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Analysis of a Crowdsourcing Markovian Queue with Phase-Type and Imperfect Service, Working Vacations, Breakdown, and Repair

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Abstract

Analysis of a crowdsourcing Markovian queue with phase-type service is considered in this paper. In this model, a customer not only receives service but also assists in delivery. In other words, in a retail environment, while some customers shop in-store, others place orders online or by phone and require home delivery. Store management can utilize online customers as couriers to complete these deliveries. However, because not every customer may agree to take part, a probabilistic element is included to capture the chances of their participation. The model also incorporates imperfect service, reflecting cases where deliveries may fail or require rework, and working breakdowns, representing partial disruptions in service capacity rather than complete stoppages. To analyze the system under steady-state conditions, matrix-analytic methods are applied. Numerical examples illustrate the significant benefits of incorporating these dynamics into traditional queueing models.

Keywords: crowdsourcing; working vacation policy; imperfect service

MSC: 60K20; 60K25

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1. Introduction

Crowdsourcing means a model where one customer receives service from another who is available to serve. Crowdsourcing has become popular in industries like food, hotels, and electronics. It has also been applied across diverse fields such as healthcare, computer science, environmental sciences, business, and marketing to improve efficiency and optimize resource allocation. For example, in healthcare, patients who have undergone treatment may share their experiences or provide support to new patients through peer mentoring programs. In computer science, crowdsourced platforms harness user contributions to improve artificial intelligence models or software development. Environmental sciences leverage crowdsourcing to collect and analyze data from volunteers, helping researchers monitor climate change or biodiversity.

The learning of queueing representations with working vacations has gained significant attention in recent years due to its practical applications in service systems, telecommunications, and manufacturing. Shrivastava and Rathore [1] investigated a Markovian queueing model with a single server, incorporating features such as distinct types of working vacations, vacation breaks, and customer abandonment. Their study offers meaningful perspectives on the system's performance under these scenarios. Similarly, Rathore and Shrivastava [2] analyzed an M/M/1 queueing system incorporating an unreliable server with partial breakdowns during working vacations, demonstrating the impact of server failures on queue performance.

A queueing system with multiple working vacations and encouraged arrivals was explored by Prakati and Julia [3], using the M/M(a,b)/1 model to derive performance measures and demonstrate real-life applications. In a related study, Singh et al. [4] investigated a transient Markovian queueing model that offers options between regular and working vacations, providing a cost analysis and numerical evaluation of system performance. These studies collectively contribute to the understanding and optimization of queueing systems with working vacation policies, highlighting their importance in modern operational research.

Ayyappan and Arulmozhi [5] introduced a priority queueing model with delayed working vacations, considering multiple real-world factors such as immediate feedback and customer impatience. Ayyappan and Karpagam [6] explored a non-Markovian single-server queueing model characterized by batch arrivals, general bulk service, server failures and repairs, a single vacation policy, and the presence of a standby server. They obtained the probability generating function for the queue length at a random time and evaluated several performance metrics of the system. For modelling and analysing crowdsourcing Markovian queues, artificial intelligence (AI) techniques can be employed to enhance both efficiency and accuracy. A relevant example is provided by Zhang et al. [7], who propose a reinforcement learning-based edge server placement strategy for the intelligent Internet of Vehicles environment.

A retrial queueing system with group service and impatient customers was analyzed by D'Arienzo et al. [8], highlighting its applicability in telecommunication systems where bulk service and impatience are significant. Strategic behavior in an M/M/1 double orbit retrial queue with imperfect service and vacations was studied by Dhibar and Jain [9], providing insights into customer decision-making and system optimization under interruptions. Furthermore, Jain et al. [10] examined a Markovian working vacation queue incorporating imperfect service, balking, and retrial phenomena, providing insights into customer behavior under such conditions.

A non-Markovian queueing model with batch arrivals, phase-type service, and multi-vacation was examined by Radha et al. [11], with applications described in cloud computing. The following survey articles, queueing models with customer's impatience by Anjali and Kolledath [12] and queueing models with discouragement policies and vacation by Sharma et al. [13] are remarkable. Haghighi et al. [14] investigated busy periods and queue length (steady-state and transient) of a single-server Poisson queue with delayed service, to set the tone for more complicated models. They analyzed their model by considering M/G/1 as well as the Erlang phase (stage) processes to set up differential difference equations for approximating a non-Markovian system.

The body of research on tandem queue analysis is extensive. Therefore, this discussion focuses exclusively on studies addressing dual tandem systems with multi-server stages and arrivals modeled by the Markov Arrival Process (MAP). Compared to the widely used stationary Poisson process—a special case of the MAP—the MAP provides a more realistic mathematical representation of bursty and correlated input processes observed in real-world systems. The MAP was originally introduced as a flexible and general framework for modeling arrival processes in [15].The model is investigated using the known results for Quasi-Birth-and-Death processes; see [16]. Chakravarthy et al. [17] used a single-server queueing framework with degradation, failures/breakdowns, and repairs, aiming to provide a realistic framework for analyzing and managing service systems under such conditions.

In [18] investigate a single-server queueing model with a Markovian arrival process (MAP) where served customers may act as temporary secondary servers. Their steady-state analysis using QBD and GI/M/1 formulations provides insights into performance measures and system dynamics. In [19] analyze a dual tandem queueing model for managing parcel pick-up networks, incorporating batch transfers, customer no-shows, and threshold-based order admissions. Their study offers insights into optimizing warehouse capacity and admission policies through steady-state performance evaluation.

2. Related Work

In Chakravarthy, [20,21] analyzed a multi-server queueing system with Poisson arrival patterns and exponential service durations, focusing on its application in crowdsourcing environments. Crowdsourcing has been applied in fields such as healthcare, computer science, and business. Recent studies have examined related queueing models like M/M/c, MAP/PH/1, and MAP/PH/c. This paper introduces vacation and working vacation concepts in a MAP/PH/1 queueing model for crowdsourcing as the baseline framework (see [20]). The proposed model extends it by incorporating imperfect service, probabilistic customer participation, and working breakdowns.

The model has broad practical applications in systems where human participation is uncertain and service reliability fluctuates. In healthcare crowdsourcing, volunteers or remote workers may contribute irregularly, introducing probabilistic participation and imperfect service outcomes. These examples demonstrate how the proposed model captures real-world variability in availability, performance, and service quality.

Shajin and Krishnamoorthy [22] propose a novel stochastic decomposition framework tailored to a combined retrial queue and inventory management model. They rigorously derive performance metrics by decoupling the queueing dynamics from inventory-level fluctuations, enabling analytical tractability. Their results illustrate how retrial behavior and restocking policy jointly affect system stability and cost, filling a gap between isolated queueing and inventory models. Shajin and Krishnamoorthy [23] extend classical queueing-inventory systems by incorporating MAP arrivals and PH-distributed service times. Their model introduces probabilistic customer behavior, item returns, and a feedback mechanism, enhancing realism. They develop new performance metrics and numerical methods to optimize revenue under a dynamic replenishment policy.

In this paper, we propose to investigate the operation of an order crowdsourcing, imperfect service, working vacation within a two-types customers. The system is a single-server queue handling two customer types: Type 1 visits the facility with limited waiting space, while Type 2 orders online with unlimited capacity. Both types arrive according to a Markovian arrival process. Service times follow phase-type distributions. During working vacations, the server serves Type 1 at a reduced rate, and service may need to be repeated if imperfect. Served Type 1 customers may assist Type 2 customers, implementing a crowdsourcing mechanism. The server is subject to breakdowns, providing slow service during failures, followed by repair. This model integrates working vacations, imperfect service, crowdsourcing, and breakdown-repair in a single framework.

2.1. Motivation

The motivation behind these models can be understood in the context of online retail and delivery services, particularly for products like books and articles. For example, consider an online bookstore such as Amazon or Flipkart. One group of customers visits physical outlets or pick-up points to purchase books, while another group orders online for home delivery. The store management can leverage in-store customers to assist in fulfilling online orders, such as by picking up books for delivery, reviewing or recommending articles, or sharing their experiences online. Since not all in-store customers may be willing or able to perform these tasks, a probability factor represents the likelihood of a customer acting as a “crowdsourced server.” Similarly, marketing campaigns like referral programs or “bring a friend” initiatives incentivize existing customers to help promote books or online articles, effectively expanding the customer base. This approach helps businesses reduce delivery and marketing costs, enhance customer engagement, and create a collaborative ecosystem between different types of customers.

2.2. Research Gap

Based on existing studies, concepts such as crowdsourcing, imperfect service, and server breakdown with repair have mostly been explored individually within traditional queueing systems, often through mechanisms like working vacations and service disruptions. To the best of our knowledge, no prior research has simultaneously considered crowdsourcing, imperfect service, and breakdown-repair in a unified Markovian arrival process in queueing model. We develop a queueing system that incorporates Markovian arrival process for Type 2 customers, phase-type service, crowdsourcing and imperfect service, and examine how the server can enhance the system’s profitability through sensitivity analysis.

2.3. Contribution of the Model

The following presumptions are to be carried out in this paper:

- This model investigates the crowdsourcing, imperfect service, and server breakdown with repair.
- This model uses the matrix analytic method to determine the proposed system's steady-state probability vector.
- The numerical illustration analyzed the system performance measures using parameter variation.

3. Model Development

This model considers a single-server queueing system with a crowdsourcing Markovian queue with phase-type and imperfect service, working vacations, breakdown and repair. It has an infinite and a finite waiting hall attached to the server.

3.1. Working Vacations

After completing customer service, the server may take a vacation when the queue is empty. If customers arrive during this vacation period, the server provides service at a reduced rate compared to the normal service rate.

3.2. Crowdsourcing

In computer networks, usually a server provides services (like hosting a website) and a client/customer uses it. But sometimes, a customer (client machine) can act as a server too. It is known as Crowdsourcing. For example, in file sharing (like BitTorrent), your computer (as a customer) downloads files from others, but at the sametime, it also uploads and shares pieces of the file to other users. So your computer is both a client (receiving data) and a server (sending data to others).

3.3. Imperfect Service

When the server's performance is imperfect, customers may request additional service if the initial service is unsatisfactory. If a new customer arrives while the server is busy, the customer may either leave the system or seek immediate re-service.

3.4. Breakdown and Repair

Server breakdowns are a critical factor to consider in practical service systems and manufacturing industries. During service delivery, the server may experience a failure that requires immediate repair. Throughout the repair process, the server suspends customer service, resuming operations only after the repair is completed to continue serving the interrupted customers.

- We examine a queueing system with a single server that handles two distinct categories of customers. Type 1 customers, denoted as T_1, arrive according to a Markovian Arrival Process (MAP) characterized by the matrix pair (D_0, D_1) of order m_1. Here, D_0 governs transitions without T_1 customers arrivals, while D_1 corresponds to transitions involving a T_1 customers arrival. The arrival rate of T_1 customers is denoted by λ_1. Similarly, Type 2 customers, denoted as T_2, follow a MAP represented by (H_0, H_1) of order m_2, where H_0 captures transitions without T_2 arrivals and H_1 captures transitions with a T_2 arrival. The arrival rate of T_2 customers is denoted by λ_2.
- If a T_1 customers arrives and encounters the server idle on working vacation, they begin service immediately, albeit at a reduced service rate. If the server is occupied, the customer enters the queue and will be served in the order of arrival (first-come-first-served) when the server becomes free. A T_1 customers is turned away if the system has reached its capacity upon arrival, based on the assumption that the waiting area for T_1 customers is limited to N spots. T_1 customers physically visit the facility (e.g., a store) to receive service, whereas T_2 customers place orders remotely (e.g., via phone or online). This distinction justifies the limited waiting area for T_1 customers and the unlimited waiting space for T_2 consumers, with no restrictions on the number

of T_2 units in the system (i.e., they have an infinite buffer). If a T_2 customers notices that the server is not busy, they will remain in the system until the server is ready to serve them.

