond^m_{I-\lambda B,B^*-\overline{\lambda}}(I)=0.
\end{aligned}$$

This proves $\delta^m_{I-\overline{\lambda}A^*,A-\lambda}(I)=0=\Diamond^m_{I-\alpha B,B^*-\overline{\alpha}}(I)$.

The conclusion $\delta^m_{I-\overline{\lambda}A^*,A-\lambda}(I)=0$ implies (by the spectral mapping theorem) that $(1-\overline{\alpha}\overline{\lambda})-(\alpha-\lambda)=0$ for all non-zero $\alpha\in\sigma_a(A)$ and $\overline{\lambda}\in\sigma_a(B^*)$. Hence there exists a non-zero scalar β such that

$$\begin{aligned}
\alpha-\lambda=\beta=1-\overline{\alpha}\overline{\lambda} &\Longrightarrow \lambda^2+\beta\lambda+(\overline{\beta}-1)=0 \\
\Longrightarrow\ \lambda &= \frac{-\beta\pm i\sqrt{4(\overline{\beta}-1)-\beta^2}}{2}\ \text{and} \\
\alpha &= \frac{\beta\pm i\sqrt{4(\overline{\beta}-1)-\beta^2}}{2}.
\end{aligned}$$

Evidently, $\sigma_a(A)=\sigma(A)$ and $\sigma_a(B)=\sigma(B)$ consist, at most, of two points (which, if $0\in\sigma(B)$, so that $\beta=1$, are respectively the sets $\{0,1\}$ and $\{-1,0\}$; observe that $I-A$ is m-nilpotent if $0\in\sigma(B)$ and $I+B$ is m-nilpotent if $0\in\sigma(A)$). □

The conclusions of Theorem 3.2 help in building a picture of the operators A,B satisfying $D^m_{d_1,d_2}(I)=0$. The following examples consider cases (iii), (iv), (vi) and (vii) of the theorem.

Example 3.3. 0 *may be in both* $\sigma_a(A)$ *and* $\sigma_a(B^*)$ *in the case in which* $\triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$*: consider, for example, the operators* A, B *such that* $\sigma_a(A) = \{0, \frac{1-e^{-i\theta}}{a}\}$, $\sigma_a(B^*) = \{0, a\}$, $A = 0I_1 \oplus \frac{1-e^{-i\theta}}{a} I_2$ *and* $B = 0I_1 \oplus aI_2$, $0 \neq a$ *real and* $0 < \theta < 2\pi$*. It is seen that*

$$\begin{aligned}
& \triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) \\
= \;& C_{\sum}\left(0I_1 \oplus \frac{1-e^{i\theta}}{a} I_2\right)^p \left(0I_1 \oplus \frac{1-e^{-i\theta}}{a} I_2\right)^k \left(0I_1 \oplus a^{p+k} I_2\right) \\
= \;& C_{\sum}\left(0I_1 \oplus (1-e^{i\theta})^p (1-e^{-i\theta})^k I_2\right) = \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} (I_1 \oplus e^{i\theta} I_2)^j (I_1 \oplus e^{-i\theta} I_2)^j \\
= \;& 0.
\end{aligned}$$

Example 3.4. *If* $\triangle^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = 0$*, then* 0 *may belong to both* $\sigma_a(A)$ *and* $\sigma_a(B^*)$ *(consider, for example the operators* $A = I_1 \oplus 0I_2$ *and* $B = 0I_1 \oplus -I_2$*), or,* 0 *may just be in one of* $\sigma_a(A)$ *and* $\sigma_a(B^*)$ *(consider the operators* $A = I_1 \oplus (1 - \sqrt{\frac{3}{2}}) I_2$ *and* $B = 0I_1 \oplus (-1 - \sqrt{\frac{3}{2}}) I_2$*).*

Example 3.5. *In the case in which* $\delta^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0, 0 \in \sigma_a(B^*)$ *implies* A *is* m*-symmetric (thus* $\sigma(A)$ *is real) and* $0 \in \sigma_a(A)$ *implies* B^* *is* m*-symmetric (thus* $\sigma(B)$ *is real). If* a, b, r_1, r_2 *are real numbers,* $\sigma(A) = \{a, r_1 + ib\}$*, and* $\sigma(B) = \{0, r_2 + ib\}$*, then* $A = aI_1 \oplus (r_1 + ib) I_2$ *and* $B = 0I_1 \oplus (r_2 + ib) I_2$ *is an example of a pair of operators satisfying* $\delta^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$*. Again, if we let* $\beta = i$ *and* $A = I_1 \oplus (-1 - i) I_2$, $B = (1+i) I_1 \oplus -I_2$ *(in the proof of case* (vii)*), then* $\delta_{A,B}(I) = -iI$, $\triangle_{A^*,B^*}(I) = iI$ *and* $\delta_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = (I - L_{iI} R_{-iI})(I) = 0$*.*

4. Equivalence $D^m_{d_1,d_2}(I) = 0 \Longleftrightarrow D_{d_1,d_2}(I) = 0$ and The Putnam-Fuglede Property

It is well known (indeed, easily proven) that if a pair (A, B) of (Banach space) operators is m-isometric (similary, m-symmetric), then it is n-isometric (resp., n-symmetric) for all integers $n \geq m$ [18]. The proposition that (A, B) is m-isometric (or, m-symmetric) implies it is n-isometric (resp., n-symmetric) for some positive integer $n < m$ fails. A prime example here is that of *strictly* m*-isometric* (resp., *strictly* m*-symmetric*) operators, where an operator A is strictly m-isometric (resp., strictly m-symmetric) if $\triangle^m_{A^*,A}(I) = 0$ and $\triangle^{m-1}_{A^*,A}(I) \neq 0$ (resp., $\delta^m_{A^*,A}(I) = 0$ and $\delta^{m-1}_{A^*,A}(I) \neq 0$). Partial results exist. Thus, if $T \in B(\mathcal{H})$ is such that $\delta^m_{T^*,T}(I) = 0$ for some even positive integer m (resp., $\triangle^m_{T^*,T}(I) = 0$ for some invertible $T \in B(\mathcal{H})$ and a positive even integer m), then $\delta^{m-1}_{T^*,T}(I) = 0$ (resp., $\triangle^{m-1}_{T^*,T}(I) = 0$); see [22], Theorem 3 and Proposition 1, respectively. Again, if T is power bounded, i.e. $\sup_n ||T^n|| < M$ for some positive real number M, then $\triangle^m_{T^*,T}(I) = 0$ if and only if $\triangle_{T^*,T}(I) = 0$ [21]. Equivalently, if T is power bounded and $\triangle^m_{T^*,T}(I) = 0$, then T is isometric. More generally, if $S \in B(\mathcal{H})$ is a power bounded operator such that $\triangle^m_{T,S}(I) = 0$ for some operator $T \in B(\mathcal{H})$ and positive integer m, then S is similar to an isometric operator (where we may choose the invertible operator effecting similarity to be a positive operator) and T is similar to S^* [15, Theorem 2.4]. The following theorem considers the equivalence $D^m_{d_1,d_2}(I) = 0$ if and only if $D_{d_1,d_2}(I) = 0$ for the operators D, d_1 and d_2 of Theorem 3.2. It is seen that just as for the cases $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$ and $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ considered in [22], a necessary and sufficient condition for the required equivalence is that the operators A and B satisfy a Putnam-Fuglede commuatativity property. Before, however, going on to state the theorem, we consider a few examples.

