Hypothesis

Structural Reparameterization of the Complex Variable s and the Fixation of the Critical Line

Shane Drake *

Posted Date: 14 November 2025

doi: 10.20944/preprints202511.0912.v1

Keywords: critical line; geometric series; Maclaurin series; complex variable; Dirichlet series; zeta function; completed L-function; symmetry

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 November 2025 doi:10.20944/preprints202511.0912.v1

Article

Structural Reparameterization of the Complex Variable *s* and the Fixation of the Critical Line

Shane Drake

Independent Researcher; shanedrake23@gmail.com

Abstract

This paper explains why the critical line sits at real part equal to one-half by treating it as an intrinsic boundary of a reparametrized complex plane ("z-space"), not a mere artifact of functional symmetry. In z-space the real part is defined by a geometric-series map that induces a rulebook for admissible analytic operations. Within this setting we rederive the classical toolkit—eta–zeta relation, Gamma reflection and duplication, theta–Mellin identity, functional equation, and the completed zeta—without importing analytic continuation from the usual *s*-variable. We show that access to the left half-plane occurs entirely through formulas written on the right, with boundary matching only along the line with real part one-half. A global Hadamard product confirms the consistency and fixed location of this boundary, and a holomorphic change of variables transports these conclusions into the classical setting.

Keywords: critical line; geometric series; Maclaurin series; complex variable; Dirichlet series; zeta function; completed L-function; symmetry

---

1. Introduction

Reflection symmetries in the complex plane arise frequently in analytic number theory. Among these, one of the most persistent is the appearance of a vertical axis of symmetry at $\Re s = 1/2$. This line emerges across a range of contexts, most famously in the study of zeta and L-functions, where analytic continuation and functional equations balance values at s and $1 - s$ [1]. For this reason, the line $\Re s = 1/2$ is commonly referred to as the critical line [2].

The presence of such symmetry is usually attributed to an associated functional equation. For example, the completed Riemann zeta function $\xi(s)$ satisfies

$$\xi(s) = \xi(1 - s), \tag{1.1}$$

from which reflection symmetry about $Rs = 1/2$ follows immediately [3].

While correct, this explanation is descriptive: it identifies the axis at $1/2$ but does not reveal why the symmetry occurs precisely on this boundary rather than along some other vertical line.

The goal of this paper is to supply a structural explanation for the location of the line $\mathrm{Rs} = 1/2$. We introduce a reparametrization of the complex variable s based on a Maclaurin–type expansion [4] that exposes geometric and analytic constraints that naturally select $1/2$ as the unique axis of symmetry. This perspective separates the mechanism that fixes the critical line from the familiar fact of symmetry encoded by a functional equation.

For clarity, the material is arranged as follows.

- **Section 1** states the motivation, the main structural statement (Theorem 1.1), and the plan of the argument.
- **Section 2** reviews the standard analytic framework for the Riemann zeta function and its completion: Dirichlet and alternating series, the Gamma factor, the functional equation, and Hadamard product. This provides the standard context against which the later reparameterization is to be compared.
- **Sections 3–4** introduce the geometric reparameterization

$$z = \varsigma(r) + it, \qquad \varsigma(r) = \frac{1}{1-r}, r \in [-1,1),$$

which forces $\Re z \geq 1/2$, and formulate the rulebook that governs admissible analytic operations in this half-plane. These sections explain why z-space is used and how access to the reflected side is to be interpreted.

- **Sections 5–8** reconstruct, entirely within the constraints of z-space, the analytic tools needed for the main Theorem: Dirichlet/eta relations, Gamma identities, theta–Mellin representation of the functional equation, completed zeta function, and the Hadamard factorization. Although these are classical constructions in the usual s-variable, they are rederived here under the geometric constraint $\Re z \geq 1/2$ imposed from the outset to confirm that each step remains valid in this setting. The components invoked in Theorem 1.1 are taken from these sections.
- **Section 9** collects the consequences of the preceding sections to complete the proof of Theorem 1.1 and, moreover, shows that the z-variable is not specific to ζ but arises as a generic geometric reparameterization. The concluding remarks indicate that the same z-space architecture applies to any function satisfying the same access and symmetry conditions, and, when available, order-one Hadamard factorization, so that the critical line appears as an intrinsic boundary of the construction rather than as a special feature of the Riemann zeta function.

In this way, Theorem 1.1 isolates the statements that are structurally responsible for fixing the critical line, while the surrounding sections supply z-space versions of the standard analytic apparatus needed to justify them.

Although one could attempt to restrict classical s-space tools to the region $\Re z \geq 1/2$, that approach would import assumptions validated only in the unrestricted s-variable. By contrast, the arguments here begin directly in z-space and adhere to its rulebook from the outset, so that every identity and limit is justified within the working half-plane, without prior appeal to the full s-domain.

Theorem 1.1 (Structural fixation of the critical line)

Let $z = \varsigma(r) + it$ *with*

$$\varsigma(r) = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, r \in [-1,1).$$

Then the vertical line $\Re z = 1/2$ *is the unique boundary that controls analytic access between the right and left half-planes in* z*-space. Under the holomorphic change of variables*

$$z \mapsto s := s(r) = \frac{1}{1-r} + it,$$

this boundary appears in s*-space as* $\Re s = 1/2$.

Why this boundary is unique:

Access boundary (z-space only).

- ***Corollaries 7.1 and 7.3.*** *The left half-plane is accessed only by formulas written in terms of the right-half plane data: Corollary 7.1 writes* $\zeta(1-z)$ *in terms of* $\zeta(z)$, Γ, *and trigonometric factors; Corollary 7.3 does the same by definition for the completed zeta* $\xi(1-z)$ *in terms of* $\xi(z)$. *Both are consistent with rules (R3)–(R6) of Lemma 4.1.*
- ***Section 3.*** *The geometric map forces* $\varsigma(r) \geq 1/2$ *(with* $\varsigma(-1) = 1/2$ *by Abel/Cesàro), so* $\Re z \geq 1/2$ *is maximal and* $\Re z = 1/2$ *is the sharp cutoff.*
- ***Corollary 7.4.*** *Non-tangential limits from both sides agree only on* $\Re z = 1/2$, *giving boundary matching* $\xi(z) = \xi(1-z)$ *exactly there.*

Structural fixation (Hadamard).

- ***Corollary 7.5.*** ξ *is entire of order one, so it admits a global Hadamard product in* z*-space.*
- ***Lemma 8.1.*** *The right-half and left-half Hadamard forms must represent the same function on their overlap. In* z*-space, this is possible only if every admissible zero satisfies* $\Re \varrho = 1/2$.
- ***Lemma 8.2.*** *The holomorphic map* $z \mapsto s$ *preserves the product and zero set, so the same boundary appears in* s*-space as* $\Re s = 1/2$.

Proof.

By §3 the geometric parameterization forces $\Re z \geq 1/2$ and identifies $\Re z = 1/2$ as the effective boundary. Corollaries 7.1 and 7.3 show that values on the reflected side are obtained solely from right-half data, and Corollary 7.4 gives boundary matching of non-tangential limits on $\Re z = 1/2$. Corollary 7.5 establishes that ξ is entire of order one. The two z-space Hadamard representations (right and reflected) must therefore coincide on their overlap; by Lemma 8.1 this forces all admissible zeros onto the common boundary $\Re \varrho = 1/2$, so no other vertical line can serve as an access boundary consistent with (R3)–(R6). Finally, Lemma 8.2 transports the boundary to s-space, yielding $\Re s = 1/2$. ∎

Remark 1.1 (Section-level references).

Sections 2–4 cite sources individually where used.

Sections 5–8 rely on standard results treated comprehensively in the references listed at each heading. These support the section as a whole rather than individual items.

2. The Analytic Structure of the Dirichlet Functions

We recall the standard analytic framework for the Riemann zeta function and its completion.

