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Not peer-reviewed version

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Posted Date: 12 November 2025

doi: 10.20944/preprints202511.0820.v1

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Non-Integrability of a Hamiltonian System and Legendre Functions

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Abstract

In this paper we are studying the meromorphic integrability of a two-dimensional Hamiltonian system with a homogeneous potential of degree 6. The approach used in this work is the theory of the Ziglin-Moralez-Ruiz-Ramis-Simo. Within the scope of this theory, the study of such systems is reduced to determining the differential Galois group of a linear differential equation, obtained as a projection onto the tangent bundle of the phase curve of its non-equilibrium solution - Variation Equations (VE). In the case of Hamiltonian systems with homogeneous potentials, the variation equations are hypergeometric. If a standard approach is used to study such a system, it is necessary to calculate a Darboux point, which is not always easy. In this paper we can skip this difficulty by reducing VE to a Legendre equation We use the results for solvability of the Galois group of the associated Legendre equation for study a Hamiltonian system with a homogeneous polynomial potential of degree 6. For the full study, the second variations are used and conditions for a non-zero logarithmic term in their solutions are found.

Keywords: Hamiltonian system; meromorphic integrability; homogeneous potential; differential Galois group

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1. Introduction

We consider Hamiltonian systems of type

$$H = \sum_i \frac{1}{2} p_i^2 + V(q), \tag{1}$$

where $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$ are canonical coordinates in a symplectic linear space $\mathbb{C}^{2n}$ and $V(q)$ is a homogeneous polynomial function. If $F = F(q, p)$ and $G = G(q, p)$ are two functions, then their Poisson bracket is defined by $\{F, G\} = \sum_i (\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i})$. The functions F and G are in involution if $\{F, G\} = 0$ is satisfied. Non-constant function F is the first integral for the Hamiltonian system with Hamiltonian H if $\{F, H\} = 0$. Since the Poisson bracket is anti-symmetric, it is clear that H is always the first integral. A Hamiltonian system is called integrable if a set of n independent first integrals in involution exists for it (see Appendix A for further details).

The study of Hamiltonian systems with a homogeneous polynomial potential is an important problem in modern mechanics and physics. In our opinion, the reason for this research interest is the quite important physical problems can be considered as Hamiltonian systems with a homogeneous potentials. The integrability of Hamiltonian systems of type (1) has been studied for a long time, but the results of this work are limited to a rather small class of solved problems. The approaches to the study of this problem are: direct method [12,13] and Painleve test [3–8]. The most remarkable and perhaps most general result was obtained in [9]. In 1982 in [14] and 1983 in [15], Ziglin reduced the study of the integrability of Hamiltonian systems to the investigation of the monodromy group of the linear equations obtained by the variation of the system in a neighborhood of its non-equilibrium solution. The Ziglin's theory gives the opportunity for the study complex Hamiltonian systems for integrability. Using this theory, Yoshida [10] was able to obtain a criterion for the integrability of two-dimensional systems with a homogeneous polynomial potential. Later in [11] Yoshida generalized his result to n degrees of freedom. In the late 20-th century, the Ziglin's theory was linked to the differential Galois theory in the works of Churchill, Baider, Rod, Singer in [16,17], Moralez-Ruiz, Ramis, and Simo in [18–20]. The idea here is to move from studying the discrete monodromy group to the continuous algebraic differential Galois group for linear variational equations. The closure of the monodromy group in the Zariski topology gives us the differential Galois group, as noted in the Picard–Vessiot theory (see Appendix A.)

The use of differential Galois theory to study the integrability of Hamiltonian systems with homogeneous potentials in [18]. In the papers [21,22], the cases of homogeneous polynomial potentials of degrees 3 and 4 are scrutinized. Homogeneous potentials with a negative degree of homogeneity are considered in [1,2].

The approach implemented to study the non-integrability of V_6 in this paper differs from [18,21,22]. Here, the search for Darboux points is skipped. With a change of variables VE_1, which is a Fuchsian differential equation with 5 regular singularities, is reduced to a Legendre equation with 3 singularities.

We scrutinize systems with Hamiltonian $H = \sum_i \frac{1}{2} p_i^2 + V(q)$, with the potential $V(q)$ shown in the form $V(q) = \sum_l V_l(q)$, here $V_l(q)$ are homogeneous polynomials of degree $l \in \mathbb{N}$, $l \geq 2$.

Every homogeneous polynomial potential $V(q)$ is represented as a sum of homogeneous polynomials of not greater than $V(q) = V_{min}(q) + \cdots + V_{max}(q)$ degrees. We call the potential $V(q)$ integrable if the Hamiltonian system $H = \sum_i \frac{1}{2} p_i^2 + V(q)$ is integrable. If a homogeneous potential $V(q) = V_{min}(q) + \cdots + V_{max}(q)$ is integrable then $V_{min}(q)$ and $V_{max}(q)$ are also integrable. This remarkable property can be seen in [13].

