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On the Characterization of the Unitary Cayley Graphs of the Upper Triangular Matrix Rings

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Abstract

There are several graphs naturally associated with rings. The *unitary Cayley graph* of a ring R is the graph with vertex set R, where two elements $x, y \in R$ are adjacent if and only if $x - y$ is a unit of R. We show that the unitary Cayley graph $C_{T_n(\mathbb{F})}$ of the ring $T_n(\mathbb{F})$ of all upper-triangular matrices over a finite field $\mathbb{F}$ is isomorphic to a semistrong product of a complete graph and the antipodal graph of a Hamming graph. In particular, when $|\mathbb{F}| = 2$, the graph $C_{T_n(\mathbb{F})}$ has a highly symmetric structure: it is the union of 2^{n-1} complete bipartite graphs. Moreover, we prove that the clique number and the chromatic number of $C_{T_n(\mathbb{F})}$ are both equal to $|\mathbb{F}|$, and we establish tight upper and lower bounds for the domination number of $C_{T_n(\mathbb{F})}$.

Keywords: unitary Cayley graph; upper triangular matrix ring; Hamming graph; semistrong product; antipodal graph

MSC: 05C25; 05E16; 15B33; 16S50

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0. Introduction

In algebraic graph theory, various graphs associated with algebraic structures play a prominent role. One notable example is the *Cayley graph*, denoted $\text{Cay}(G, S)$, of a group G with respect to a subset $S \subseteq G$. The graph $\text{Cay}(G, S)$ is connected if and only if S generates G. Moreover, $\text{Cay}(G, S)$ is regular, and its properties provide important insights into the structure of finitely generated groups. We refer to the books [13] on algebraic graph theory and [15,24] on geometric group theory for applications of Cayley graphs to groups.

Let R be a finite ring with identity. The *unitary Cayley graph* of R, denoted by C_R, is a simple graph with vertex set R, where two elements $x, y \in R$ are adjacent if and only if $x - y$ is a unit of R. In other words, it is the Cayley graph of the additive group of R with respect to the set of units of R.

The idea of graph representations modulo integers was introduced by Erdös and Evans [10]. The definition of unitary Cayley graphs was first presented by Dejter and Giudici in [7] for rings $\mathbb{Z}_n$. In particular, they proved that if $n = 2^s$ then $C_{\mathbb{Z}_n}$ is isomorphic to a complete bipartite graph $K_{2^{s-1},2^{s-1}}$. Unitary Cayley graphs have been studied intensively for $\mathbb{Z}_n$ (see, for example, [3,7,11,17,21]). Later, the diameter, girth, eigenvalues, vertex and edge connectivity, and vertex and edge chromatic number were described for any finite commutative ring R [1].

Let $\mathbb{F}$ be a finite field, n be a positive integer, $n \geq 2$. In their works [19,20] Kiani, Mollahajiaghaei, and Aghaei described the chromatic number, clique number, independence number, diameter of unitary Cayley graph $C_{M_n(\mathbb{F})}$ of matrix ring $M_n(\mathbb{F})$ over a finite field $\mathbb{F}$ and proved that the graph $C_{M_n(\mathbb{F})}$ is regular and strongly regular for $n = 2$. For some other recent papers on unitary Cayley graphs of the matrix ring $M_n(\mathbb{F})$ and matrix semiring, we refer the reader to [5,9,16,26,29].

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In a related study [27] Rattanakangwanwong and Meemark computed the clique number, the chromatic number, and the independence number of the subgraph of the unitary Cayley graph of a matrix algebra over a finite field induced by the set of idempotent matrices.

In this paper, we characterize the properties of the unitary Cayley graph $C_{T_n(\mathbb{F})}$ of the ring $T_n(\mathbb{F})$ consisting of all upper triangular square matrices of order n over a field $\mathbb{F}$.

Since the ring of all upper triangular $n \times n$ matrices $T_n(\mathbb{F})$ over $\mathbb{F}$ is a subring of the full matrix ring $M_n(\mathbb{F})$, the unitary Cayley graph $C_{T_n(\mathbb{F})}$ is a subgraph of the unitary Cayley graph $C_{M_n(\mathbb{F})}$ of the matrix ring $M_n(\mathbb{F})$ over the finite field $\mathbb{F}$.