For $\Re s > 1$, the Riemann zeta function [1] is defined by the absolutely convergent Dirichlet series

$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} n^{-s}. \tag{2.1}$$

The Dirichlet eta function is given by the alternating series

$$\eta(s) \coloneqq \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}, \qquad \Re(s) > 0, \tag{2.2}$$

which converges by Dirichlet's test and hence defines a holomorphic function in the half-plane $\Re s > 0$ [5]. In the region $\Re s > 1$, the two series are related by

$$\eta(s) = (1 - 2^{1-s})\,\zeta(s), \tag{2.3}$$

where

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}} \tag{2.4}$$

provides a meromorphic continuation of $\zeta(s)$ to $\Re s > 0$, with a single singularity at $s = 1$.

The Gamma function is defined for $\Re s > 0$ by Euler's integral

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, dt, \tag{2.5}$$

and satisfies the standard analytic continuation and functional identities [6], which are employed in deriving the classical functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\,\Gamma(1-s)\zeta(1-s), \tag{2.6}$$

This reflection formula extends $\zeta(s)$ meromorphically to the entire complex plane $\mathbb{C}$, with a single simple pole at $s = 1$ [1,3].

Symmetry is most naturally expressed through the completed zeta function

$$\xi(s) \coloneqq \frac{1}{2} s(s-1)\pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s). \tag{2.7}$$

The prefactors cancel the pole at $s = 1$ and the trivial zeros at $s = 0, -2, -4, \ldots$, so ξ is entire and satisfies the symmetric functional equation [1,7]

$$\xi(s) = \xi(1-s). \tag{2.8}$$

Because $\xi(s)$ is entire of order, its global growth can be described completely by a Hadamard factorization [1,8],

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \exp\left(\frac{s}{\rho}\right), \tag{2.9}$$

where the product extends over all nontrivial zeros ρ of $\zeta(s)$, and $A, B \in \mathbb{C}$ are constants.

This product converges absolutely and uniformly on compact subsets of $\mathbb{C}$, and encodes both the distribution of zeros and the global analytic structure of ξ.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 November 2025 doi:10.20944/preprints202511.0912.v1

The normalization $\xi(0) = 1/2$ determines $e^A = 1/2$, and the symmetry (2.8) implies $B = 0$, yielding the symmetric form

$$\xi(s) = \frac{1}{2}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)\exp\left(\frac{s}{\rho}\right), \tag{2.10}$$

which manifests the reflectional symmetry of ξ about the critical line $\Re s = 1/2$.

The structural origin of this axis will be developed in the following sections. For additional historical and expository context, see Edwards (1974) [2] or Hardy (1949) [9].

3. Reparametrizing the Real Part of Complex Numbers via Geometric Series

Consider the geometric series

$$\varsigma(r) \coloneqq \sum_{k=0}^{\infty} r^k, \qquad |r| < 1, \tag{3.1}$$

which converges to the closed form

$$\varsigma(r) = \frac{1}{1-r}. \tag{3.2}$$

For real $r \in (-1,1)$, the range of $\varsigma(r)$ covers the interval $(1/2, \infty)$, since

$$\lim_{r\to -1^+} \varsigma(r) = \frac{1}{1-(-1)} = \frac{1}{2}, \lim_{r\to 1^-} \varsigma(r) = +\infty. \tag{3.3}$$

Hence there is a bijection

$$r \in (-1,1) \ \leftrightarrow\ \varsigma \in (1/2, \infty). \tag{3.4}$$

By Abel and Cesàro summation (see Remark 3.1), the endpoint $r = -1$ is interpreted consistently as $\varsigma(-1) = 1/2$, thereby extending the domain of the geometric series to

$$r \in [-1,1) \ \leftrightarrow\ \varsigma \in [1/2, \infty). \tag{3.5}$$

We embed this into the complex plane by introducing a vertical parameter $t \in \mathbb{R}$ and defining

$$z \coloneqq z(r) = \varsigma(r) + it. \tag{3.6}$$

Thus z lies on the closed half-plane $\Re z = \varsigma(r) \geq 1/2$.

The power-series identity

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k, \quad r \in [-1,1), \tag{3.7}$$

suggests a natural analytic continuation of $\varsigma(r)$ to all $r \in \mathbb{R} \setminus \{1\}$ via

$$\sigma(r) \coloneqq \frac{1}{1-r}. \tag{3.8}$$

We then define the corresponding complex parameter

$$s(r) := \sigma(r) + it, \tag{3.9}$$

interpreting $s(r)$ as the Dirichlet variable used in §2 and as the analytically continued image of the geometric-series parameter r.

In this construction, the s-plane represents the analytic continuation of the z-plane beyond the disk of convergence for the original geometric series [4]. Henceforth, all structural definitions will be formulated in z-space, without further analytic continuation into s-space. This provides a purely geometric explanation for the fixed location and emergence of the critical line $R(s) = 1/2$.

Remark 3.1 (Endpoint summability at $\mathbf{r = -1}$ [9])

Abel Summation: For $0 \leq x < 1$,

$$\sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x} \quad \Rightarrow \quad \lim_{x\to 1^-} \sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{2}. \tag{3.10}$$

Cesàro Summation: Let $S_N = \sum_{k=0}^{N} (-1)^k$ so $S_N = 1$ for even N and $S_N = 0$ for odd N. The Cesàro means

$$\sigma_N \coloneqq \frac{1}{N+1} \sum_{n=0}^{N} S_n \tag{3.11}$$

satisfy $\sigma_N \to 1/2$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 November 2025 doi:10.20944/preprints202511.0912.v1

Therefore, both Abel and Cesàro summation assign the endpoint value $\varsigma(-1) = 1/2$, justifying the extension in (3.5).

Remark 3.2 (No further extension by Abel/Cesàro on the real line [9])

Abel Summation: For $a_k = r^k$, the Abel sum is

$$\sum_{k\geq 0} a_k x^k = \sum_{k\geq 0} (rx)^k = \frac{1}{1-rx}, \qquad |rx| < 1. \tag{3.12}$$

If $|r| > 1$, then for all x sufficiently close to 1^- we have $|rx| < 1$, so the defining power series fails to converge in any neighborhood of $x = 1$. Hence the Abel limit $\lim_{x\to 1^-} \sum (rx)^k$ is undefined.

Cesàro Summation: A necessary condition for Cesàro $(C, 1)$ summability is $a_k \to 0$. For $a_k = r^k$ with $|r| \geq 1$ and $r \neq -1$, we have $a_k \nrightarrow 0$, so the series is not Cesàro-summable.

Hence, no further real values of r beyond -1 yield a consistent Abel or Cesàro limit, and the extended interval $r \in [-1, 1)$ is maximal for real summability.

4. Rulebook for z-Space Operations

Lemma 4.1 (Rulebook)

Let $U \subset C$ be a domain. The following principles govern analytic operations in z-space.

(R1) Series of holomorphic functions. If $\sum f_n$ converges uniformly on compact subsets of U (normal convergence), then $\sum f_n$ is holomorphic on U. Finite algebraic operations and termwise differentiation are valid on U [10].

(R2) Identity theorem / uniqueness. If two holomorphic (or meromorphic) functions agree on a nonempty open subset of a connected domain U, then they agree on all U where both are defined [8].

(R3) Locality. A derivation carried out inside U is valid provided every intermediate expression is defined and holomorphic on U. Evaluation outside U is neither required nor permitted [8].

(R4) Restriction of global identities. If $F(w) = G(w)$ holds as an identity of meromorphic functions on $\mathbb{C}$, then for any open $U \subset \mathbb{C}$ and any holomorphic map $h: U \to \mathbb{C}$ avoiding poles,

$$F \circ h = G \circ h \text{ on } U. \tag{4.1}$$

[11]

(R5) Boundary domains and extensions. Let

$$H_{1/2} = \{\, z \in \mathbb{C}: \Re z > 1/2 \,\}, \overline{H}_{1/2} = \{\, z \in \mathbb{C}: \Re z \geq 1/2 \,\}. \tag{4.2}$$

All analytic operations governed by (R1)–(R4) are confined to open domains such as $H_{1/2}$ where the relevant functions are holomorphic.

The closure $\overline{H}_{1/2}$ is used only for limit evaluation or boundary continuity, as justified by Abel–Cesàro extension (Remarks 3.1–3.2). No differentiation, contour deformation, or analytic continuation may be performed on the boundary itself.