In this paper, our study will be limited to assessing the non-integrability of such problem. We assume that $\mathbb{C}^4$ is a symplectic linear space with canonical variables $q = (q_1, q_2)$ and $p = (p_1, p_2)$. We are studying the Hamiltonian systems defined by the Hamiltonian $H = \frac{1}{2} \sum_1^2 p_i^2 + V_l(q)$, where $V_l(q)$ is a homogeneous function of degree l. The equations of motion of the above system are

$$\begin{aligned} \dot{q}_i &= p_i, \\ \dot{p}_i &= -\frac{\partial V(q_1, q_2)}{\partial q_i}, \; i = 1, 2. \end{aligned}$$

We say that a Hamiltonian system with 2 degrees of freedom is integrable in the sense of Liouville if there exists a second (other than H) first integral F for which the gradients dH and dF are linearly independent (see [1] for details).

In this paper we are studying a two dimensional model with sixth-order homogeneous potential

$$H = \frac{1}{2}(p_r^2 + p_z^2) + r^6 + Ar^2z^4 + Dr^3z^3 + Br^4z^2 + Er^5z + Cz^6, \tag{2}$$

where A, B, C, D and E are appropriate real constants for existing additional meromorphic integral of motion.

Let us justify the absence of the coefficient in front of $r^5 z$: If we assume that this coefficient in V_6 is α, i.e. we have

$$V_6 = r^6 + \alpha r^5 z + Ar^2z^4 + Dr^3z^3 + Br^4z^2 + Er^5z + Cz^6.$$

Now let us change the variables $r \to \frac{6r - \alpha z}{36}$ and $z \to \frac{z}{6}$ (a linear change, we can also define a canonical one of the Hamiltonian system). This is enough to change the coefficient α to 0 (see [12]).

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The Hamiltonian equations are:

$$\dot{r} = p_r,\ \dot{p}_r = -(2Arz^4 + 4Br^3z^2 + 3Dr^2z^3 + 5Er^4z + 6r^5),$$
$$\dot{z} = p_z,\ \dot{p}_z = -(4Ar^2z^3 + 2Br^4z + 6Cz^5 + 3Dr^3z^2 + Er^5), \tag{3}$$

(here as usual $\dot{} = \frac{d}{dt}$).

The organization of the text is as follows: In Section 2 we are investigating the solvability of Legendre's equation. In Section 3 we are trying to find a suitable partial solution, we so, we have identified the variational equations VE_1, and have related them to Legendre's equation. In Section 4 we are going through cases that have not been achieved using the results of Section 3. In this section the main theorem of our paper is proved. In the Appendix A we have given some facts from the Differential Galois theory necessary for our text.

2. The Associated Legendre Equation

In this section, we are studying for solvability of the associated Legendre equation

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left(p(p+1) - \frac{q^2}{1-z^2}\right)w = 0,\ p,\ q \in \mathbb{R},\ p+q \neq -1, -2, -3\ldots, \tag{4}$$

and we have applied the result to study the potential

$$V_6(r,z) = r^6 + Ar^2z^4 + Dr^3z^3 + Br^4z^2 + Er^5z + Cz^6,$$

(A, B, C, D, $E \in \mathbb{R}$) for integrability within the scope of Liouville.

Let us write down the known facts about the solutions of (4) and outline some properties required for solvability. (We follow [23].) With

$$\begin{aligned} P_p^q(z) &= \left(\frac{1+z}{1-z}\right)^{q/2} {}_2F_1\left(p+1, -p; 1-q, \frac{1}{2} - \frac{z}{2}\right), \\ Q_p^q(z) &= \frac{\pi}{2\sin(q\pi)}(\cos(q\pi)\left(\frac{1+z}{1-z}\right)^{q/2} {}_2F_1\left(p+1, -p; 1-q, \frac{1}{2} - \frac{z}{2}\right) \\ &- \left(\frac{1-z}{1+z}\right)^{q/2} \frac{\Gamma(p+q+1)}{\Gamma(p-q+1)} {}_2F_1\left(p+1, -p; 1+q, \frac{1}{2} - \frac{z}{2}\right)) \end{aligned} \tag{5}$$

we can show the solutions of (4) (P and Q Legendre functions), shown via the use of the hypergeometric function ${}_2F_1\left(a, b; c, z\right)$. The singularities of equation (4) are the points $z = -1$, $z = 1$ and $z = \infty$, which are regular. The following equalities are valid

$$\begin{aligned} P_p^{-q}(ze^{si\pi}) &= e^{spi\pi}P_p^{-q}(z) + \frac{2i\sin\left(p+1/2\right)s\pi}{\cos p\pi\Gamma(q-p)}Q_p^q(z) \\ Q_p^q(ze^{si\pi}) &= (-1)^s e^{-spi\pi}Q_p^q(z). \end{aligned} \tag{6}$$

The indicative equations for the singular points ± 1 are $\rho^2 - \rho + \dfrac{1-q^2}{4} = 0$ (see [26]) with a roots (exponents) $\rho_{1,2} = \dfrac{1 \pm q}{2}$, and for ∞ it is $\lambda^2 + \lambda - p^2 - p = 0$ with an exponents $\lambda_1 = -p-1$, $\lambda_2 = p$.