We will show that when $|\mathbb{F}| = 2$, the unitary Cayley graph $C_{T_n(\mathbb{F})}$ of the ring of all upper triangular matrices $T_n(\mathbb{F})$ over $\mathbb{F}$ has a quite simple structure.

Theorem 1. *If $|\mathbb{F}| = 2$, then the graph $C_{T_n(\mathbb{F})}$ has 2^{n-1} connected components, and each component is isomorphic to the complete bipartite graph $K_{m,m}$, where $m = 2^{\frac{n(n-1)}{2}}$.*

In the case of $|\mathbb{F}| > 2$, the unitary Cayley graph $C_{T_n(\mathbb{F})}$ is connected. Let S be a set of p elements, p and l be positive integers. The *Hamming graph* $H(l, p)$ is defined on the set of ordered l-tuples of elements of S with two vertices being adjacent if they differ in precisely one coordinate. Hamming graphs find active applications in information theory and computer science. Specifically, we will consider the Hamming graph $H(n, |\mathbb{F}|)$ for $S = \mathbb{F}$.

The antipodal graph of a graph G, denoted by $A(G)$, has the same vertex set as G with two vertices u and v are adjacent if the distance between u and v is equal to the diameter of G (see [2,18]). The antipodal graph $A(H(n, |\mathbb{F}|))$ of the Hamming graph $H(n, |\mathbb{F}|)$ is defined on the set of all n-tuples of elements of the field $\mathbb{F}$, and two n-tuples $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are adjacent provided $u_i \neq v_i$ for every $1 \leq i \leq n$.

The semistrong or strong tensor product $G \bullet H$ of two simple graphs G and H (see, for example, [12], [23]) is the graph with vertex set $V(G \bullet H) = V(G) \times V(H)$ and edge set $E(G \bullet H) = \{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)\}$.

For the case of $|\mathbb{F}| > 2$, we prove the following result that describes the structure of graph $C_{T_n(\mathbb{F})}$.

Theorem 2. *Let $|\mathbb{F}| > 2$. Then*

$$C_{T_n(\mathbb{F})} \simeq K_m \bullet A(H(n, |\mathbb{F}|)), \tag{1}$$

where $m = |\mathbb{F}|^{\frac{n(n-1)}{2}}$.

We also prove that the clique number of the graph $C_{T_n(\mathbb{F})}$ and the chromatic number of this graph equal the number of elements of the field $\mathbb{F}$. We establish tight upper and lower bounds for the domination number $\gamma(C_{T_n(\mathbb{F})})$ of $C_{T_n(\mathbb{F})}$.

Theorem 3. *For any finite field $\mathbb{F}$ and a positive integer n, $n \geq 2$*

$$n + 1 \leq \gamma(C_{T_n(\mathbb{F})}) \leq 2^n.$$

These bounds are tight.

In addition, we characterize the diameter and the triameter and describe the bounds of the independence number of $C_{T_n(\mathbb{F})}$.

1. Preliminaries

In this paper, we consider only finite simple graphs. Let $G = (V(G), E(G))$ be a finite connected simple graph. Define a metric d_G on the set of vertices $V(G)$ as follows: for any $u, v \in V(G)$ the distance $d_G(u, v)$ is defined as the length of the shortest path between u and v.

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The *diameter* of a connected graph G is the value

$$diam(G) = \max\{d_G(u,v) : u,v \in V(G)\}.$$

For every triplet of vertices $u,v,w \in V(G)$, we define

$$d_G(u,v,w) = d_G(u,v) + d_G(u,w) + d_G(v,w).$$

The *triameter* of a connected graph G is defined as the value

$$triam(G) = \max\{d_G(u,v,w) : u,v,w, \in V(G)\}.$$

The triplet of vertices $u,v,w \in V(G)$ is called *triametral* if $d_G(u,v,w) = triam(G)$. The main motivation for studying $triam(G)$ comes from its appearance in lower bounds on the radio k-chromatic number of a graph and the total domination number of a connected graph ([6,14,22,28]).