- The service durations for both customer types are modeled using phase-type distributions, represented by (α_1, U_1) and (α_2, U_2) of orders n_1 and n_2, respectively. After a service is completed and no customers remain in the system, the server begins a vacation. The duration of this vacation is modeled by an exponential distribution with rate η. A vacation is terminated prematurely if an T_1 customers arrival occurs during this period. During a vacation, the server offers service to arriving T_1 customers but at a reduced service rate. It is important to note that only T_1 customers can be served during vacation mode.
- During a vacation, the service times for T_1 customers are modeled by a phase-type distribution, represented by $(\alpha_1, \theta_1 U_1)$ of order n_1, where $0 \leq \theta_1 \leq 1$. The server operates at this reduced service rate throughout the remaining vacation period. After the vacation ends, the server re-enters the system. If any customers are waiting, the server immediately switches to the normal service rate and continues serving until the system becomes empty. If no customers are present, the server remains idle.

  Let μ_1 and μ_2 denote the regular service rates for T_1 and T_2 customers, respectively:

$$\mu_1 = \left[\alpha_1(-U_1)^{-1}e\right]^{-1}, \quad \mu_2 = \left[\alpha_2(-U_2)^{-1}e\right]^{-1},$$

  and the service rate during a working vacation is $\theta_1\mu_1$. After being served during a working vacation, a T_1 customer may not be satisfied with the service quality. In such cases, the server repeats the same service (re-service) with probability p_1. If the customer is satisfied, they depart the system with probability q_1, where $p_1 + q_1 = 1$.
- While T_1 customers are being served by the server, T_2 customers may be served either by the system server or by a T_1 customers who has already completed their own service and is available to act as a temporary server. A T_2 customers can be served by a T_1 customers under the following conditions:
  - The T_1 customers must have just completed their service and must choose to assist in serving a T_2 customers.
  - There must be at least one T_2 customers waiting for service at the moment the T_1 customer becomes available. In other words, the T_2 customers must not have already commenced service with the system server.
  - After completing service for a T_1 customers, the T_1 customers can instantaneously provide service to a T_2 customers. However, at any given time, a T_1 customers can serve only one T_2 customers.
- With probability p_2, we suppose that a served T_1 customers will be available to service a T_2 customers under the previously mentioned conditions, where $0 \leq p_2 \leq 1$. With a probability of $q_2 = 1 - p_2$, the consumers who was served T_1 will exit the system. The system server provides service to a T_1 customers when a service is finished. The server will serve a T_2 customers if one is in the system, but only if no T_1 customers are waiting. Customers with T_1 are presumed to have non-preemptive precedence over those with T_2. For analytical purposes, if a T_1 customers chooses to serve a T_2 customers, that T_2 client is instantly eliminated from the system. This assumption simplifies the analysis by eliminating the need to track T_2 customers once they are assigned to a T_1 customers for service.
- When a breakdown occurs during a regular service period, we assume that breakdowns are generated according to an exponential distribution with rate ψ. During a breakdown, the server continues delivering service to the current customer, but at a reduced service rate. After completing the ongoing slow service, the server undergoes a repair period. During the working breakdown period, the server provides slow service to customers, where the service times follow phase-type distributions represented by $(\alpha_1, \theta_2 U_1)$ for T_1 customers, with $0 < \theta_2 < 1$ and of order

n_1, and $(\alpha_2, \theta_3 U_2)$ for T_2 customers, with $0 < \theta_3 < 1$ and of order n_2. The corresponding slow service rates are given by $\mu_{BD(1)} = \left[\alpha_1(-\theta_2 U_1)^{-1}e\right]^{-1}$, $\mu_{BD(2)} = \left[\alpha_2(-\theta_3 U_2)^{-1}e\right]^{-1}$. The repair times follow a phase-type distribution with representation (γ, S) of order l, and the repair rate is $\tau = \left[\gamma(-S)^{-1}e\right]^{-1}$. A diagram of the model is shown in Figure 1.

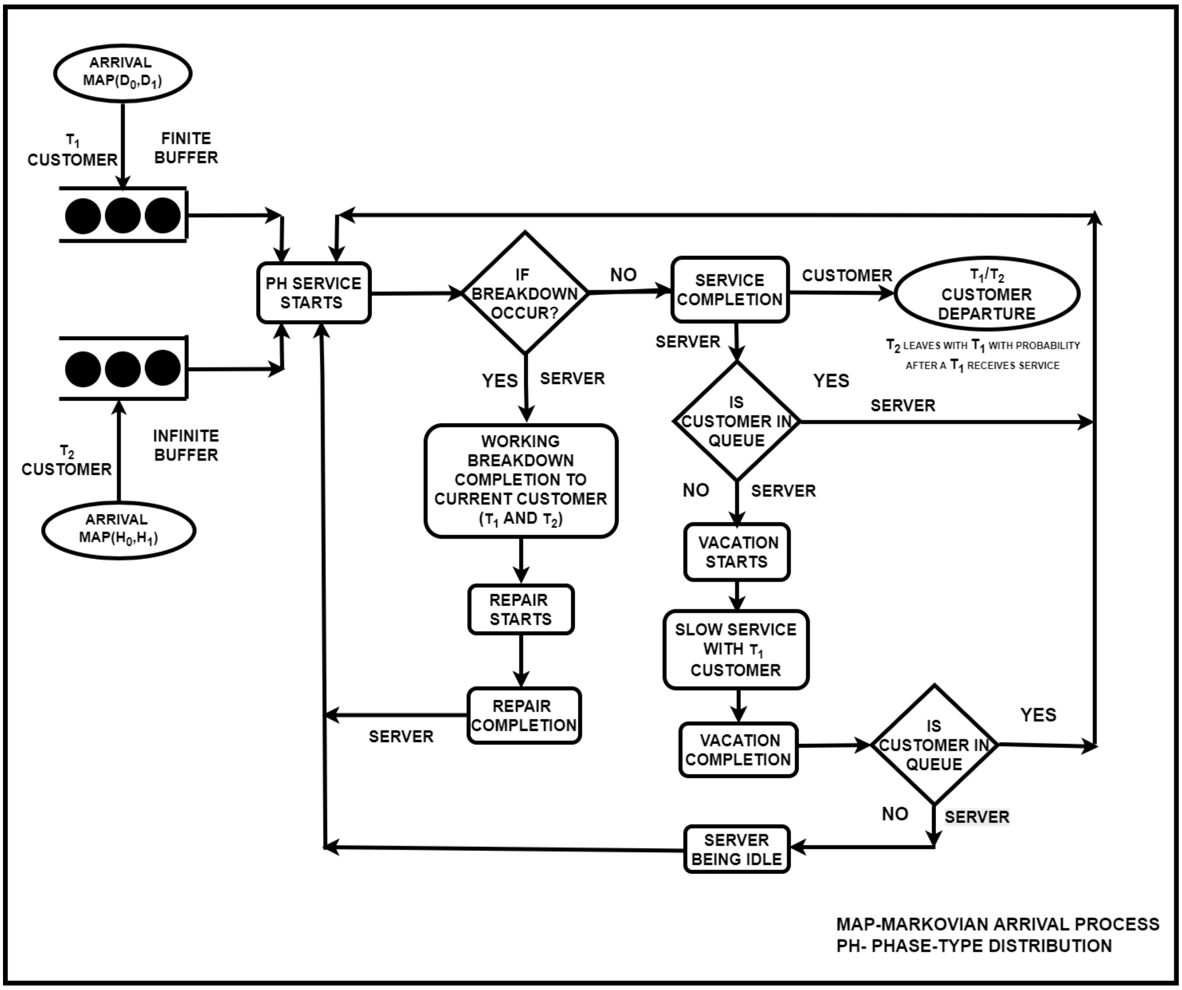

Figure 1. Schematic representation

4. Matrix Form of the QBD Generator

4.1. Assumptions

We will use the following assumptions throughout the model:

- $e_0 = 2m_1m_2 + 4Nn_1m_1m_2 + (N+1)lm_1m_2$.
- $e_1 = m_1m_2 + 4Nn_1m_1m_2 + 2(N+1)n_2m_1m_2 + (N+1)lm_1m_2$.
- $N_1(t)$: Total Type-2 customers in the system at epoch t.
- $N_2(t)$: Total Type-1 customers in the system at epoch t.

- $J(t)$ represents the server's status at epoch t.

$$J(t)=\begin{cases}0, & \text{in case the server is inactive in working vacation,}\\ 1, & \text{in case the server is busy in working vacation mode,}\\ 2, & \text{in case the server reservice in working vacation,}\\ 3, & \text{in case the server is idle in normal mode,}\\ 4, & \text{in case the server is busy (T1) in normal mode,}\\ 5, & \text{in case the server is busy (T2) in normal mode ,}\\ 6, & \text{in case the server is busy (T1) in working breakdown,}\\ 7, & \text{in case the server is busy (T2) in working breakdown,}\\ 8, & \text{in case the server is repair.}\end{cases}$$

- $R(t)$ denotes the phase of the repair process at time t.
- $S_1(t)$ represents the service phase for a T_1 customer at time t.
- $S_2(t)$ represents the service phase for a T_2 customer at time t.
- $A_1(t)$ refers to the phase of the T_1 arrival process at time t.
- $A_2(t)$ refers to the phase of the T_2 arrival process at time t.

Let $Y(t)=\{N_1(t),N_2(t),J(t),R(t),S_1(t),S_2(t),A_1(t),A_2(t)\}$, where Y=$\{Y(t):t\geq 0\}$, is a $CTMC$ with state space

$$\chi=\chi(0)\bigcup_{i=1}^{\infty}\chi(i), \tag{1}$$

where

$$\begin{aligned}&\chi(0)=\{(0,0,0,b_1,b_2):\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(0,a,1,k_2,b_1,b_2):\ 1\leq a\leq N,\ 1\leq k_2\leq n_1,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(0,a,2,k_2,b_1,b_2):\ 1\leq a\leq N,\ 1\leq k_2\leq n_1,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(0,0,3,b_1,b_2):\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(0,a,4,k_2,b_1,b_2):\ 1\leq a\leq N,\ 1\leq k_2\leq n_1,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(0,a,6,k_2,b_1,b_2):\ 1\leq a\leq N,\ 1\leq k_2\leq n_1,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(0,a,8,k_1,b_1,b_2):\ 0\leq a\leq N,\ 1\leq k_1\leq l,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\},\end{aligned}$$

and for $i\geq 1$,

$$\begin{aligned}&\chi(i)=\{(i,0,0,b_1,b_2):\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(i,a,1,k_2,b_1,b_2):\ 1\leq a\leq N,\ 1\leq k_2\leq n_1,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(i,a,2,k_2,b_1,b_2):\ 1\leq a\leq N,\ 1\leq k_2\leq n_1,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(i,a,4,k_2,b_1,b_2):\ 1\leq a\leq N,\ 1\leq k_2\leq n_1,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(i,a,5,k_3,b_1,b_2):\ 1\leq a\leq N,\ 1\leq k_3\leq n_2,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(i,a,6,k_2,b_1,b_2):\ 1\leq a\leq N,\ 1\leq k_2\leq n_1,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}\\ &\cup\{(i,a,7,k_3,b_1,b_2):\ 0\leq a\leq N,\ 1\leq k_3\leq n_2,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\},\\ &\cup\{(i,a,8,k_2,b_1,b_2):\ 0\leq a\leq N,\ 1\leq k_1\leq l,\ 1\leq b_1\leq m_1,\ 1\leq b_2\leq m_2\}.\end{aligned}$$