Therefore, we can specify the generators of the local monodromy for the points ± 1. These are the matrices $\begin{pmatrix} e^{(1+q)i\pi} & \alpha_1 \\ 0 & e^{-(1+q)i\pi} \end{pmatrix}$, and $\begin{pmatrix} e^{(1-q)i\pi} & \alpha_2 \\ 0 & e^{-(1-q)i\pi} \end{pmatrix}$.

For the point ∞, they are $\begin{pmatrix} e^{-2(1+p)i\pi} & \alpha_3 \\ 0 & e^{2(1+p)i\pi} \end{pmatrix}$, and $\begin{pmatrix} e^{2pi\pi} & \alpha_4 \\ 0 & e^{-2pi\pi} \end{pmatrix}$ ($\alpha_j \in \mathbb{C}$).

It is known from theory (see Appendix A) that a necessary condition for the solvability of a linear differential equation is that the monodromy group should be commutative. With the generators found, we can conclude that our equation does not have Liouville solutions (is not solvable) if p *or* $q \notin \mathbb{Q}$.

Proposition 1. *The equation (4) is non solvable if at least one of* $p, q \notin \mathbb{Q}$.

In the next of this paper, we assume that $p, q \in \mathbb{Q}$. We now have an expressions for the solution of (5) utilizing the P and Q Legendre functions $P_p^q(z)$ and $Q_p^q(z)$, which otherwise we can express using the Hypergeometric function ${}_2F_1\left(p+1, -p; 1-q, \frac{1}{2} - \frac{z}{2}\right)$. Here we can apply Kimura's conditions [24,25]. Thus we get the following

Theorem 1. *Let* $p, q \in \mathbb{Q}$, *then the equation (4) is non solvable if:*
(i) all of numbers $2p+1$, $2(q-p)+1$ *and* $2(p+q)+1$ *are not odd integer;*
(ii) at least one of $p \neq 1/2(-1 \pm (1/2+m))$ *or* $q \neq \pm(1/2+l)$, *for* $l, m \in \mathbb{Z}$;
(iii) at least one of $p \neq 1/2(-1 \pm (1/3+m))$ *or* $q \neq \pm(2/3+l)$, *for* $l, m \in \mathbb{Z}$, m *is odd;*
(iv) at least one of $p \neq 1/2(-1 \pm (2/5+m))$ *or* $q \neq \pm(2/5+l)$, *for* $l, m \in \mathbb{Z}$, m *is even;*
(v) at least one of $p \neq 1/2(-1 \pm (1/5+m))$ *or* $q \neq \pm(4/5+l)$, *for* $l, m \in \mathbb{Z}$, m *is odd.*

It is not difficult to conclude that if $q = 0$, then Legendre's the equation 4 is not solvable for $p \notin \mathbb{Z}$.

3. Sixth-Order Homogeneous Potential

First, we are finding a non equilibrium partial solution for (3). Let we put $r = p_r = 0$ in (3) and we have

$$\ddot{z} = -6Cz^5,$$

multiplied by $\dot{z}$ and integrated by the time t we get

$$\dot{z}^2 = -2(Cz^6 + Ch^3), \tag{7}$$

where h is a real constant.To this end, it we need to find a non branching solution, so, we get $w = z^2$ (finite ramified covering of the curve $y^2 = -2(Cz^6 + h)$). We obtain $\dot{w} = 2z\dot{z} = \dfrac{dz^2}{dt}$, then we have

$$\begin{aligned} \dot{w}^2 &= -8(Cw^4 + Ch^3 w), \\ \ddot{w} &= -4C(4w^3 + h^3). \end{aligned} \tag{8}$$

Further, we follow the procedures for Ziglin-Morales-Ramis theory and we detect an invariant manifold here w is the solution of (8). According to theory, the solution of (8) must be a rational function of Weierstrass $\wp$-function. It is convenient to choose for field of constants $K = \mathbb{C}(w)$ - rational functions over a complex variable w. After having found the Variation Equations (VE) we have $\xi_{11} = dr$, $\eta_{11} = dp_r$, $\xi_{12} = dz$, $\eta_{12} = dp_z$, and we achieve:

$$\begin{aligned} \ddot{\xi}_{11} &= -2Az^4\xi_{11}, \\ \ddot{\xi}_{12} &= -30Cz^4\xi_{12}, \end{aligned} \tag{9}$$

and after changing $w = z^2$ we have

$$\begin{aligned} \ddot{\xi}_{11} &= -2Aw^2\xi_{11}, \\ \ddot{\xi}_{12} &= -30Cw^2\xi_{12}, \end{aligned} \tag{10}$$