Example 4.1. *Given* $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$*, let* $\sigma_a(B^*) = \{0, \mu\}$ *for somel non-zero real number* μ *. Then*

$$\begin{aligned}
\sigma_a(A - \mu) = \sigma_a(A - (0 + \mu)) &= \frac{\mu \pm i\sqrt{4 - \mu^2}}{2} \\
\Longrightarrow \quad \sigma_a(A) &= \{\frac{-\mu \pm i\sqrt{4 - \mu^2}}{2}\} = \{e^{i\theta}, -e^{-i\theta}\}
\end{aligned}$$

for some $0 \leq \theta \leq 2\pi$. *Letting*

$$A = e^{i\theta} I_1 \oplus -e^{-\theta} I_2 \text{ and } B = 0I_1 \oplus \mu I_2,$$

it is seen that

$$\begin{aligned} \triangle^1_{\delta_{A^*,B^*},\delta_{A,B}}(I) &= I - \Big((e^{-i\theta} I_1 \oplus -(e^{i\theta}+\mu)I_2)(e^{i\theta} I_1 \oplus -(e^{-i\theta}+\mu)I_2)\Big) \\ = \quad & I - \Big(I_1 \oplus (1+\mu(e^{i\theta}+e^{-i\theta})\mu^2)I_2\Big) = 0, \end{aligned}$$

since $e^{i\theta} + e^{-i\theta} = -\mu$. *Define operators* $E, F \in B(B(\mathcal{H}))$ *by* $E = \delta_{A^*,B^*}$ *and* $F = \delta_{A,B}$. *Let* $N \in B(\mathcal{H})$ *be an n-nilpotent operator,* $N = N_1 I_{\oplus} N_2 I_2$; *let* $L_{N^*} = \mathcal{N}_1$ *and* $L_N = \mathcal{N}_2$. *Then* $\delta_{A^*+N^*,B^*} = \delta_{A^*,B^*} + L_{N^*} = E + \mathcal{N}_1$ *and* $\delta_{A+N,B} = F + \mathcal{N}_2$, *where* $\mathcal{N}_i$, $1 \leq i \leq 2$, *are n-nilpotent operators. Trivially,* $\triangle^1_{A^*-\mu,A-\mu}(I) = 0$ *and, since N is n-nilpotent,* $\triangle^{2n-1}_{A^*+N^*-\mu,A+N-\mu}(I) = 0$ *(see [7,18]). Similary,* $\triangle^{2n-1}_{B+N-e^{-i\theta},B^*+N^*-e^{-i\theta}}(I) = 0$. *However, neither of* $\triangle^1_{E+\mathcal{N}_1,F+\mathcal{N}_2}(I)$, $\triangle^1_{A^*+N^*-\mu,A+N-\mu}(I)$ *and* $\triangle^1_{B+N-e^{i\theta},B^*+N^*-e^{-i\theta}}(I)$ *is the* 0 *operator.*

Example 4.2. *For some non-zero real numbers* r, r_1, r_2 *and* $0 \leq \theta < 2\pi$, *let* $\sigma(A) = \{\alpha_1, \alpha_2\} = \{r_1 e^{-i\theta}, r_2 e^{-i\theta}\}$, $\sigma(B) = \{\lambda\} = \{re^{i\theta}\}$, $A = re^{-i\theta} I_1 \oplus r_2 e^{-i\theta} I_2$ *and* $B = re^{i\theta}(I_1 \oplus I_2)$.*Then*

$$\delta^1_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = (I - A^*B^*) - (I - AB) = 0.$$

Let $N = N(I_1 \oplus I_2)$ *be an n-nilpotent operator in* $B(\mathcal{H})$. *Let* $\triangle_{A^*,B^*} = E$, $\triangle_{A,B} = F$, $\triangle_{A^*+N^*,B^*} = I - L_{A^*}R_{B^*} - L_{N^*}R_{B^*} = E + \mathcal{N}_1$ *and* $\triangle_{A+N,B} = F + \mathcal{N}_2$. *Then,* $\mathcal{N}_i$, $1 \leq i \leq 2$, *are n-nilpotent operators such that* $\mathcal{N}_1$ *commutes with E,* $\mathcal{N}_2$ *commutes with F and , see [18],* $\delta^{2n-2}_{E+\mathcal{N}_1,F+\mathcal{N}_2}(I) = 0$. *Evidently,*

$$\delta^1_{I-\overline{\lambda}A^*,I-\lambda A}(I) = 0 = \delta^1_{I-\alpha B,I-\overline{\alpha}B^*}(I),$$

hence

$$\delta^{2n-1}_{I-\overline{\lambda}(A^*+N^*),I-\lambda(A+N)}(I) = 0 = \delta^{2n-1}_{I-\alpha_1 B,I-\overline{\alpha_2}B^*}(I).$$

However, neither of $\delta^1_{I-\overline{\lambda}(A^*+N^*),I-\lambda(A+N)}(I)$ *and* $\delta^1_{\triangle_{A^*+N^*,B^*},\triangle_{A+N,B}}(I)$ *is the* 0 *operator.*

Observe here that the operator $A + N$ is not normal in either of the above examples.

Theorem 4.3. *If* $A, B \in B(\mathcal{H})$ *are such that* $\sigma(A) \neq \{0\} \neq \sigma(B)$, $\alpha \in \sigma_a(A)$ *and* $\overline{\lambda} \in \sigma_a(B^*)$, *then:*

(i) $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$ *implies* $\triangle_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$ *if and only if the pairs of operators* $(A^* - \overline{\lambda}, A - \lambda)$ *and* $(B - \alpha, B^* - \overline{\alpha})$ *are* (PF)*-pairs.*

(ii) $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ *implies* $\delta_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ *if and only if the pairs of operators* $(I - \overline{\lambda}A^*, I - \lambda A)$ *and* $(I - \alpha B, I - \overline{\alpha}B^*)$ *are* (PF)*-pairs.*

(iii) $\triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ *implies* $\triangle_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ *if and only if the pairs of operators* $(I - \overline{\lambda}A^*, I - \lambda A)$ *and* $(I - \alpha B, I - \overline{\alpha}B^*)$ *are* (PF)*-pairs.*

(iv) $\triangle^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = 0$ *implies* $\triangle_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = 0$ *if and only if the pairs of operators* $(I - \overline{\lambda}A^*, A - \lambda)$ *and* $(I - \alpha B, \overline{\alpha} - B^*)$ *are* (PF)*-pairs.*