A *clique* is a subgraph of a graph G that is isomorphic to a complete graph. The *clique number* of G is the size of the largest clique in G, denoted by $\omega(G)$.

The chromatic number, $\chi(G)$, of a graph G is the smallest number of colours for $V(G)$ so that adjacent vertices are coloured by different colours.

An independent set of a graph G is a subset of the vertices such that no two vertices in the subset are connected by an edge in G. A maximum independent set is an independent set of the largest possible size. The number of vertices of the maximum independent set is called *the independence number* of the graph G and denoted by $\alpha(G)$.

A dominating set of a graph G is a subset of the vertices S such that any vertex v of G is in S or adjacent to a vertex in S. A minimum dominating set is a dominating set of the smallest possible size. The number of vertices of the minimum dominating set is called *the domination number* of the graph G and denoted by $\gamma(G)$.

We refer the reader to [8] for general background and for all undefined notions on graph theory used in the paper.

2. The Properties of the Unitary Cayley Graph of $T_n(\mathbb{F})$

Denote by $(T_n(\mathbb{F}))^*$ the set of all invertible upper triangular $n \times n$ matrices. The vertices of the graph $C_{T_n(\mathbb{F})}$ are upper triangular $n \times n$ matrices of $T_n(\mathbb{F})$ over $\mathbb{F}$ and two vertices a and b are adjacent if and only if $a - b$ is an element of $(T_n(\mathbb{F}))^*$.

For any unital finite ring R the unitary Cayley graph C_R is $|R^*|$-regular [1]. As $|(T_n(\mathbb{F}))^*| = (|\mathbb{F}| - 1)^n \cdot |\mathbb{F}|^{\frac{n^2-n}{2}}$, the next proposition becomes evident.

Proposition 1. *The graph $C_{T_n(\mathbb{F})}$ is s-regular, where*

$$s = (|\mathbb{F}| - 1)^n |\mathbb{F}|^{\frac{n^2-n}{2}}.$$

From the structure of upper triangular matrices, we deduce a simple condition for the existence of an edge in the graph $C_{T_n(\mathbb{F})}$.

Proposition 2. *Two matrices $a = \{a_{ij}\}_{1\le i,j\le n}$ and $b = \{b_{ij}\}_{1\le i,j\le n}$ are adjacent in $C_{T_n(\mathbb{F})}$ if and only if $a_{ii} - b_{ii} \neq 0$ for any $1 \le i \le n$.*

Proof. Indeed, $a - b$ is invertible if and only if $det(a - b) \neq 0$. As a and b are upper triangular,

$$det(a - b) = \prod_{i=1}^{n} (a_{ii} - b_{ii}).$$

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This completes the proof of the proposition. □

Theorem 4. *If $|\mathbb{F}| \neq 2$, then the graph $C_{T_n(\mathbb{F})}$ is connected.*

First, we prove Theorem 1.

Proof of Theorem 1. Let $|\mathbb{F}| = 2$ and let $a = \{a_{ij}\}_{1 \leq i,j \leq n}$ be a matrix of $T_n(\mathbb{F})$. For the vector $(a_{11}, a_{22}, \ldots, a_{nn})$ there exists only one vector $(b_{11}, b_{22}, \ldots, b_{nn})$ such that

$$a_{ii} - b_{ii} = 1 \mod 2, \qquad \text{for any } 1 \leq i \leq n.$$

Proposition 2 implies, that the matrix a is adjacent with any matrix $b = \{b_{ij}\}_{1 \leq i,j \leq n}$ from $T_n(\mathbb{F})$ with the elements $(b_{11}, b_{22}, \ldots, b_{nn})$ on the main diagonal. This is true for all matrices with the elements $(a_{11}, a_{22}, \ldots, a_{nn})$ on the main diagonal. However, any matrix with the elements $(b_{11}, b_{22}, \ldots, b_{nn})$ on the main diagonal is connected by an edge with any matrix with the elements $(a_{11}, a_{22}, \ldots, a_{nn})$ on the main diagonal and is not adjacent with any other matrix from $T_n(\mathbb{F})$.