Now let us change the variables to (10) $t \to w(t)$, and we obtain VE_1-equations. Let us write denote with $' = \frac{d}{dw}$ then we get for the VE_1 two Fuchsian linear differential equations:

$$\xi_{11}'' + \frac{4w^3 + h^3}{2w(w^3 + h^3)}\xi_{11}' - \frac{A}{4C}\frac{w}{(w^3 + h^3)}\xi_{11} = 0,$$
$$\xi_{12}'' + \frac{4w^3 + h^3}{2w(w^3 + h^3)}\xi_{12}' - \frac{15}{4}\frac{w}{(w^3 + h^3)}\xi_{12} = 0. \tag{11}$$

The equations (11) are a Fuchsian and have five regular singularities 0, $-h$, $\frac{h}{2}(1 \pm \sqrt{3}i)$ and ∞. There are three possible ways to investigate (11) the solvability: The first is to apply the Kovacic's algorithm [27] to the first equation (it is not really simple, but it is still an option). The second – is studying the existence of a hyperexponential solution [28,29]. The third one - is to change of variables

$$z^2 := (1 + \frac{w^3}{h^3}),\ \tilde{\xi}_{11} := \frac{\xi_{11}}{(h^3(z^2 - 1))^{1/12}}, \tag{12}$$

we get an easy to investigate equation (for simplicity, we have omitted the tilde).

$$\frac{d^2\xi_{11}}{dz^2} - \frac{2z}{1 - z^2}\frac{d\xi_{11}}{dz} + \left(\frac{2A - 5C}{36C} - \frac{1/6}{1 - z^2}\right)\xi_{11} = 0. \tag{13}$$

This change is a finite branched covering map $\mathbb{CP}^1 \to \mathbb{CP}^1$. In general, the differential Galois group is changed under such transformation, but the identity component remains unchanged (see [18], p. 28).

The equation (13) is associated Legendre equation with $p = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{4C + 2A}{9C}}$ and $q = \frac{1}{6}$. We note $\tau = \pm\sqrt{\frac{2A+4C}{9C}}$ $(C \neq 0)$ and apply the results of Proposition 1 and Theorem 1:

Proposition 2. *The system (3) is non-integrable for* $p = -\frac{1}{2} + \frac{1}{2}\tau \notin \mathbb{Q}$.

Proposition 3. *For* $p \in \mathbb{Q}$, *the system (3) is non-integrable for* $p = -\frac{1}{2} + \frac{1}{2}\tau \neq \pm(k - \frac{1}{6})$, $k \in \mathbb{Z}$.

Now we need to go through the case $p = -\frac{1}{2} + \frac{1}{2}\tau = \pm(k - \frac{1}{6})$, $k \in \mathbb{Z}$, (which is equivalent to $\tau = -2k + \frac{4}{3}$, and $\tau = 2k + \frac{2}{3}$ for $k \in \mathbb{Z}$) to obtain additional non-integrability conditions.

4. Additional Cases

In this section we are considering cases $\tau = -2k + \frac{4}{3}$, and $\tau = 2k + \frac{2}{3}$ for $k \in \mathbb{Z}$ that complete the research from the previous section.

Let us find the second variations of the Hamiltonian system with Hamiltonian (2). We denote it with

$$\begin{aligned} r &= \varepsilon\xi_{11} + \varepsilon^2\xi_{21} + \ldots, \\ z &= z(t) + \varepsilon\xi_{12} + \varepsilon^2\xi_{22} + \ldots, \\ p_r &= \varepsilon\eta_{11} + \varepsilon^2\eta_{21} + \ldots, \\ p_z &= \dot{z}(t) + \varepsilon\eta_{12} + \varepsilon^2\eta_{22} + \ldots, \end{aligned}$$

here $(p_r, r, p_z, z) = (0, 0, \dot{z}(t), z(t))$ is an invariant manifold of the system (3).

We do the replacement in the system (3) and we compare the coefficients at ε^2. After changing the variables $t \to z(t) \to w(t)$ we obtain

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$$\begin{aligned}\xi_{21}'' &+ \frac{4w^3+h^3}{2w(w^3+h^3)}\xi_{21}' - \left(\frac{9\tau^2-4}{8}\right)\frac{w}{(w^3+h^3)}\xi_{21} = K_2^{(1)} \\ \xi_{22}'' &+ \frac{4w^3+h^3}{2w(w^3+h^3)}\xi_{22}' - \frac{15}{4}\frac{w}{(w^3+h^3)}\xi_{22} = K_2^{(2)},\end{aligned} \tag{14}$$

and we have

$$K_2^{(1)} = \left(\frac{9\tau^2-4}{2}\right)\frac{w^{1/2}\xi_{11}\xi_{12}}{(w^3+h^3)} - \left(\frac{3D}{8C}\right)\frac{w^{1/2}(\xi_{11})^2}{(w^3+h^3)},$$

$$K_2^{(2)} = \left(\left(\frac{9\tau^2-4}{4}\right)\frac{w^{1/2}(\xi_{11})^2}{(w^3+h^3)} + \frac{15}{2}\frac{w^{1/2}(\xi_{12})^2}{(w^3+h^3)}\right).$$