Hence $H_{1/2}$ is the working analytic domain, while $\overline{H}_{1/2}$ serves only for evaluating limiting values such as $\xi(1/2 + it)$.

(R6) Compatibility of convergence with the working domain (access rule). Let $U \subset \mathbb{C}$ be the working analytic domain in z-space (typically an open set with $U \subset H_{1/2}$), and let f be specified on some open set $V \subset \mathbb{C}$ by a convergent representation (e.g., a Dirichlet/geometric series, integral, or canonical product) that makes f holomorphic on V.

1. **Operative region.** All valid manipulations with f in z-space are confined to

$$E \coloneqq U \cap V.$$

Conclusions must be stated on E or on its boundary via limits as in (R5).

2. **When U is larger than V.** If $U \not\subseteq V$, the working domain is automatically restricted to E. No evaluation or analytic operation is permitted at points of $U \setminus V$unless a meromorphic/analytic continuation of f is explicitly established there.
3. **When V is larger than U.** If $V \not\subseteq U$, the part of V outside U is inaccessible within the z-space framework by (R3) and the construction of z-space from the geometric-series parameter, cf. §3. Only values in E and boundary limits per (R5) may be used in derivations.

4. **Boundary use and continuation.** Boundary values may be taken as non-tangential limits from E as in (R5). Analytic continuation of f may be invoked only insofar as the continued function remains within U. Values outside U are accessible only by relating them to values inside U solely via global identities valid on $\mathbb{C}$ (e.g., functional symmetries) composed as in (R4), followed—if needed—by boundary limiting in the sense of (R5).
5. **Effective boundary.** For proofs in z-space, the effective boundary for f is $\overline{E}\cap\overline{U}\subset\overline{H}_{1/2}$: that is, the smaller boundary determined by the intersection of the domain of convergence of f and the working domain of U.

Notation. When extending a z-space identity for a function $f(z)$ to a domain D larger than the working region $H_{1/2}$ (or to its closure $\overline{H}_{1/2}$ when invoking (R5)), we do not write $z\in\mathbb{C}$, since z is evaluated only on $H_{1/2}$. Instead, when (R2) or (R4) is used to appeal to an identity valid beyond $z\in H_{1/2}$, we denote this by

$$z\in D\big[H_{1/2}\big]\ \Big(\text{or } z\in D\big[\bar{H}_{1/2}\big] \text{ when using (R5)}\Big)$$

To mean: "the identity holds on the ambient domain D, while access/evaluation is restricted to $H_{1/2}$ (respectively $\overline{H}_{1/2}$)."

Notation. To indicate the provenance of a constraint, we place a small label directly above a relation or membership symbol. The marker "V" (e.g., "$\overset{V}{<}$") signals that the bound comes from the convergence domain of the function f. The marker "U" denotes a bound imposed by the working domain. The absence of a marker indicates that the bound is independent of domain constraints.

This annotation is purely declarative: it does not change the underlying logical statement; it simply records the source of the constraint for reference.

5. Zeta and Eta Definitions and Identity in z-Space [5]

Definition 5.1 (Dirichlet functions in z-space)

For complex z, define the Riemann zeta function

$$\zeta(z):=\sum_{n=1}^{\infty}n^{-z}\tag{5.1}$$

which is an absolutely convergent Dirichlet series on $\Re z\overset{V}{>}1$.

For $\Re(z)\overset{U}{>}1/2$, define the Dirichlet eta function

$$\eta(z):=\sum_{n=1}^{\infty}(-1)^{n-1}n^{-z}\tag{5.2}$$

which converges by Dirichlet's test and is holomorphic on $\Re z\overset{U}{>}1/2$.

In lieu of invoking analytic continuation, note that the reflection $z\mapsto 1-z$ maps η to a region confined to the strip

$$0\overset{V}{<}\Re(1-z)\overset{U}{<}1/2,$$

Lemma 5.1 (η–ζ relation on the right half-plane)

For $\Re z>1$,

$$\eta(z)=\sum_{n\geq 1}(-1)^{n-1}n^{-z}=(1-2^{\,1-z})\,\zeta(z).\tag{5.3}$$

Hence the meromorphic identity

$$\frac{\eta(z)}{1-2^{1-z}}=\zeta(z)\tag{5.4}$$

extends to the half-plane $\Re z\overset{U}{\geq}1/2$.

Proof.

On $\Re z>1$, the Dirichlet series for ζ and η converge uniformly on compact sets, so term wise algebra and regrouping are justified by (R1), yielding (5.3). Since η is holomorphic on $\Re z\overset{U}{>}1/2$ by (5.2), the quotient (5.4) extends to a meromorphic function there via (R2), with the following qualification:

The denominator vanishes precisely at

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 November 2025 doi:10.20944/preprints202511.0912.v1

$$z = 1 - \frac{2\pi i k}{log\ 2}, k \in \mathbb{Z}, \tag{5.5}$$

which lie on the vertical line $\Re z = 1$. On $\Re z > 1$, (5.3) gives $\eta(z) = (1 - 2^{1-z})\, \zeta(z)$ where ζ is holomorphic. Hence $\eta(z) = 0$ at every point (5.5).

Thus, the potential singularities of (5.5) are removable, except at $z = 1$ where ζ has a simple pole. Consequently, (5.4) defines a meromorphic function on $\Re z \overset{U}{>} 1/2$ with at most a simple pole at $z = 1$, and the identity extends to the boundary $\Re z = 1/2$ by (R5). ∎

6. Gamma Definition and Identities in z-Space [4,6]

Definition 6.1 (Gamma in z-space).

Define the Gamma function by Euler's integral in z-space

$$\Gamma(z) = \int_0^\infty t^{\,z-1} e^{-t}\, dt\,, \tag{6.1}$$

which converges and is holomorphic on $\Re z \overset{U}{>} 1/2$.

By reflection $z \mapsto 1 - z$, admissible use of $\Gamma(1 - z)$ is confined to

$$0 \overset{V}{<} \Re(1 - z) \overset{U}{<} 1/2,$$

or equivalently

$$1/2 \overset{U}{<} \Re z \overset{V}{<} 1,$$

and, by (R5), the non-tangential boundary limits $\Re(1 - z) = \Re z = 1/2$ are permitted.

Proposition 6.1 (Beta–Gamma identity on the working strip)

Let $x, y \in \mathbb{C}$ with

$$\Re x \overset{U}{>} 1/2, \qquad \Re y \overset{U}{>} 1/2.$$

Define the beta integral

$$B(x, y) := \int_0^1 t^{\,x-1} (1 - t)^{\,y-1}\, dt\,. \tag{6.2}$$

Then

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \tag{6.3}$$

Proof.

First, $B(x, y)$ converges absolutely on the working strip $H_{1/2}$: near $t = 0$ the integrand is $t^{\Re x - 1}$ and near $t = 1$ it is $(1 - t)^{\Re y - 1}$, both integrable when $Rx, Ry > 0$.

For $\Re x \overset{U}{>} 1/2$ and $\Re y \overset{U}{>} 1/2$, both $\int_0^\infty u^{x-1} e^{-u}\, du$ and $\int_0^\infty v^{y-1} e^{-v}\, dv$, converge by Definition 6.1, so using Fubini/Tonelli,

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty u^{\,x-1} v^{\,y-1} e^{-(u+v)}\, du\, dv\,.$$

Making the change of variables

$$u = rt, \qquad v = r(1 - t), \qquad r \in (0, \infty), \qquad t \in (0,1),$$

whose Jacobian is r. Then

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^\infty \int_0^\infty (rt)^{x-1} (r(1 - t))^{y-1} e^{-r}\, r\, dt\, dr \\ &= \int_0^\infty r^{x+y-1} e^{-r}\, dr \int_0^1 t^{\,x-1} (1 - t)^{\,y-1}\, dt \\ &= \Gamma(x + y)\, B(x, y), \end{aligned}$$

which rearranging yields (6.3). All interchanges and substitutions are justified by absolute convergence in the working domain and by (R1) and (R3), extending to $\mathbb{C}[H_{1/2}]$ by (R2). ∎

Proposition 6.2 (Euler reflection and meromorphic continuation of Γ)

For $1/2 \overset{U}{<} \Re z < 1$,

$$\Gamma(z)\, \Gamma(1 - z) = \pi / \sin(\pi z)\,. \tag{6.4}$$

By (R2), the identity extends meromorphically to all $\mathbb{C}[H_{1/2}]$. In particular, Γ extends meromorphically with simple poles at $z = 0, -1, -2, \ldots$.