4.2. The Infinitesimal Generator Matrix

Suppose that the generator of the CTMC is of the quasi-birth-and-death (QBD) type in the matrix, $\{N_1(t),N_2(t),J(t),R(t),S_1(t),S_2(t),A_1(t),A_2(t):t\geq 0\}$, such that the generator is of the form gives by:

$$Q = \begin{bmatrix} A_{00} & A_{01} & 0 & 0 & 0 & 0 & \dots \\ A_{10} & J_1 & J_0 & 0 & 0 & 0 & \dots \\ 0 & J_2 & J_1 & J_0 & 0 & 0 & \dots \\ 0 & 0 & J_2 & J_1 & J_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \dots \end{bmatrix}, \tag{2}$$

where the square matrices $A_{00}, A_{01}, A_{10}, J_0, J_1$ and J_2 are such that :

$$A_{00}e + A_{01}e = A_{10}e + J_1 e + J_0 e = J_2 e + J_1 e + J_0 e = 0.$$

The following describes Markov chain transitions and the corresponding rates:

- A_{00} contains transitions within level 0.
- Since Q is a generator, its diagonal elements A_{00}^{ii} are negative. The modules of these entries are the intensity of Markov chain leaving its states.
- When a Type 1 customer arrives on the system and the server starts the service. The corresponding intensities are the diagonal elements of the matrix $\alpha_1 \otimes (D_1 \otimes I_{m_2})$.
- Vacation completion for the server. The corresponding intensities are the diagonal elements of the matrix $\eta I_{m_1} I_{m_2}$.
- A server finishes slow service for T_1 customers with probability q_1. The corresponding intensities are the diagonal elements of the matrix $q_1 \theta_1 U_1^0 \otimes I_{m_1 m_2}$.
- A server is interrupted due to breakdown. The intensities of this event are defined by the matrix $I_{n_1} \otimes \alpha_1 \otimes \psi I_{m_1} I_{m_2}$. The matrix A_{00} governs,

$$A_{00} = \begin{bmatrix} A_{00}^{11} & A_{00}^{12} & \mathbf{0} & A_{00}^{14} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{00}^{21} & A_{00}^{22} & A_{00}^{23} & \mathbf{0} & A_{00}^{25} & \mathbf{0} & \mathbf{0} \\ A_{00}^{31} & A_{00}^{32} & A_{00}^{33} & \mathbf{0} & \mathbf{0} & A_{00}^{36} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{00}^{44} & A_{00}^{45} & \mathbf{0} & \mathbf{0} \\ A_{00}^{51} & A_{00}^{52} & \mathbf{0} & \mathbf{0} & A_{00}^{55} & A_{00}^{56} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{00}^{66} & A_{00}^{67} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{00}^{74} & A_{00}^{75} & \mathbf{0} & A_{00}^{77} \end{bmatrix},$$

$$A_{00}^{11} = (D_0 \oplus H_0) - \eta I_{m_1 m_2}\, , A_{00}^{12} = \begin{bmatrix} \alpha_1 \otimes (D_1 \otimes I_{m_2}) & \mathbf{0} \end{bmatrix}, A_{00}^{14} = \eta I_{m_1 m_2},$$

$$A_{00}^{21} = \begin{bmatrix} q_1 \theta_1 U_1^0 \otimes I_{m_1 m_2} \\ \mathbf{0} \end{bmatrix}, A_{00}^{22} = \begin{bmatrix} M_1 & M_2 & 0 & \dots & 0 & 0 \\ M_3 & M_1 & M_2 & \dots & 0 & 0 \\ 0 & M_3 & M_1 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & M_1 & M_2 \\ 0 & 0 & 0 & \dots & M_3 & M_1 + M_2 \end{bmatrix},$$

$$M_1 = (\theta_1 U_1 \otimes I_{m_1}) \oplus D_0 \otimes I_{m_2} + I_{n_1 m_1} \otimes H_0 - \eta I_{n_1 m_1 m_2},$$

$$M_2 = I_{n_1} \otimes (D_1 \otimes I_{m_2}), M_3 = q_1 \theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2},$$

$$A_{00}^{23} = I_N \otimes \left(p_1 \theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2}\right), A_{00}^{25} = I_N \otimes \left(\eta I_{n_1 m_1 m_2}\right),$$

$$A_{00}^{31} = \begin{bmatrix} \theta_1 U_1^0 \otimes I_{m_1 m_2} \\ \mathbf{0} \end{bmatrix}, A_{00}^{32} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_{N-1} \otimes \left(\theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2}\right) & \mathbf{0} \end{bmatrix},$$

$$A_{00}^{33} = \begin{bmatrix} M_4 & M_5 & 0 & \dots & 0 & 0 \\ 0 & M_4 & M_5 & \dots & 0 & 0 \\ 0 & 0 & M_4 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & M_4 & M_5 \\ 0 & 0 & 0 & \dots & 0 & M_4 + M_5 \end{bmatrix},$$

$$M_4 = (\theta_1 U_1 \otimes I_{m_1} + I_{n_1} \otimes D_0) \oplus H_0 - \eta I_{n_1 m_1 m_2},$$

$$M_5 = I_{n_1} \otimes (D_1 \otimes I_{m_2}),\ A_{00}^{36} = I_N \otimes (\eta I_{n_1 m_1 m_2}),$$

$$A_{00}^{44} = D_0 \otimes I_{m_2} + I_{m_1} \otimes H_0,\ A_{00}^{45} = \begin{bmatrix} \alpha_1 \otimes (D_1 \otimes I_{m_2}) & \mathbf{0} \end{bmatrix},$$

$$A_{00}^{51} = \begin{bmatrix} U_1^0 \otimes I_{m_1 m_2} \\ \mathbf{0} \end{bmatrix},\ A_{00}^{52} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_{N-1} \otimes (U_1^0 \alpha_1 \otimes I_{m_1 m_2}) & \mathbf{0} \end{bmatrix},$$

$$A_{00}^{55} = \begin{bmatrix} M_6 & M_7 & 0 & \dots & 0 & 0 \\ 0 & M_6 & M_7 & \dots & 0 & 0 \\ 0 & 0 & M_6 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & M_6 & M_7 \\ 0 & 0 & 0 & \dots & 0 & M_6 + M_7 \end{bmatrix}$$

$$M_6 = (U_1 \otimes I_{m_1} + D_0 \otimes I_{n_1}) \oplus H_0 - \psi I_{n_1 m_1 m_2},$$

$$M_7 = I_{n_1} \otimes (D_1 \otimes I_{m_2}),\ A_{00}^{56} = I_N \otimes (\mathbf{I}_{n_1} \otimes \alpha_1 \otimes \psi I_{m_1 m_2}),$$

$$A_{00}^{66} = \begin{bmatrix} M_8 & M_9 & 0 & \dots & 0 & 0 \\ 0 & M_8 & M_9 & \dots & 0 & 0 \\ 0 & 0 & M_8 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_8 & M_9 \\ 0 & 0 & 0 & \dots & 0 & M_8 + M_9 \end{bmatrix},$$

$$M_8 = (\theta_2 U_1 \otimes I_{m_1} + I_{n_1} \otimes D_0) \oplus H_0,\ M_9 = I_{n_1} \otimes (D_1 \otimes I_{m_2}),$$

$$A_{00}^{67} = [I_N \otimes (\theta_2 U_1^0 \gamma \otimes I_{m_1 m_2}) \quad \mathbf{0}],$$

$$A_{00}^{74} = \begin{bmatrix} S^0 \otimes I_{m_1 m_2} \\ \mathbf{0} \end{bmatrix},\ A_{00}^{75} = \begin{bmatrix} \mathbf{0} \\ I_N \otimes (S^0 \alpha_1 \otimes I_{m_1 m_2}) \end{bmatrix},$$

$$A_{00}^{77} = \begin{bmatrix} M_{10} & M_{11} & 0 & \dots & 0 & 0 \\ 0 & M_{10} & M_{11} & \dots & 0 & 0 \\ 0 & 0 & M_{10} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{10} & M_{11} \\ 0 & 0 & 0 & \dots & 0 & M_{10} + M_{11} \end{bmatrix},$$

$$M_{10} = (S \otimes I_{m_1} + I_l \otimes D_0) \oplus H_0,\ M_{11} = I_l \otimes (D_1 \otimes I_{m_2}).$$

- A_{01} contains transitions within level 0 to 1.
- When a Type 2 customer arrives on the system with finite capacity N. The corresponding intensities are the diagonal elements of the matrix $I_N \otimes (I_{n_1 m_1} H_1)$.The matrix A_{01} governs

$$A_{01} = \begin{bmatrix} A_{01}^{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{01}^{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{01}^{33} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{01}^{45} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{01}^{54} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{01}^{66} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{01}^{78} \end{bmatrix}$$

$$A_{01}^{11} = I_{m_1} \otimes H_1,\ A_{01}^{22} = I_N \otimes (I_{n_1 m_1} \otimes H_1),\ A_{01}^{33} = I_N \otimes (I_{n_1 m_1} \otimes H_1)$$

$$A_{01}^{45} = \begin{bmatrix} \alpha_2 \otimes (I_{m_1} \otimes H_1) & \mathbf{0} \end{bmatrix},\ A_{01}^{54} = I_N \otimes (I_{n_1 m_1} \otimes H_1),$$

$$A_{01}^{66} = I_N \otimes (I_{n_1 m_1} \otimes H_1),\ A_{01}^{78} = I_{N+1} \otimes (I_{l m_1} \otimes H_1),$$

- A_{10} contains transitions within level 1 to 0.
- A server finishes slow service for U_1 customers with probability q_1 and after service completion will start the fresh service for waiting customers. The corresponding intensities are the diagonal elements of the matrix $p_2 q_1 \theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2}$.
- A server completes the service for T_2 customers. The rates of this event occurrence are defined by the matrix $U_2^0 \alpha_1 \otimes I_{m_1 m_2}$. The matrix A_{10} governs,

$$A_{10} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{10}^{21} & A_{10}^{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{10}^{31} & A_{10}^{32} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{10}^{41} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{10}^{45} & \mathbf{0} & \mathbf{0} \\ A_{10}^{51} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{10}^{55} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{10}^{67} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{10}^{77} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$A_{10}^{21} = \begin{bmatrix} p_2 q_1 \theta_1 U_1^0 \otimes I_{m_1 m_2} \\ \mathbf{0} \end{bmatrix},\ A_{10}^{22} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_{N-1} \otimes (p_2 q_1 \theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2}) & \mathbf{0} \end{bmatrix},$$

$$A_{10}^{31} = \begin{bmatrix} p_2 \theta_1 U_1^0 \otimes I_{m_1 m_2} \\ \mathbf{0} \end{bmatrix},\ A_{10}^{32} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_{N-1} \otimes (p_2 \theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2}) & \mathbf{0} \end{bmatrix},$$

$$A_{10}^{41} = \begin{bmatrix} p_2 U_1^0 \otimes I_{m_1 m_2} \\ \mathbf{0} \end{bmatrix},\ A_{10}^{45} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_{N-1} \otimes (p_2 U_1^0 \alpha_1 \otimes I_{m_1 m_2}) & \mathbf{0} \end{bmatrix},$$

$$A_{10}^{51} = \begin{bmatrix} U_2^0 \otimes I_{m_1 m_2} \\ \mathbf{0} \end{bmatrix},\ A_{10}^{55} = \begin{bmatrix} \mathbf{0} \\ I_N \otimes (U_2^0 \alpha_1 \otimes I_{m_1 m_2}) \end{bmatrix},$$

$$A_{10}^{67} = \begin{bmatrix} I_N \otimes (p_2 \theta_2 U_1^0 \gamma \otimes I_{m_1 m_2}) & \mathbf{0} \end{bmatrix},\ A_{10}^{77} = I_{N+1} \otimes (\theta_3 U_2^0 \gamma \otimes I_{m_1 m_2}),$$