We will focus on the singular point ∞. To this end, we change variables $w = \frac{1}{x}$ into (11) and (14) (here we assume $' = \frac{d}{dx}$). We have

$$\begin{aligned}\xi_{11}'' &+ \frac{3h^3x^2}{2(h^3x^3+1)}\xi_{11}' - \left(\frac{9\tau^2-4}{16}\right)\frac{\xi_{11}}{x^2(h^3x^3+1)} = 0, \\ \xi_{12}'' &+ \frac{3h^3x^2}{2(h^3x^3+1)}\xi_{12}' - \frac{15}{4}\frac{\xi_{12}}{x^2(h^3x^3+1)} = 0,\end{aligned} \tag{15}$$

$$\begin{aligned}\xi_{21}'' &+ \frac{3h^3x^2}{2(h^3x^3+1)}\xi_{21}' - \left(\frac{9\tau^2-4}{16}\right)\frac{\xi_{21}}{x^2(h^3x^3+1)} \\ &= \left(\frac{9\tau^2-4}{2}\right)\frac{\xi_{11}\xi_{12}}{x^{3/2}(h^3x^3+1)} + \left(\frac{3D}{8C}\right)\frac{(\xi_{11})^2}{x^{3/2}(h^3x^3+1)}, \\ \xi_{22}'' &+ \frac{3h^3x^2}{2(h^3x^3+1)}\xi_{22}' + \frac{15}{4}\frac{\xi_{22}}{x^2(h^3x^3+1)} \\ &= \left(\frac{9\tau^2-4}{4}\right)\frac{(\xi_{11})^2}{x^{3/2}(h^3x^3+1)} + \frac{15}{2}\frac{(\xi_{12})^2}{x^{3/2}(h^3x^3+1)}.\end{aligned} \tag{16}$$

We would like to underline some known facts we will need for our further research. Solutions should be provided for the equations (15) or (16) where Wronsky determinant is a constant. This happens exactly when the coefficient in front of the first derivative in the differential equation disappears. This can be achieved with the following standard manipulation. We have

$$\xi(x)'' + a(x)\xi(x)' + b(x)\xi(x) = K_2(x),$$

and let $\xi(x) = \zeta(x)e^{-\frac{1}{2}\int a(x)dx}$, then we obtain equation in normal form

$$\zeta(x)'' - r(x)\zeta = -K_2(x)e^{\frac{1}{2}\int a(x)dx},$$

where $r(x) = \frac{1}{2}a(x)' + \frac{1}{4}(a(x))^2 - b(x)$.

Now we change the form of (11) and (14). Let us replace ξ with ζ by

$$\xi = \zeta e^{-\frac{1}{2}\int \frac{3h^3x^2}{2(h^3x^3+1)}dx} = \zeta.(h^3x^3+1)^{-\frac{1}{4}},$$

and we obtain

$$\begin{aligned} \zeta_{11}'' - r_1(x)\zeta_{11} &= 0, \\ \zeta_{12}'' - r_2(x)\zeta_{12} &= 0, \end{aligned} \tag{17}$$

$$\begin{aligned} \zeta_{21}'' - r_1(x)\zeta_{21} &= \tilde{K_2}^{(1)}, \\ \zeta_{22}'' - r_2(x)\zeta_{22} &= \tilde{K_2}^{(2)}, \end{aligned} \tag{18}$$

here

$$\begin{aligned} r_1(x) &= \frac{-h^6x^6 + h^3(9\tau^2+20)x^3 + 9\tau^2 - 4}{16x^2(h^3x^3+1)^2}, \\ r_2(x) &= \frac{-h^6x^6 + 84h^3x^3 + 60}{16x^2(h^3x^3+1)^2}. \end{aligned}$$

We also need to give the solutions of (17) in series near 0.

$$\begin{aligned} \zeta_{11}^{(1)}(x) &= x^{\frac{1}{2}+\frac{3\tau}{4}}\left(1 - \frac{(9\tau^2-28)h^3}{144+72\tau}x^3 + \dots\right), \\ \zeta_{11}^{(2)}(x) &= x^{\frac{1}{2}-\frac{3\tau}{4}}\left(1 + \frac{(9\tau^2-28)h^3}{-144+72\tau}x^3 + \dots\right), \end{aligned} \tag{19}$$

$$\begin{aligned} \zeta_{12}^{(1)}(x) &= x^{\frac{5}{2}}\left(1 - \frac{3h^3}{28}x^3 + \dots\right), \\ \zeta_{12}^{(2)}(x) &= x^{-\frac{3}{2}}\left(-144 - 108h^3x^3 + \dots\right). \end{aligned} \tag{20}$$

Now we have

$$\begin{aligned} \tilde{K_2}^{(1)}(\zeta_{11}, \zeta_{12}) &= -K_2^{(1)}(h^3x^3+1)^{1/4} \\ &= \left(-\frac{(9\tau^2-4)}{2}\frac{\zeta_{11}\zeta_{12}}{x^{3/2}(h^3x^3+1)^{3/4}} - \left(\frac{3D}{8C}\right)\frac{(\zeta_{11})^2}{x^{3/2}(h^3x^3+1)^{3/4}}\right) \\ \tilde{K_2}^{(2)}(\zeta_{11}, \zeta_{12}) &= -K_2^{(2)}(h^3x^3+1)^{1/4} \\ &= \left(-\frac{(9\tau^2-4)}{2}\frac{(\zeta_{11})^2}{x^{3/2}(h^3x^3+1)^{3/4}} - \frac{15}{2}\frac{(\zeta_{12})^2}{x^{3/2}(h^3x^3+1)^{3/4}}\right). \end{aligned}$$