(v) $\triangle^m_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0$ *implies* $\triangle_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0$ *if and only if the pairs of operators* $(A^* - \overline{\lambda}, I - \lambda A)$ *and* $(B - \alpha, \overline{\alpha}B^* - I)$ *are* (PF)*-pairs.*

(vi) $\delta^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$ *implies* $\delta_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$ *if and only if the pairs of operators* $(A^* - \overline{\lambda}, A - \lambda)$ *and* $(B - \alpha, B^* - \overline{\alpha})$ *are* (PF)*-pairs.*

(vii) $\delta^m_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0$ *implies* $\delta_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0$ *if and only if the pairs of operators* $(A^* - \overline{\lambda}, I - \lambda A)$ *and* $(B - \alpha, \overline{\alpha}B^* - I)$ *are* (PF)*-pairs.*

(viii) $\delta^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = 0$ *implies* $\delta_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = 0$ *if and only if the pairs of operators* $(I - \overline{\lambda}A^*, A - \lambda)$ *and* $(I - \alpha B, \overline{\alpha} - B^*)$ *are* (PF)*-pairs.*

Proof. Before going on to prove the theorem, we make a few observations (which we will use in the sequel without further reference). As seen in the proofs of Proposition 3.1 and Theorem 3.2, if $D^m_{d_1,d_2}(I) = 0$ for any of the choices $D = \triangle$ or δ, $d_1 = \triangle_{A^*,B^*}$ or δ_{A^*,B^*} and $d_2 = \triangle_{A,B}$ or $\delta_{A,B}$, then the spectra $\sigma(A)$ and $\sigma(B)$ consist at most of two points. Consequently, $\sigma_a(A) = \sigma(A)$ and $\sigma_a(B) = \sigma(B)$; also, if $I - \lambda A$ or $\lambda - A$, similarly $I - \lambda B$ or $\lambda - B$, is left invertible for some scalar λ, then it is invertible. Furthermore, if A (similarly, B) is normal, then the spectral points of A (resp., B) are (simple poles of the resolvent of the operator, hence) normal eigenvalues of the operator. The operator A (resp., B) has a direct sum decomposition $A = \alpha_1 I_1 \oplus \alpha_2 I_2$ (resp., $B = \lambda_1 I_1 \oplus \lambda_2 I_2$), $I = I_1 \oplus I_2$, and to every $x \in \mathcal{H}$ there corresponds an $\alpha = \alpha_1$ or α_2 (resp., $\lambda = \lambda_1$ or λ_2) such that $Ax = \alpha x$ (resp., $Bx = \lambda x$).

The proof for almost all the cases is similar, execept for differences in detail: we prove below cases (i) to (iii) in some detail and provide a brief outline of the proof for cases (v) and(vi).

(i). If $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$ implies $\triangle_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$, then $\triangle_{A^*-\overline{\lambda},A-\lambda}(I) = 0 = \triangle_{B-\alpha,B^*-\overline{\alpha}}(I)$ for all $\alpha \in \sigma(A)$ and $\lambda \in \sigma(B)$ (see the proof of Proposition 3.1). The conclusion

$$\triangle_{A^*-\overline{\lambda},A-\lambda}(I) = 0 \Longrightarrow A - \lambda \text{ is left invertible} \Longrightarrow A - \lambda \text{ is invertible}$$

and

$$\triangle_{B-\alpha,B^*-\overline{\alpha}}(I) = 0 \Longrightarrow B - \alpha \text{ is right invertible} \Longrightarrow B - \alpha \text{ is invertible.}$$

(Indeed $A - \lambda$ and $B - \alpha$ are unitary.) Consequently,

$$\begin{aligned} &(A^* - \overline{\lambda})(A - \lambda) = I = (A - \lambda)(A^* - \overline{\lambda}) \Longrightarrow A \text{ is normal and} \\ &(B - \alpha)(B^* - \overline{\alpha}) = I = (B^* - \overline{\alpha}) \Longrightarrow B \text{ is normal.} \end{aligned}$$

Hence $(A^* - \overline{\lambda}, A - \lambda)$ and $(B^* - \overline{\alpha}, B - \alpha)$ are (PF)-pairs.

To prove the sufficiency, we recall from Proposition 3.1 that

$$\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0 \Longrightarrow \triangle^m_{A^*-\overline{\lambda},A-\lambda}(I) = 0 = \triangle^m_{B-\alpha,B^*-\overline{\alpha}}(I)$$

for all $\alpha \in \sigma(A)$ and $\lambda \in \sigma(B)$. The hypothesis on the (PF)-property implies (that the ascent is less than or equaltoone,hence)

$$\triangle_{A^*-\overline{\lambda},A_\lambda}(I) = 0 = \triangle_{A-\lambda,A^*-\overline{\lambda}}(I)$$

and

$$\triangle_{B-\alpha,B^*-\overline{\alpha}}(I) = 0 = \triangle_{B^*-\overline{\alpha},B-\alpha}(I).$$

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Consequently, A, B are normal, $A = \oplus_{i=1}^{2}\alpha_i I_i$, $B = \oplus_{i=1}^{2}\lambda_i I_{1i}$ ($I_1 \oplus I_2 = I = I_{11} \oplus I_{12}$), and for each $x \in \mathcal{H}$ there exists a λ ($= \lambda_1$ or λ_2) such that

$$\begin{aligned} & \langle \triangle_{\delta_{A^*,B^*},\delta_{A,B}}(I)x, x\rangle = \langle (I - A^*A + A^*B - B^*A + B^*B)x, x\rangle \\ = \quad & \langle (I - A^*A + \lambda A^* - \overline{\lambda}A + |\lambda|^2)x, x\rangle = \langle \triangle_{A^*-\overline{\lambda},A-\lambda}(I)x, x\rangle \\ = \quad & 0. \end{aligned}$$

This proves the sufficiency. (Observe that we also have $\triangle_{\delta_{A,B},\delta_{A^*,B^*}}(I) = 0$.)

(ii). If $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ implies $\delta_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$, then $\delta_{I-\overline{\lambda}A^*,I-\lambda A}(I) = 0 = \delta_{I-\alpha B,I-\overline{\alpha}B^*}(I)$ for all $\alpha \in \sigma(A)$ and $\lambda \in \sigma(B)$. Hence $I - \lambda A$ and $I - \alpha B$ are self-adjoint. Consequently, A and B are normal, and the necessity is proved.