- J_1 contains transitions within level n for $n \geq 1$.
- Its diagonal elements J_1^{ii} are negative. The modules of these entries are the intensity of Markov chain leaving its states.
- When a Type 1 customer arrives on the system and the server starts the service. The corresponding intensities are the diagonal elements of the matrix $\alpha_1 \otimes (D_1 \otimes I_{m_2})$.
- After vacation completion, the server start the service for Type 2 customers. The corresponding intensities are the diagonal elements of the matrix $\alpha_2 \otimes \eta I_{m_1} I_{m_2}$.
- A server finishes slow service for U_1 customers with probability q_1. The corresponding intensities are the diagonal elements of the matrix $q_1 q_2 \theta_1 U_1^0 \otimes I_{m_1 m_2}$.
- A server is interrupted due to breakdown. The intensities of this event are defined by the matrix $I_{n_1} \otimes \psi I_{m_1} I_{m_2}$. The matrix J_1 governs,

$$J_1 = \begin{bmatrix} J_1^{11} & J_1^{12} & \mathbf{0} & \mathbf{0} & J_1^{15} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ J_1^{21} & J_1^{22} & J_1^{23} & J_1^{24} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ J_1^{31} & J_1^{32} & J_1^{33} & J_1^{34} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J_1^{44} & J_1^{45} & J_1^{46} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_1^{55} & \mathbf{0} & J_1^{57} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_1^{66} & \mathbf{0} & J_1^{68} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_1^{77} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J_1^{84} & J_1^{85} & \mathbf{0} & \mathbf{0} & J_1^{88} \end{bmatrix},$$

$$J_1^{11} = D_0 \oplus H_0 - \eta I_{m_1 m_2},\ J_1^{12} = \begin{bmatrix} \alpha_1 \otimes (D_1 \otimes I_{m_2}) & 0 \end{bmatrix},$$

$$J_1^{15} = \begin{bmatrix} \alpha_2 \otimes (\eta I_{m_1 m_2}) & 0 \end{bmatrix},\ J_1^{21} = \begin{bmatrix} (q_2 q_1 \theta_1 U_1^0) \otimes I_{m_1 m_2} \\ 0 \end{bmatrix},$$

$$J_1^{22} = \begin{bmatrix} M_{12} & M_{13} & 0 & \dots & 0 & 0 \\ M_{14} & M_{12} & M_{13} & \dots & 0 & 0 \\ 0 & M_{14} & M_{12} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{12} & M_{13} \\ 0 & 0 & 0 & \dots & M_{14} & M_{12} + M_{13} \end{bmatrix},$$

$$M_{12} = \theta_1 U_1 \oplus D_0 \oplus H_0 - \eta I_{n_1 m_1 m_2},\ M_{13} = I_{n_1} \otimes (D_1 \otimes I_{m_2}),$$

$$M_{14} = (q_2 q_1 \theta_1 U_1^0 \alpha_1) \otimes I_{m_1 m_2},\ J_1^{23} = I_N \otimes (p_1 \theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2}),$$

$$J_1^{24} = I_N \otimes (I_{n_1} \otimes \eta I_{m_1 m_2}),\ J_1^{31} = \begin{bmatrix} (q_2 \theta_1 U_1^0) \otimes I_{m_1 m_2} \\ 0 \end{bmatrix},$$

$$J_1^{32} = \begin{bmatrix} 0 & 0 \\ I_{N-1} \otimes (q_2 \theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2}) & 0 \end{bmatrix},$$

$$J_1^{33} = \begin{bmatrix} M_{15} & M_{16} & 0 & \dots & 0 & 0 \\ 0 & M_{15} & M_{16} & \dots & 0 & 0 \\ 0 & 0 & M_{15} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{15} & M_{16} \\ 0 & 0 & 0 & \dots & 0 & M_{15} + M_{16} \end{bmatrix},$$

$$M_{15} = \theta_1 U_1 \oplus D_0 \oplus H_0 - \eta I_{n_1 m_1 m_2},\ M_{16} = I_{n_1} \otimes (D_1 \otimes I_{m_2}),$$

$$J_1^{34} = I_N \otimes (I_{n_1} \otimes \eta I_{m_1 m_2}),\ J_1^{44} = \begin{bmatrix} M_{17} & M_{18} & 0 & \dots & 0 & 0 \\ M_{19} & M_{17} & M_{18} & \dots & 0 & 0 \\ 0 & M_{19} & M_{17} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{17} & M_{18} \\ 0 & 0 & 0 & \dots & M_{19} & M_{17} + M_{18} \end{bmatrix},$$

$$M_{17} = U_1 \oplus D_0 \oplus H_0 - \psi I_{n_1 m_1 m_2},\ M_{18} = I_{n_1} \otimes (D_1 \otimes I_{m_2}),\ M_{19} = (q_2 U_1^0 \alpha_1) \otimes I_{m_1 m_2},$$

$$J_1^{45} = \begin{bmatrix} (q_2 U_1^0 \alpha_2) \otimes I_{m_1 m_2} & 0 \\ 0 & 0 \end{bmatrix},\ J_1^{46} = I_N \otimes (I_{n_1} \otimes \psi I_{m_1 m_2}),$$

$$J_1^{55} = \begin{bmatrix} M_{20} & M_{21} & 0 & \dots & 0 & 0 \\ 0 & M_{20} & M_{21} & \dots & 0 & 0 \\ 0 & 0 & M_{20} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{20} & M_{21} \\ 0 & 0 & 0 & \dots & 0 & M_{20}+M_{21} \end{bmatrix},$$

$M_{20} = U_2 \oplus D_0 \oplus H_0 - \psi I_{n_2 m_1 m_2}$, $M_{21} = I_{n_2} \otimes (D_1 \otimes I_{m_2})$,

$$J_1^{57} = I_{N+1} \otimes (I_{n_2} \otimes \psi I_{m_1 m_2}),\ J_1^{66} = \begin{bmatrix} M_{22} & M_{23} & 0 & \dots & 0 & 0 \\ 0 & M_{22} & M_{23} & \dots & 0 & 0 \\ 0 & 0 & M_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{22} & M_{23} \\ 0 & 0 & 0 & \dots & 0 & M_{22}+M_{23} \end{bmatrix},$$

$M_{22} = \theta_2 U_1 \oplus D_0 \oplus H_0$, $M_{23} = I_{n_1} \otimes (D_1 \otimes I_{m_2})$,

$$J_1^{68} = I_N \otimes (q_2 \theta_2 U_1^0 \gamma \otimes I_{m_1 m_2}),\ J_1^{77} = \begin{bmatrix} M_{24} & M_{25} & 0 & \dots & 0 & 0 \\ 0 & M_{24} & M_{25} & \dots & 0 & 0 \\ 0 & 0 & M_{24} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{24} & M_{25} \\ 0 & 0 & 0 & \dots & 0 & M_{24}+M_{25} \end{bmatrix},$$

$M_{24} = \theta_3 U_2 \oplus D_0 \oplus H_0$, $M_{25} = I_{n_2} \otimes (D_1 \otimes I_{m_2})$,

$$J_1^{84} = \begin{bmatrix} 0 \\ I_N \otimes (S^{)} \alpha_1 \otimes I_{m_1 m_2}) \end{bmatrix},\ J_1^{85} = \begin{bmatrix} S^0 \alpha_2 \otimes I_{m_1 m_2} & 0 \\ 0 & 0 \end{bmatrix},$$

$$J_1^{88} = \begin{bmatrix} M_{26} & M_{27} & 0 & \dots & 0 & 0 \\ 0 & M_{26} & M_{27} & \dots & 0 & 0 \\ 0 & 0 & M_{26} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{26} & M_{27} \\ 0 & 0 & 0 & \dots & 0 & M_{26}+M_{27} \end{bmatrix},$$

$M_{26} = S \oplus D_0 \oplus H_0$, $M_{27} = I_l \otimes (D_1 \otimes I_{m_2})$,

- J_0 contains transitions represents transitions from n to $n+1$ for $n \geq 1$. When a Type 2 customer arrives on the system with finite capacity N. The corresponding intensities are the diagonal elements of the matrix $I_N \otimes (I_{n_1 m_1} H_1)$.
- The matrix J_0 governs,

$$J_0 = \begin{bmatrix} J_0^{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_0^{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & J_0^{33} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J_0^{44} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_0^{55} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_0^{66} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_0^{77} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_0^{88} \end{bmatrix},$$

$$J_0^{11} = I_{m_1} \otimes H_1,\ J_0^{22} = I_N \otimes (I_{n_1 m_1} \otimes H_1),\ J_0^{33} = I_N \otimes (I_{n_1 m_1} \otimes H_1),$$

$$J_0^{44} = I_N \otimes (I_{n_1 m_1} \otimes H_1),\ J_0^{55} = I_{N+1} \otimes (I_{n_2 m_1} \otimes H_1),\ J_0^{66} = I_N \otimes (I_{n_1 m_1} \otimes H_1),$$

$$J_0^{77} = I_{N+1} \otimes (I_{n_2 m_1} \otimes H_1),\ J_0^{88} = I_{N+1} \otimes (I_{l m_1} \otimes H_1),$$

- J_2 contains transitions represents transitions from n to $n-1$ for $n \geq 1$.
- A server finishes slow service for U_1 customers with probability q_1 and after service completion will start the fresh service for waiting customers. The corresponding intensities are the diagonal elements of the matrix $p_2 q_1 \theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2}$.
- A server completes the service for T_2 customers and after service completion will start the fresh service for waiting T_2 customers. The rates of this event occurrence are defined by the matrix $U_2^0 \alpha_2 \otimes I_{m_1 m_2}$. The matrix J_2 governs,

$$J_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ J_2^{21} & J_2^{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ J_2^{31} & J_2^{32} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2^{44} & J_2^{45} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2^{54} & J_2^{55} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2^{68} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2^{78} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$J_2^{21} = \begin{bmatrix} p_2 q_1 \theta_1 U_1^0 \otimes I_{m_1 m_2} \\ \mathbf{0} \end{bmatrix},\ J_2^{22} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_{N-1} \otimes (p_2 q_1 \theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2}) & \mathbf{0} \end{bmatrix},$$

$$J_2^{31} = \begin{bmatrix} p_2 \theta_1 U_1^0 \otimes I_{m_1 m_2} \\ \mathbf{0} \end{bmatrix},\ J_2^{32} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_{N-1} \otimes (p_2 \theta_1 U_1^0 \alpha_1 \otimes I_{m_1 m_2}) & \mathbf{0} \end{bmatrix},$$

$$J_2^{45} = \begin{bmatrix} p_2 U_1^0 \alpha_2 \otimes I_{m_1 m_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ J_2^{44} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_{N-1} \otimes (p_2 U_1^0 \alpha_1 \otimes I_{m_1 m_2}) & \mathbf{0} \end{bmatrix},$$

$$J_2^{55} = \begin{bmatrix} U_2^0 \alpha_2 \otimes I_{m_1 m_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ J_2^{54} = \begin{bmatrix} \mathbf{0} \\ I_N \otimes (U_2^0 \alpha_1 \otimes I_{m_1 m_2}) \end{bmatrix},$$

$$J_2^{68} = \begin{bmatrix} I_N \otimes (p_2 \theta_2 U_1^0 \gamma \otimes I_{m_1 m_2}) & \mathbf{0} \end{bmatrix},\ J_2^{78} = I_{N+1} \otimes (\theta_3 U_2^0 \gamma \otimes I_{m_1 m_2}).$$

The matrices R is given by $R = \lim_{j \to \infty} R_j$ where the sequence of matrices $\{R_j\}$ is defined as:

$$R_0 = 0, \tag{3}$$

$$R_{j+1} = -J_0 J_1^{-1} - R_n^2 J_1^{-1} J_2, \quad j = 0, 1, 2, \ldots \tag{4}$$

For stable systems, the sequence $\{R_j\}$ is monotonically increasing and converges to R. Hence, R can be evaluated by successive substitutions using equation 4, until a desired level of convergence is achieved.