Without loosing a community, we can assume that $\zeta_{11}^{(1)}(\zeta_{11}^{(2)})' - \zeta_{11}^{(2)}(\zeta_{11}^{(1)})' = 1$ and $\zeta_{12}^{(1)}(\zeta_{12}^{(2)})' - \zeta_{12}^{(2)}(\zeta_{12}^{(1)})' = 1$. Then the fundamental matrix of (17) and its inverse are

$$X(z) = \begin{pmatrix} \zeta_{11}^{(1)} & \zeta_{11}^{(2)} & 0 & 0 \\ (\zeta_{11}^{(1)})' & (\zeta_{11}^{(2)})' & 0 & 0 \\ 0 & 0 & \zeta_{12}^{(1)} & \zeta_{12}^{(2)} \\ 0 & 0 & (\zeta_{12}^{(1)})' & (\zeta_{12}^{(2)})' \end{pmatrix}, \tag{21}$$

$$X^{-1}(z) = \begin{pmatrix} (\zeta_{11}^{(2)})' & -\zeta_{11}^{(2)} & 0 & 0 \\ -(\zeta_{11}^{(1)})' & \zeta_{11}^{(1)} & 0 & 0 \\ 0 & 0 & (\zeta_{12}^{(2)})' & -\zeta_{12}^{(2)} \\ 0 & 0 & -(\zeta_{12}^{(1)})' & \zeta_{12}^{(2)} \end{pmatrix}. \tag{22}$$

We show that a logarithmic term appears in local solution of (VE_2). For this purpose, it is sufficient to show that at least one component of $X^{-1}f_2$ has a nonzero residue at 0. We are calculating of $X^{-1}f_2$, which looks like

$$(-\zeta_{11}^{(2)}\tilde{K_2}^{(1)}, \zeta_{11}^{(1)}\tilde{K_2}^{(1)}, -\zeta_{12}^{(2)}\tilde{K_2}^{(2)}, -\zeta_{12}^{(1)}\tilde{K_2}^{(2)})^T.$$

 doi:10.20944/preprints202511.0820.v1

We have $\tau \in \{-2k + 4/3, 2k + 2/3\}$ for $k \in \mathbb{Z}$. The presence of a non-zero residue in the above expression is possible in the cases;

1) $\zeta_{11}^{(2)} \widetilde{K_2^{(1)}}(\zeta_{11}^{(1)}, \zeta_{12}^{(1)})$, $\zeta_{11}^{(2)} \widetilde{K_2^{(1)}}(\zeta_{11}^{(1)}, \zeta_{12}^{(2)})$ and $\zeta_{11}^{(1)} \widetilde{K_2^{(1)}}(\zeta_{11}^{(2)}, \zeta_{12}^{(1)})$, then we obtain conditions for non-integrability k is odd and $D \neq 0$;

2) $\zeta_{11}^{(1)} \widetilde{K_2^{(1)}}(\zeta_{11}^{(1)}, \zeta_{12}^{(2)})$, then we have k is even and $k \neq 0$;

3) $\zeta_{11}^{(2)} \widetilde{K_2^{(1)}}(\zeta_{11}^{(2)}, \zeta_{12}^{(2)})$, then $k \in \mathbb{Z} \setminus \{1\}$.

For sub-cases $k = 0$ we obtain non zero residue in the case $\zeta_{11}^{(1)} \widetilde{K_2^{(2)}}(\zeta_{11}^{(2)}, \zeta_{12}^{(1)})$, and for $k = 1$ we have the condition $D \neq 0$.

We proved the following

Theorem 2. *Let* $\tau = \pm\sqrt{\frac{2A+4C}{9C}}$ *($C \neq 0$), then the system (3) is non integrable if at least one of the conditions is available:*

(i) $\tau \notin \mathbb{Q}$;

(ii) $\tau \in \mathbb{Q} \setminus \{-2k + 4/3, 2k + 2/3\}$, *for* $k \in \mathbb{Z}$;

(iii) $\tau \in \{-2k + 4/3, 2k + 2/3\}$, *for* $k \in \mathbb{Z} \setminus \{1\}$;

(iiii) for $\tau = -2/3$ *($k = 1$) and* $D \neq 0$.