For sufficiency, we start by recalling $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ implies $\delta^m_{I-\overline{\lambda}A^*,I-\lambda A}(I) = 0 = \delta^m_{I-\alpha B,I-\overline{\alpha}B^*}(I)$ for all $\alpha \in \sigma(A)$ and $\lambda \in \sigma(B)$. The (PF)-property hypothesis thus implies $\delta_{I-\overline{\lambda}A^*,I-\lambda A}(I) = 0 = \delta_{I-\lambda A,I-\overline{\lambda}A^*}(I)$ and $\delta_{I-\alpha B,I-\overline{\alpha}B^*}(I) = 0 = \delta_{I-\overline{\alpha}B^*,I-\alpha B}(I)$, A and B are normal operators with direct sum representaions of type $A = \oplus_{i=1}^{2}\alpha_i I_i$ and $B = \oplus_{i=1}^{2}\lambda_i I_{1i}$. Furthermore, for each $x \in \mathcal{H}$, there exists an λ ($= \lambda_1$ or λ_2) such that $Bx = \lambda x$. Since this implies

$$\begin{aligned} & \langle \left(\delta_{\triangle_{A^*,B^*},\triangle_{A,B}}(I)\right)x, x\rangle = \langle (AB - A^*B^*)x, x\rangle \\ = \quad & \langle (\lambda A - \overline{\lambda}A^*)x, x\rangle = \langle \delta_{I-\overline{\lambda}A^*,I-\lambda A}(I)x, x\rangle = 0, \end{aligned}$$

the proof is complete. (Remark here that $\delta_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = -\delta_{\triangle_{A,B},\triangle_{A^*,B^*}}(I)$, hence also $\delta_{\triangle_{A,B},\triangle_{A^*,B^*}}(I) = 0$.)

(iii). We start by proving the sufficiency of the conditions. If $\alpha \in \sigma_a(A)$ and $\overline{\lambda} \in \sigma_a(B^*)$, then (see Theorem 3.2)

$$\triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0 \Longrightarrow \triangle^m_{I-\overline{\lambda}A^*,I-\lambda A} = 0 = \triangle^m_{I-\alpha B,I-\overline{\alpha}B^*}(I),$$

hence by the (PF)-property

$$\begin{aligned} & \triangle_{I-\overline{\lambda}A^*,I-\lambda A}(I) = 0 = \triangle_{I-\alpha B,I-\overline{\alpha}B^*}(I) \text{ and} \\ & \triangle_{I-\lambda A,I-\overline{\lambda}A^*}(I) = 0 = \triangle_{I-\overline{\alpha}B^*,I-\alpha B}(I). \end{aligned}$$

Consequently,

$$\begin{aligned} & I - (I - \overline{\lambda}A^*)(I - \lambda A) = 0 = I - (I - \lambda A)(I - \overline{\lambda}A^*) \\ \Longleftrightarrow \quad & \lambda A + \overline{\lambda}A^* = |\lambda|^2 A^*A = |\lambda|^2 AA^* \end{aligned}$$

and

$$\begin{aligned} & I - (I - \alpha B)(I - \overline{\alpha}B^*) = 0 = I - (I - \overline{\alpha}B^*)(I - \alpha B) \\ \Longleftrightarrow \quad & \alpha B + \overline{\alpha}B^* = |\alpha|^2 BB^* = |\alpha|^2 B^*B. \end{aligned}$$

Thus A, B are normal operators, $A = \alpha_1 I_1 \oplus \alpha_2 I_2$, $B = \lambda_1 I_{11} \oplus \lambda_2 I_{12}$ (for some not necessarily distinct scalars $\alpha_i, \lambda_i; 1 \leq i \leq 2$). Furthermore, to each $x \in \mathcal{H}$, there corresponds a $\lambda(= \lambda_1$ or $\lambda_2)$ such that $Bx = \lambda x$. We have

$$\langle \left(\triangle^1_{\triangle_{A,B},\triangle_{A^*,B^*}}(I)\right)x, x\rangle = \langle \left(\overline{\lambda}A^* + \lambda A - |\lambda|^2 A^*A\right)x, x\rangle = 0$$

for all $x \in \mathcal{H}$. Hence, also, $\triangle^1_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$. (Remark here that

$$\begin{aligned}\triangle^1_{\triangle_{A^*,B^*}.\triangle_{A,B}}(I) &= (L_{A^*}R_{B^*} + L_A R_B - L_{A^*}L_A R_{B^*}R_B)(I)\\ &= A^*B^* + AB - -A^*AB^*B\\ &= AB + A^*B^* - AA^*BB^* = (L_A R_B + L_{A^*}R_{B^*} - L_A L_{A^*} R_B R_{B^*})(I)\\ &= \triangle^1_{\triangle_{A,B},\triangle_{A^*,B^*}}(I),\end{aligned}$$

hence $\triangle^1_{\triangle_{A,B},\triangle_{A^*,B^*}}(I) = 0$.)

To prove the necessity of the condition, we start,by observing that if $\triangle^1_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ and $0 \in \sigma_a(B^*)$, then $(I - A) = 0$ (argue as in the proof of Theorem 3.2), hence A is normal, and $\triangle_{\triangle_{I,B^*},\delta_{I,B}}(I) = I - (I^*_B)(I - B) = 0$. Thus, $I - B$ is left invertible, hence invertible. But then $(I - B^*)(I - B) = I = (I - B)(I - B^*)$, which implies B is (also) normal. A similar argument shows that if $0 \in \sigma_a(A)$, then A and B are normal. Considering now non-zero $\alpha \in \sigma_a(A)$ and $\overline{\lambda} \in \sigma_a(B^*)$, $\triangle^1_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ iimplies

$$\begin{gathered}\triangle_{I-\overline{\lambda}A^*,I-\lambda A}(I) = 0,\\ \triangle_{I-\alpha B,I-\overline{\alpha}B^*}(I) = 0.,\end{gathered}$$

i.e., $I - \lambda A$ and $I - \overline{\lambda}B^*$ are left invertible, hence invertible. We have

$$\begin{aligned}& (I - \overline{\lambda}A^*)(I - \lambda A) = I = (I - \lambda A)(I - \overline{\lambda}A^*)\\ \iff\ & \overline{\lambda}A^* + \lambda A - |\lambda|^2 A^*A = \overline{\lambda}A^* + \lambda A - |\lambda|^2 AA^*, \text{ and}\\ & (I - \alpha B)(I - \overline{\alpha}B^*) = I = (I - \overline{\alpha}B^*)(I - \alpha B)\\ \iff\ & \alpha B + \overline{\alpha}B^* - |\alpha|^2 BB^* = \alpha B + \overline{\alpha}B^* - |\alpha|^2 B^*B,\end{aligned}$$

i.e., once again the operators A and B are normal. Hence $(I - \overline{\lambda}A^*, I - \lambda A)$ and $(I - \overline{\lambda}B^*, I - \lambda B)$ are (PF)-pairs for all α and λ..