Hence, all matrices with the elements $(a_{11}, a_{22}, \ldots, a_{nn})$ or $(b_{11}, b_{22}, \ldots, b_{nn})$ on the main diagonal form the connected component of the graph $C_{T_n(\mathbb{F})}$. Since we choose matrix a arbitrarily, any connected component of graph $C_{T_n(\mathbb{F})}$ is defined by two vectors $(c_{11}, c_{22}, \ldots, c_{nn})$ and $(d_{11}, d_{22}, \ldots, d_{nn})$, such that

$$c_{ii} - d_{ii} = 1 \mod 2, \text{ for any } 1 \leq i \leq n.$$

Therefore, the graph $C_{T_n(\mathbb{F})}$ has 2^{n-1} connected components and any component is isomorphic to $K_{m,m}$ where m equals the number of upper triangular $n \times n$ matrices over field $\mathbb{F}_2$ with elements $(c_{11}, c_{22}, \ldots, c_{nn})$ on the main diagonal, i.e.

$$m = 2^{\frac{n(n-1)}{2}}.$$

□

As the ring of all upper triangular matrices $T_n(\mathbb{F})$ is a subring of the matrix ring $M_n(\mathbb{F})$, we have the next result.

Corollary 1. *Let $\mathbb{F}$ be a finite field, $|\mathbb{F}| = 2$. Then the unitary Cayley graph $C_{M_n(\mathbb{F})}$ contains at least 2^n different subgraphs that are isomorphic to the complete bipartite graph $K_{m,m}$, where*

$$m = 2^{\frac{n(n-1)}{2}}.$$

Proof. The ring $M_n(\mathbb{F})$ contains two subrings of upper and lower triangular matrices. We can define the unitary Cayley graph $C_{LT_n(\mathbb{F})}$ of the ring of all lower triangular $n \times n$ matrices over the field $\mathbb{F}$. It is clear that the graphs $C_{T_n(\mathbb{F})}$ and $C_{LT_n(\mathbb{F})}$ have the same properties and both are subgraphs of $C_{M_n(\mathbb{F})}$. From Theorem 1, we get the corollary assertion. □

Proof of Theorem 4. Let now $|\mathbb{F}| > 2$. Assume, that matrices $a = \{a_{ij}\}_{1 \leq i,j \leq n}$ and $b = \{b_{ij}\}_{1 \leq i,j \leq n}$ are not adjacent. Then we can construct a matrix $c = \{c_{ij}\}_{1 \leq i,j \leq n}$ in the following way:

- for all i, $1 \leq i \leq n$, we choose c_{ii} such that $c_{ii} \neq a_{ii}$, $c_{ii} \neq b_{ii}$ (we can choose c_{ii}, because $|F| > 2$);
- for all $1 \geq i, j \geq n$, $i \neq j$ we set $c_{ij} = 0$.

Proposition 2 implies that the matrix c is connected by an edge with the matrices a and b. So, for any matrices a and b that are not adjacent, there exists a matrix c which is connected by an edge with both a and b simultaneously. Therefore, the graph $C_{T_n(\mathbb{F})}$ is connected. □

This theorem directly implies the following.

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Corollary 2. *If* $|\mathbb{F}| > 2$, *then*

$$diam(C_{T_n(\mathbb{F})}) = 2.$$

Proof. The proof of this corollary directly follows from the proof of Theorem 4. □

The next proposition describes the triameter of the graph $C_{T_n(\mathbb{F})}$.

Proposition 3. *If* $|\mathbb{F}| > 2$, *then*

$$triam(C_{T_n(\mathbb{F})}) = 6.$$

Proof. It is clear that $triam(C_{T_n(\mathbb{F})}) \leq 6$, because $triam(G) = \max\{d_G(u,v,w) : u,v,w,\in V(G)\}$ and $d_G(u,v,w) = d_G(u,v) + d_G(u,w) + d_G(v,w) \leq 3 \cdot diam(G)$. Since $|F| > 2$, there exist three pairwise different elements $a,b,c \in F$. Consider three diagonal matrices $d_1 = diag(a,a,a,\ldots,a)$, $d_2 = diag(a,b,b,\ldots,b)$ and $d_3 = diag(a,c,c,\ldots,c)$. Then we have $d_{C_{T_n(\mathbb{F})}}(d_1,d_2,d_3) = 2+2+2 = 6$. This completes the proof of the proposition. □

The next proposition characterizes the clique number $\omega(C_{T_n(\mathbb{F})})$ of the graph $C_{T_n(\mathbb{F})}$.