5. Analysis

5.1. Condition for Stability

Let $J = J_0 + J_1 + J_2$ be an irreducible infinitesimal generator matrix with order $m_1 m_2 + 4N n_1 m_1 m_2 + 2(N+1) n_2 m_1 m_2 + (N+1) l m_1 m_2$.

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$$J = \begin{bmatrix} J^{11} & J^{12} & \mathbf{0} & \mathbf{0} & J^{15} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ J^{21} & J^{22} & J^{23} & J^{24} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ J^{31} & J^{32} & J^{33} & J^{34} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J^{44} & J^{45} & J^{46} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J^{54} & J^{55} & \mathbf{0} & J^{57} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J^{66} & \mathbf{0} & J^{68} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J^{77} & J^{78} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J^{84} & J^{85} & \mathbf{0} & \mathbf{0} & J^{88} \end{bmatrix},$$

where
$J^{11} = J_1^{11} + J_2^{11}$, $J^{12} = J_1^{12}$, $J^{15} = J_1^{15}$, $J^{21} = J_1^{21} + J_2^{21}$, $J^{22} = J_1^{22} + J_2^{22} + J_0^{22}$, $J^{24} = J_1^{24}$, $J^{31} = J_1^{31} + J_2^{31}$, $J^{32} = J_1^{32} + J_2^{32}$, $J^{33} = J_1^{33} + J_0^{33}$, $J^{34} = J_1^{34}$, $J^{44} = J_1^{44} + J_0^{44} + J_2^{44}$, $J^{45} = J_1^{45} + J_2^{45}$, $J^{46} = J_1^{46}$, $J^{54} = J_1^{54}$, $J^{55} = J_1^{55} + J_2^{55}$, $J^{57} = J_1^{57}$, $J^{66} = J_1^{66} + J_0^{66}$, $J^{68} = J_1^{68} + J_2^{68}$, $J^{77} = J_1^{77} + J_0^{77}$, $J^{78} = J_2^{78}$, $J^{84} = J_1^{84}$, $J^{85} = J_1^{85}$, $J^{88} = J_1^{88} + J_0^{88}$.

Let $\boldsymbol{\xi}$ denote the steady-state probability vector of the matrix F, satisfying the conditions $\boldsymbol{\xi}F = 0$ and $\boldsymbol{\xi}e = 1$. The vector $\boldsymbol{\xi}$ is partitioned as follows:

$$\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7),$$

where the dimensions of each component are given by:

- ξ_0 has size $m_1 m_2$,
- ξ_1, ξ_2, and ξ_3 each have size $N n_1 m_1 m_2$,
- ξ_4 has size $(N+1) n_2 m_1 m_2$,
- ξ_5 has size $N n_1 m_1 m_2$,
- ξ_6 has size $(N+1) n_2 m_1 m_2$,
- ξ_7 has size $(N+1) l m_1 m_2$.

The steady-state probability vector $\boldsymbol{\xi}$ is obtained by solving the following system of equations:

$$\begin{aligned} \xi_0 J^{11} + \xi_1 J^{21} + \xi_2 J^{31} &= 0, \\ \xi_0 J^{12} + \xi_1 J^{22} + \xi_2 J^{32} &= 0, \\ \xi_1 J^{23} + \xi_2 J^{33} &= 0, \\ \xi_1 J^{24} + \xi_2 J^{34} + \xi_3 J^{44} + \xi_4 J^{54} + \xi_7 J^{84} &= 0, \\ \xi_0 J^{15} + \xi_3 J^{45} + \xi_4 J^{55} + \xi_7 J^{85} &= 0, \\ \xi_3 J^{46} + \xi_5 J^{66} &= 0, \\ \xi_4 J^{57} + \xi_6 J^{77} &= 0, \\ \xi_5 J^{68} + \xi_6 J^{78} + \xi_7 J^{88} &= 0, \end{aligned}$$

subject to normalizing condition

$$\begin{aligned} \xi_0 e_{m_1 m_2} + \sum_{a=1}^{N} \xi_{1a} e_{n_1 m_1 m_2} + \sum_{a=1}^{N} \xi_{2a} e_{n_1 m_1 m_2} + \sum_{d=1}^{N} \xi_{3d} e_{n_1 m_1 m_2} + \sum_{d=0}^{N} \xi_{4d} e_{n_2 m_1 m_2} \\ + \sum_{a=1}^{N} \xi_{5a} e_{n_1 m_1 m_2} + \sum_{a=0}^{N} \xi_{6a} e_{n_2 m_1 m_2 m} + \sum_{d=0}^{N} \xi_{7d} e_{l m_1 m_2 m} = 1. \end{aligned}$$

The LIQBD description of the model indicates that the queueing system is stable if and only if the left drift exceeds the right drift (see Theorem 3.1.1 of Neuts [16]), that is,

$$\boldsymbol{\xi} J_0 e < \boldsymbol{\xi} J_2 e. \tag{5}$$

$$\begin{aligned}\xi J_0 e &= \boldsymbol{\xi_0} J_0^{11} e + \boldsymbol{\xi_1} J_0^{22} e + \boldsymbol{\xi_2} J_0^{33} e + \boldsymbol{\xi_3} J_0^{44} e + \boldsymbol{\xi_4} J_0^{55} e + \boldsymbol{\xi_5} J_0^{66} e + \boldsymbol{\xi_6} J_0^{77} e + \boldsymbol{\xi_7} J_0^{88} e \\ &= \sum_{i=0}^{7} \boldsymbol{\xi_i} J_0^{ii} e. \end{aligned} \tag{6}$$

$$\begin{aligned}\xi J_2 e = \big[\, & \xi_1(J_2^{21}) + \xi_2(J_2^{31}),\ \xi_1(J_2^{22}) + \xi_2(J_2^{32}),\ 0, \xi_3(J_2^{44}) + \xi_4(J_2^{54}), \\ & \xi_3(J_2^{45}) + \xi_4(J_2^{55}),\ 0,\ 0, \xi_5(J_2^{68}) + \xi_6(J_2^{78}) \,\big] e. \end{aligned} \tag{7}$$

Multiplying by $(\boldsymbol{e}_{n_1} \boldsymbol{e}_{m_1} \otimes H_1)$ in Equation 6 and substituting Equations 6 and 7 into Equation 5, we obtain

$$\begin{aligned} & \xi_0(e_{m_1} \otimes H_1) + \sum_{a=1}^{N} \xi_{1d}(e_{n_1 m_1} \otimes H_1) + \sum_{a=1}^{N} \xi_{2d}(e_{n_1 m_1} \otimes H_1) + \sum_{a=1}^{N} \xi_{3d}(e_{n_1 m_1} \otimes H_1) \\ & + \sum_{a=0}^{N} \xi_{4d}(e_{n_2 m_1} \otimes H_1) + \sum_{a=1}^{N} \xi_{5d}(e_{n_1 m_1} \otimes H_1) + \sum_{a=0}^{N} \xi_{6d}(e_{n_2 m_1} \otimes H_1) \\ & + \sum_{a=0}^{N} \xi_{7d}(e_{l m_1} \otimes H_1) < \sum_{a=1}^{N} \xi_{1d}(p_2 q_1 \theta_1 U_1^0 \otimes e_{m_1 m_2}) + \sum_{a=1}^{N} \xi_{2d}(p_2 \theta_1 U_1^0 \otimes e_{m_1 m_2}) \\ & + \sum_{a=1}^{N} \xi_{3d}(p_2 U_1^0 \otimes e_{m_1 m_2}) + \sum_{a=0}^{N} \xi_{4d}(U_2^0 \otimes e_{m_1 m_2}) + \sum_{a=1}^{N} \xi_{5d}(p_2 \theta_2 U_1^0 \otimes e_{m_1 m_2}) \\ & + \sum_{a=0}^{N} \xi_{6d}(\theta_3 U_2^0 \otimes e_{m_1 m_2}). \end{aligned}$$

5.2. The Stationary Probability Vector

Let Z be the solution for the infinitesimal generator Q of the process {Z(t): $t \geq 0$}. This Z is splitted up depending on status of the server $(z_0, z_1, z_2, ...)$, where z_0 is of size $2m_1m_2 + 4Nn_1m_1m_2 + (N+1)lm_1m_2$ and $z_1, z_2, ...$ are of size $m_1m_2 + 4Nn_1m_1m_2 + 2(N+1)n_2m_1m_2 + (N+1)lm_1m_2$. As Z is a vector satisfies the condition

$$ZQ = 0 \ \ and \ \ Ze = 1. \tag{8}$$

The probability vector Z follows a matrix geometric structure under the steady state is

$$z_j = z_1 R^{j-1},\ j \geq 2, \tag{9}$$

where R is the quadratic equation's lowest non-negative solution

$$R^2 J_2 + R J_1 + J_0 = 0, \tag{10}$$

and the vector z_0, z_1 are obtained with the help of succeeding equations:

$$z_0 A_{00} + z_1 A_{10} = 0, z_0 A_{01} + z_1 [J_1 + R J_2] = 0, \tag{11}$$

subject to a condition normalization

$$z_0 e_0 + z_1 [I - R]^{-1} e_1 = 1. \tag{12}$$

As an outcome, we can employ the structure of something like the coefficient matrices to compute the vector Z and the matrix R using the Logarithmic Reduction Algorithm in Latouche and Ramaswami [24].

6. Busy Period Analysis

- A busy period is commonly described as the interval of time between when customers join an empty system with positive inventory and when they exit the system empty after obtaining their services in a queueing inventory system of single-server demonstration. Consequently, this marks the beginning of the shift from level 1 to level 0. The first return time of level zero, followed by at least one visit to any subsequent level, is an analogy for the busy cycle.
- A notable concept introduced by Latouche and Ramaswami [25] is the fundamental period, which pertains to the time taken to transition from level i to $i-1$, where $i \geq 2$ within the framework of a QBD.
- A notable feature of this approach is that for each level i, $i \geq 2$, there are $m_1 m_2 + 4Nn_1 m_1 m_2 + 2(N+1)n_2 m_1 m_2 + (N+1)lm_1 m_2$ states associated with it. This quantitative representation captures the complexity and variability present within the system across different levels.
- The QBD stream conditional probability originates in the state (u,x) at time $t=0$ and goes to the level $(i-1)$ but not earlier time x, allowing for changes. The variable $G_{jj'}(u,x)$ represents the u transition to the left and reaching the state (i,j').

The transition matrix

$$\widehat{G}_{jj'}(z,y) = \sum_{u=1}^{\infty} z^u \int_0^{\infty} e^{-yx} dG_{jj'}(u,x) \;\; ;|z| \leq 1,\ Re(y) \geq 0, \tag{13}$$

and the matrix is shown as $\widehat{G}(z,y) = \widehat{G}_{jj'}(z,y)$, satisfying

$$\widehat{G}(z,y) = z(yI - J_1)^{-1} J_2 + (yI - J_1)^{-1} J_0 \widehat{G}^2(z,y). \tag{14}$$

Let $G = G_{jj'} = \widehat{G}(1,0)$, is an initial passage time without the boundary states.

$$G = -(J_1 + RJ_2)^{-1} J_2. \tag{15}$$

Otherwise, the G matrix's values could be calculated using the concept of a logarithmic reduction procedure [24].