If we use the homogeneity of the potential V_6, and we perform the transformation $z \leftrightarrow r$, we may get a similar result. Since the coefficient in front of r^6 is previously assumed to be 1, the non-integrability condition reduces to $D \neq 0$. Thus we proved the main theorem in this work:

Theorem 3. *Let the condition* $D \neq 0$ *is held for the potential* V_6*, then the Hamiltonian system (3) is non-integrable.*

Examples of integrable homogeneous potentials of degree 6 can be found in [13]. There, the potential $V_6 = r^6 + 80r^2z^4 + 24r^4z^2 + 64z^6$ has been taken into account. From the presentation in [2], we can come to the conclusion that $V_6 = r^6 + 3r^2z^4 + 3r^4z^2 + z^6$ is integrable. In both examples, $D = 0$ is satisfied.

Author Contributions: Conceptualization, D.N. and G.G.; methodology, G.G.; software, G.G.; validation, D.N. and G.G.; formal analysis, D.N. and G.G.; investigation, G.G. and D.N.; writing—original draft preparation, D.N. and G.G.; writing—review and editing, G.G. and D.N.; visualization, G.G.; supervision, G.G.; project administration, G.G. All authors have read and agreed to the published version of the manuscript.

Funding: The authors declared that no funding was received for this article.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work was partially supported with grant 80-10-10 / 21.5.2025 of the Sofia University Science Foundation.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Differential Galois Theory

Here we are introducing some classical and more advanced results that we used for our research.

We say that a system with n degree of freedom is integrable in the sense of Liouville, if it has a complete set of n independent first integrals $f_1 = H, f_2, \ldots f_n$ in involution (the Poison brackets $\{f_i, f_j\}$ have vanished for $i \neq j$). We would like to remind some facts about integrability of Hamiltonian systems in the complex domain, the Ziglin–Morales–Ramis theory and its relations to differential Galois groups of linear equations. We will go through [18–20,30] .

We assume that a Hamiltonian system

$$\dot{x} = X_H(x), \quad t \in \mathbb{C}, \quad x \in M \tag{A1}$$

corresponds to an analytic Hamiltonian H, defined on the complex $2n$-dimensional manifold M. If we suppose the system (A1) has a non-equilibrium solution $\Psi(t)$, we denote by Γ its phase curve. We can write the equation in variation (VE) near this solution

$$\dot{\xi} = DX_H(\Psi(t))_\xi, \quad \xi \in T_\Gamma M. \tag{A2}$$

Further, we consider the normal bundle of Γ, $F := T_\Gamma M/TM$ and let $\pi : T_\Gamma M \to F$ be the natural projection. The equation (A2) leads to an equation on F

$$\dot{\eta} = \pi_*(DX_H(\Psi(t))(\pi^{-1}\eta), \quad \eta \in F. \tag{A3}$$

which is called a normal variational equation (NVE) around Γ. The (NVE) (A3) recognizes a first integral dH, linear on the fibers of F. The level set $F_r := \{\eta \in F|dH(\eta) = r\}, r \in \mathbb{C}$, is $(2n-2)$-dimensional affine bundle over Γ. We will call F_r the reduced phase space of (A3) and the restriction of the (NVE) on F_r is called the reduced normal variational equation.

Then the main result of the Morales–Ramis [18] theory is:

Theorem A1. *Let's assume that the Hamiltonian system (A1) has n meromorphic first integrals in involution, then the identity component G^0 of the Galois group of the variational equation is abelian.*

Next we consider a linear system

$$y' = A(x)y, \quad y \in \mathbb{C}^n, \tag{A4}$$

or linear homogeneous differential equation, which is essentially the same

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0, \tag{A5}$$

with $x \in \mathbb{CP}^1$ and $A \in \mathrm{gl}(n, \mathbb{C}(x))$, $(a_j(x) \in \mathbb{C}(x))$. Let $S := \{x_1, \ldots, x_s\}$ be the set of singular points of (A4) (or (A5)) and let $Y(x)$ be a fundamental solution of (A4) (or (A5)) at $x_0 \in \mathbb{C} \setminus S$. According to existence theorem this solution is analytic near of x_0. The continuation of $Y(x)$ along a nontrivial loop on $\mathbb{CP}^1$ defines a linear automorphism of the space of the solutions, called the monodromy. Analytically this transformation can be presented as a follows: the linear automorphism Δ_γ, associated with a loop $\gamma \in \pi_1(\mathbb{CP}^1 \setminus S, x_0)$ corresponds to multiplication of $Y(x)$ from the right by a constant matrix M_γ, called monodromy matrix

$$\Delta_\gamma Y(x) = Y(x)M_\gamma.$$

The set of these matrices forms the monodromy group.

We add another object to the (A4) (or (A5)) - a differential Galois group. We have a differential field K, that is a field with a derivation $\partial ='$, i.e. an additive mapping complying with the derivation rule. Differential automorphism of K is an automorphism commuting with the derivation.