Proposition 4. *Let* $\mathbb{F}$ *be a finite field. Then*

$$\omega(C_{T_n(\mathbb{F})}) = |\mathbb{F}|.$$

Proof. Let S be a clique of the maximal size of G_{T_n}. The Proposition 2 implies that

$$a_{ii} \neq b_{ii}$$

for any two different elements $a = \{a_{ij}\}_{1\leq i,j\leq n}$ and $b = \{b_{ij}\}_{1\leq i,j\leq n}$ from S and for any i, $1 \leq i \leq n$. But for any i we can choose only $|\mathbb{F}|$ different elements a_{ii}. So, $|S| = |\mathbb{F}|$. □

Lemma 1. *[20] Let F be a finite field, n be a positive integer.Then*

$$\alpha(C_{M_n(\mathbb{F})}) = |F|^{n^2-n}.$$

Proposition 5. *For any finite field* $\mathbb{F}$ *and a positive integer* n, $n \geq 2$

$$|\mathbb{F}|^{\frac{n^2+n-2}{2}} \leq \alpha(C_{T_n(\mathbb{F})}) \leq |F|^{n^2-n}. \tag{2}$$

The lower bound is tight.

Proof. The upper bounds of $\alpha(C_{T_n(\mathbb{F})})$ in the formula (2) follows from Lemma 1, because the graph $C_{T_n(\mathbb{F})}$ is an induced subgraph of $C_{M_n(\mathbb{F})}$.

To proof lower bound we define a subset S of the set of vertices of the graph $C_{T_n(\mathbb{F})}$ as a set of all matrices $a = \{a_{ij}\}_{1\leq i,j\leq n}$ with $a_{11} = 0$.

First, note that by Proposition 2 there is no an edge between any two different elements $a = \{a_{ij}\}_{1\leq i,j\leq n}$ and $b = \{b_{ij}\}_{1\leq i,j\leq n}$ from S, because $a_{11} = b_{11} = 0$. Hence, S is an independent set. So,

$$\alpha(C_{T_n(\mathbb{F})}) \geq |S| = |\mathbb{F}|^{\frac{n^2+n-2}{2}},$$

and our proposition is proved.

Let us now $|\mathbb{F}| = 2$. Theorem 1 implies that the graph $C_{T_n(\mathbb{F}_2)}$ is a union of 2^{n-1} subgraphs, each of them is a complete bipartite graph. So, any independent set of $C_{T_n(\mathbb{F}_2)}$ is a union of independent

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sets of $K_{m,m}$, where $m = 2^{\frac{n(n-1)}{2}}$. Hence, the independence number of $C_{T_n(\mathbb{F}_2)}$ is equal to the sum of the independence number of its connected components:

$$\alpha(C_{T_n(\mathbb{F}_2)}) = 2^{n-1} \cdot 2^{\frac{n(n-1)}{2}} = 2^{\frac{n^2+n-2}{2}}.$$

□

3. The Domination Number of $C_{T_n(\mathbb{F})}$

Lemma 2. *For any finite field $\mathbb{F}$ and positive integer n, $n \geq 2$*

$$\gamma(C_{T_n(\mathbb{F})}) > n.$$

Proof. Let S be of subset of $T_n(\mathbb{F})$, $|S| = n$. We have to show, that S is not a dominating set.