Proof.

On the open strip $1/2 \overset{U}{<} \Re z < 1$ the Beta integral

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 November 2025 doi:10.20944/preprints202511.0912.v1

$$B(z,1-z)=\int_0^1 t^{z-1}(1-t)^{-z}\,dt \tag{6.5}$$

converges absolutely, since near $t=0$ the integrand behaves like $t^{\Re z-1}$ $(\Re z \overset{U}{>} 1/2)$, and near $t=1$ like $(1-t)^{-\Re z}$ $(\Re z<1)$.

By Proposition 6.1,

$$B(z,1-z)=\Gamma(z)\Gamma(1-z). \tag{6.6}$$

Evaluate $B(z,1-z)$ via the substitution

$$t=\frac{u}{1+u},\qquad u\in(0,\infty),$$

so that

$$dt=(1+u)^{-2}\,du,\qquad 1-t=(1+u)^{-1},\qquad t/1-t=u.$$

Substituting into (6.5) gives

$$B(z,1-z)=\int_0^\infty\left(\frac{u}{1+u}\right)^{z-1}\left(\frac{1}{1+u}\right)^{-z}(1+u)^{-2}\,du=\int_0^\infty\frac{u^{z-1}}{1+u}\,du\,. \tag{6.7}$$

To computed the last integral directly for $1/2\overset{U}{<}\Re z<1$, split $(0,\infty)$ at $u=1$, set $u\mapsto 1/u$ on $(1,\infty)$, and combine to obtain:

$$\int_0^\infty\frac{u^{z-1}}{1+u}\,du=\int_0^1\frac{u^{z-1}+u^{-z}}{1+u}\,du=\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{z+n}=\frac{\pi}{sin(\pi z)}. \tag{6.8}$$

where the last equality is the standard partial fraction expansion of $\pi/\sin(\pi z)$. All interchanges are justified by absolute convergence on $1/2\overset{U}{<}\Re z<1$.

Combining (6.6) and (6.8) yields (6.4) on $1/2\overset{U}{<}\Re z<1$. Since both sides are meromorphic in z and agree on a nonempty open strip, (R2) extends the identity to all $\mathbb{C}[H_{1/2}]$.

Because $\sin(\pi z)$ vanishes simply at each integer, Γ has simple poles at $z=0,-1,-2,\ldots.$ ■

Corollary 6.1 (Nonvanishing)

$\Gamma(z)\neq 0$ *for all* $z\in\mathbb{C}[H_{1/2}]$. *Equivalently,* $1/\Gamma$ *is entire.*

Proof.

If $z\notin\mathbb{Z}[H_{1/2}]$, then $\sin(\pi z)\neq 0$ and (6.4) shows $\Gamma(z)\Gamma(1-z)$ is finite and nonzero, so neither factor vanishes. If $z\in\mathbb{Z}[H_{1/2}]$, then $\pi/\sin(\pi z)$ has a pole, hence $\Gamma(z)\Gamma(1-z)$ cannot be zero. Hence Γ has no zeros anywhere.

Since Γ has only simple poles (at $z=0,-1,-2,\ldots$) and no zeros, the reciprocal $1/\Gamma$ has simple zeros at those points and is holomorphic elsewhere, and the poles of Γ become removable for $1/\Gamma$. Therefore $1/\Gamma$ extends to an entire function. ■

Proposition 6.3 (Legendre's duplication).

For $\Re z>1/2$,

$$\Gamma(z)\Gamma(z+1/2)=2^{1-2z}\sqrt{\pi}\,\Gamma(2z). \tag{6.9}$$

By (R2), this identity extends meromorphically to all $\mathbb{C}[H_{1/2}]$.

Proof.

On the open strip $1/2\overset{U}{<}\Re z<1$, by Proposition 6.1 and the trigonometric form of the Beta function (obtained by the substitution $t=\sin^2\theta$, valid under (R1)–(R3)), we have:

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt=2\int_0^{\frac{\pi}{2}}\sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta\,,\qquad \Re x,\Re y\overset{U}{>}1/2.$$

Apply this with $x=z$, $y=1/2$ (interpreted as a non-tangential boundary value via (R5)) and with $x=y=z$. Then

$$B(z,1/2)=2\int_0^{\pi/2}\sin^{2z-1}\theta\,d\theta,\qquad B(z,z)=2\int_0^{\pi/2}\sin^{2z-1}\theta\cos^{2z-1}\theta\ d\theta\,. \tag{6.10}$$

Set $\varphi=2\theta$ in the second integral and use $\sin\varphi=2\sin\theta\cos\theta$ and $d\varphi=2\,d\theta$. A direct calculation (permitted by (R1)–(R3)) gives the trigonometric identity

$$B(z,1/2)=2^{2z-1}B(z,z),\qquad 1/2\overset{U}{<}\Re z<1. \tag{6.11}$$

Invoking the Beta–Gamma identity (6.3) on the same strip,

$$B(z,1/2)=\frac{\Gamma(z)\Gamma(1/2)}{\Gamma(z+1/2)},\qquad B(z,z)=\frac{\Gamma(z)^2}{\Gamma(2z)}. \tag{6.12}$$

Substitute (6.12) into (6.11) and use $\Gamma(1/2)=\sqrt{\pi}$ to obtain

$$\frac{\Gamma(z)\sqrt{\pi}}{\Gamma(z+1/2)} = 2^{\,2z-1}\frac{\Gamma(z)^2}{\Gamma(2z)}.$$

which rearranges to

$$\Gamma(z)\,\Gamma(z+1/2) = 2^{\,1-2z}\,\sqrt{\pi}\,\Gamma(2z),$$

establishing (6.9) for $1/2 \overset{U}{<} \Re z < 1$. Since both sides are meromorphic in z and agree on the nonempty open strip, (R2) extends the identity meromorphically to all $z \in \mathbb{C}[H_{1/2}]$. ■

Proposition 6.4 (Scaling law for the Gamma kernel)

Let $\alpha > 0$ *and* z *satisfy* $\Re z \overset{U}{>} 1/2$. *Then*

$$\int_0^\infty e^{-\alpha t} t^{\,z-1} dt = \alpha^{-z}\Gamma(z) \tag{6.18}$$

and the identity is holomorphic in $z \in H_{1/2}$.

Proof.

For $t \to \infty$, $e^{-\alpha t}$ gives exponential decay; for $t \downarrow 0$, the integrand behaves like $t^{\Re z-1}$ with $\Re z \overset{U}{>} 1/2$. Hence the integral converges absolutely, and by dominated convergence on compact subsets of $H_{1/2}$, it defines a holomorphic function of z (R1)–(R3).

Set $u = \alpha t$ (so $t = u/\alpha$, $dt = du/\alpha$). Then we get (6.18)

$$\int_0^\infty e^{-\alpha t}\, t^{z-1}\, dt = \int_0^\infty e^{-u}\left(\frac{u}{\alpha}\right)^{z-1}\frac{du}{\alpha} = \alpha^{-z}\int_0^\infty e^{-u}\, u^{\,z-1}\, du = \alpha^{-z}\,\Gamma(z),$$

where $\Gamma(z)$ is Euler's integral (Definition 6.1) on $H_{1/2}$. All steps are justified by the absolute convergence already noted and the legality of substitutions in (R3). ■

7. Functional Equation and Completed Zeta Function in z-Space [1,6]

Lemma 7.1 (Theta–Mellin identity and symmetry on the strip)

Let

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}, \tag{7.1}$$

and for $1 < \Re z < 2$ *define*

$$I(z) := \frac{1}{2}\int_0^\infty (\theta(t)-1)\, t^{\frac{z}{2}}\frac{dt}{t}, \tag{7.2}$$

Then

$$I(z) = \frac{1}{2}\int_1^\infty (\theta(t)-1)\, t^{\frac{z}{2}}\,\frac{dt}{t} + \frac{1}{2}\int_1^\infty (\theta(t)-1)\, t^{\frac{1-z}{2}}\,\frac{dt}{t} = I(1-z) \tag{7.3}$$

and

$$I(z) = \pi^{-z/2}\Gamma(z/2)\zeta(z). \tag{7.4}$$

on $\mathbb{C}[H_{1/2}] \setminus \{1\}$

Proof.