The coefficient field in (A4) (and (A5)) is $K = \mathbb{C}(x)$. Let y_{ij} be elements of the fundamental matrix $Y(x)$. Let $L(y_{ij})$ be the extension of K generated by K and y_{ij} – a differential field. This extension is called a Picard–Vessiot's extension. Similarly to classical Galois Theory we define the Galois group $G := Gal_K(L) = Gal(L/K)$ to be the group of all differential automorphisms of L leaving the elements of K fixed. Galois group is an algebraic group. It has an unique connected component G^0 which contains the identity and is a normal subgroup of finite index. Galois group G can be represented as an algebraic linear subgroup of $\mathrm{GL}(n, \mathbb{C})$ by

$$\sigma(Y(x)) = Y(x)R_\sigma,$$

where $\sigma \in G$ and $R_\sigma \in \mathrm{GL}(n, \mathbb{C})$.

We can do the same locally at $a \in \mathbb{CP}^1$, replacing $\mathbb{C}(x)$ by the field of germs of meromorphic functions at a. Hence we can speak of a local differential Galois group G_a of (A4) at $a \in \mathbb{CP}^1$, defined in the same way for Picard-Vessiot extensions of the field $\mathbb{C}\{x-a\}[(x-a)^{-1}]$.

It is worth here that by its definition the monodromy group is contained in the differential Galois group of the corresponding system.

Next, we are presenting some facts from the theory of linear systems with singularities. We call a singular point x_i regular if any of the solutions of (A4) (or of (A5)) has at most polynomial growth in arbitrary sector with a vertex at x_i. Otherwise the singular point is called irregular.

We say that the system (A4) has a singularity of the Fuchs type at x_i if $A(x)$ has a simple pole at $x = x_i$. For the equation (A5) the Fuchs type singularity at x_i means that the functions $(x-x_i)^j a_j(x)$ are holomorphic in a neighborhood of x_i.

If the system (A4) has a singularity of the Fuchs type, then this singularity is regular. The opposite is not true. However, for the equation (A5) the regular singularities coincide with the singularities of Fuchs type.

A system with only regular singularities is called Fuchsian system. For such systems we have :

Theorem A2. (**Schlesinger**) *The differential Galois group coincides with the Zariski closure in* $\mathrm{GL}(n, \mathbb{C})$ *of the monodromy group.*

The fact that G^0 is abelian doesn't imply necessarily integrability of the Hamiltonian system. There is a method which, in the case of abelian Galois group, can draw conclusion when the system (A1) is non-integrable. This method based on the higher variational equations has been introduced in [18] and the Theorem A1 has been extended in [20]. What is the idea of higher variational equations? For the system (A2) with a particular solution $\Psi(t)$ we put

$$x = \Psi(t) + \varepsilon\xi^{(1)} + \varepsilon^2\xi^{(2)} + \ldots + \varepsilon^k\xi^{(k)} + \ldots, \tag{A6}$$

where ε is a small parameter. When substituting the above expression into eq. (A2) and comparing terms with the same order in ε we obtain the following chain of linear non-homogeneous equations

$$\dot{\xi}^{(k)} = A(t)\xi^{(k)} + f_k(\xi^{(1)}, \ldots, \xi^{(k-1)}), \quad k = 1, 2, \ldots, \tag{A7}$$

where $A(t) = DX_H(\Psi(t))$ and $f_1 \equiv 0$. The equation (A7) is called k-th variational equation (VE_k). Let $X(t)$ be the fundamental matrix of (VE_1)

$$\dot{X} = A(t)X.$$

Then the solutions of $(\mathrm{VE}_k), k > 1$ can be can be expressed with

$$\xi^{(k)} = X(t)c(t), \tag{A8}$$

where $c(t)$ is a solution of

$$\dot{c} = X^{-1}(t)f_k. \tag{A9}$$

Although (VE_k) are not actually homogeneous equations, they can be placed in this framework, and therefore, successive extensions $K \subset L_1 \subset L_2 \subset \ldots \subset L_k$ can be outlined, where L_k is the extension obtained by adjoining the solutions of (VE_k). The differential Galois groups $Gal(L_1/K), \dots, Gal(L_k/K)$ can be defined accordingly. The following result is proven in [20].

Theorem A3. (**Morales-Ruiz, Ramis, Simó**) *If the Hamiltonian system (A2) is integrable in Liouville sense then the identity component of each Galois group $Gal(L_k/K)$ is abelian.*

Pay attention that we apply Theorem A3 to the situation where the identity component of the Galois group $Gal(L_1/K)$ is abelian. This means that the first variational equation is solvable. Once we have the solution of (VE_1), then the solutions of (VE_k) can be found by the method of constant variations as explained above. Hence, the differential Galois groups $Gal(L_k/K)$ are solvable. One possible way to show that some of them are not commutative is to find a logarithmic term in the corresponding solution. We need to explain why the existence of a non-zero logarithmic term in VE_k around some singular point guarantees us non-integrability. The Galois group $Gal(L_k/K)$ is abelian, if and only if, the local monodromy of the (VE_k) around the singular point of the coefficients is identity. If for some k, we obtain non-zero residue in the Laurent expansions of the expressions of $X^{-1}(t)f_k$, near singularity point, then the local monodromy will be represented by a lower (or upper) triangular matrix which is not an identity, i. e. the Galois group $Gal(L_k/K)$ is not abelian (see detailed descriptions and explanations in [18–20]).

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