An outline proof of (vi) and (vii). If $\delta^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$ implies $\delta_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$, then

$$\delta_{A^*-\overline{\lambda},A-\lambda}(I) = 0 = \delta_{B-\alpha,B^*-\overline{\alpha}}(I) \Longrightarrow A - \lambda \text{ and } B - \alpha \text{ self} - \text{adjoint}.$$

Hence A, B are normal and the necessity follows. For sufficiency,

$$\begin{aligned}& \delta^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0 \Longrightarrow \delta^m_{A^*-\overline{\lambda},A-\lambda}(I) = 0 = \delta^m_{B-\alpha,B^*-\overline{\alpha}}(I)\\ \Longrightarrow\ & \delta_{A^*-\overline{\lambda},A-\lambda}(I) = 0 = \delta_{B-\alpha,B^*-\overline{\alpha}}(I)\\ \Longrightarrow\ & A, B \text{ normal} \Longrightarrow \langle\Big(\delta_{\delta_{A^*,B^*},\delta_{A,B}}(I)\Big)x, x\rangle,\ x \in \mathcal{H},\\ \Longrightarrow\ & \langle\delta_{A^*-\overline{\lambda},A-\lambda}(I)x, x\rangle = 0\\ \Longrightarrow\ & \delta_{\delta_{A^*,B^*}\delta_{A,B}}(I) = 0.\end{aligned}$$

If $\delta^m_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0$ implies $\delta_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0$, then

$$\begin{aligned}& \delta_{I-\overline{\lambda}A^*,A-\lambda}(I) = 0 = \diamondsuit_{I-\alpha B,B^*-\overline{\alpha}}(I)\\ \Longrightarrow\ & (I - \overline{\lambda}A^*)(\lambda - A) = (\lambda - A)(I - \overline{\lambda}A^*), (I - \alpha B)(B^* - \overline{\alpha}) = (B^* - \overline{\alpha})(I - \alpha B)\\ \Longrightarrow\ & A, B \text{ are normal}.\end{aligned}$$

(Notice here that if $0 \in \sigma(A)$, then $B = I$ and $A^* = A$, and if $0 \in \sigma(B)$, then $A = I$ and $B = B^*$.) This proves the necessity. For sufficiency,

$$\begin{aligned} & \delta^m_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0 \Longrightarrow \delta^m_{I-\overline{\lambda}A^*,A-\lambda}(I) = 0 = \Diamond^m_{I-\alpha B,B^*-\overline{\alpha}}(I) \\ \Longrightarrow \quad & \delta_{I-\overline{\lambda}A^*,A-\lambda}(I) = 0 = \Diamond_{I-\alpha B,B^*-\overline{\alpha}}(I) \\ \Longrightarrow \quad & A, B \text{ normal} \Longrightarrow \langle\Big(\delta_{\delta_{A^*,B^*},\triangle_{A,B}}(I)\Big)x, x\rangle,\ x \in \mathcal{H}, \\ \Longrightarrow \quad & \langle\delta_{I-\overline{\lambda}A^*,A-\lambda}(I)x, x\rangle = 0 \\ \Longrightarrow \quad & \delta_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0. \end{aligned}$$

□

Remark 4.4. If $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$, then Theorem 3.2 implies $0 \notin \sigma(A-\lambda) \wedge \sigma(B^* - \overline{\alpha})$ for all $\alpha \in \sigma(A)$ and $\lambda \in \sigma(B)$, and $A - \lambda$ and $B^* - \overline{\alpha}$ are invertible m-isometries. As such, $A - \alpha$ and $B^* - \overline{\lambda}$ (hence, also $B - \lambda$) are generalised scalar operators (see [19, Lemma 2.9], [9]). Since generalised scalar operators are polaroid, and, since (for a non-constant) polynomial $p(.)$, $p(T)$ is polaroid for an operator T if and only if T is polaroid [5], A, B are polaroid. If $\sigma(A) = \{\alpha_1, \alpha_2\}$ and $\sigma(B) = \{\lambda_1, \lambda_2\}$, then there exists a positive integer n (equal to the maximum of the order of the poles at the points α_i and λ_i, $1 \le i \le 2$) and nilpotent operators N_i, $1 \le i \le 2$, such that

$$A = \Big(A|_{(A-\alpha_1)^{-n}(0)} \oplus A|_{(A-\alpha_2)^{-n}(0)}\Big) + N_1$$

and

$$B = \Big(B|_{(B-\lambda_1)^{-n}(0)} \oplus B|_{(B-\lambda_2)^{-n}(0)}\Big) + N_2.$$

(Here, either of the components in the representation of A, resp. B, may be absent.) This known representation of operators A and B, see [19], extends to operators A, B such that $\triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$.

Proposition 4.5. *Let $A, B \in B(\mathcal{H})$ be such that $\sigma(A) \ne \{0\} \ne \sigma(B)$. If $\triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$, then there exist scalars α_i and λ_i, nilpotents N_i ($1 \le i \le 2$) and a positive integer n such that*

$$A = \Big(A|_{(A-\alpha_1)^{-n}(0)} \oplus A|_{(A-\alpha_2)^{-n}(0)}\Big) + N_1$$

and

$$B = \Big(B|_{(B-\lambda_1)^{-n}(0)} \oplus B|_{(B-\lambda_2)^{-n}(0)}\Big) + N_2.$$

(Either of the components in the representation of A, resp. B, may be absent.)

Proof. The hypothesis $\triangle^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ implies $I - \lambda A$ and $I - \overline{\alpha}B^*$ are m-isometric left invertible operators, hence invertible operators (since the spectrum consists at most of two poits). This implies $I - \lambda A$ and $I - \overline{\alpha}B^*$ are generalized scalar, hence polaroid, operators. Consequently, A and$B$ are polaroid, and if $\sigma(A) = \{\alpha_1, \alpha_2\}$ and $\sigma(B) = \{\lambda_1, \lambda_2\}$, then A and B have the representation of the statement of the proposition. □

Do operators A and B satisfying the hypotheses of the remaining cases of Theorem 3.2 have analogous representations? We do not know the answer to this question. However, the following proposition shows that there is an answer in the affirmative for certain choices of the spectral points α and λ, and m (for the remaining cases). In the following proposition, the scalar β shall refer to the scalar β (of the corresponding case) of the statement of Theorem 3.2. The scalar n in the statement of the proposition is not necessarily the same in the direct sum representations of the operators A and B.

Proposition 4.6. *Let $A, B \in B(\mathcal{H})$ be such that $\sigma(A) \ne \{0\} \ne \sigma(B)$.*

(a) If either $\triangle^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = 0$, *or* $\triangle^m_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0$, $-1 \in \sigma(B)$ *(or* $1 \in \sigma(A)$ *or* $|\beta| = 1$*), then*

$$A = \left(A|_{(A-1)^{-n}(0)} \oplus A|_{(A+1-e^{i\theta})^{-n}(0)}\right) + N_1 \text{ and}$$
$$B = \left(B|_{(B+1)^{-n}(0)} \oplus B|_{(B-1+e^{i\theta})^{-n}(0)}\right) + N_2$$

for some positive integer n*, nilpotents* N_i *(*$1 \le i \le 2$*) and* $0 \le \theta < 2\pi$*.*

(b) Let $m \le 4$*.*

(i) If either $\delta^m_{\triangle_{A,B^*},\triangle_{A,B}}(I) = 0$ *or* $\delta^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$*, then then there exist scalars* α_i *and* λ_i*, nilpotents* N_i *(*$1 \le i \le 2$*) and a positive integer* n *such that*

$$A = \left(A|_{(A-\alpha_1)^{-n}(0)} \oplus A|_{(A-\alpha_2)^{-n}(0)}\right) + N_1$$

and

$$B = \left(B|_{(B-\lambda_1)^{-n}(0)} \oplus B|_{(B-\lambda_2)^{-n}(0)}\right) + N_2.$$