As $t \to \infty$, $\theta(t) - 1 = O(e^{-\pi t})$. As $t \downarrow 0$, the modular relation

$$\theta(t) = t^{-1/2}\theta(1/\mathrm{t}). \tag{7.5}$$

implies

$$\theta(t) - 1 = t^{-1/2} + O(e^{-\pi/t})$$

Hence

$$\int_0^\infty |\theta(t)-1|\, t^{\frac{\Re z}{2}-1} dt < \infty \text{ for } 1 < \Re z < 2, \tag{7.6}$$

so (7.2) converges absolutely and defines a holomorphic function on the strip by dominated convergence (R1), (R3).

Split at $t = 1$ and use (7.5). On $(0,1)$, substitute $t \mapsto 1/t$. For $1 < \Re z < 2$ this gives

$$I(z) = \frac{1}{2}\int_1^\infty (\theta(t)-1)\, t^{\frac{z}{2}}\,\frac{dt}{t} + \frac{1}{2}\int_1^\infty (\theta(t)-1)\, t^{\frac{1-z}{2}}\,\frac{dt}{t} = I(1-z).$$

which is (7.3)

Since $\theta(t) - 1 = 2\sum_{n\ge 1} e^{-\pi n^2 t}$, absolute convergence on $1 < \Re z < 2$ permits exchanging sum and integral via (R1), (R3), yielding

doi:10.20944/preprints202511.0912.v1

$$I(z) = \sum_{n\geq 1} \int_0^\infty e^{-\pi n^2 t}\, t^{\frac{z}{2}-1}\, dt = \sum_{n\geq 1} (\pi n^2)^{-\frac{z}{2}}\, \Gamma\left(\frac{z}{2}\right) = \pi^{-\frac{z}{2}}\, \Gamma\left(\frac{z}{2}\right) \sum_{n\geq 1} n^{-z} = \pi^{-\frac{z}{2}}\, \Gamma\left(\frac{z}{2}\right) \zeta(z),$$

by the scaling law (Proposition. 6.4) with $\alpha = \pi n^2$and $z \mapsto z/2$. Thus, we prove (7.4) on $1 < \Re z < 2$, and extend it meromorphically to $z \in \mathbb{C}[H_{1/2}] \setminus \{1\}$ by (R2) ■

Lemma 7.2 (Classical functional equation in *z*-space)

For $z \in \mathbb{C}[H_{1/2}] \setminus \{1\}$

$$\zeta(z) = 2^{z} \pi^{z-1} \sin(\pi z/2)\, \Gamma(1-z) \zeta(1-z). \tag{7.7}$$

Proof.

From (7.3)–(7.4) we have on $1 < \Re z < 2$,

$$\pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z). \tag{7.8}$$

Rearrange to get

$$\zeta(z) = \pi^{z-\frac{1}{2}} \frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma(z/2)} \zeta(1-z). \tag{7.9}$$

Now use the global Gamma identities from §6, admissible by (R4), with arguments avoiding poles on $1 < \Re z < 2$:

1. Duplication with $w = (1-z)/2$ gives

$$\Gamma\left(\frac{1-z}{2}\right) \Gamma\left(1 - \frac{z}{2}\right) = 2^{1-2w} \sqrt{\pi}\, \Gamma(2w) = 2^{z} \sqrt{\pi}\, \Gamma(1-z). \tag{7.10}$$

2. Reflection with $w = z/2$ gives

$$\Gamma(z/2)\Gamma(1-z/2) = \pi/\sin(\pi z/2)\,. \tag{7.11}$$

Divide (7.10) by (7.11) to eliminate $\Gamma(1-z/2)$:

$$\frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma(z/2)} = \frac{2^{z} \sqrt{\pi}\, \Gamma(1-z)}{\pi/\sin\,(\pi z/2)} = 2^{z} \pi^{-1/2} \sin(\pi z/2)\, \Gamma(1-z). \tag{7.12}$$

Insert (7.12) into (7.9):

$$\zeta(z) = \pi^{z-1/2} [2^{z} \pi^{-1/2} \sin(\pi z/2)\, \Gamma(1-z)]\, \zeta(1-z) = 2^{z} \pi^{z-1} \sin(\pi z/2)\, \Gamma(1-z) \zeta(1-z),$$

Which is (7.7) on $1 < \Re z < 2$.

Both sides of (7.7) are meromorphic in z and agree on a nonempty open strip, so by (R2) together with the composition rule (R4), the identity extends to $\mathbb{C}[H_{1/2}] \setminus \{1\}$. ■

Corollary 7.1 (Functional continuation without the Identity Theorem)

Beginning with the functional equation (7.7), already valid for $\mathbb{C}[H_{1/2}] \setminus \{1\}$.

Then, by algebraic rearrangement and identities admissible under (R4), we obtain,

$$\zeta(1-z) = 2^{1-z} \pi^{-z} \cos(\pi z/2)\, \Gamma(z)\, \zeta(z), \tag{7.13}$$

Thus ζ *admits a meromorphic continuation to* $\mathbb{C}[\overline{H}_{1/2}] \setminus \{1\}$ *within the z-space framework by (R4)-(R5).*

Proof.

Starting from (7.7), on $\mathbb{C}[H_{1/2}] \setminus \{1\}$,

$$\zeta(z) = 2^{z} \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z),$$

solve for $\zeta(1-z)$:

$$\zeta(1-z) = \frac{\zeta(z)}{2^{z} \pi^{z-1} \sin(\pi z/2)\, \Gamma(1-z)}. \tag{7.14}$$

Apply Euler's reflection formula (6.4),

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$$

and the trigonometric identity

$$\sin(\pi z) = 2\sin(\pi z/2)\cos(\pi z/2),$$

both admissible by (R4). These eliminate $\Gamma(1-z)$ from (7.14) and yield (7.13).

Every step uses a global identities composed with z-space variables, so (R4) preserves validity on $\mathbb{C}[H_{1/2}] \setminus \{1\}$, and (R5) allows passage to the boundary $\mathbb{C}[\overline{H}_{1/2}] \setminus \{1\}$.■

Corollary 7.2 (Access to the entire complex plane realized in *z*-space)

By Definition 5.1, $\zeta(z)$, via the η-quotient, is meromorphic of $\Re z \overset{U}{>} 1/2$ with at most a simple pole at $z = 1$. and $\zeta(1-z)$ converges for $1/2 \overset{U}{<} \Re z \overset{V}{<} 1$ (equivalently $0 \overset{U}{<} \Re(1-z) \overset{V}{<} 1/2$).

Formula (7.13),

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\cos\left(\frac{\pi z}{2}\right)\Gamma(z)\,\zeta(z),$$

expresses $\zeta(1-z)$ entirely in terms of data evaluated on the half-plane $\Re z > 1/2$, extending to the reflected region $\Re(1-z) < 1/2$ using only operations permitted by (R4). Thus (7.13) satisfies the access rule (R6), and reaches $z \in \mathbb{C}[\bar{H}_{1/2}] \setminus \{1\}$ completely within the z-space framework by (R4)-(R5), without invoking analytic continuation in s-space. ■

Definition 7.1 (Completed zeta in z-space)

For $\Re z > 1$ *define*

$$\xi(z) := \frac{1}{2}\, z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z). \tag{7.15}$$

Then ξ *extend to be an entire function on* $\mathbb{C}\left[\bar{H}_{1/2}\right]$ *and satisfies the functional identity*

$$\xi(z) = \xi(1-z). \tag{7.16}$$

Justification

The condition $\Re z > 1$ ensures that $\zeta(z)$ and $\Gamma(z/2)$ are holomorphic on the working half-plane $H_{1/2}$, so the product in (7.15) is well defined there by (R1)–(R3).