Assume that $S = \{a^1, \ldots, a^n\}$, $a^k = \{a^k_{ij}\}_{1\leq i,j\leq n}$, $1 \leq k \leq n$. Let $\mathbb{S}_n$ be a set of all permutations on the set $\{1, 2, \ldots, n\}$. Consider the set U of all matrices

$$\begin{pmatrix} a_{11}^{\pi(1)} & 0 & \ldots & 0 \\ 0 & a_{22}^{\pi(2)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & a_{nn}^{\pi(n)} \end{pmatrix},$$

where $\pi \in \mathbb{S}_n$. Note that $|U| = n! > |S|$. Hence, there exists a matrix $b \in U$, $b \notin S$. Moreover, b is not connected by an edge with any matrix from S, which proves the lemma. □

Theorem 5. *Let $|\mathbb{F}| > n$. Then*

$$\gamma(C_{T_n(\mathbb{F})}) = n + 1.$$

Proof. Lemma 2 implies that it is sufficient to show the existence of a dominating set with $n + 1$ elements. Let $u_0, \ldots, u_n$ be elements of the field $\mathbb{F}$. Let S be a set of matrices of the form:

$$\begin{pmatrix} u_0 & 0 & \ldots & 0 \\ 0 & u_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & u_0 \end{pmatrix}, \begin{pmatrix} u_1 & 0 & \ldots & 0 \\ 0 & u_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & u_1 \end{pmatrix}, \ldots, \begin{pmatrix} u_n & 0 & \ldots & 0 \\ 0 & u_n & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & u_n \end{pmatrix}.$$

Any matrix $a \in T_n(\mathbb{F})$, $a = \{a_{ij}\}_{1\leq i,j\leq n}$, has only n elements on its main diagonal. So, there exists an element $u_j \in \mathbb{F}$, $0 \leq j \leq n$, such that $a_{ii} \neq u_j$ for all $1 \leq i \leq n$. Then the matrix a is adjacent to the matrix

$$\begin{pmatrix} u_j & 0 & \ldots & 0 \\ 0 & u_j & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & u_j \end{pmatrix},$$

that is an element of S. Therefore, S is a dominating set. □

Proposition 6. *Let $\mathbb{F}$ be a finite field. Then*

$$\gamma(C_{T_n(\mathbb{F})}) \leq 2^n. \quad (3)$$

Proof. We have to show, that there exists a domination set with 2^n elements.

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Let S be the set of all matrices $a = \{a_{ij}\}_{1\le i,j\le n}$ with $a_{ii} \in \{0,1\}$, and $a_{ij} = 0, i \neq j, 1 \le i,j \le n$. We would like to show that S is a dominating set of $C_{T_n(\mathbb{F})}$. Indeed, let $b = \{b_{ij}\}_{1\le i,j\le n}$ be a matrix and b not be an element of S. We have to show that there is an element $c \in S$ which is adjacent to b.

Assume that I is a set of indices $I \subseteq \{1,\dots n\}$, such that $\{b_{jj}\}$ are 0 or 1, $j \in I$. Define a matrix $c = \{c_{ij}\}_{1\le i,j\le n}$ as follows:

- $c_{jj} = b_{jj} - 1 \mod 2$, for all $j, j \in I$;
- $c_{jj} = 1$ for all $j, j \notin I$
- $c_{ij} = 0$, for all $1 \le i,j \le n, i \neq j$.

By definition $c \in S$. Proposition 2 implies that the matrix c is adjacent to the matrix b. Hence, S is a dominating set.

Note that the set S contains 2^n elements. Therefore, inequality (3) holds. □

Proposition 7. *Let $\mathbb{F} = \mathbb{F}_2$ be a field with two elements. Then*

$$\gamma(C_{T_n(\mathbb{F}_2)}) = 2^n.$$

Proof. Theorem 1 implies that the graph $C_{T_n(\mathbb{F}_2)}$ is a union 2^{n-1} subgraph, each of them being a complete bipartite graph. So, any dominating set of $C_{T_n(\mathbb{F}_2)}$ contains at least two elements from each connected component. Hence,

$$\gamma(C_{T_n(\mathbb{F}_2)}) \ge 2 \cdot 2^{n-1} = 2^n.$$

Then the inequality (3) proves our proposition. □

Proof of Theorem 3. The proof of this theorem directly follows from Lemma 2, Theorem 5 and Propositions 6 and 7. □

4. Connection with Hamming Graphs

Proof of Theorem 2. We have to show that the unitary Cayley graph $C_{T_n(\mathbb{F})}$ of the ring of all upper triangular matrices $T_n(\mathbb{F})$ over $\mathbb{F}$ is isomorphic to the semistrong product of the complete graph K_m and the antipodal graph of the Hamming graph $A(H(n,|\mathbb{F}|))$, where $m = |\mathbb{F}|^{\frac{n(n-1)}{2}}$.