For boundary levels 1 and 0, we get the equations provided by,

$$\widehat{G}^{(1,0)}(z,y) = z(yI - J_1)^{-1} A_{10} + (yI - J_1)^{-1} J_0 \widehat{G}(z,y) \widehat{G}^{(1,0)}(z,y), \tag{16}$$

$$\widehat{G}^{(0,0)}(z,y) = (yI - A_{00})^{-1} A_{01} \widehat{G}^{(1,0)}(z,y). \tag{17}$$

Due to the stochastic character of G, $\widehat{G}^{(0,0)}(1,0)$ and $\widehat{G}^{(1,0)}(1,0)$ the matrices are utilised to calculate the subsequent cases. The following are the instants that can be calculated at $z=1$ and $y=0$.

$$\vec{\mathcal{N}}_1 = -\frac{\partial}{\partial x} \widehat{G}(z,y)e = -[J_1 + J_0(I+G)]^{-1} e, \tag{18}$$

$$\vec{\mathcal{N}}_2 = \frac{\partial}{\partial z} \widehat{G}(z,y)e = -[J_1 + J_0(I+G)]^{-1} J_2 e, \tag{19}$$

$$\vec{\mathcal{N}}_1^{(1,0)} = -\frac{\partial}{\partial s} \widehat{G}^{(1,0)}(z,y)e = -[J_1 + J_0 G]^{-1} (J_0 \vec{\mathcal{N}}_1 + e), \tag{20}$$

$$\vec{\mathcal{N}}_2^{(1,0)} = \frac{\partial}{\partial z} \widehat{G}^{(1,0)}(z,y)e = -[J_1 + J_0 G]^{-1} (J_0 \vec{\mathcal{N}}_2 + A_{10} e), \tag{21}$$

$$\vec{\mathcal{N}}_1^{(0,0)} = -\frac{\partial}{\partial s} \widehat{G}^{0,0}(z,y)e = -A_{00}^{-1} [A_{01} \vec{\mathcal{N}}_1^{(1,0)} + e], \tag{22}$$

$$\vec{\mathcal{N}}_2^{(0,0)} = \frac{\partial}{\partial z} \widehat{G}^{(0,0)}(z,y)e = -A_{00}^{-1} [A_{01} \vec{\mathcal{N}}_2^{(1,0)}]. \tag{23}$$

7. Evaluating System Operations

In this section, we calculate some system performance measures useful in qualitative interpretation of the model under study. We shall use the following representations:

7.1. Estimated Number of (T_2) Customers Currently in the System

$$E_{T_2} = \sum_{i=1}^{\infty} i z_i e.$$

7.2. Probability of the Server Being Inactive During a Working Vacation Period

Let $z_{i,d,0,b_1,b_2}$ denotes the invariant probability of the number of T_2 customers in the queue is i for $i \geq 1$, the number of T_1 customers in the system is d for $1 \leq d \leq N$, the server is inactive in working vacation is 0, and the number of arrival phases is b_1 for $1 \leq b_1 \leq m_1$ and b_2, for $1 \leq b_2 \leq m_2$.

$$P_{idle-vacation} = \sum_{b_1=1}^{m_1} \sum_{b_2=1}^{m_2} z_{0,0,0,b_1,b_2} + \sum_{i=1}^{\infty} \sum_{b_1=1}^{m_1} \sum_{b_2=1}^{m_2} z_{i,0,0,b_1,b_2}.$$

7.3. Probability of the Server Being Active During a Working Vacation Period

Let $z_{i,d,1,k_1,b_1,b_2}$ denotes the invariant probability of the number of T_2 customers in the queue is 0, the number of T_1 customers in the system is d for $1 \leq d \leq N$, the server is active in working vacation is 1, the number of T_1 customers service phase is k_1 for $1 \leq k_1 \leq n_1$ and the number of arrival phases is b_1 for $1 \leq b_1 \leq m_1$ and b_2, for $1 \leq b_2 \leq m_2$.

$$P_{BT_1}^{WV} = \sum_{i=0}^{\infty} \sum_{a=1}^{N} \sum_{k_2=1}^{n_1} \sum_{b_1=1}^{m_1} \sum_{b_2=1}^{m_2} z_{i,a,1,k_2,b_1,b_2}.$$

7.4. Probability That the Server Is Reservice in Working Vacation

$$P_{RBT_1}^{WV} = \sum_{i=0}^{\infty} \sum_{a=1}^{N} \sum_{k_2=1}^{n_1} \sum_{b_1=1}^{m_1} \sum_{b_2=1}^{m_2} z_{i,a,2,k_2,b_1,b_2}.$$

7.5. Probability of the Server Being Inactive During a Normal Operation

Let $z_{i,d,3,b_1,b_2}$ denotes the invariant probability of the number of T_2 customers in the queue is 0, the number of T_1 customers in the system is d for $1 \leq d \leq N$, the server is idle in normal period is 3, and the number of arrival phases is b_1 for $1 \leq b_1 \leq m_1$ and b_2, for $1 \leq b_2 \leq m_2$.

$$P_{idle-systme} = \sum_{b_1=1}^{m_1} \sum_{b_2=1}^{m_2} z_{0,0,3,b_1,b_2}.$$

7.6. Probability of the Server Handling T_1 Customers in Normal Mode

$$P_{BT_1}^{NM} = \sum_{i=0}^{\infty} \sum_{a=1}^{N} \sum_{k_2=1}^{n_1} \sum_{b_1=1}^{m_1} \sum_{b_2=1}^{m_2} z_{ia4k_2b_1b_2}.$$

7.7. Probability That the Server Is Active $(T_2\ )$ in Normal Mode

$$P_{BT_2}^{NM} = \sum_{i=0}^{\infty} \sum_{a=0}^{N} \sum_{k_3=1}^{n_2} \sum_{b_1=1}^{m_1} \sum_{b_2=1}^{m_2} z_{(i,a,5,k_3,b_1,b_2)}.$$

7.8. *Probability That the Server Is Busy*

$$P_{busy} = P^{WV}_{BT_1} + P^{WV}_{RBT_1} + P^{NM}_{BT_1} + P^{NM}_{BT_2}.$$

7.9. *Probability That the Server Is Idle*

$$P_{idle} = P_{idle-vacation} + P_{idle-system}.$$

7.10. *Particular Case*

When parameters representing imperfect service, working breakdowns, and probabilistic participation are deactivated (i.e., set to ideal or null values), the model simplifies to the Chakravarthy [20] framework without those additional complexities. This reduction confirms structural validity—showing that the extended model is a true generalization rather than an unrelated formulation.

Furthermore, the analytical results—such as steady-state probabilities, mean queue length, and system utilization—were cross-checked to ensure they converge to the same values as those derived from the baseline model when the extension parameters are neutralized. Such consistency provides mathematical assurance that the new features have been integrated correctly into the queueing framework.

8. Numerical Implementation

To generate numerical results, we utilized various MAP representations for incoming arrivals, ensuring that their mean values are equal to 1, following the approach suggested by Chakravarthy [26].

- T_1 **customers with Erlang arrival distribution-** (ERA)**:**

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

- T_2 **customers with Erlang arrival distribution -** (ERA)**:**

$$H_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \; H_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

- **Exponential arrival pattern -**T_1 (EXA)**:**

$$D_0 = [-1], \; D_1 = [1].$$

- **Exponential arrival pattern-**$T2$ (EXA)**:**

$$H_0 = [-1], \; H_1 = [1].$$

- **Hyper exponential arrival-**T_1 $(HEXA)$**:**

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \; D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}.$$

- **Hyper exponential arrival-**T_2 $(HEXA)$**:**

$$H_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \; H_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}.$$

For the servicing and repair progression, take into account the following PH-distributions:

- **Erlang service** (ERS):

$$\alpha_1 = \alpha_2 = [1, 0], \quad U_1 = U_2 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$$

- **Erlang repair** (ERR):

$$\gamma = [1, 0], \quad S = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$$

- **Exponential service** (EXS):

$$\alpha_1 = \alpha_2 = [1], \quad U_1 = U_2 = [-1].$$

- **Exponential repair** (EXR):

$$\gamma = [1], \quad S = [-1].$$

- **Hyper exponential service** $(HEXS)$:

$$\alpha_1 = \alpha_2 = [0.8, 0.2], \quad U_1 = U_2 = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

- **Hyper exponential repair** $(HEXR)$:

$$\gamma = [0.8, 0.2], \quad S = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

Illustrative 1. We investigated the implications of the the T_1 service rate (μ_1) on the determined system size (E_{system}). We define $\lambda_1 = 1$, $\lambda_2 = 1.2$, $\mu_2 = 20$, and $\eta = 3$, $\theta_1 = 0.6$, $\theta_2 = 0.4$, $\tau = 3$, $\theta_3 = 0.6$, $p_1 = q_1 = 0.6$, $p_2 = q_2 = 0.6$, $N = 5$, such that the system remains stable.

- The matrix R using the Logarithmic Reduction Algorithm in Latouche and Ramaswami [24]. Numerical results were computed iteratively up to $I = 4$ for Erlang distribution, $I = 5$ for exponential distribution and $I = 7$ for hyperexponential distribution. At this level, the solution demonstrated convergence with an accuracy of up to six significant digits. The computations were performed using MATLAB software (R2019b), ensuring precise numerical implementation and consistency. This confirms the stability and reliability of the obtained results.
- By incorporating different arrival and service time categories, we analyze the relationship between the T_1 service (μ_1) and the expected system size (E_{system}), as presented in Tables 1–3.
- As the T_1 service rate (μ_1) increases, the system processes customers or orders more efficiently, thereby reducing congestion and queue lengths. The service rate (μ_1) represents the average number of customers or jobs a server can handle per unit of time. It directly governs how quickly the system can process incoming tasks and therefore plays a crucial role in determining system congestion and stability.
- In relation to all other arrival times, the E_{system} drops down quickly for $HEXA$ and slowly for ERA. Similarly, in terms of service times, the E_{system} drops rapidly in ERS and more slowly with $HEXS$. This suggests that systems exposed to variable (heterogeneous) arrival processes benefit more significantly from an increase in service rate, as higher μ_1 compensates for irregular inflow patterns.

Table 1. E_{system} vs. service rate μ_1-*ERS*

μ_1	ERA	EXA	HEXA
11	0.273323	0.272934	0.327740
12	0.265213	0.267634	0.320544
13	0.258952	0.263574	0.315105
14	0.254005	0.260387	0.310875
15	0.250021	0.257834	0.307507
16	0.246757	0.255752	0.304772
17	0.244045	0.254029	0.302514
18	0.241763	0.252584	0.300622
19	0.239822	0.251360	0.299017
20	0.238153	0.250310	0.297641

Table 2. E_{system} vs. service rate μ_1-*EXS*

μ_1	ERA	EXA	HEXA
11	0.264056	0.289999	0.344393
12	0.257478	0.282631	0.334970
13	0.252446	0.277002	0.327868
14	0.248502	0.272595	0.322372
15	0.245343	0.269071	0.318024
17	0.240639	0.263832	0.311646
18	0.238852	0.261847	0.309260
19	0.237336	0.260165	0.307251
20	0.236037	0.258726	0.305543

Table 3. E_{system} vs. service rate μ_1-*HEXS*

μ_1	ERA	EXA	HEXA
11	0.541994	0.416627	0.366188
12	0.507018	0.392839	0.353202
13	0.479533	0.374753	0.344100
14	0.457402	0.360665	0.337594
15	0.439211	0.349462	0.332868
16	0.423996	0.340394	0.329391
17	0.411079	0.332939	0.326806
18	0.399972	0.326728	0.324870
19	0.390314	0.321490	0.323411
20	0.381834	0.317027	0.322309

Illustrative 2. We analyzed the effect of the T_1 server's service rate, denoted as μ_1, influences the probability that the server is busy during normal mode, represented by $P^{NM}_{BT_1}$ in Tables 4–6. To ensure system stability during this analysis, the following parameters were defined: $\lambda_1 = 1$, $\lambda_2 = 1.2$, $\mu_2 = 20$, $\eta = 3$, $\theta_1 = 0.6$, $\theta_2 = 0.4$, $\theta_3 = 0.6$, $\tau = 3$, $p_1 = q_1 = 0.6$, $p_2 = q_2 = 0.6$, and $N = 5$.