From Lemma 7.1, we have on $1 < \Re z < 2$,

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z),$$

Multiplying both sides by $\frac{1}{2}z(z-1)$, which cancels the simple pole at $z = 1$ coming from ζ and the pole at $z = 0$ from $\Gamma(z/2)$, we obtain

$$\xi(z) = \xi(1-z).$$

By (R2) and (R5) this symmetry extends meromorphically to all $z \in \mathbb{C}\left[\overline{\mathrm{H}}_{1/2}\right]$. The trivial zeros of ζ at $z = 0, -2, -4, \ldots$ cancel the further poles of $\Gamma(z/2)$ at these points, so ξ is entire. ■

Corollary 7.3 (Access to the reflected side from the right half-plane).

Let $\xi(z)$ be defined as in Definition 7.1. Then for all $z \in \mathbb{C}[\bar{H}_{1/2}]$,

$$\xi(1-z) = \xi(z).$$

In particular, the value of ξ at the reflected point $1-z$ does not require any additional analytic information beyond what is already available in the right half-plane $\Re z \geq 1/2$. The completed zeta function evaluated at $1-z$ is already expressed in terms of the data used to define $\xi(z)$ on $\Re z > 1/2$, and the passage to the boundary $\Re z = 1/2$ is justified by (R5). Thus, within the z-space framework, the reflected half-plane is accessed entirely through formulas written on the right, in accordance with (R6), without any appeal to analytic continuation in the s-variable. ■

Corollary 7.4 (Restatement of Boundary equality on the critical line (R5))

Let $\xi(z) = \xi(1-z)$ *be defined as in Definition 7.1.Then taking non-tangential limits from the right and from the left half-plane gives*

$$\lim_{\Re z\downarrow 1/2}\xi(z) = \lim_{\Re(1-z)\uparrow 1/2}\xi(1-z) = \xi(1/2+it), t \in \mathbb{R}. \tag{7.17}$$

Thus, the analytic continuations from $\Re z > 1/2$ *and from* $\Re(1-z) < 1/2$ *agree on the boundary, adjoining the two analytic pieces at the critical line, such that*

$$\xi(z) = \xi(1-z), \qquad \Re z = 1/2. \tag{7.18}$$

Proof.

By Corollary 7.3, the equality $\xi(z) = \xi(1-z)$ holds on $\mathbb{C}[H_{1/2}]$.

Approaching the line $\Re z = 1/2$ from either side yields equal non-tangential boundary values on the whole line by (R5). Hence, ξ is continuous across the critical line. ■

Corollary 7.5 (Order one via Theta–Mellin within z-space).

Let $z = \varsigma(r) + it$ *with* $\varsigma(r) \geq 1/2$, *then*

$$\xi(z) = \frac{1}{2}z(z-1)\int_1^{\infty}(\theta(t)-1)\left(t^{\frac{z}{2}} + t^{\frac{(1-z)}{2}}\right)\frac{dt}{t}. \tag{7.19}$$

is entire by Definition 7.1, and is accessible entirely within the z-space framework by (R4)-(R6).

The right side defines a holomorphic function for $\varsigma(r) > 1/2$, *with non-tangential boundary limits at* $\varsigma(r) = 1/2$ *by (R5). Consequently, there exist constants* $A, B > 0$ *such that for every* $z = \varsigma(r) + it$ *with* $\varsigma(r) \geq 1/2$,

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 November 2025 doi:10.20944/preprints202511.0912.v1

$$|\xi(z)| \le \exp(A\,(|\varsigma(r)| + |t|)\,log(2 + |\varsigma(r)| + |t|) + B)\,. \tag{7.20}$$

Using $\xi(1-z) = \xi(z)$ *from (7.16), and generalizing both sides as* $\xi(w)$*, this bound holds for any* $w \in \mathbb{C}$ *by applying (7.19) to either* $w = z$ *if* $\Re w \ge 1/2$ *or to* $w = 1 - z$ *(if* $\Re w < 1/2$*). Hence* ξ *is of order one on* $\mathbb{C}$*.*

Proof.

Write $z = \varsigma + it$ with $\varsigma := \varsigma(r) \ge 1/2$. From (7.19) and $|\theta(t) - 1| \ll e^{-\pi t}$,

$$|\xi(z)| \le \frac{1}{2}|z(z-1)| \int_1^\infty e^{-\pi t}\left(t^{\frac{\varsigma}{2}} + t^{\frac{(1-\varsigma)}{2}}\right)\frac{dt}{t}. \tag{7.21}$$

Make the change $t = e^{2u}$ (so $dt/t = 2\,du$) and define

$$\Phi(u) := \frac{1}{2}(\theta(e^{2u}) - 1), \tag{7.22}$$

so that $|\Phi(u)| \ll 1$ as $u \downarrow 0$ and $|\Phi(u)| \ll \exp(-\pi e^{2u})$ as $u \to \infty$. Then (7.21) becomes

$$|\xi(z)| \le |z(z-1)| \int_0^\infty |\Phi(u)|\,\left|e^{uz} + e^{u(1-z)}\right| du\,. \tag{7.23}$$

Since $\varsigma \ge 1/2$,

$$\left|e^{uz} + e^{u(1-z)}\right| \le e^{u(\varsigma + |t|)} + e^{u(1-\varsigma+|t|)} \le 2\,e^{u(\max\{1,\varsigma\}+|t|)}.$$

Let

$$M := \log(2 + |t|)\,. \tag{7.24}$$

On $0 \le u \le M$, we have $|\Phi(u)| \ll 1$, so

$$\int_0^M |\Phi(u)| e^{u(\max\{1,\varsigma\}+|t|)} du \ll \exp\big(M(\max\{1,\varsigma\} + |t|)\big) = \exp\big((\max\{1,\varsigma\} + |t|)\,\log(2 + |t|)\big)\,. \tag{7.25}$$

On $u \ge M$: with $|\Phi(u)| \ll \exp(-\pi e^{2u})$,

$$\int_M^\infty |\Phi(u)|\,e^{u(\max\{1,\varsigma\}+|t|)}\,du \ll \int_M^\infty e^{u(\max\{1,\varsigma\}+|t|)-\pi e^{2u}}\,du \ll 1, \tag{7.26}$$

since for $u \ge M$ we have $e^{2u} \ge (2 + |t|)^2$, and the term $-\pi e^{2u}$ dominates.

Combining both ranges and using $|z(z-1)| \ll (|\varsigma| + |t|)^2$ we obtain (7.20).

Finally, for any $w \in \mathbb{C}$: if $\Re w \ge 1/2$, apply this bound directly with $w = z$; if $\Re w < 1/2$, apply (7.23) with $w = 1 - z$ and use $\xi(z) = \xi(1 - z)$. Thus ξ obeys an order-one growth bound on all $w \in \mathbb{C}$. ■

Remark 7.1 (Standard nature of the auxiliary estimates)

Every inequality used in the proof of Corollary 7.5 is independent of the z-space parameterization and of the access rules (R1)–(R6). In particular:

- the theta tail bound on $[1, \infty)$, $|\theta(t) - 1| \ll e^{-\pi t}$;
- the change of variables $t = e^{2u}$ with $dt/t = 2\,du$ and the definitions $\Phi(u) = \frac{1}{2}(\theta(e^{2u}) - 1)$, together with the bounds $|\Phi(u)| \ll 1$ as $u \downarrow 0$ and $|\Phi(u)| \ll e^{-\pi e^{2u}}$as $u \to \infty$;
- the elementary envelope $\left|e^{uz} + e^{u(1-z)}\right| \le e^{u(\Re z+|\Im z|)} + e^{u(1-\Re z+|\Im z|)}$;
- and the quadratic estimate $|z(z-1)| \ll (|\Re z| + |\Im z|)^2$;

are all classical and hold for an arbitrary complex variable z. Thus, the growth bound in Corollary 7.5 rests only on standard analytic estimates, applied verbatim in the present setting. ■

8. The Completed Zeta function for Order One Use [1,6,8]

Notation: The symbol ϱ (Greek rho-variant) denotes a nontrivial zero of ζ (equivalently of ξ) in z-space,

$$\xi(\varrho) = 0, \qquad \Re\varrho \ge 1/2, \tag{8.1}$$

where "nontrivial" means that ϱ is not one of the zeros forced by the normalizing prefactor in Definition 7.1 (i.e. not $z = 1$ from ζ, nor $z = 0, -2, -4, \dots$ coming from the interaction of $\Gamma(z/2)$ with the trivial zeros of ζ). In other words, ϱ denotes a genuine zero of ξ arising from the analytic content of ζ, not from normalizing factors.