(ii) If either $\delta^m_{\delta_{A,B^*},\triangle_{A,B}}(I) = 0$*, or* $\delta^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = 0$*, the operators* $A - I, B + I$ *are not nilpotent and* $-1 \in \sigma(B)$ *(or* $1 \in \sigma(A)$ *or* $\beta \neq 1$ *is real), then*

$$A = \left(A|_{(A-1)^{-n}(0)} \oplus A|_{(A+1-\beta)^{-n}(0)}\right) + N_1 \text{ and}$$
$$B = \left(B|_{(B+1)^{-n}(0)} \oplus B|_{(B-1+\beta)^{-n}(0)}\right) + N_2$$

for some positive integer n *and nilpotents* N_i *(*$1 \le i \le 2$*).*

Proof. (a). If either of the hypotheses on $\sigma(A)$, $\sigma(B)$ and β, holds, then necessarily

$$\sigma(A) = \{-1 + e^{i\theta}, 1\}, \sigma(B) = \{-1, 1 - e^{i\theta}\} \text{ and } \triangle^m_{A^*+I,A+I}(I) = 0 = \triangle^m_{I-B,I-B^*}(I)$$

(see Theorem 3.2). The operator $A + I$ and $I - B^*$ being invertible m-isometric are generealised scalar, hence polaroid. This implies A, B are polaroid, hence have the repersentation of the statement of the proposition.

(b). Before going on to considering the operators $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$ and $\delta^m_{\delta_{A^*,B^*},\delta_{A,B}}(I) = 0$, $m \le 4$, we recall some facts. It is known, see [29,37], that an operator $T \in B(\mathcal{H})$ is 3-symmetric if and only if it is unitarily equivalent to the restriction to a closed invariant subspace of the operator

$$S = \begin{bmatrix} V & E \\ 0 & V \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{H}),\ [V, E] = 0,\ V \text{ self adjoint.}$$

Let $H_0(p)$, p a positive integer, denote the class of operators $T \in B(\mathcal{H})$ such that

$$H_0(T - \lambda) = (T - \lambda)^{-p}(0),\ \lambda \in \mathbb{C}.$$

(Recall, $H_0(T)$ denotes the quasi-nilpotent part

$$H_0(T) = \{x \in \mathcal{H} : \lim_{n\to\infty} ||T^n x||^{\frac{1}{n}} = 0\}$$

of T [5, Page 119].)

Normal operators are $H_0(1)$ operators [5, Theorem 4.46]. Since

$$||(S-\lambda)^n(x\oplus y)||^{\frac{1}{n}} = ||(V-\lambda)^n(x) + nE(V-\lambda)^{n-1}y||^{\frac{1}{n}} + ||(V-\lambda)^n x||^{\frac{1}{n}}$$

for all $x \oplus y \in \mathcal{H} \oplus \mathcal{H}$, $H_0(V-\lambda) = (V-\lambda)^{-1}(0)$ implies

$$H_0(S-\lambda) \subseteq (S-\lambda)^{-2}(0).$$

The reverse inclusion $(S-\lambda)^{-n}(0) \subseteq H_0(S-\lambda)$ being always true,

$$H_0(S-\lambda) = (S-\lambda)^{-2}(0), \text{ all } \lambda \in \mathtt{C}.$$

Property $H_0(p)$ is inherited by restrictions to closed invariant subspaces [5, Theorem 4.36]. Hence "a 3-symmetric operator $T \in H_0(2)$", i.e., $H_0(T-\lambda) = (T-\lambda)^{-2}(0)$ for all $\lambda \in \mathtt{C}$.

(i). If $\delta^m_{\triangle_{A^*,B^*},\triangle_{A,B}}(I) = 0$, then $\delta^m_{I-\overline{\lambda}A^*,I-\lambda A}(I) = 0 = \delta^m_{I-\alpha B,I-\overline{\alpha}B^*}(I)$ for all $\alpha \in \sigma(A)$ and $\lambda \in \sigma(B)$. Recall from [19, Theorem 3] that if an operator $T \in B(\mathcal{H})$ is m-symmetric for an even positive integer m, then it is $(m-1)$-symmetric. Hence we may assume that our operators $1-\lambda A$ and $I-\overline{\alpha}B^*$ are 3-symmetric (recall $m \leq 4$), and therefore $H_0(2)$ operators. Since $H_0(p)$ operators are polaroid (of index p [5, Corollary 4.37]), $1-\lambda A$ and $I-\overline{\alpha}B^*$, hence also A and B, are polaroid operators. This implies that if $\sigma(A) = \{\alpha_1, \alpha_2\}$ and $\sigma(B) = \{\lambda_1, \lambda_2\}$, then A, B have the representation of the statement of the proposition.

A similar argument applied to $\delta^m_{\delta_{A,B^*},\delta_{A,B}}(I) = 0$ implies $A-\lambda$ and $B^*-\overline{\alpha}$, hence A and B , are polaroid operators. Consequently, A and B have the representation of the statement of the proposition.

(ii). If $\delta^m_{\delta_{A^*,B^*},\triangle_{A,B}}(I) = 0$ (or, $\delta^m_{\triangle_{A^*,B^*},\delta_{A,B}}(I) = 0$), then $\delta^m_{A^*-\lambda,I-\lambda A}(I) = 0 = \delta^m_{B-\alpha,\overline{\alpha}B^*-I}(I)$ (resp., $\delta^m_{I-\overline{\lambda}A^*,A-\lambda}(I) = 0 = \delta^m_{I-\alpha B,\overline{\alpha}-B^*}(I)$); see Theorem 3.2. Furthermore, if $\lambda = -1$ (or $\alpha = 1$ or β is real), then

$$\sigma(A) = \{-1+\beta, 1\} \text{ and } \sigma(B) = \{-1, 1-\beta\}$$

(in both the cases). Observe here that $\beta \neq 1$: for if it were then we would have $0 \in \sigma(A)$ and $0 \in \sigma(B)$, which then forces $A-1$ and $B+1$ to be nilpotents (and by assumption $\sigma(A) \neq \{0\} \neq \sigma(B)$). Evidently, if $\alpha = 1$, or $\lambda = -1$, then $\delta^4_{A^*+I,A+I}(I) = 0 = \delta^4_{B-I,B^*-I}(I)$ implies that the operators A, B are polaroid and have the stated representations. □

5. The Compact Case

We consider in the following the structure of operators A, B such that the operator $D^m_{d_1,d_2}$ is compact. The argument below is patterned on that used to consider the cases $\triangle^m_{\delta_{A^*,B^*},\delta_{A,B}}$ and $\delta^m_{\triangle_{A^*,B^*}\triangle_{A,B}}$ in [22]. We start by recalling our standing hypothesis that A, B are not nilpotent (thus, $\sigma(A) \neq \{0\} \neq \sigma(B)$).If $D^m_{d_1,d_2}$ is compact, then the Fredholm spectrum $\sigma_e(D^m_{d_1,d_2})$ of $D^m_{d_1,d_2}$ equals 0 [5,32], and this by the spectral mapping theorem implies

$$\sigma_e(D_{d_1,d_2}) = (\sigma_e(d_1) - \sigma(d_2)) \cup (\sigma(d_1) - \sigma_e(d_2)) = \{0\},\ D = \delta$$

and

$$\sigma_e(D_{d_1,d_2}) = 1 - (\sigma_e(d_1)\sigma(d_2) \cup \sigma(d_1)\sigma_e(d_2)) = \{0\},\ D = \triangle$$