- The matrix R using the Logarithmic Reduction Algorithm in Latouche and Ramaswami [24]. Numerical results were computed iteratively up to $I = 4$ for Erlang distribution, $I = 6$ for exponential distribution and $I = 7$ for hyperexponential distribution. At this level, the solution demonstrated convergence with an accuracy of up to six significant digits. The computations were performed using MATLAB software (R2019b), ensuring precise numerical implementation and consistency. This confirms the stability and reliability of the obtained results.
- As the Type-1 service rate (μ_1) increases, the system's ability to handle incoming tasks improves, allowing the server to process requests more quickly. Consequently, the proportion of time that

the server remains busy during normal operation, denoted as $P_{BT_1}^{NM}$, decreases. A higher service rate thus alleviates congestion and reduces queue lengths, as tasks are completed faster than they arrive, leading to fewer intervals in which the server is continuously occupied.

- When comparing different arrival distributions, the rate at which $P_{BT_1}^{NM}$ decreases varies depending on the variability of arrivals. Under *hyper-exponential arrivals*, which exhibit high variability and burstiness, an increase in μ_1 produces a more pronounced reduction in $P_{BT_1}^{NM}$, since the system benefits substantially from faster service during sudden demand spikes. In contrast, for *Erlang arrivals*, where inter-arrival times are more regular, the decline in $P_{BT_1}^{NM}$ is more gradual because the arrival process is smoother and less sensitive to service rate adjustments.
- Similarly, for different service-time distributions, the responsiveness of $P_{BT_1}^{NM}$ to changes in μ_1 depends on service variability. When service times follow an *Erlang distribution*, which is more uniform, increasing μ_1 rapidly reduces $P_{BT_1}^{NM}$, as predictable and shorter service durations allow the server to become idle more frequently. However, under a *hyper-exponential service* pattern, characterized by higher variability, the reduction in $P_{BT_1}^{NM}$ is slower, since occasional long service times can still keep the server occupied even at higher service rates.

Table 4. $P_{BT_1}^{NM}$ vs. service rate μ_1-ERS

μ_1	ERA	EXA	HEXA
11	0.062016	0.054459	0.048891
12	0.056528	0.049764	0.044835
13	0.051871	0.045785	0.041364
14	0.047881	0.042375	0.038368
15	0.044432	0.039423	0.035761
16	0.041425	0.036846	0.033473
17	0.038784	0.034578	0.031453
18	0.036448	0.032568	0.029657
19	0.034369	0.030775	0.028050
20	0.032508	0.029166	0.026606

Table 5. $P_{BT_1}^{NM}$ vs. service rate μ_1-EXS

μ_1	ERA	EXA	HEXA
11	0.057451	0.054327	0.048716
12	0.052635	0.049782	0.044798
13	0.048525	0.045896	0.041416
14	0.044982	0.042544	0.038475
15	0.041901	0.039627	0.035901
16	0.039200	0.037068	0.033633
17	0.036815	0.034809	0.031622
18	0.034696	0.032800	0.029828
19	0.032800	0.031003	0.028220
20	0.031096	0.029388	0.026770

Table 6. $P_{BT_1}^{NM}$ vs. service rate μ_1-$HEXS$

μ_1	ERA	EXA	HEXA
11	0.047689	0.047659	0.062945
12	0.044663	0.044522	0.056734
13	0.041942	0.041719	0.051656
14	0.039491	0.039209	0.047412
15	0.037280	0.036952	0.043806
16	0.035278	0.034917	0.040699
17	0.033461	0.033074	0.037993
18	0.031806	0.031400	0.035614
19	0.030295	0.029875	0.033505
20	0.028911	0.028479	0.031623

Illustrative 3.

- In Figures 2–4, we analyze the impact of the Type 2 service rate (μ_2) on the probability that the server remains idle during normal operation, denoted as $P_{idle-system}$, to evaluate how variations in service efficiency influence overall system utilization and stability.
- We fix the parameters as $\lambda_1 = 1$, $\lambda_2 = 1.2$, $\mu_1 = 20$, $\eta = 3$, $\theta_1 = 0.6$, $\theta_2 = 0.4$, $\tau = 3$, $\theta_3 = 0.6$, $p_1 = q_1 = 0.6$, $p_2 = q_2 = 0.6$, $N = 5$ and $\mu_2 = 11, 13, 15, 17, 19, 21, 23, 25, 27, 29$ to ensure that the system remains within the stability region, where the total service capacity exceeds the effective arrival rate.
- A close examination of Figures 2–4 indicates that, under all combinations of arrival and service time distributions, the probability that the server is idle during normal operation $P_{idle-system}$ increases with the Type 2 service rate (μ_2).
- This behavior can be interpreted as follows: as the Type 2 service rate (μ_2) increases, the server completes tasks more rapidly, leading to shorter busy periods and more frequent idle intervals. In essence, a higher μ_2 enhances the system's processing efficiency, reducing congestion and waiting times. However, when the service rate grows significantly beyond the arrival rate, the server experiences extended idle times, reflecting potential over-provisioning of service capacity.

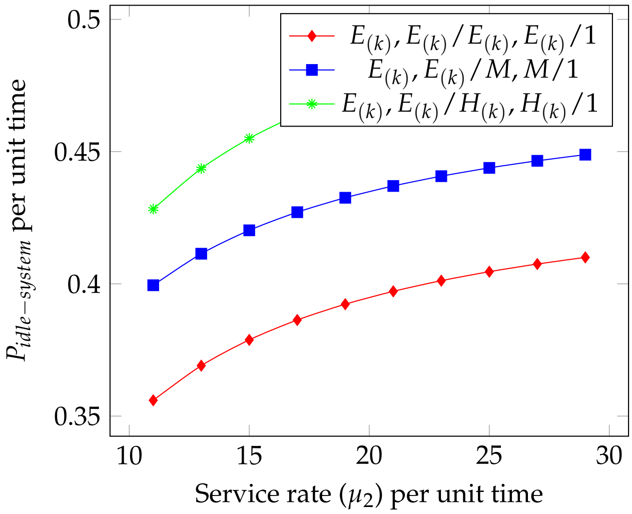

Figure 2. $P_{idle-system}$ vs. Service rate (μ_2) - ERA.

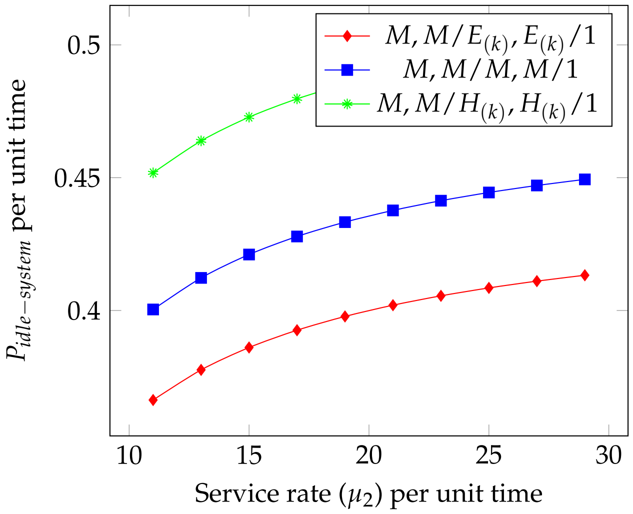

Figure 3. $P_{idle-system}$ vs. Service rate (μ_2) - EXA.

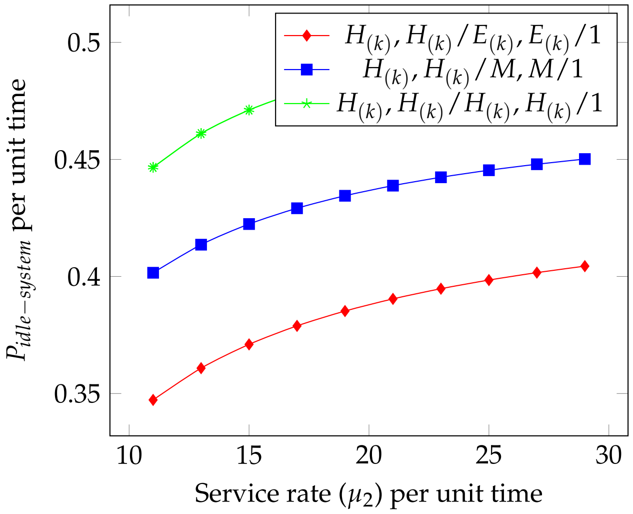

Figure 4. $P_{idle-system}$ vs. Service rate (μ_2) - HEXA.

When the arrival rate, breakdown rate, and probability of dissatisfaction increase, the average queue size tends to grow because the system experiences higher input and reduced efficiency. A higher arrival rate means more customers enter the system per unit of time, while an increased breakdown rate decreases the effective service capacity as servers become unavailable more often. Likewise, a higher probability of dissatisfaction can lead to more re-service requests or inefficiencies, effectively increasing the workload. Together, these factors cause congestion, leading to longer queues. Conversely, when the service rate, repair rate, and probability of satisfaction increase, the average queue size decreases because the system processes customers more efficiently. A higher service rate allows faster completion of tasks, an improved repair rate reduces downtime, and a higher satisfaction probability lowers the likelihood of repeated service or complaints. As a result, the system clears customers more smoothly, leading to shorter queues and better overall performance.

9. Conclusion

This paper investigates a queueing model applicable to scenarios like crowdsourced operations, working vacations, service imperfections, functional breakdowns, and phase-type repair mechanisms. The framework permits the server to continue attending to a specific group of customers during vacation periods, though at a diminished service efficiency. By adopting a versatile class of arrival processes and representing service durations through phase-type distributions, we highlight the benefits of integrating these elements into traditional queueing theory. These advantages are further illustrated with comprehensive numerical simulations.

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Notations

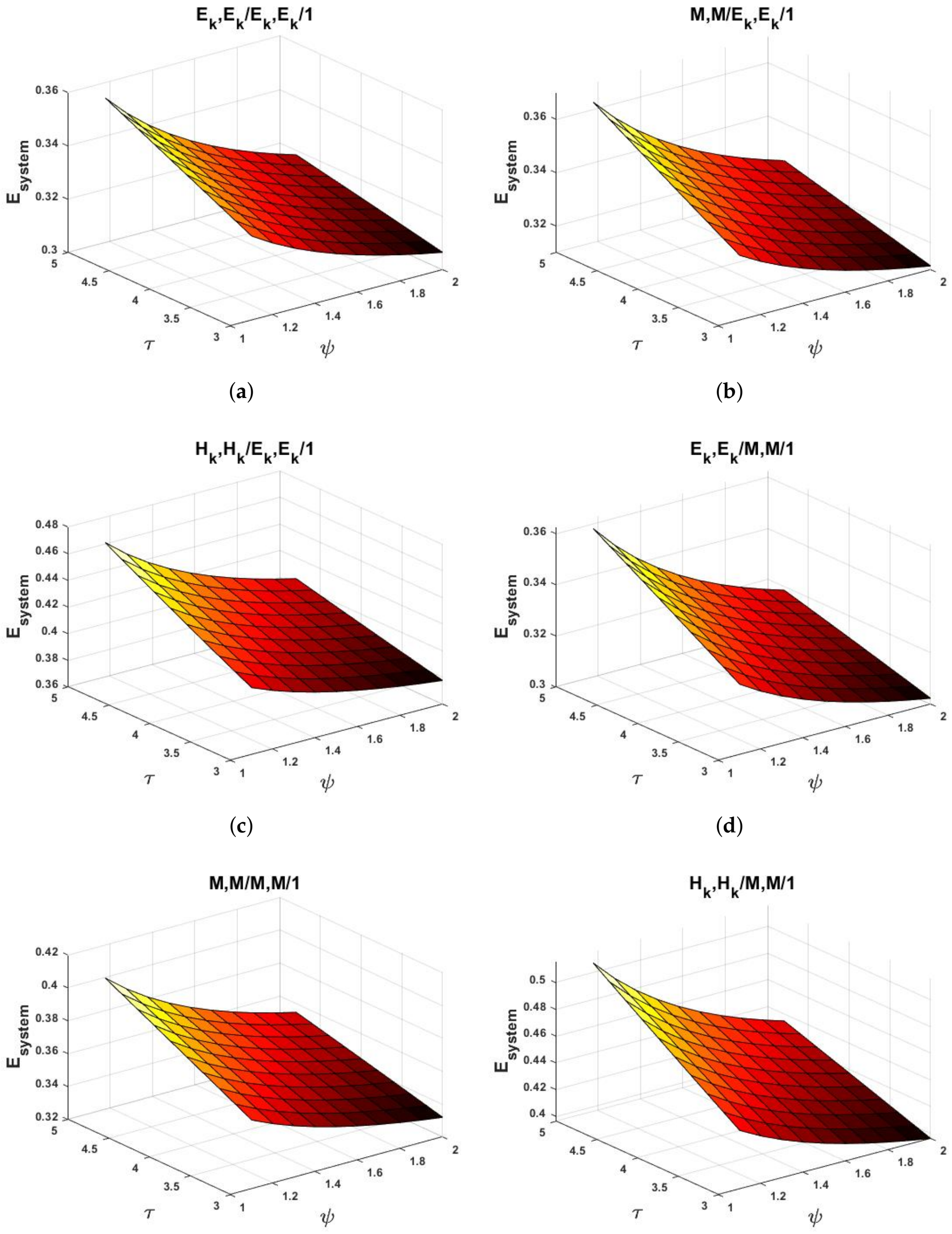
The following notations are used in this manuscript:

$\otimes$	Kronecker product, producing a block matrix from two matrices of appropriate sizes.
$\oplus$	Kronecker sum, generates a block matrix by combining two matrices of compatible sizes.
I_m	Refers to the identity matrix of size $m \times m$.
e	A column vector of appropriate length, with all entries equal to one.
MAP	Markovian Arrival Process.
PH	Phase type distributions.