Write

$$\varrho := \varsigma(q) + i\gamma, \gamma \in \mathbb{R}, \tag{8.2}$$

so that its real part can be expressed in the geometric parameter of §3 by

$$\varsigma(q) = \sum_{k=0}^{\infty} \mathrm{q}^{\mathrm{k}} = \frac{1}{1-q}, q \in [-1,1), \tag{8.3}$$

and the real parts of zeros may be indexed by the geometric variable q, with $q = -1$ corresponding to the boundary value $\varsigma(-1) = 1/2$ obtained via (R5).

The symbol ρ (standard rho) denotes the corresponding zero in s-space obtained by the analytic change of variables $z \mapsto s$ introduced in §3.

Definition 8.1 (The Hadamard form of the completed zeta function in z-space)

Since $\xi(z)$ *is entire of order one (Corollary 7.5), Hadamard's factorization theorem gives a global product for* ξ*. Within the* z*-space framework, where zeros are indexed only when they are accessible by* $R\varrho \geq 1/2$*, we write*

$$\xi(z) = \frac{1}{2} \prod_{\varrho} \left(1 - \frac{z}{\varrho}\right) exp\left(\frac{z}{\varrho}\right), \qquad \Re z, \Re\varrho \geq \frac{1}{2}. \tag{8.4}$$

This means we are evaluating the global Hadamard product on the right half of z*-space, indexing zeros by those* ϱ *that lie in the accessible half-plane determined by the geometric parametrization of §3.*

By the symmetry $\xi(1-\mathrm{z}) = \xi(z)$ *from (7.16), the same product can be written on the reflected side as*

$$\xi(1-z) = \frac{1}{2} \prod_{\varrho} \left(1 - \frac{1-z}{\varrho}\right) exp\left(\frac{1-z}{\varrho}\right), \qquad \Re(1-z), \Re(1-\varrho) \leq \frac{1}{2}, \tag{8.5}$$

Justification.

Because $\xi(z)$ is entire of order one, the general Hadamard's factorization has the form

$$\xi(z) = \mathrm{e}^{\mathrm{A+Bz}} \prod_{\varrho} \left(1 - \frac{\mathrm{z}}{\varrho}\right) \exp\left(\frac{\mathrm{z}}{\varrho}\right), \tag{8.6}$$

with absolute and locally uniform convergence. The normalization $\xi(0) = 1/2$ gives $e^{A} = 1/2$, and the symmetry $\xi(z) = \xi(1-z)$ forces $B = 0$, yielding (8.4). The reflected form (8.5) is obtained via functional symmetry *(7.16)*. By (R5), both representations admit non-tangential limits to $\Re z = 1/2$, as in Corollary 7.4. ■

Corollary 8.1 (Nontrivial zeros and the z-space Hadamard product).

Let $\xi(z)$ *be defined as in Definition 7.1,*

$$\xi(z) := \frac{1}{2}\, z(z-1)\, \pi^{-z/2}\, \Gamma(z/2)\, \zeta(z), Rz > 1.$$

Then every zero that appears in the Hadamard factorization of $\xi(z)$ *in Definition 8.1 is a nontrivial zero of* ζ*; conversely, every nontrivial zero of* ζ *appears as a zero of* ξ*. Thus, the zero set of the* z*-space Hadamard product coincides only with the nontrivial zeros.*

Proof.

By Definition 7.1, the prefactor

$$\frac{1}{2}\, z(z-1)\, \pi^{-z/2}\, \Gamma(z/2)$$

is chosen so that:

1. the simple pole of $\zeta(z)$ at $z = 1$ and the pole of $\Gamma(z/2)$ at $z = 0$ are both removed by the factor $\mathrm{z}(z-1)$;
2. the poles of $\Gamma(z/2)$ at $z = -2, -4, -6, \ldots$ are cancelled by the trivial zeros of $\zeta(z)$ at those same negative even integers;
3. the factor $\pi^{-z/2}$ never vanishes and therefore introduces no new zeros.

Hence $\xi(z)$ has no zeros coming from the normalizing prefactor; forced singularities and trivial zero are cancelled at the level of ξ. What remains are the zeros coming from $\zeta(z)$ that are not among its trivial zeros.

In Definition 8.1 we express $\xi(z)$ by its Hadamard product

$$\xi(z) = \frac{1}{2} \prod_{\varrho} \left(1 - \frac{z}{\varrho}\right) exp\left(\frac{z}{\varrho}\right), \qquad \Re z, \Re\varrho \geq \frac{1}{2},$$

so this product enumerates only the zeros of $\xi(z)$, which are the zeros of $\zeta(z)$. Therefore, any statements about the zeros of the z-space Hadamard product are equivalent to statements about the nontrivial zeros of ζ. ■

Lemma 8.1 (The z-space Hadamard structure on the location of admissible zeros)

The z-space Hadamard products in (8.4) and (8.5) are both intended to represent the same entire function ξ, each written only in terms of zeros that are admissible in its respective half-plane (right: $\Re z \geq 1/2$, left: $\Re(1-z) \leq 1/2$). Under this z-space framework, consistency of the two representations forces all admissible nontrivial zeros to lie on the common boundary

$$\Re\varrho = 1/2. \tag{8.6}$$

Proof.

Assume, for contradiction, that there is an admissible nontrivial zero ϱ with $R\varrho > 1/2$. By functional symmetry $\xi(z) = \xi(1-z)$ from (7.16), the reflected point $1-\varrho$ is also a zero.

The right-hand Hadamard product (8.4) includes only zeros that are accessible in the half-plane $\Re\varrho \geq 1/2$, in accordance with the geometric parametrization $\varsigma(q) = \sum_{k=0}^{\infty} \mathrm{q}^{\mathrm{k}}$. Conversely, the reflected product (8.4) includes only those zeros accessible from $\Re(1-\varrho) \leq 1/2$ $(R\varrho \geq 1/2)$. The two Hadamard products are both intended to represent the same entire function ξ; yet, if any zero ϱ lay strictly off the critical line, the two products would have non-matching zero sets, contradicting uniqueness under the identity theorem (R2).

Hence, structural consistency is maintained only if the two products coincide term-by-term, which occurs precisely when all admissible nontrivial zeros lie on the shared boundary line identified in 7.19. Thus, within the z-space framework, a consistent Hadamard structure requires

$$\Re\varrho = 1/2. \blacksquare$$

Lemma 8.2 (Hadamard consistency and extension into s-space)

Let the Hadamard product in z-space be defined as in (8.4)–(8.5). By Lemma 8.1, the two z-space Hadamard forms are mutually consistent only when every admissible nontrivial zero satisfies

$$\Re\varrho = 1/2.$$

Recalling the geometric parametrization (8.3)

$$\varsigma(q) = \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \qquad q \in [-1,1), \tag{8.8}$$

and define its holomorphic continuation as

$$s(q) := \sigma(q) + it, \qquad \sigma(q) := \frac{1}{1-q}, q \in \mathbb{R} \setminus \{1\}. \tag{8.9}$$

By (R4), any identity for $\xi(z)$ obtained in z-space remains valid after the substitution $z \mapsto s(q)$. Thus, in s-space we may write

$$\xi(s) = \frac{1}{2} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) exp\left(\frac{s}{\rho}\right), \tag{8.10}$$

where each ρ is the image in s-space of a z-space zero ϱ. Hence the continued function $\xi(s)$ has the same set of nontrivial zeros, lying on

$$\Re s = 1/2.$$

Proof.

By Definition 8.1, ξ admits the z-space Hadamard factorizations (8.4)-(8.5).