[11]. In either case, there exists a non-zero scalar β (not necessarily the same for the two cases) such that

$$\sigma_e(d_1) = \sigma(d_1) = \{\beta\} = \sigma_e(d_2) = \sigma(d_2) \quad \text{if } D = \delta, \quad \text{and ...}$$

$$\sigma_e(d_1) = \sigma(d_1) = \{\beta\}, \sigma_e(d_2) = \sigma(d_2) = \{\frac{1}{\beta}\}, \text{ if } D = \triangle.$$

We prove that the operators A, B are algebraic.

Proposition 5.1. *If $D^m_{d_1,d_2}$ is compact, then A, B are algebraic.*

Proof. By definition

$$D^m_{d_1,d_2} = \delta^m_{d_1,d_2} = \sum_{j=0}^{m}(-1)^j \left(\begin{array}{c} m \\ j \end{array}\right) L^{m-j}_{d_1} R^j_{d_2}, \text{ and}$$
$$D^m_{d_1,d_2} = \triangle^m_{d_1,d_2} = \sum_{j=0}^{m}(-1)^j \left(\begin{array}{c} m \\ j \end{array}\right) L^j_{d_1} R^j_{d_2}.$$

We claim that the sequences

$$\{L^{m-j}_{d_1}\}^m_{j=0}, \ \{R^j_{d_2}\}^m_{j=0} \text{ if } D = \delta \text{ and}$$
$$\{L^j_{d_1}\}^m_{j=0}, \ \{R^j_{d_2}\}^m_{j=0} \text{ if } D = \triangle$$

are linearly dependent. Suppose, for example, that the sequence $\{R^j_{d_2}\}^m_{j=0}$ is linearly independent. Then, since $\sum_{j=0}^m (-1)^j \left(\begin{array}{c} m \\ j \end{array}\right) L^{m-j}_{d_1} R^j_{d_2}$ (similarly, $\sum_{j=0}^m (-1)^j \left(\begin{array}{c} m \\ j \end{array}\right) L^j_{d_1} R^j_{d_2}$) is compact, the sequence $\{L^{m-j}_{d_1}\}^m_{j=0}$ (resp., the sequence $\{L^j_{d_1}\}^m_{j=0}$) is compact. In particular, $L^0_{d_1} = I$ is compact. This being a contradiction, our claim is proved. The linear dependence of the sequences implies that the operators L_{d_1} and R_{d_2} are algebraic operators. Since the operator L_E (or, R_E) is algebraic if and only if E is algebraic, and since (for non-nilpotent operators A, B) $\delta_{A,B} = L_A - R_B$ and $\triangle_{A,B} = I - L_A R_B$ are algebraic if and only if A and B are algebraic [16, Proposition 2.6], the operators A, B are algebraic. □

If A, B are algebraic, and $\sigma(\delta_{A,B}) = \{\beta\}$ (resp., $\sigma(\triangle_{A,B}) = \{\beta\}$) for some scalar β, then there exist finite sequences $a = \{a_j\}^n_{j=1}$ and $b = \{b_j\}^n_{j=1}$ such that

$$\sigma(\delta_{A,B}) = \sigma(A) - \sigma(b) = \{a_j - b_j\}^n_{j=1} \ (= a - b)$$
$$(\text{resp.}, \sigma(\triangle_{A,B}) = 1 - \sigma(A)\sigma(B) = \{1 - a_j b_j\}^n_{j=1} \ (= 1 - ab).$$

The following table lists spectra $\sigma(D_{d_1,d_2})$ for various choices of D, d_1 and d_2.

There exists finite scalar sequences sequences $\{a_j\}^n_{j=1}, \{b_j\}^n_{j=1}$ and a scalar β such that if:
$(A)(i)$ $D = \delta, d_1 = \delta_{A^*,B^*}$ *and:*
(a) $d_2 = \delta_{A,B}$, *then* $\sigma(d_1) = \{\overline{a_j - b_j}\}^n_{j=1} = \{\beta\} = \{a_j - b_j\}^n_{j=1} = \sigma(d_2)$;
(b) $d_2 = \triangle_{A,B}$, *then* $\sigma(d_1) = \{\overline{a_j - b_j}\}^n_{j=1} = \{\beta\} = \{1 - a_j b_j\}^n_{j=1} = \sigma(d_2)$.

$(A)(ii)$ $D = \delta, d_1 = \triangle_{A^*,B^*}$ *and:*
(a) $d_2 = \delta_{A,B}$, *then* $\sigma(d_1) = \{1 - \overline{a_j b_j}\}^n_{j=1} = \{\beta\} = \{a_j - b_j\}^n_{j=1} = \sigma(d_2)$;
(b) $d_2 = \triangle_{A,B}$, *then* $\sigma(d_1) = \{1 - \overline{a_j b_j}\}^n_{j=1} = \{\beta\} = \{1 - a_j b_j\}^n_{j=1} = \sigma(d_2)$.

$(B)(i)$ $D = \triangle, d_1 = \triangle_{A^*,B^*}$ *and:*
(a) $d_2 = \triangle_{A,B}$, *then* $\sigma(d_1) = \{1 - \overline{a_j b_j}\}^n_{j=1} = \{\beta\}, \sigma(d_2) = \{1 - a_j b_j\}^n_{j=1} = \{\frac{1}{\beta}\}$;
(b) $d_2 = \delta_{A,B}$, *then* $\sigma(d_1) = \{1 - \overline{a_j b_j}\}^n_{j=1} = \{\beta\}, \sigma(d_2) = \{a_j - b_j\}^n_{j=0} = \{\frac{1}{\beta}\}$.

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$(B)(ii)$ $\quad D = \triangle, d_1 = \delta_{A^*,B^*}$ *and:*
(a) $d_2 = \triangle_{A,B}$*, then* $\sigma(d_1) = \{\overline{a_j - b_j}\}_{j=1}^{n} = \{\beta\}, \sigma(d_2) = \{1 - a_j b_j\}_{j=1}^{n} = \{\frac{1}{\beta}\}$*;*
(b) $d_2 = \delta_{A,B}$*, then* $\sigma(d_1) = \{\overline{a_j - b_j}\}_{j=1}^{n} = \{\beta\}, \sigma(d_2) = \{a_j - b_j\}_{j=1}^{n} = \{\frac{1}{\beta}\}$*.*

The following theorem characterises operators A, B such that $D^m_{d_1,d_2}$ is compact. Recall from [5, Page 128] that a point λ is a pole of A if and only if $H_0(A-\lambda) = (A-\lambda)^{-n}(0)$ for some positive integer n.