Appendix A

Illustrative 4.

- To examine the effect of the expected system size (E_{system}) on both the breakdown rate (ψ) and the repair rate (τ), we refer to Figures A1 and A2. The parameters are set as $\lambda_1 = 1$, $\lambda_2 = 1.2$, $\mu_2 = 20$, $\eta = 3$, $\theta_1 = 0.6$, $\theta_2 = 0.4$, $\tau = 3$, $\theta_3 = 0.6$, $p_1 = q_1 = 0.6$, $p_2 = q_2 = 0.6$, and $N = 5$ to ensure that the system remains stable throughout the analysis.
- When examining the influence of service-time distributions, E_{system} increases sharply under the Erlang service (ERS) scenario, where the regularity of service times amplifies the impact of repeated breakdowns—each interruption extends queue lengths more significantly. In contrast, under the Exponential service (EXS) case, the increase in E_{system} is more moderate, as the stochastic nature of service times allows for some inherent flexibility in handling interruptions.
- Similarly, considering arrival-time distributions, E_{system} grows slowly under Erlang arrivals (ERA) due to their low variability and predictable inflow, which helps the system absorb disturbances more effectively. However, under Hyper-exponential arrivals (HEXA), where inter-arrival times are highly variable and bursty, E_{system} increases rapidly. In such cases, bursts of arrivals coinciding with breakdowns intensify congestion and elevate system size significantly.

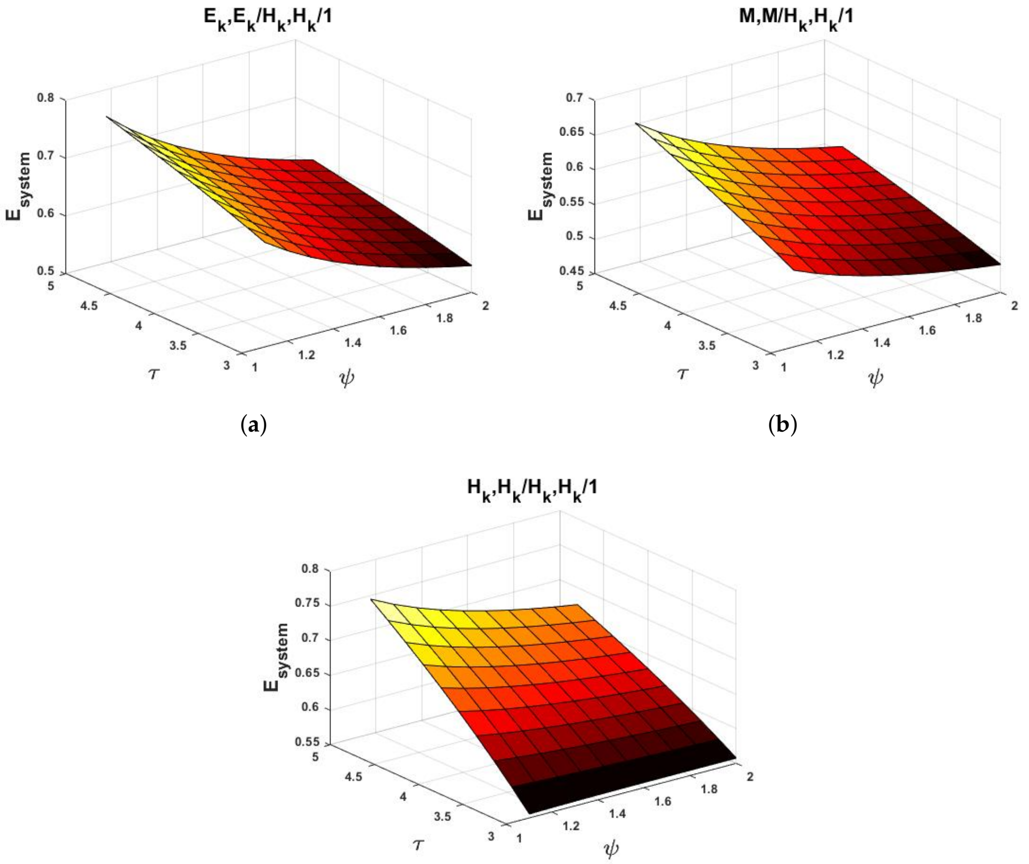


(**a**) (**b**)

(**c**) (**d**)

(**e**) (**f**)

Figure A1. E_{system} vs. breakdown and repair rates



(**a**) (**b**)

(**c**)

Figure A2. E_{system} vs. breakdown and repair rates

References

1. Shrivastava, R. K.; Rathore, R. Analysis of Single Server Markovian Queueing Model with Differentiated Working Vacation, Vacation Interruption, Soft Failure, Reneging of Customers. *Int. J. Global Acad. Sci. Res.* **2024**, *5* (2), 78–92.
2. Rathore, R.; Shrivastava, R. K. Analysis of M/M/1 Queueing Model with Removable and Unreliable Server, Partial Breakdown during Working Vacation, Setup with Repair. *J. Math. Probl. Equ. Stat.* **2024**, *5* (2), 141–155.
3. Prakati, P.; Julia, R. M. K. Analysis of Multiple Working Vacations Queuing System with Encouraged Arrival Using M/M(a,b)/1 Model. *Baghdad Sci. J.* **2024**, *6* (3), 9162–9175.
4. Singh, M.; Jain, M.; Azhagappan, A. Cost Analysis of a Transient Markovian Queueing Model with Provision of Options between Regular and Working Vacation. *Int. J. Math. Oper. Res.* **2024**, *10* (1), 138906–138920.
5. Ayyappan, G.; Arulmozhi, N. Analysis of M, MAP/PH1, PH2/1 Non-preemptive Priority Queueing Model with Delayed Working Vacations, Immediate Feedback, Impatient Customers, Differentiate Breakdown, and Phase-Type Repair. *Reliab. Theory Appl.* **2023**, *18* (4), 64–79.
6. Ayyappan, G.; Karpagam, S. An $M^{[X]}/G(a,b)/1$ queueing system with server breakdown and repair, stand-by server, and single vacation. *Int. J. Math. Oper. Res.* **2019**, *14* (2), 221–235.
7. Zhang, Y.; Li, X.; Chen, H.; Wang, J. Reinforcement Learning-Based Edge Server Placement in the Intelligent Internet of Vehicles Environment. *IEEE Trans. Intell. Transp. Syst.* **2025**.
8. D'Arienzo, M. P.; Dudin, A. N.; Dudin, S. A.; Manzo, R. Analysis of a retrial queue with group service of impatient customers. *J. Ambient Intell. Humaniz. Comput.* **2020**, *11*, 2591–2599.
9. Dhibar, S.; Jain, M. Strategic Behavior for M/M/1 Double Orbit Retrial Queue with Imperfect Service and Vacation. *Int. J. Math. Oper. Res.* **2023**, *15* (3), 245–260.
10. Jain, M.; Dhibar, S.; Sanga, S. S. Markovian Working Vacation Queue with Imperfect Service, Balking, and Retrial. *J. Ambient Intell. Humaniz. Comput.* **2022**, *13* (5), 1–20.

11. Radha, S.; Maragathasundari, S.; Swedheetha, C. Analysis on a non-Markovian batch arrival queuing model with phases of service and multi vacations in cloud computing services. *Int. J. Math. Oper. Res.* **2023**, *24* (3), 425–449.
12. Anjali, C. K.; Kolledath, S. Survey on queuing models with discouragement, policies, and vacation. *Int. J. Math. Oper. Res.* **2024**, *28* (1), 105–145.
13. Sharma, S.; Kumar, R.; Soodan, B. S.; Singh, P. Queuing models with customers' impatience: a survey. *Int. J. Math. Oper. Res.* **2023**, *26* (4), 523–547.
14. Haghighi, A. M.; Mishev, D. P. A time-dependent tandem BMAP with balking and batch service with possible breakdown and delayed service. *Queueing Models and Service Management* 2025, 8(2), 35—63.
15. Neuts, M. F. A versatile Markovian point process. *Journal of Applied Probability* 1979, 14, 764–779.
16. Neuts, M. F. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*; Courier Corporation: Chelmsford, MA, USA, 1994.
17. Choudhary, A.; Chakravarthy, S. R.; Sharma, D. C. Impact of the degradation in service rate in MAP/PH/1 queueing system with phase type vacations, breakdowns, and repairs. *Annals Operation Research* **2023**, *331* (2), 1207–1248.
18. Chakravarthy, S. R., Dudin, A. N., Dudin, S. A., & Dudina, O. S. (2023). *Queueing system with potential for recruiting secondary servers*. *Mathematics, 11*(3), 624.
19. Dudin, A. N., Dudina, O. S., Dudin, S. A., & Melikov, A. (2025). *A dual tandem queue as a model of a pick-up point with batch receipt and issue of parcels*. *Mathematics*, 13(3).
20. Chakravarthy, S. R.; Ozkar, S. MAP/PH/1 queueing model with working vacation and crowdsourcing. *Math. Applicanda* **2016**, *44* (2), 263.
21. Chakravarthy, S. R.; Dudin, A. N. A queueing model for crowdsourcing. *J. Oper. Res. Soc.* **2017**, *68* (3), 221–236.
22. Shajin, D.; Krishnamoorthy, A. Stochastic decomposition in retrial queueing-inventory system. *RAIRO Oper. Res.* **2020**, *54*, 81–99.
23. Shajin, D.; Melikov, A. Queueing inventory system with return of purchased items and customer feedback. *RAIRO Oper. Res.* **2025**, *59*, 1443–1473.
24. Latouche, G.; Ramaswami, V. A Logarithmic Reduction Algorithm for Quasi-Birth-Death Processes. *J. Appl. Probab.* **1993**, *30*, 650–674.
25. Latouche, G.; Ramaswami, V. *Introduction to Matrix Analytic Methods in Stochastic Modeling*; SIAM: Philadelphia, PA, 1999.
26. Chakravarthy, S. R. 2022. *Introduction to Matrix-Analytic Methods in Queues 1: Analytical and Simulation Approach—Basics*; ISTE Ltd.: London, UK; John Wiley & Sons: New York, NY, USA, 2022.