First, we define a complete graph on the set of all strictly upper triangular matrices. This graph is isomorphic to the graph K_m, where $m = |\mathbb{F}|^{\frac{n(n-1)}{2}}$.

Now, determine a bijection φ from the set of vertices of the graph $C_{T_n(\mathbb{F})}$ to the Cartesian product of the set of vertices of K_m and the set of vertices of the graph $A(H(n,|\mathbb{F}|))$. Let $a = \{a_{ij}\}_{1\le i,j\le n}$ be a matrix from $T_n(\mathbb{F})$. Define

$$\varphi(a) = (\hat{a}, (a_{11}, \dots, a_{nn})),$$

where

$$\hat{a} = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{(n-1),n} \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

We would like to show that two matrices $a = \{a_{ij}\}_{1\le i,j\le n}$ and $b = \{b_{ij}\}_{1\le i,j\le n}$ are adjacent in the graph $C_{T_n(\mathbb{F})}$ if and only if they are connected by edge in the semistrong product of graphs K_m and $A(H(n,|\mathbb{F}|))$. Assume that there exist the edge ab in the graph $C_{T_n(\mathbb{F})}$. Proposition 2 implies

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that $a_{ii} \neq b_{ii}$ for any i, $1 \leq i \leq n$. Then the vectors $(a_{11}, \dots, a_{nn})$ and $(b_{11}, \dots, b_{nn})$ are adjacent in $A(H(n, |\mathbb{F}|))$. If the matrices

$$\hat{a} = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{(n-1),n} \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad \hat{b} = \begin{pmatrix} 0 & b_{12} & \dots & b_{1n} \\ 0 & 0 & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{(n-1),n} \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

are different, then they are connected by an edge in K_m. Therefore, from the definition of semistrong product of graphs follows, that there exists the edge ab in the graph $K_m \bullet A(H(n, |\mathbb{F}|))$.

Assume there is no edge ab in $C_{T_n(\mathbb{F})}$. Proposition 2 implies that there exist i, $1 \leq i \leq n$, such that $a_{ii} = b_{ii}$. Consequently, the vectors $(a_{11}, \dots, a_{nn})$ and $(b_{11}, \dots, b_{nn})$ are not adjacent in $A(H(n, |\mathbb{F}|))$. Thus, there is no edge ab in $K_m \bullet A(H(n, |\mathbb{F}|))$.

□

We can describe the connection between $C_{T_n(\mathbb{F})}$ and $A(H(n, |\mathbb{F}|))$ in another way. Define on $T_n(\mathbb{F})$ equivalence relation $\equiv$:

a matrix $a = \{a_{ij}\}_{1 \leq i,j \leq n}$ is equivalent to a matrix $b = \{b_{ij}\}_{1 \leq i,j \leq n}$ if and only if $a_{ii} = b_{ii}$ for all i, $1 \leq i \leq n$.

That is, each equivalence class is determined by the main diagonal of matrices.

Let $\hat{C}_{T_n}$ be a graph induced by a graph $C_{T_n(\mathbb{F})}$ on the set $T_n(\mathbb{F})|_{\equiv}$. Then we have the next description of $\hat{C}_{T_n}$.

Proposition 8. *$\hat{C}_{T_n} \simeq A(H(n, |\mathbb{F}|))$, for all $n \geq 2$.*

The proof of this proposition follows directly from Theorem 2.

5. The Chromatic Number of the Graph $C_{T_n(\mathbb{F})}$

We will use the next lemma to discuss the chromatic number of the graph $C_{T_n(\mathbb{F})}$.

Lemma 3 ([12])**.** *Let G_1, G_2 be finite simple graphs. Then*

$$\chi(G_1 \bullet G_2) = \chi(G_2).$$

Theorem 6. *Let $\mathbb{F}$ be a finite field. Then*

$$\chi(C_{T_n(\mathbb{F})}) = |\mathbb{F}|.$$

Proof.