Theorem 5.2. *Let* $A, B \in B(\mathcal{H})$ *be such that* $\sigma(A) \neq \{0\} \neq \sigma(B)$*. The operator* $D^m_{d_1,d_2}$ *is compact if and only if the following conditions are satisfied.*

(i) There exist finite sequences $a = \{a_j\}_{j=1}^n \in \sigma(A)$ *and* $b = \{b_j\}_{j=1}^n \in \sigma(B)$*, not necessarily the same for the two cases under consideration, such that*

$$\sigma(d_1) = \{\overline{a_j - b_j}\}_{j=1}^n \text{ if } d_1 = \delta_{A^*,B^*} \text{ and } \sigma(d_1) = \{1 - \overline{a_j b_j}\}_{j=1}^n \text{ if } d_1 = \triangle_{A^*,B^*},$$
$$\sigma(d_2) = \{a_j - b_j\}_{j=1}^n \text{ if } d_2 = \delta_{A,B} \text{ and } \sigma(d_2) = \{1 - a_j b_j\}_{j=1}^n \text{ if } d_2 = \triangle_{A,B}.$$

(ii) There exist decompositions

$$\mathcal{H} = \oplus_{j=1}^n H_0(A - a_j) = \oplus_{j=1}^n (A - a_j)^{-r_j}(0),\ \mathcal{H} = \oplus_{j=1}^n H_0(B - b_j) = \oplus_{j=1}^n (B - b_j)^{-s_j}(0),$$
$$A = \oplus_{j=1}^n A_j = \oplus_{j=1}^n (A|_{H_0(A-a_j)}),\ B = \oplus_{j=1}^n B_j = \oplus_{j=1}^n (B|_{H_0(B-b_j)})$$

such that

$$A - a = \oplus_{j=1}^n (A_j - a_j) \text{ and } B - b = \oplus_{j=1}^n (B_j - b_j)$$

are nilpotent operators (of order $r = \max\{r_j : 1 \le j \le n\}$ *and* $s = \max\{s_j : 1 \le j \le n\}$*, respectively).*

Proof. The necessity of condition (i) having already been seen, we prove the necesity of condition (ii). The operators A, B being algebraic are polaroid (i.e., all isolated points of the spectrum are poles of the resolvent of the operator [5, Page 299]). Since $\sigma(A), \sigma(B)$ are finitely countable, (i) follows.

The proof of the sufficiency of the conditions requires some argument. However, the argument being very similar in all cases, we restrict ourselves to considering cases $D = \triangle, d_1 = \delta_{A^*,B^*}, d_2 = \triangle_{A,B}$ and $D = \delta, d_1 = \triangle_{A^*,B^*}, d_2 = \delta_{A,B}$. (The cases $D = \triangle, d_1 = \delta_{A^*,B^*}, d_2 = \delta_{A,B}$ and $D = \delta, d_1 = \triangle_{A^*,B^*}, d_2 = \triangle_{A,B}$ are proved in [22].)

The hypotheses imply that the spectral points a_j and b_j are poles of A, B respectively. Hence A and B, consequently also L_A and R_B, are polaroid operators. This, see [16, Proposition 3.1], implies that the operator $L_A R_B$ and $L_A - R_B$ are polaroid. The functional calculus for polaroid operators [5, Page 305] now tells us that the operators $\triangle_{A,B} - \mu$ and $\delta_{A,B} - \mu$ (hence, also $\triangle_{A^*,B^*} - \mu$ and $\delta_{A^*,B^*} - \mu$) are polaroid for all scalars μ.

Considering first the case $D = \triangle, d_1 = \delta_{A^*,B^*}$ and $d_2 = \triangle_{A,B}$, we have from the above that the operators $\delta_{A,B^*} - \beta$ and $\triangle_{A,B} - \frac{1}{\beta}$ are polaroid. Since $\sigma(\delta_{A^*,B^*}) = \{\beta\}$ and $\sigma(\triangle_{A,B}) = \{\frac{1}{\beta}\}$,

$$\sigma(\delta_{A^*,B^*} - \beta) = \{0\} = \sigma(\triangle_{A,B} - \frac{1}{\beta})$$

and the operators $\delta_{A^*,B^*} - \beta$ and $\triangle_{A,B} - \frac{1}{\beta}$ are nilpotents of some orders r and s, respectively. Set $r + s - 1 = t$. Then

$$\begin{aligned} & \triangle_{\delta_{A^*,B^*},\triangle_{A,B}} = \left(I - L_{\delta_{A^*,B^*}} R_{\triangle_{A,B}}\right) \\ = \quad & (-1)\left(L_{\delta_{A^*,B^*}-\beta} R_{\triangle_{A,B}} + L_\beta R_{\triangle_{A,B}-\frac{1}{\beta}}\right) \end{aligned}$$

and

$$\begin{aligned} & \triangle^t_{\delta_{A^*,B^*},\triangle_{A,B}} = (-1)^t \sum_{p=0}^{t} \left(\begin{array}{c} t \\ p \end{array} \right) R^{t-p}_{\triangle_{A,B}} L^p_\beta L^{t-p}_{\delta_{A^*,B^*}-\beta} R^p_{\triangle_{A,B}-\frac{1}{\beta}} \\ = \quad & 0, \end{aligned}$$

since $R^p_{\triangle_{A,B}-\frac{1}{\beta}} = 0$ for all $p \geq s$, and if $p \leq s - 1$ then $t - p \geq t - (s - 1) = r$ and this implies $L^{t-p}_{\delta_{A^*,B^*}-\beta} = 0$. Trivially, being nilpotent, $\triangle_{\delta_{A^*,B^*},\triangle_{A,B}}$ is compact.

To complete the proof of the theorem, we observe that if $D = \delta, d_1 = \triangle_{A^*,B^*}$ and $d_2 = \delta_{A,B}$, then there exist positive integers r and s such that

$$(\triangle_{A^*,B^*} - \beta)^r = 0 = (\delta_{A,B} - \beta)^s$$

and

$$\begin{aligned} & \left(\delta_{\triangle_{A^*,B^*},\delta_{A,B}}\right)^{r+s-1} = \left(L_{\triangle_{A^*,B^*}} - R_{\delta_{A,B}}\right)^{r+s-1} \\ = \quad & \left(L_{\triangle_{A^*,B^*}-\beta} - R_{\delta_{A,B}-\beta}\right)^{r+s-1} \\ = \quad & \sum_{p=0}^{r+s-1} (-1)^p \left(\begin{array}{c} r+s-1 \\ p \end{array} \right) L^{r+s-1-p}_{\triangle_{A^*,B^*}-\beta} R^p_{\delta_{A,B}-\beta} \\ & = 0 \quad , \end{aligned}$$

since $R^p_{\delta_{A,B}-\beta} = 0$ for all $p \geq s$, and if $p \leq s - 1$, equivalently $r + s - 1 - p \geq r$, then $L^r_{\triangle_{A^*,B^*}-\beta} = 0$.
$\square$

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