Lemma 8.1 shows that these two representations are consistent only when all admissible nontrivial zeros satisfy $\Re\varrho = 1/2$.

Now consider the holomorphic change of variables

$$z \mapsto s(q) = 1/(1-q) + it$$

defined on the domain of the geometric parameter. By (R4), composing an identity valid in z-space with this holomorphic map preserves the identity, and therefore preserves the zero set of ξ.

Hence the continued function $\xi(s)$ has the same set of nontrivial zeros as $\xi(z)$, now viewed in the s-variable, and these lie on the same fixed boundary

$$\Re s = 1/2. \blacksquare \tag{8.11}$$

Corollary 8.2 (All Nontrivial Zeros and the Critical Line)

In z-space, the Hadamard product of Definition 8.1 enumerates only the nontrivial zeros of ξ (Corollary 8.1). By Lemma 8.1, the two Hadamard representations can be simultaneously valid only when every admissible nontrivial zero satisfies

$$\Re\varrho = 1/2$$

By Lemma 8.2, the holomorphic change of variables $z \mapsto s$ carries this zero set into s-space without altering their location, so the corresponding zeros satisfy

$$\Re s = 1/2.$$

Hence, within the z-space framework developed in §§3–8, the critical line is the unique vertical line on which the nontrivial zeros appearing in the derived Hadamard product can occur. Consistency (and thus uniqueness) of the product is required by the identity theorem (R2), and is preserved under the holomorphic change of variable $z \mapsto s$ by (R4), where the geometric restrictions are no longer explicitly obvious. Under this change, we recover the classical Hadamard product (2.10) that enumerates exactly the nontrivial zeros of ξ. Therefore, the z-space Hadamard product is global by the s-space formulation, and the shared set of all nontrivial zeros [2,12] lie on

$$\Re s = 1/2.$$

9. Conclusion

Remark 9.1 (Structural interpretation within z-space and conclusion of Theorem 1.1).

On the working domain $\Re z \geq 1/2$, the identity $\xi(z) = \xi(1-z)$ from (7.16) gives reflection symmetry about the vertical line $\Re z = 1/2$. By the geometric reparameterization of §3, this line is not arbitrary: it is exactly the boundary value $\varsigma(-1) = 1/2$ of

$$\varsigma(r) = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, r \in [-1,1),$$

so $\Re z = 1/2$ is the unique vertical boundary coming from the image of the geometric parameter.

Corollary 7.1 and 7.3 show that the reflected half-plane is accessed solely through formulas written on the right, in accordance with the access rule (R6). Thus $\Re z = 1/2$ is the sharp cutoff.

Lemma 8.1 shows that the z-space Hadamard structure of ξ enforces this same axis analytically: compatibility of the right- and left-half Hadamard products on their overlap requires all admissible zeros to satisfy $\Re\varrho = 1/2$. Lemma 8.2 then confirms that this constraint is preserved under the holomorphic change of variables $z \mapsto s$: the z-space access rules (R6) and boundary matching (R5) transfer to s-space, even though the geometric origin of the symmetry is no longer explicit there.

Hence, the geometric and analytic structures coincide, and the critical line $\Re z = 1/2$ is fixed uniquely as the axis of symmetry in z-space, manifesting in s-space as the critical line.

This completes the proof of Theorem 1.1. ■

Remark 9.2 (Generality of z-space and scope of the framework).

The complex variable z used in this paper is not tied to any single function. It is produced by geometric reparameterization of the real part

$$z = \varsigma(r) + it, \qquad \varsigma(r) = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, r \in [-1,1),$$

which forces $\Re z \geq 1/2$ with boundary value $\varsigma(-1) = 1/2$ (Abel/Cesàro). All the analytic arguments in §§3–8 use only this geometric origin of z together with the access rules of §4, not any special property of ζ. Thus, the same scheme applies to any function Ξ satisfying the conditions below.

Geometric parameterization. There is a representation $z = \varsigma(r) + it$ with $\varsigma(r) = 1/(1-r)$, $r \in [-1,1)$, so that $\Re z \geq 1/2$ and $\Re z = 1/2$ is a genuine endpoint obtained by Abel/Cesàro. Analytic work is conducted in $H_{1/2} = \{\Re z > 1/2\}$; the boundary $\Re z = 1/2$ is used only for non-tangential limits, as in (R5).

Reflection accessed from the right. The function Ξ satisfies a symmetry of the form

$$\Xi(z) = \varepsilon\, \Xi(1-z), |\varepsilon| = 1,$$

and the term $\Xi(1-z)$ is obtained *only* by writing it in terms of right-half data (values of $\Xi(z)$, admissible gamma factors, and global identities) in accordance with (R1)–(R6). This "access-from-the-right" condition makes $\Re z = 1/2$ the non-movable boundary of the construction; replacing $1 - z$ by $a - z$ does not change the geometric cutoff $\Re z \geq 1/2$.

Optional Hadamard strengthening. If, in addition, Ξ is entire of order one and admits a canonical Hadamard product, then the right-hand and reflected products can coincide on their overlap only when they enumerate the same zeros. Under the z-space admissibility restriction, this forces all nontrivial zeros (i.e., zeros not coming from normalizing prefactors) to lie on $\Re z = 1/2$.

Preservation under the s-variable. For the holomorphic change of variables

$$z \mapsto s(r) = \frac{1}{1-r} + it,$$

allowed by (R4), the boundary $\Re z = 1/2$ is carried to $\Re s = 1/2$, so both the access property and, when present, the zero-location property persist in s-space.

In this sense, "z-space" is a general analytic device for functions whose real part can be written via the geometric series. Within this device, $\Re z = 1/2$ is an intrinsic access boundary, and—when order-one growth and Hadamard factorization are available—the same boundary is enforced by the zero structure. Thus, we may apply the framework to any function that satisfies the same geometric access and symmetry conditions, so that the critical line appears as an intrinsic boundary rather than a merely descriptive one [3].

Acknowledgements: I am deeply grateful to Kate Drake for her encouragement, support, perseverance, and patience. I also appreciate the YouTube community for making mathematics accessible—especially Brady Haran and his many interviewees, Grant Sanderson of 3Blue1Brown, and Burkard Polster and the Mathologer team. Additional thanks to Zetamath, Quanta Magazine, Stand-up Maths, and so many others who contribute to the broader mathematical conversation. I appreciate Dr. Gary Lawlor of the BYU Department of Mathematics for his encouragement and for reviewing earlier, fruitless manuscripts. I also thank Dr. Matthew Asplund and Dr. David Cardon for their kindness.

Conflicts of Interest: There is nothing to declare. No conflicts, no affiliations.

References

1. Titchmarsh, E. C. The Theory of the Riemann Zeta-Function, 2nd ed., revised by D. R. Heath-Brown, Oxford Univ. Press, Oxford, 1986.
2. Edwards, H. M. Riemann's Zeta Function, Dover Publications, New York, 1974.
3. Iwaniec, H. and Kowalski, E. Analytic Number Theory, American Mathematical Society Colloquium Publications, Vol. 53, Providence, RI, 2004.
4. Whittaker, E. T. and Watson, G. N. A Course of Modern Analysis, 4th ed., Cambridge Univ. Press, Cambridge, 1927.
5. Apostol, T. M. Introduction to Analytic Number Theory, Springer–Verlag, New York, 1976.
6. NIST Digital Library of Mathematical Functions, Release 1.1.9, 2023. Available at: https://dlmf.nist.gov.
7. Ivić, A. The Riemann Zeta-Function: Theory and Applications, Dover Publications, Mineola, NY, 1985.
8. Conway, J. B. Functions of One Complex Variable I, 2nd ed., Graduate Texts in Mathematics, Vol. 11, Springer–Verlag, New York, 1978.
9. Hardy, G. H. Divergent Series, Clarendon Press, Oxford, 1949.
10. Ahlfors, L. V. Complex Analysis, 3rd ed., McGraw–Hill, New York, 1979.
11. Remmert, R. Theory of Complex Functions; Springer: New York, NY, USA, 1991.
12. Riemann, B. Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, Monatsberichte der Berliner Akademie, 1859.