Case 1. Let $|\mathbb{F}| = 2$. Then the graph $C_{T_n(\mathbb{F})}$ is an union of a bipartite graphs, so $\chi(C_{T_n(\mathbb{F})}) = 2$.

Case 2. Let now $|\mathbb{F}| = |\mathbb{F}| > 2$. By Proposition 4 we have $\chi(C_{T_n(\mathbb{F})}) \geq |\mathbb{F}|$. So, we have to show that $\chi(C_{T_n(\mathbb{F})}) \leq |\mathbb{F}|$. Theorem 4 and Lemma 3 imply that for this purpose, it is enough to show that the chromatic number of the graph $A(H(n, |\mathbb{F}|))$ is less or equal to $|\mathbb{F}|$.

Define subsets $S_0, \dots, S_{|\mathbb{F}|-1}$ of the set of vertices of the graph $A(H(n, |\mathbb{F}|))$ in the next way:

$$\begin{aligned} S_0 &= \{(0, x_1, \dots, x_n) \mid x_i \in \mathbb{F},\ 2 \leq i \leq n\} \\ S_1 &= \{(1, x_1, \dots, x_n) \mid x_i \in \mathbb{F},\ 2 \leq i \leq n\} \\ &\dots\dots \\ S_{|\mathbb{F}|-1} &= \{(|\mathbb{F}| - 1, x_1, \dots, x_n) \mid x_i \in \mathbb{F},\ 2 \leq i \leq n\}. \end{aligned}$$

Colour the sets $S_0, \ldots, S_{|\mathbb{F}|-1}$ by different colours $\alpha_1, \ldots, \alpha_{|\mathbb{F}|}$. Then, any two elements u and v of one colour are not adjacent. Indeed, if $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vertices coloured by the same colour, then there exists i, $0 \leq i \leq |\mathbb{F}| - 1$ such that u and v are elements of S_i. But it is mean that $u_1 = v_1$. So, u and v are not adjacent in $A(H(n, |\mathbb{F}|))$. Hence, $\chi(A(H(n, |\mathbb{F}|))) \leq |\mathbb{F}|$. This completes the proof of the theorem. □

6. Discussion

Note that Proposition 4 and Theorem 2 imply that the complete graph K_m, where $m = |\mathbb{F}|^{\frac{n(n-1)}{2}}$, is not a subgraph of $C_{T_n(\mathbb{F})}$. Thus, we obtain an example in which the semistrong product of two graphs does not contain a subgraph isomorphic to the first factor. This observation naturally leads to the following problem:

In which cases does the semistrong product of graphs contain an isomorphic copy of the first graph?

In Proposition 5, we established upper and lower bounds for the independence number of the graph $C_{T_n(\mathbb{F})}$. The lower bound is tight. What can be said about the upper bound?

Determine the tight upper bound for the independence number of the graph $C_{T_n(\mathbb{F})}$.

Since the graph $C_{T_n(\mathbb{F})}$ can be represented as a product of well-known graph constructions, it is plausible that its automorphism group can also be described as a product of well-known groups. This motivates the following question:

Describe the automorphism group of the graph $C_{T_n(\mathbb{F})}$.

7. Conclusions

We show that the graph $C_{T_n(\mathbb{F})}$ is isomorphic to the semistrong product of the complete graph K_m and the antipodal graph of the Hamming graph $A(H(n, |\mathbb{F}|))$ for some m. In particular, when the field $\mathbb{F}$ contains only two elements, the graph $C_{T_n(\mathbb{F})}$ has 2^{n-1} connected components, each of which is isomorphic to a complete bipartite graph. We further prove that the clique number and the chromatic number of $C_{T_n(\mathbb{F})}$ are both equal to the number of elements in the field $\mathbb{F}$, and we establish tight upper and lower bounds for the domination number of $C_{T_n(\mathbb{F})}$. Additionally, we show that the diameter of $C_{T_n(\mathbb{F})}$ equals 2, and that its triameter equals 6 in the case $|\mathbb{F}| > 2$. Several open problems are also formulated in this paper.

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